PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 9, Pages 2543–2547
S 0002-9939(05)08011-1
Article electronically published on April 8, 2005

ON FINITE GROUPS ADMITTING A SPECIAL NONCOPRIME ACTION

GÜLİN ERCAN

(Communicated by Jonathan I. Hall)

Abstract. An important result of Turull (1984) is the following:

Let GA be a finite solvable group, $G \lhd GA$ and $(|G|, |A|) = 1$. Then $f(G) \leq f(C_G(A)) + 2\ell(A)$, where f denotes the Fitting height and ℓ denotes the composition length.

The purpose of this work is to give a treatment of the minimal configuration in this framework with additional conditions, yet without the coprimeness condition.

Here we will prove (see Theorem 2) the following:

Let G be a finite solvable group and let α be an automorphism of G of order p for some prime p. Assume that the orders of elements of $H = G\langle\alpha\rangle$ lying outside of G are not divisible by p^2. If $C_{\mathcal{S}}(x)$ is nilpotent for any $x \in H - G$ of order p and for any x-invariant section $\mathcal{S}$ of G, then $f(G) \leq 3$. Furthermore, if the nilpotency condition is replaced by abelianness, then $f(G) \leq 2$.

An immediate consequence of this theorem is a particular case of Turull's result (see also [1] and [4]):

Let G be a finite solvable group and let α be an automorphism of G of order p for some prime p where $(|G|, |\alpha|) = 1$. If $C_G(\alpha)$ is nilpotent, then $f(G) \leq 3$. Furthermore if $C_G(\alpha)$ is abelian, then $f(G) \leq 2$.

Although our main purpose is the proof of Theorem 2, and Theorem 1 below makes its appearence as an auxiliary, it should be pointed out that Theorem 1 is of independent interest, too. Theorem 1 is, in its turn, a generalization of the following Lemma.

Lemma ([3, Lemma 1]). *Let $G = ST$ be a group where $S \lhd G$, S is a p-group and T is a t-group for distinct primes p and t, and let α be an automorphism of G of order p^n which leaves T invariant. Assume that $C_{T/T_0}(z) = 1$, where $T_0 = C_T(S)$ and $z = \alpha^{p^{n-1}}$. Let V be a $kG\langle\alpha\rangle$-module on which S acts faithfully and k is a field of characteristic different from p. If $[C_V(z), C_S(z)] = 1$, then $[S, T] = 1$.*

Theorem 1. *Let $\langle\alpha\rangle$ be a cyclic group of order p^n for some prime p, and let G be a group acted on by $\langle\alpha\rangle$. Suppose that $S \lhd G\langle\alpha\rangle$ is an s-group and T is an $\langle\alpha\rangle$-invariant t-subgroup of G for distinct primes s and t, such that $[S, T] \neq 1$. Let V be a $kG\langle\alpha\rangle$-module on which S acts faithfully, where k is a field of characteristic*

Received by the editors May 17, 2004.
2000 *Mathematics Subject Classification.* Primary 20D10, 20F28.
Key words and phrases. Noncoprime action, fitting height.

not dividing s. *Let* $z=\alpha^{p^{n-1}}$. *Then either* $[C_V(z),C_S(z)]\neq 1$
or $[C_V(z),C_T(z)]\neq 1$
or $[C_S(x),C_{T/T_0}(x)]\neq 1$ *for some* $\overline{x}\in(T/T_0)\langle\alpha\rangle-(T/T_0)$ *of order* p, *where* $T_0=C_T(S)$.

Proof. Set $H=G\langle\alpha\rangle$ and use induction on $|H|+\dim_k V$. We may assume that $n=1$ and $G=ST$.

(1) $\Phi(T/T_0)=1$ *and* $\langle\alpha\rangle$ *acts irreducibly on* T/T_0.

This is an immediate consequence of induction argument applied to $ST_1\langle\alpha\rangle$ on V for a minimal $\langle\alpha\rangle$-invariant subgroup T_1/T_0 of T/T_0.

(2) $t\neq p$.

Assume the contrary. Then T/T_0 is centralized by any $1\neq\overline{x}\in(T/T_0)\langle\alpha\rangle-(T/T_0)$. Let U be an irreducible $T\langle\alpha\rangle$-submodule of $S/\Phi(S)$ on which T acts nontrivially, and let $\overline{t}\in T/T_0$ such that $[U,\overline{t}]\neq 1$. Then $C_U(\overline{t})=1$. This yields a contradiction as $U=\langle C_U(a)|1\neq a\in\langle\overline{t},\alpha\rangle\rangle$ by ([6, 5.3.16]) and $[C_U(\overline{x}),T/T_0]=1$ for any $1\neq\overline{x}\in\langle\overline{t},\alpha\rangle-\langle\overline{t}\rangle$.

(3) $S/\Phi(S)$ *is an irreducible* $T\langle\alpha\rangle$*-module with* $[S,T]=S$, $[\Phi(S),T]=1$ *and* S *is special.*

Let S_1 be a normal subgroup of H properly contained in S on which T acts nontrivially. Put $T_1=C_T(S_1)$. By induction, there exists $\overline{x}\in(T/T_1)\langle\alpha\rangle-(T/T_1)$ such that $[C_{S_1}(x),C_{T/T_1}(x)]\neq 1$. As $t\neq p$, this yields that $[C_{S_1}(x),C_{T/T_0}(x)]\neq 1$ which is not the case. Thus $T\langle\alpha\rangle$ acts irreducibly on $S/\Phi(S)$, $[S,T]=S$, $[\Phi(S),T]=1$ and S is special.

(4) $[T,\alpha]=1$.

Assume the contrary. Then $C_{T/T_0}(\alpha)=1$ and so $C_{S/\Phi(S)}(\alpha)\neq 1$. Now $s\neq p$, because otherwise $[C_V(\alpha),C_S(\alpha)]\neq 1$ by the Lemma.

Let M be an irreducible $ST\langle\alpha\rangle$-submodule of V on which S acts nontrivially. Then $[M,S]=M$ and so $[M,T]\neq 1$. Set $\overline{S}=S/C_S(M)$. By Clifford's theorem aplied to $\overline{S}T$ on M, we have that $M=W_1\oplus\cdots\oplus W_r$, where the W_i's are homogeneous $\overline{S}T$-modules. Here $N_{\langle\alpha\rangle}(W_1)=N_{\langle\alpha\rangle}(W_i)$ for each $i=1,\cdots,r$ and so either $r=1$ or $r=p$. If the latter holds, then $[W_i,C_{\overline{S}}(\alpha)]=1$ for each i, because $[C_M(\alpha),C_S(\alpha)]=1$ and $s\neq p$. It follows that $C_S(\alpha)\leq C_S(M)$ and so $C_S(\alpha)\leq\Phi(S)$ which is not the case. Thus M is a homogeneous $\overline{S}T$-module, i.e. $M=M_1\oplus\cdots\oplus M_i$ with $M_i\cong M_1$ irreducible $\overline{S}T$-modules.

If $\overline{S}$ is nonabelian, then $[\Phi(\overline{S}),\alpha]=1$ and so $C_M(\alpha)\leq C_M(\Phi(S))=1$. This shows that char $k\neq p$. Observe that $[\overline{S},\alpha]\neq 1$, because otherwise $[\overline{S},T]=1$ by the three subgroup lemma. By [5] applied to the action of both $[\overline{S},\alpha]\langle\alpha\rangle$ and $T\langle\alpha\rangle$ on M, we conclude that $s=2=t$, which is impossible.

Thus $\overline{S}$ is abelian. The number of homogeneous components of $M_1|_{\overline{S}}$ is a power of t and so the number of homogeneous components of $M|_{\overline{S}}$ is also a power of t. Since $t\neq p$, α fixes a homogeneous component W of $M|_{\overline{S}}$. If U is a homogeneous component of $M|_{\overline{S}}$ which is α-invariant and different from W, then $W=U^a$ for some $a\in T\langle\alpha\rangle$. Now $[a,\alpha]\in N_T(W)$ and so $1\neq C_{T/N_T(W)}(\alpha)\cong C_{(T/T_0)/(N_T(W)/T_0)}(\alpha)$, i.e. $C_{T/T_0}(\alpha)\neq 1$, which is not the case. Thus α fixes exactly one homogeneous component W of $M|_{\overline{S}}$. Observe that either $N_T(W)=T$ or $N_T(W)\leq T_0$. If the first holds, then $[[\overline{S},T],W]=[\overline{S},W]=1$ implying that $C_M(\overline{S})\neq 1$, which is not the case. Hence $N_T(W)\leq T_0$. Also note that $[\overline{S},\alpha]\neq 1$, because otherwise $[\overline{S},T]=1$ by the three subgroup lemma. Now $[[\overline{S},\alpha],W]=1$, and so there exists a homogeneous component U of $M|_{\overline{S}}$ such that $U \neq U^\alpha$. Here note that

$$C_{\overline{S}}(\alpha) \leq \mathrm{Ker}(\overline{S} \text{ on } U),$$

as $[C_U(\alpha), C_S(\alpha)] = 1$, and so

$$C_{\overline{S}}(\alpha) \cap \mathrm{Ker}(\overline{S} \text{ on } W) \leq \mathrm{Ker}(\overline{S} \text{ on } M) = 1.$$

Then $C_{\overline{S}}(\alpha) \cap \langle C_{\overline{S}}(\alpha)^{\overline{t}} | 1 \neq \overline{t} \in \overline{T}\rangle = 1$, where $\overline{T} = T/C_T(M)$ and so $C_{\overline{S}}(\alpha)^{\overline{x}} \cap \langle C_{\overline{S}}(\alpha)^{\overline{t}} | \quad \overline{x} \neq \overline{t}\rangle = 1$. Now $\sum_{\overline{t}\in\overline{T}} C_{\overline{S}}(\alpha)^{\overline{t}} = \bigoplus_{\overline{t}\in\overline{T}} C_{\overline{S}}(\alpha)^{\overline{t}} = \overline{S}$ since $\overline{S}$ is an irreducible $T\langle\alpha\rangle$-module. It follows that $|\overline{S}| = |C_{\overline{S}}(\alpha)|^{|\overline{T}|}$. On the other hand $[\overline{S}, [\overline{T}/\overline{T}_0, \alpha]] = \overline{S}$ and so $|\overline{S}| = |C_{\overline{S}}(\alpha)|^p$ by Lemma 4.5 in [7]. As $t \neq p$, we get a contradiction. Therefore $[T/T_0, \alpha] = 1$, i.e. $C_T(\alpha)T_0 = T$. By induction we see that $C_T(\alpha) = T$.

(5) $[S, \alpha] = S$ *and so* $s \neq p$.

$[S, \alpha]$ is either trivial or the whole of S. If it is trivial, then $[S, T] = 1$ as $[C_S(\alpha), C_{T/T_0}(\alpha)] = 1$, a contradiction.

(6) S *is abelian.*

Assume the contrary. Then $1 \neq \Phi(S) = Z(S)$. Let M be an irreducible $ST\langle\alpha\rangle$-submodule of V on which $\Phi(S)$ acts nontrivially. Set $\overline{S} = S/C_S(M)$. We consider $M|_{\overline{S}T} = W_1 \oplus \cdots \oplus W_r$, where W_i's are homogeneous $\overline{S}T$-components of M. If $r = p$, then $[W_i, T] = 1$ for each i, as $[C_M(\alpha), T] = 1$, and so $[M, T] = 1$, which is not the case. Then $r = 1$. It follows that $[\Phi(\overline{S}), \alpha] = 1$ implying that $C_M(\alpha) \leq C_M(\Phi(\overline{S})) = 1$. If $\Phi(\overline{S})$ is not cyclic, then there exists $1 \neq a \in \Phi(\overline{S})$ such that $C_M(a) \neq 1$, by ([6, 5.3.16]), implying that $C_{\overline{S}}(M) \neq 1$, a contradiction. Hence $\Phi(\overline{S})$ is cyclic and so $\overline{S}$ is estraspecial, where $|\overline{S}| = 2^{2n+1}$ and $p = 2^n + 1$ for some $n \geq 1$, by [5].

By [6, 5.5.2], the number of distinct cyclic subgroups of order 4 in $\overline{S}$ is

$$\frac{1}{2}(2^{2n} \mp (-2)^n).$$

Since each cyclic group of order 4 contains two elements of order 4, and distinct cyclic subgroups of order 4 have no element of order 4 in common, there are $2^{2n} \mp (-2)^n = 2^n(2^n \mp 1)$ elements of order 4 in $\overline{S}$. As $T\langle\alpha\rangle$ acts irreducibly on $\overline{S}/\Phi(\overline{S})$ and $[\overline{S}, T] = \overline{S}$, we have $C_{\overline{S}}(T) \leq \Phi(\overline{S})$. It follows that $C_{\overline{S}}(T) = \Phi(\overline{S})$, since $[\Phi(\overline{S}), T] = 1$. Now $\Phi(\overline{S})$ contains no element of order 4, since it is cyclic of order 2. Thus $T\langle\alpha\rangle$ permutes the elements of $\overline{S}$ of order 4, without fixing any, in orbit of length $|(T\langle\alpha\rangle)/(\Phi(T))| = tp$. Therefore tp divides $2^n(2^n \pm 1)$. But as $t \neq s = 2$ and $p = 2^n + 1$, tp divides $2^n + 1 = p$ which yields that $t = 1$, a contradiction.

(7) Finally, let M be an irreducible $ST\langle\alpha\rangle$-submodule of V on which S acts nontrivially. Set $\overline{S} = S/C_S(M)$. Let $\Omega = \{W_1, \cdots, W_r\}$ be the set of all homogeneous $\overline{S}$-components of M. Since $[S, \alpha] = S$, no W_i is α-invariant. Because otherwise as $[S, W_i] = 1$ for each i, we have $C_M(S) \neq 1$, a contradiction.

Let $\mathcal{O} = \{W, W^\alpha, \cdots, W^{\alpha^{p-1}}\}$ be an α-orbit. Set $\overline{T} = T/C_T(M)$ and $X = \bigoplus_{i=0}^{p-1} W^{\alpha^i}$. As $[C_X(\alpha), T\langle\alpha\rangle] = 1$, we have $[W, N_{T\langle\alpha\rangle}(W)] = 1$. Let $t \in T$. If $Y = X^t$, then $C_Y(\alpha) = C_X(\alpha)^t = C_X(\alpha)$ and so $X \cap Y \neq 0$, i.e. $X = Y$. Hence T acts on $\mathcal{O}$ and $\mathcal{O} = \Omega$. This gives that $p = |\Omega| = |T\langle\alpha\rangle : N_{T\langle\alpha\rangle}(W)|$. Then $N_{T\langle\alpha\rangle}(W) = T$ because T is the unique subgroup of $T\langle\alpha\rangle$ of index p. This yields that $[W, T] = 1$ and so $[M, T] = 1$, a contradiction which completes the proof of Theorem 1. □

As a consequence of Theorem 1, we have

Theorem 2. *Let G be a finite solvable group and let α be an automorphism of G of order p for some prime p. Assume that the orders of elements of $H = G\langle\alpha\rangle$ lying outside G are not divisible by p^2. If $C_{\mathcal{S}}(x)$ is nilpotent for any $x \in H - G$ of order p and for any x-invariant section $\mathcal{S}$ of G, then $f(G)$ is at most 3. Furthermore, if the nilpotency condition is replaced by abelianness, then $f(G) \leq 2$.*

Proof. Let $H = G\langle\alpha\rangle$ be a minimal counterexample to the theorem. We may assume that $f(G) = 4$. Then by Lemma 1 in [2] there exist subgroups C_i of G and subgroups $D_i \lhd C_i$ for $i = 1, 2, 3, 4$ and an element $x \in H - G$ of order p such that the following are satisfied:

(i) C_i is a p_i-subgroup for some prime p_i, i.e. $\pi(C_i) = \{p_i\}$ for any i and $p_i \neq p_{i+1}$ for $i = 1, 2, 3$.

(ii) C_i and D_i are $(\prod_{j>i} C_j)$ $\langle\alpha\rangle$-invariant for any i.

(iii) $\overline{C}_i = C_i/D_i$ is a special group on the Frattini factor group of which $(\prod\limits_{j>i} C_j)\langle\alpha\rangle$ acts irreducibly and C_{i+1} acts trivially on $\Phi(\overline{C}_i)$ for any i.

(iv) $[C_i, C_{i+1}] = C_i$ for $i = 1, 2, 3$.

(v) $C_{C_{i+1}}(\overline{C}_i/\Phi(\overline{C}_i)) = C_{C_{i+1}}(\overline{C}_i)$ is contained in $\Phi(C_{i+1} \bmod D_{i+1})$ for $i = 1, 2, 3$.

(vi) $[C_j, C_i]$ is not contained in $\Phi(C_j \bmod D_j)$ for any $i = 2, 3, 4$ and any $1 \leq j < i$.

Put $K = C_1C_2C_3C_4$. Now $K\langle x\rangle$ satisfies the hypothesis of the theorem.

Applying Theorem 1 to the action of $\overline{C}_3C_4\langle x\rangle$ on the Frattini factor group $\tilde{C}_2$ of $\overline{C}_2$ we see that $[C_{\tilde{C}_2}(x), C_{C_4}(x)] \neq 1$ with the requirement $\pi(C_2) = \pi(C_4)$. Also applying Theorem 1 to the action of $\overline{C}_2C_3\langle x\rangle$ on C_1 we see that $[C_{C_1}(x), C_{C_3}(x)] \neq 1$ with the requirement $\pi(C_1) = \pi(C_3)$. Now $D_4 = C_{C_4}(\overline{C}_2)$ and so $C_{C_4}(x) \nleq D_4$, i.e. $[\overline{C}_4, x] = 1$ This forces that $C_{\overline{C}_3}(x) \leq \Phi(\overline{C}_3)$, because otherwise $[\overline{C}_3\overline{C}_4, x] = 1$, which is not the case. Then $C_{\overline{C}_3}(x) \leq Z(\overline{C}_3C_4\langle x\rangle)$ and so $C_{\tilde{C}_2}(C_{\overline{C}_3}(x))$ is either trivial or $\tilde{C}_2$. If it is trivial, then $C_{\tilde{C}_2}(x) = 1$, which is not the case. Hence $C_{\overline{C}_3}(x) = 1$, i.e. $C_{C_3}(x) \leq D_3 = C_{C_3}(C_1)$ as $\pi(C_1) = \pi(C_3)$, a contradiction. This completes the proof of the first claim.

The last claim can be easily shown by an application of Theorem 1 to $C_1C_2C_3\langle x\rangle$, where C_i are subgroups of H and $D_i \lhd C_i$, $i = 1, 2, 3$, satisfying (i)–(vi). ☐

Corollary. *Let G be a finite solvable group and let α be an automorphism of G of order p for some prime p where $(|G|, |\alpha|) = 1$. If $C_G(\alpha)$ is nilpotent, then $f(G) \leq 3$. Furthermore if $C_G(\alpha)$ is abelian, then $f(G) \leq 2$.*

References

1. Asar, A.O.: Automorphism of prime order of soluble groups whose subgroups of fixed points are nilpotent. Journal of Algebra 88, 178-189 (1984). MR0741938 (85k:20069)
2. Ercan G., Güloğlu, İ.: On the Fitting length of $H_n(G)$. Rend. Sem. Mat. Univ. Padova, 89 (1993). MR1229051 (94f:20035)
3. Ercan, G, Güloğlu, İ.: On finite groups admitting a fixed point free automorphism of order *pqr*, J. Group Theory **7** (2004), no. 4, 437–446. MR2080444
4. Feldman, A.: Fitting height of soluble groups admitting an automorphism of prime order with abelian fixed point subgroup, Journal of Algebra 68, 97-108 (1981). MR0604296 (83b:20021)
5. Gagola, S., Jr.: Solvable groups admitting an almost fixed point free automorphism of prime order. Illinois J. Math. 22, 191-207 (1978). MR0473007 (57:12686)

6. Gorenstein, D.: Finite Groups, New York (1968). MR0231903 (38:229)
7. Hartley, B., Turau, V.: Finite solvable groups admitting an automorphism of prime power order with few fixed points. Math. Proc. Camb. Phil. Soc., 431-441 (1987). MR0906617 (88i:20041)
8. Turull, A.: Fitting height of groups and of fixed points. Journal of Algebra 86, 555-566 (1984). MR0732266 (85i:20021)

Department of Mathematics, Middle East Technical University, 06531, Ankara, Turkey