

HOMOMORPHISMS FROM MAPPING CLASS GROUPS

WILLIAM HARVEY AND MUSTAFA KORKMAZ

ABSTRACT. This paper concerns rigidity of the mapping class groups. We show that any homomorphism $\varphi : \text{Mod}_g \rightarrow \text{Mod}_h$ between mapping class groups of closed orientable surfaces with distinct genera $g > h$ is trivial if $g \geq 3$ and has finite image for all $g \geq 1$. Some implications are drawn for more general homomorphisms of these groups.

1. INTRODUCTION

The mapping class groups resemble lattices in higher rank Lie groups in various ways. Along these lines, in parallel with the classic rigidity theorems of Mostow and Margulis, there are results by various authors restricting the existence of nontrivial morphisms between lattices and mapping class groups; for a discussion of this the reader may consult the extensive review article by N.V. Ivanov [Iv].

There are natural inclusion morphisms between mapping class groups of finite-type surfaces stemming from inclusions of bordered surfaces, and similarly one may show that certain ramified coverings of surfaces give rise to inclusions between mapping class groups of closed surfaces. However, the intuition remains that the closed surface mapping class groups are in some definite sense as distinct, for differing values of g , as possible.

A recent result of M.R. Bridson and K. Vogtmann on the analogous question for automorphism groups of free groups shows that there is no nontrivial morphism from the group $\text{Aut } F_n$ into $\text{Aut } F_m$ when $n > m > 1$. In this paper we exploit some well-known facts about Dehn twists, together with a specific twist description of a torsion element of maximal order in the mapping class group Mod_g , to show that this same result holds true for Mod_g . In fact our methods show that only one or two genus-dependent algebraic properties of these groups are needed, so that morphisms into arbitrary groups lacking the relevant properties are automatically trivial.

Date: October 31, 2018.

1991 Mathematics Subject Classification. Primary 57M20; Secondary 20E25, 30F10.

Key words and phrases. Mapping class groups, torsion elements, Dehn twists, Torelli subgroup.

2. BASIC RESULTS ON SUBGROUPS OF MAPPING CLASS GROUPS

We fix a closed oriented surface S of genus g as shown in Figure 1. Let Mod_g denote the mapping class group of S , the group of orientation preserving diffeomorphisms of S considered up to isotopy.

The Torelli group \mathcal{T}_g is, by definition, the kernel of the action of the mapping class group on the first homology group $H_1(S; \mathbb{Z})$ of S . The choice of a canonical basis for $H_1(S; \mathbb{Z})$ gives rise to an epimorphism from the mapping class group Mod_g onto the symplectic group $Sp(2g, \mathbb{Z})$ and one obtains a short exact sequence

$$1 \rightarrow \mathcal{T}_g \rightarrow \text{Mod}_g \xrightarrow{\eta} Sp(2g, \mathbb{Z}) \rightarrow 1.$$

Notationally, the same letter will be used for a diffeomorphism and its isotopy class. Similarly, we make no distinction between isotopic curves in a surface.

We introduce some basic types of mapping classes and state some fundamental facts about them. For a simple closed curve a on S , we denote by t_a the (isotopy class of a) right Dehn twist along a . Two obvious simple properties of these mappings have proved to be crucial in building up more intricate surface mappings and also in recognising relations between them. Firstly, Dehn twists along disjoint loops commute (up to isotopy): this makes it easy to find free abelian subgroups of rank $3g - 3$ in Mod_g , by choosing maximal systems of disjoint homotopically distinct loops which determine a pair of pants decomposition of S . Secondly, if f is any surface homeomorphism, then (again, up to isotopy) $t_{f(a)} = f \circ t_a \circ f^{-1}$.

It is elementary that if a is a separating (i.e. null-homologous) loop then the Dehn twist t_a is contained in the Torelli group \mathcal{T}_g . Furthermore, if a and b are two non-separating simple closed curves whose union bounds a subsurface of S , then $t_a t_b^{-1}$ is also in the group \mathcal{T}_g ; moreover, it follows from Theorem 2 in [J] that if one of the two subsurfaces bounded by the union $a \cup b$ is of genus 1, then \mathcal{T}_g is the normal closure of $t_a t_b^{-1}$ in Mod_g . A precise converse result by Vautaw will be discussed and used in the last section.

We shall also need to focus attention on certain torsion elements of Mod_g . According to the classical Hurwitz-Nielsen Realisation Problem, any finite subgroup of Mod_g should be isomorphic to a group of automorphisms of some compact Riemann surface of genus g , and a theorem of Hurwitz states that the order of such a group is at most $84(g - 1)$. The Realisation Problem was proved by S.P. Kerckhoff [Ke], but the case of finite cyclic groups had been settled much earlier by J. Nielsen and W. Fenchel. This characterisation of the finite cyclic subgroups of the mapping class group by surface mappings leads to a bound on their order. We state the basic facts we shall need about torsion subgroups below [W, H, H2]. For more details on automorphisms of Riemann surfaces, the reader might consult [B, FK].

Theorem 1.

(a) *The order of any finite subgroup of Mod_g is at most $84(g - 1)$ if $g \geq 2$.*

- (b) The order of a finite cyclic subgroup of Mod_g is at most $4g + 2$. This bound is achieved in every genus.
- (c) If Mod_g has an element of prime order p , then either $p \leq g + 1$ or $p = 2g + 1$.
- (d) There is no element of order $4g + 1$ in Mod_g .

Certain geometrically defined torsion elements, including a specific element of maximal order in Mod_g , first documented by Wiman [W] in 1895, will feature prominently in the sequel.

3. HOMOMORPHISMS BETWEEN MAPPING CLASS GROUPS

As background reference and source for unexplained facts, we refer the reader to Ivanov’s survey paper [Iv].

Consider the system of simple closed curves (loops) $a_0, a_1, a_2, \dots, a_{2g+1}$ on S as shown in Figure 1 and write t_j for the twist along the curve a_j . We let $\delta = t_1 t_2 \cdots t_{2g}$. Note that for the composition of two functions α and β , we use the following notation: $\alpha\beta$ means $\alpha \circ \beta$, so that β is applied first.

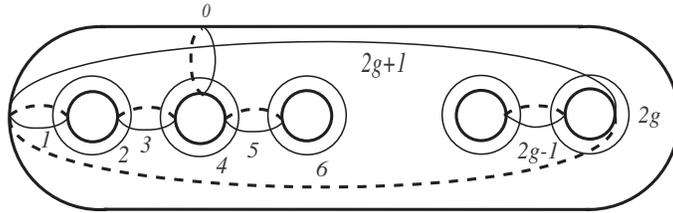


FIGURE 1. A system of simple loops in S .
The curve labelled i denotes a_i .

Lemma 2. *The order of δ in Mod_g is $4g + 2$.*

Proof. Using the elementary properties of twist maps mentioned above, it is easy to verify that $\delta(a_i) = a_{i+1}$ for $1 \leq i \leq 2g$ and $\delta(a_{2g+1}) = a_1^{-1}$: recall that equality for closed loops in S means they are freely homotopic. It follows that the order of δ is at least $4g + 2$. On the other hand, from Theorem 1 we know that the order is at most $4g + 2$. The fact that $\delta^{4g+2} = 1$ can also be proved by a more elaborate direct computation with twists. \square

Lemma 3. *Let $g \geq 1$ and let a and b be two non-separating simple closed curves on S . Let N be the normal closure of $t_a^{-1}t_b$ in Mod_g . Suppose either that a intersects b transversely at one point, or that a is disjoint from b and the complement of $a \cup b$ in S is connected. Then N is the commutator subgroup $[\text{Mod}_g, \text{Mod}_g]$. In particular, $N = \text{Mod}_g$ for $g \geq 3$.*

Proof. The proof follows similar lines to a result of McCarthy and Papadopoulos on involutions in Mod_g , [MP]. It is well-known that for $g \geq 2$ the mapping class group Mod_g is generated by the Dehn twists t_0, t_1, \dots, t_{2g} about the curves a_0, a_1, \dots, a_{2g} of Figure 1. When $g = 1$, the curve a_0 does not exist, and Mod_1 is generated by the two twists t_1, t_2 . We denote by Γ the quotient group Mod_g/N .

Suppose first that a intersects b at one point. Clearly, for each $i = 1, 2, \dots, 2g - 1$ there is a diffeomorphism $f_i : S \rightarrow S$ such that $f_i(a) = a_i$ and $f_i(b) = a_{i+1}$. Hence, since $t_a^{-1}t_b$ is in N , so is the element

$$t_i^{-1}t_{i+1} = t_{f_i(a)}^{-1}t_{f_i(b)} = f_i t_a^{-1} t_b f_i^{-1}.$$

Similarly, one sees that $t_0^{-1}t_4$ belongs to N . Consequently, all generators of Mod_g represent the same element in Γ , which implies that the group Γ is cyclic, hence abelian. Therefore, N contains $[\text{Mod}_g, \text{Mod}_g]$. On the other hand, since both a and b are non-separating, there exists a diffeomorphism $k : S \rightarrow S$ such that $k(a) = b$. Then $t_a^{-1}t_b = t_a^{-1}k t_a k^{-1}$ is a commutator, which shows that N is contained in $[\text{Mod}_g, \text{Mod}_g]$. Thus we have $N = [\text{Mod}_g, \text{Mod}_g]$.

Suppose next that a is disjoint from b and that the complement of $a \cup b$ is connected. In this case, g must be at least 2, of course. By cutting S open along b , it is easy to see that there exists a non-separating simple loop c on S such that c intersects a transversely at one point, c is disjoint from b and the complement of $b \cup c$ is connected. By the classification of surfaces, there is a diffeomorphism $f : S \rightarrow S$ such that $f(a) = b$ and $f(b) = c$. From this, it follows that

$$t_a^{-1}t_c = t_a^{-1}t_b t_b^{-1}t_c = t_a^{-1}t_b f t_a^{-1} t_b f^{-1}$$

is contained in N . Now the conclusion of the lemma follows from the first case.

For the final statement, we have $g \geq 3$, so that the first homology group $H_1(\text{Mod}_g; \mathbb{Z})$ of Mod_g is trivial by Powell's theorem. This means that $[\text{Mod}_g, \text{Mod}_g]$, which coincides with N , is equal to Mod_g . \square

Our first theorem determines the normal closure of each power of δ in Mod_g .

Theorem 4. (a) *Let k be an integer with $1 \leq k \leq 2g$. The normal subgroup of Mod_g generated by δ^k is the full group Mod_g if $g \geq 3$. It has index 2 if $g = 2$ or $(g, k) = (1, 1)$ and index 4 if $(g, k) = (1, 2)$.*

(b) *If $g \geq 3$, then the normal subgroup of Mod_g generated by δ^{2g+1} contains the Torelli group \mathcal{T}_g as a subgroup of index 2. In fact it is the kernel of the natural morphism $\psi : \text{Mod}_g \rightarrow \text{PSP}(2g, \mathbb{Z})$, given by $\psi = P \circ \eta$ where $P : \text{Sp} \rightarrow \text{PSP} = \text{Sp}/\{\pm I\}$.*

Proof. For each positive integer $k \leq 2g$, let N_k denote the normal subgroup of Mod_g generated by δ^k . We prove first that each N_k contains the commutator subgroup of Mod_g .

We observe that for any k the element

$$t_1^{-1} \delta^k t_1 \delta^{-k} = t_1^{-1} t_{\delta^k(a_1)} = t_1^{-1} t_{k+1}$$

is contained in N_k . Next we note that a_1 intersects $a_{k+1} = \delta^k(a_1)$ at one point if $k = 1$ or $k = 2g$, whereas if $2 \leq k \leq 2g - 1$ then a_1 is disjoint from a_{k+1} and the complement of $a_1 \cup a_{k+1}$ in S is connected. By Lemma 3, N_k contains the commutator subgroup of Mod_g , the normal closure of $t_1^{-1} t_{k+1}$. It follows that the quotient Mod_g/N_k is abelian and hence a quotient of $H_1(\text{Mod}_g; \mathbb{Z})$.

For $g \geq 3$, the first homology group $H_1(\text{Mod}_g; \mathbb{Z})$ of Mod_g is trivial by Powell's theorem. Therefore, $N_k = \text{Mod}_g$ in this case. If $g = 2$, the group $H_1(\text{Mod}_2; \mathbb{Z})$ is isomorphic to $\mathbb{Z}/10\mathbb{Z}$ by a result of Mumford and is generated by the class of the Dehn twist about any non-separating simple closed curve. The element δ^k represents the $4k$ -th power of this generator of $H_1(\text{Mod}_2; \mathbb{Z})$. Hence the quotient group Mod_2/N_k is cyclic of order 2. If $g = 1$, then $H_1(\text{Mod}_1; \mathbb{Z})$ is isomorphic to $\mathbb{Z}/12\mathbb{Z}$, again generated by the class of the Dehn twist about any non-separating simple closed curve. The element δ^k represents the $2k$ -th power of this generator of $H_1(\text{Mod}_1; \mathbb{Z})$, so it follows that Mod_1/N_k is a cyclic group, of order 2 if $k = 1$ and of order 4 if $k = 2$. The conclusion (a) of the theorem follows.

To prove (b), let N denote the normal closure of δ^{2g+1} in Mod_g . It can easily be shown (e.g. by another twist calculation) that δ^{2g+1} is (up to isotopy) a hyperelliptic involution of S , taking each loop a_j , $1 \leq j \leq 2g$ to its inverse. Thus, δ^{2g+1} acts as minus the identity on the first homology of S . Therefore, N is contained in the kernel of $\psi : \text{Mod}_g \rightarrow \text{PSp}(2g, \mathbb{Z})$. On the other hand, the element

$$t_0^{-1} \delta^{2g+1} t_0 \delta^{-(2g+1)} = t_0^{-1} t_{\delta^{2g+1}(a_0)}$$

is contained in N . Now, a_0 and $\delta^{2g+1}(a_0)$ are disjoint non-separating curves whose union bounds a subsurface of genus one, and it follows from Theorem 2 in [J] that the Torelli group is the normal closure of such an element. Hence, \mathcal{T}_g is contained in N . But the hyperelliptic involution δ^{2g+1} is not in \mathcal{T}_g and the index of \mathcal{T}_g in the kernel of ψ is 2. Therefore N is equal to the kernel of ψ , proving (b).

This completes the proof of the theorem. □

Remark 5. If $g = 2$, then since δ^5 is the hyperelliptic involution, which is central in Mod_2 , the normal closure of δ^5 is cyclic of order 2. Thus, Theorem 4(b) is false in this case. For $g = 1$, of course, the Torelli group is trivial.

Lemma 6. *Let $g \geq 3$ and let r denote either of the rotations r_1 and r_2 illustrated in Figure 2. Then the normal closure of r in Mod_g is the full mapping class group Mod_g .*

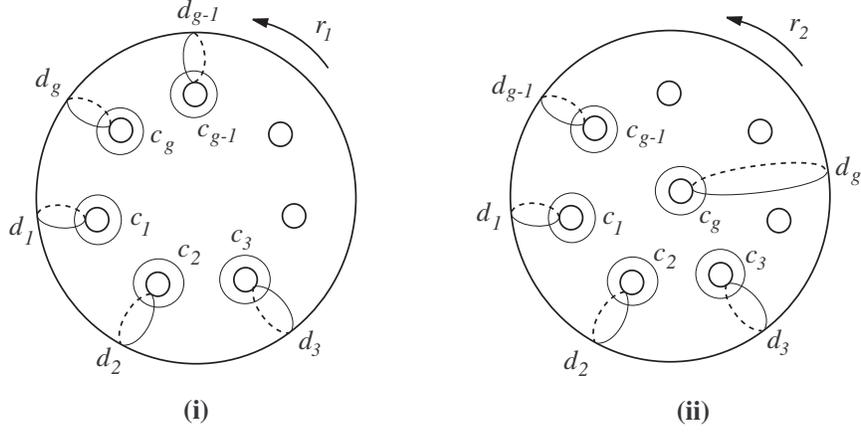


FIGURE 2. Two finite order surface mappings, induced by rotations r_1 and r_2 through $2\pi/g$ in (i) and $2\pi/(g-1)$ in (ii).

Proof. In order to avoid double indices, we let d denote the curve d_1 . Note that the curves d and $r(d) = d_2$ are disjoint, the complement of their union is connected and the element $t_d^{-1} t_{r(d)} = t_d^{-1} r t_d r^{-1}$ is contained in the normal closure of r . The lemma now follows from Lemma 3, after applying a homeomorphism which identifies the surface in Figure 1 with that of Figure 2; for instance we may take one which carries the curves a_{2j-1} to c_j for $j = 1, \dots, g$. \square

We are now ready to prove our main result.

Theorem 7. *Let $g > h$ and let $\varphi : \text{Mod}_g \rightarrow \text{Mod}_h$ be a homomorphism. The image of φ is trivial if $g \neq 2$ and has order at most two if $g = 2$.*

Proof. There is nothing to prove when $h = 0$, since the group Mod_0 is trivial. Hence we assume that $h \geq 1$, so that $g \geq 2$.

The subgroup $\langle \delta \rangle$ of Mod_g generated by δ is cyclic of order $4g + 2$ as we saw earlier. But by Theorem 1 (b), the maximal cyclic subgroups of Mod_h have order at most $4h + 2 < 4g + 2$, and so there must be an integer k with $1 \leq k \leq 2g + 1$ such that $\varphi(\delta^k) = 1$. We remark also that if $g \geq 3$, the kernel of φ contains the Torelli group. Therefore, φ factors through the natural map $\eta : \text{Mod}_g \rightarrow \text{Sp}(2g, \mathbb{Z})$. That is, φ induces a map $\Phi : \text{Sp}(2g, \mathbb{Z}) \rightarrow \text{Mod}_h$ such that $\Phi\eta = \varphi$.

Now if the integer k , for which $\varphi(\delta^k) = 1$, is less than $2g + 1$, it follows from Theorem 4 that the image of φ is trivial if $g \geq 3$ and has order at most 2 if $g = 2$, because the normal closure of δ^k is contained in the kernel of φ .

The theorem follows immediately from this observation in the (simplest possible) case where $2g + 1$ is prime. For since $2g + 1$ is greater than $2h + 1$, by Theorem 1 (c) the mapping class group Mod_h has no element of order $2g + 1$. Therefore we must have $\varphi(\delta^k) = 1$ for some $1 \leq k \leq 2g$. This

completes the proof for infinitely many values of g , in particular for $g = 2, 3$ and 5 .

Suppose next that $g > 5$. Consider a symplectic basis $c_1, d_1, \dots, c_g, d_g$ of $H_1(S; \mathbb{Z})$ (with appropriate choice of orientations) as shown in Figure 2 (i) if g is odd and Figure 2 (ii) if g is even. There is a subgroup Σ_g of $Sp(2g, \mathbb{Z})$, isomorphic to the symmetric group on g letters, which represents all permutations of the hyperbolic pairs (c_i, d_i) . Let A_g denote the alternating subgroup of Σ_g . Let r denote r_1 if g is odd and r_2 if g is even in Figure 2. Then the image $\eta(r)$ of the surface mapping r is contained in A_g .

The group Mod_h cannot contain a subgroup of order $g!/2$ when $g > 5$, because $g!/2$ is greater than $84(h-1)$. Therefore, the restriction of Φ to A_g is not injective, and since A_n is simple for $n > 5$, A_g must be contained in the kernel of Φ . Therefore r is contained in the kernel of φ and, since the normal closure of r is the full group Mod_g , φ is trivial.

Suppose finally that $g = 4$. If $h = 1$ then $4h + 2$ is less than $2g + 1$. If $h = 2$ then $4h + 1 = 2g + 1$. Therefore, by Theorem 1 we have $\varphi(\delta^k) = 1$ for some $1 \leq k \leq 2g$. The conclusion of the theorem follows in this case. Now let $h = 3$. Again, φ factors through the map η . Consider the element $\gamma = t_1 t_2 t_3 t_4$. A twist calculation shows that γ^{10} is the Dehn twist about a nullhomologous simple closed curve, the boundary of S cut open along all these curves. Hence it is contained in the Torelli group. Moreover, the order of $\eta(\gamma)$ is 10 in $Sp(8, \mathbb{Z})$. Since the prime 5 is greater than $h + 1$ and different from $2h + 1$, there must be a positive integer $k < 5$ such that $\eta(\gamma)^k$ is contained in the kernel of Φ , and so γ^k is contained in the kernel of φ . By considering the curves a_1 and $\gamma^k(a_1) = a_{k+1}$, as in the proof of Lemma 3, we see that the normal closure of γ^k is the full group Mod_4 . Therefore, the image of φ is trivial in this case too.

This completes the proof. □

Remark 8. The method of proof for the general case $g > 5$ in the above theorem is a sharpening of our original one and was suggested to us by a conversation with M.R. Bridson.

Corollary 9. *Let $g > 5$ be an integer. Suppose that H is any group with the property that it contains no element of order $4g + 2$ and no subgroup of order $g!$. Then any homomorphism $\varphi : \text{Mod}_g \rightarrow H$ has trivial image.*

Proof. The proof of Theorem 7 applies verbatim. □

4. HOMOMORPHISMS FROM MAPPING CLASS GROUPS

In this section we complete the paper by drawing further implications about arbitrary homomorphic images of mapping class groups. The first amounts to no more than an abstraction of Theorem 7.

Theorem 10. *Let g be a positive integer. Suppose that H is a group with the property that it has no element of order $2g + 1$. If $\varphi : \text{Mod}_g \rightarrow H$ is a*

homomorphism, then the image of φ is trivial if $g \geq 3$ and is a subgroup of a cyclic group of order 2 (resp. 4) if $g = 2$ (resp. $g = 1$).

Proof. The hypotheses imply that there is an integer k with $1 \leq k \leq 2g$ such that $\varphi(\delta^k) = 1$. The normal closure N_k of δ^k is contained in the kernel of φ . But by Theorem 4, it is the full group Mod_g if $g \geq 3$, whereas it has index 2 when $g = 2$ and index 2 or 4 if $g = 1$. Since the quotient Mod_g/N_k is cyclic, the result follows. \square

We note that if $g = 2$ (resp. $g = 1$), then there is a homomorphism from the mapping class group Mod_g onto a cyclic group of order 2 (resp. 4).

By the same method, but bringing into play the rotations r_i , we obtain the next result.

Theorem 11. *Let $g \geq 3$ be a positive integer. If $\varphi : \text{Mod}_g \rightarrow H$ is a homomorphism, then the image of φ is trivial unless H contains elements of order $g - 1$ and g .*

Proof. Suppose that H contains no element of order g . Consider the rotation r_1 in Figure 2 (i). Since the order of r_1 is g , there must be an integer k with $1 \leq k < g$ such that $\varphi(r_1^k) = 1$. The proof of Lemma 6 also shows that the normal closure of r_1^k is equal to Mod_g . It follows that the kernel of φ is Mod_g .

If H has no element of order $g - 1$, the same argument applies, mutatis mutandis, with the rotation r_2 of Figure 2 (ii) in place of r_1 . \square

A further criterion for recognising homomorphic images of Mod_g stems from restrictions on the rank of free abelian subgroups. Recall that if a is null-homologous then $t_a \in \mathcal{T}_g$, and that if b and c are two disjoint non-separating simple closed curves, whose union bounds a subsurface of S , then $t_b^{-1}t_c$ is in the Torelli group \mathcal{T}_g . A converse criterion, concerning the intersection of any free abelian twist subgroup with the Torelli group, is proved by Vautaw in [V], Theorem 3.1. His result can be stated as follows.

Theorem 12. *Let c_1, c_2, \dots, c_k be a set of homotopically nontrivial, pairwise disjoint, non-separating, simple closed curves on S such that for each pair $i \neq j$ the surface obtained by cutting S along c_i and c_j is connected. Then an element $t_{c_1}^{n_1} t_{c_2}^{n_2} \cdots t_{c_k}^{n_k}$ is contained in the Torelli group if and only if $n_i = 0$ for all i .*

As mentioned in section 1, it follows immediately from the definitions that Dehn twists about homotopically distinct loops commute up to isotopy, providing a natural source of free abelian subgroups, though not the only one. There is, however, an upper bound for the rank of an abelian subgroup of the mapping class group, determined by Birman, Lubotzky and McCarthy [BLM].

Theorem 13. *The rank of a free abelian subgroup of the mapping class group Mod_g is at most $3g - 3$.*

Certain specific subgroups of finite index in Mod_g will also be significant here. We recall that, for any positive integer m , the *congruence subgroup* $N(2g, m)$ of level m in the genus g symplectic group $Sp(2g, \mathbb{Z})$ is the kernel of the natural epimorphism from $Sp(2g, \mathbb{Z}) \rightarrow Sp(2g, \mathbb{Z}/m\mathbb{Z})$ given by the mod m reduction. The congruence subgroup problem for $Sp(2g, \mathbb{Z})$ was solved by Mennicke [M].

Theorem 14. *If G is a normal subgroup of $Sp(2g, \mathbb{Z})$ distinct from the trivial subgroup and the center, then G contains a congruence subgroup $N(2g, m)$ for some m . In particular, G is of finite index in $Sp(2g, \mathbb{Z})$.*

In the present context it is meaningful to ask whether the mapping class groups possess a similar co-finitality property in relation to some natural infinite family of finite index subgroups. However, the range of possible subgroup families is considerably broader than in the case of linear groups such as the symplectic modular group and so far, in the absence of any faithful linear representation for Mod_g , there is no natural choice.

With this preamble we can state our final result about morphisms from mapping class groups, which says essentially that in order for a group to be an (infinite) homomorphic image of Mod_g , it must either have a torsion element of maximal order or the abelian rank must be at least $3g - 3$, that of Mod_g .

Theorem 15. *Let $g \geq 3$ be an integer. Suppose that H is any group with the following properties: it contains no element of order $4g + 2$ and any free abelian subgroup has rank $< 3g - 3$. If $\varphi : \text{Mod}_g \rightarrow H$ is a homomorphism, then the image of φ is finite.*

Proof. Since H has no element of order $4g + 2$, there must be an integer k with $1 \leq k \leq 2g + 1$ such that $\varphi(\delta^k) = 1$. If $\varphi(\delta^k) = 1$ for some $1 \leq k \leq 2g$, then the image of φ is trivial by Theorem 4 and the conclusion is obvious in this case. If $\varphi(\delta^k) \neq 1$ for all $k \leq 2g$, then $\varphi(\delta^{2g+1}) = 1$. But then the kernel of φ contains the Torelli group, the kernel of η . Therefore there is a homomorphism $\Phi : Sp(2g, \mathbb{Z}) \rightarrow H$ such that $\Phi\eta = \varphi$.

It is an elementary exercise to choose a system of non-separating, pairwise disjoint, non-isotopic, simple closed curves $c_1, c_2, \dots, c_{3g-3}$ on S_g such that (i) the complement $S \setminus \bigcup_{j=1}^{3g-3} c_j$ is a union of $2g - 2$ spheres-with-three-holes, and

(ii) for each $i \neq j$ the complement of $c_i \cup c_j$ is connected.

Now the (abelian) subgroup A of Mod_g generated by the Dehn twists about $c_1, c_2, \dots, c_{3g-3}$ has rank $3g - 3$. Also, since the intersection $A \cap \mathcal{T}_g$ is trivial by Theorem 12, the restriction of η to A is injective. Hence, $\eta(A)$ is a free abelian group of rank $3g - 3$. Since $\Phi(\eta(A))$ is abelian in H and H does not contain any free abelian subgroup of rank $3g - 3$, there must be a non-trivial element $h \in \eta(A)$ such that $\Phi(h) = 1$. Clearly, $\eta(A)$ does not contain the central element $-I$ in $Sp(2g, \mathbb{Z})$. Therefore, the kernel of Φ is a nontrivial normal subgroup of $Sp(2g, \mathbb{Z})$ different from the center and so, by

Theorem 14, it contains a congruence subgroup. In particular, the image of Φ (and, hence, of φ) is finite.

This completes the proof. \square

5. CONCLUDING REMARKS

Our results imply restrictions on the index of any normal subgroup of finite index in a mapping class group for $g \geq 3$: let $G < \text{Mod}_g$ be a normal subgroup of finite index n , then n is divisible by $g-1$, g and $2g+1$. Results of this type may have some bearing on the notion of subgroup growth for mapping class groups.

As N.V. Ivanov has pointed out, it is natural to ask a more general question of rigidity type, whether a non-trivial morphism (or more generally a morphism with infinite image) exists from some *finite index subgroup of* Mod_g to Mod_h with $g > h$. The abelian rank aspect of our final result may have some bearing on answering this problem. We note that there are subgroups of finite index in Mod_2 admitting a homomorphism onto a free group of finite rank [Ko]: since Mod_1 is virtually free, we must thus assume in this problem that g is at least 3. It should also be made clear that a negative answer to this problem would imply a positive answer to Problem 2.11 (A) in [Ki].

Acknowledgements: This paper grew from a discussion between the first author and M.R. Bridson on the comparison between $\text{Out } F_n$ and Mod_g , prompting a question which the first author posed in communication with Nikolai V. Ivanov, who included it in a problem session at the AMS meeting in Ann Arbor in March 2002, which the second author attended. Our joint work started when the second author (M.K.) visited King's College London on 2002; M.K thanks the Mathematics Department of King's College London for hospitality and financial support. Both authors are grateful to the Warwick Mathematics Institute for hospitality and to the London Mathematical Society for travel to (and financial support during) the Symposium on Geometric Topology at Warwick in July 2002.

REFERENCES

- [BLM] J. S. Birman, A. Lubotzky, J. McCarthy, *Abelian and solvable subgroups of the mapping class groups*, Duke Math. J. **50** (1983), 1107–1120.
- [B] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Mathematical Society Lecture Note Series, 280. Cambridge University Press, Cambridge, 2000.
- [BV] M. R. Bridson, K. Vogtmann, *Homomorphisms from automorphism groups of free groups*, To appear in Bull. London Math. Soc.
- [FM] B. Farb, H. Masur, *Superrigidity of mapping class groups*, Topology **37** (1998), 1169–1176.
- [FK] H. M. Farkas, I. Kra, *Riemann Surfaces*, 2nd ed., Graduate Texts in Mathematics, vol 71, Springer-Verlag, Berlin Heidelberg New York, 1992.
- [H] W. J. Harvey, *Cyclic groups of automorphisms of a compact Riemann surface*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 86–97.

- [H2] W. J. Harvey, *On branch loci in Teichmüller space*, Trans. Amer. Math. Soc. **153** (1971), 387 - 399.
- [Iv] N. V. Ivanov, *Mapping Class Groups*, Chapter 12 in Handbook of Geometric Topology, (Editors R.J. Daverman & R.B. Sher), Elsevier Science (2002), 523-633.
- [J] D. Johnson, *Homeomorphisms of a surface which act trivially on homology*, Proc. Amer. Math. Soc. **75** (1979), 119-125.
- [KM] V. A. Kaimanovich, H. Masur, *The Poisson boundary of the mapping class group*, Invent. Math. **125** (1996), 221-264.
- [Ke] S. P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. (2) **117** (1983), 235-265.
- [Ki] R. Kirby, *Problems in low-dimensional topology*, in Geometric Topology (W. Kazez ed.) AMS/IP Stud. Adv. Math. vol 2.2, American Math. Society, Providence 1997.
- [Ko] M. Korkmaz, *On cofinite subgroups of mapping class groups*, Proceedings of 9th Gökova Geometry-Topology Conference, and Turkish Journal of Mathematics, to appear.
- [MP] J. D. McCarthy, A. Papadopoulos, *Involutions in surface mapping class groups*, L'Enseignement Mathématique **33** (1987), 275-290.
- [M] J. Mennicke, *Zur Theorie der Siegelschen Modulgruppe*, Math. Ann. **159** (1965), 115-129.
- [V] W. R. Vautaw, *Abelian subgroups of the Torelli group*, Alg. Geom. Topol. **2** (2002), 157-170.
- [W] A. Wiman, *Ueber die hyperelliptischen Curven und diejenigen vom Geschlechte $p = 3$, welche eindeutigen Transformationen in sich zulassen*, Bihang Kongl. Svenska Vetenskaps-Akademiens Handl. (Stockholm 1895-6) Vol. **21**, 1-23.

DEPARTMENT OF MATHEMATICS, KING'S COLLEGE, LONDON, WC2R 2LS.

E-mail address: bill.harvey@kcl.ac.uk

DEPARTMENT OF MATHEMATICS, MIDDLE EAST TECHNICAL UNIVERSITY, 06531 ANKARA, TURKEY

E-mail address: korkmaz@arf.math.metu.edu.tr