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Remote Estimation of the Wiener Process over a Channel with Random Delay

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***Abstract*—In this paper, we consider a problem of sampling a Wiener process, with samples forwarded to a remote estimator via a channel that consists of a queue with random delay. The estimator reconstructs a real-time estimate of the signal from causally received samples. Motivated by recent research on age-of-information, we study the optimal sampling strategy that minimizes the mean square estimation error subject to a sampling frequency constraint. We prove that the optimal sampling strategy is a threshold policy, and find the optimal threshold. This threshold is determined by the sampling frequency constraint and how much the Wiener process varies during the channel delay. An interesting consequence is that even in the absence of the sampling frequency constraint, the optimal strategy is *not* zero-wait sampling in which a new sample is taken once the previous sample is delivered; rather, it is optimal to wait for a non-zero amount of time after the previous sample is delivered, and then take the next sample. Further, if the sampling times are independent of the observed Wiener process, the optimal sampling problem reduces to an age-of-information optimization problem that has been recently solved. Our comparisons show that the estimation error of the optimal sampling policy is much smaller than those of age-optimal sampling, zero-wait sampling, and classic uniform sampling.**

I. Introduction

Consider a system with two terminals (see Fig. 1): An observer measuring a Wiener process W_t and an estimator, whose goal is to provide the best-guess $\hat{W}_t$ for the current value of W_t. These two terminals are connected by a channel that transmits time-stamped samples of the form (S_i, W_{S_i}), where the sampling times S_i satisfy $0 \leq S_1 \leq S_2 \leq \ldots$ The channel is modeled as a work-conserving FIFO queue with random *i.i.d.* delay Y_i, where $Y_i \geq 0$ is the channel delay (i.e., transmission time) of sample i.[1] The observer can choose the sampling times S_i causally subject to an average sampling frequency constraint

$$\liminf_{n\to\infty} \frac{1}{n}\mathbb{E}[S_n] \geq \frac{1}{f_{\max}},$$

where $f_{\max}$ is the maximum allowed sampling frequency.

Unless it arrives at an empty system, sample i needs to wait in the queue until its transmission starts. Let G_i be the transmission starting time of sample i such that $S_i \leq G_i$. The delivery time of sample i is $D_i = G_i + Y_i$. The initial value $W_0 = 0$ is known by the estimator for free, represented by $S_0 = D_0 = 0$. At time t, the estimator forms $\hat{W}_t$ using causally received samples with $D_i \leq t$. By minimum mean square error (MMSE) estimation,

$$\begin{aligned}\hat{W}_t =& \mathbb{E}[W_t | W_{S_j}, D_j \leq t] \\ =& W_{S_i}, \text{ if } t \in [D_i, D_{i+1}),\ i = 0, 1, 2, \ldots, \end{aligned} \quad (1)$$

as illustrated in Fig. 2. We measure the quality of remote estimation via the MMSE:

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (W_t - \hat{W}_t)^2 dt\right].$$

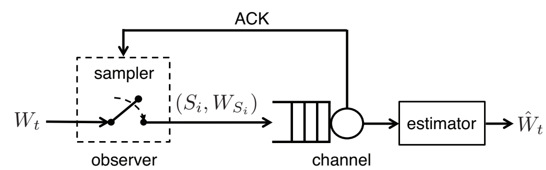

Fig. 1: System model.

In this paper, we study the optimal sampling strategy that achieves the fundamental tradeoff between $f_{\max}$ and MMSE. The contributions of this paper are summarized as follows:

- The optimal sampling problem for minimizing the MMSE subject to the sampling frequency constraint is solved exactly. We prove that the optimal sampling strategy is a threshold policy, and find the optimal threshold. This threshold is determined by $f_{\max}$ and the amount of signal variation during the channel delay (i.e., random transmission time of a sample). Our threshold policy has an important difference from the previous threshold policies studied in, e.g., [1]–[10]: In our model, each sample waiting in the queue for its transmission opportunity unnecessarily becomes stale. We have proven that it is suboptimal to take a new sample when the channel is busy. Consequently, the threshold should be *disabled* whenever there is a packet in transmission.
- An unexpected consequence of our study is that even in the absence of the sampling frequency constraint

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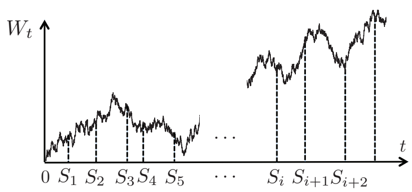
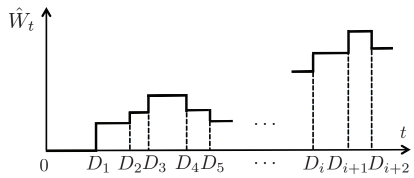
[1] By "work-conserving", we meant that the channel is kept busy whenever there exist some generated samples that are not delivered to the estimator.

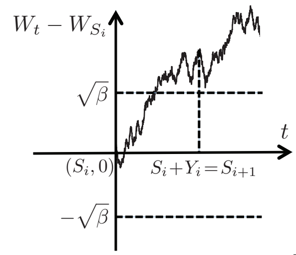


(a) Wiener process W_t and its samples.

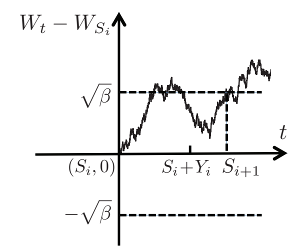


(b) Estimate process $\hat{W}_t$ using causally received samples.

Fig. 2: Sampling and remote estimation of the Wiener process.



(a) If $|W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}$, sample $i+1$ is taken at time $S_{i+1} = S_i + Y_i$.



(b) If $|W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}$, sample $i+1$ is taken at time t that satisfies $t \geq S_i + Y_i$ and $|W_t - W_{S_i}| = \sqrt{\beta}$.

Fig. 3: Illustration of the threshold policy (4).

(i.e., $f_{\max} = \infty$), the optimal strategy is *not* zero-wait sampling in which a new sample is generated once the previous sample is delivered; rather, it is optimal to wait a positive amount of time after the previous sample is delivered, and then take the next sample.

- If the sampling times are independent of the observed Wiener process, the optimal sampling problem reduces to an age-of-information optimization problem solved in [11], [12]. The asymptotics of the MMSE-optimal and age-optimal sampling policies at low/high channel delay or low/high sampling frequencies are studied.
- Our theoretical and numerical comparisons show that the MMSE of the optimal sampling policy is much smaller than those of age-optimal sampling, zero-wait sampling, and classic uniform sampling.

II. Related Work

On the one hand, the results in this paper are closely related to the recent age-of-information studies, e.g., [11]–[20], where the focus was on queueing and channel delay, without a signal model. The discovery that the zero-wait policy is not always optimal for minimizing the age-of-information can be found in [11]–[13]. The sub-optimality of a work-conserving scheduling policy was also observed in [19], which considered scheduling updates to different users with unreliable channels. One important observation in our study is that the behavior of the optimal update policy changes dramatically after adding a signal model.

On the other hand, the paper can be considered as a contribution to the rich literature on remote estimation, e.g., [1]–[10], [21], by adding a queueing model. Optimal transmission scheduling of sensor measurements for estimating a stochastic process was recently studied in [9], [10], where the samples are transmitted over a channel with additive noise. In the absence of channel delay and queueing (i.e., $Y_i = 0$), the problems of sampling Wiener process and Gaussian random walk were addressed in [1], [7], [8], where the optimality of threshold policies was established. To the best of our knowledge, [7] is the closest study with this paper. Because there is no queueing and channel delay in [7], the problem analyzed therein is a special case of ours.

III. Main Result

Let $\pi = (S_0, S_1, \ldots)$ represent a sampling policy, and Π be the set of *causal* sampling policies which satisfy the following conditions: (i) The *information* that is available for determining the sampling time S_i includes the history of the Wiener process $(W_t : t \in [0, S_i])$, the history of channel states $(I_t : t \in [0, S_i])$, and the sampling times of previous samples $(S_0, \ldots, S_{i-1})$, where $I_t \in \{0, 1\}$ is the idle/busy state of the channel at time t. (ii) The inter-sampling times $\{T_i = S_{i+1} - S_i, i = 0, 1, \ldots\}$ form a *regenerative process* [22, Section 6.1]: There exist integers $0 \leq k_1 < k_2 < \ldots$ such that the post-k_j process $\{T_{k_j+i}, i = 0, 1, \ldots\}$ has the same distribution as the post-k_1 process $\{T_{k_1+i}, i = 0, 1, \ldots\}$ and is independent of the pre-k_j process $\{T_i, i = 0, 1, \ldots, k_j - 1\}$; in addition, $\mathbb{E}[S_{k_1}^2] < \infty$ and $0 < \mathbb{E}[(S_{k_{j+1}} - S_{k_j})^2] < \infty$ for $j = 1, 2, \ldots$[2] We assume that the Wiener process W_t and the channel delay Y_i are determined by two external processes, which are mutually independent and do not change according to the sampling policy $\pi \in \Pi$. We also assume that the Y_i's are *i.i.d.* with $\mathbb{E}[Y_i^2] < \infty$.

A sampling policy $\pi \in \Pi$ is said to be *signal-independent* (*signal-dependent*), if π is (not) independent of the Wiener process $\{W_t, t \geq 0\}$. Example policies in Π include:

1. *Uniform sampling* [23], [24]: The inter-sampling times are constant, such that for some $\beta \geq 0$,
$$S_{i+1} = S_i + \beta. \tag{2}$$
2. *Zero-wait sampling* [11]–[14]: A new sample is generated once the previous sample is delivered, i.e.,
$$S_{i+1} = S_i + Y_i. \tag{3}$$
3. *Threshold policy in signal variation:* The sampling times are given by
$$S_{i+1} = \inf\left\{t \geq S_i + Y_i : |W_t - W_{S_i}| \geq \sqrt{\beta}\right\}, \tag{4}$$
which is illustrated in Fig. 3. If $|W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}$, sample $i+1$ is generated at the time $S_{i+1} = S_i + Y_i$ when

[2] Really, we assume that T_i is a regenerative process because we analyze the time-average MMSE in (6), but operationally a nicer definition is $\limsup_{n\to\infty} \mathbb{E}[\int_0^{D_n} (W_t - \hat{W}_t)^2 dt]/\mathbb{E}[D_n]$. These two definitions are equivalent when T_i is a regenerative process.

sample i is delivered; otherwise, if $|W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}$, sample $i+1$ is generated at the earliest time t such that $t \geq S_i+Y_i$ and $|W_t - W_{S_i}|$ reaches the threshold $\sqrt{\beta}$. It is worthwhile to emphasize that even if there exists time $t \in [S_i, S_i+Y_i)$ such that $|W_t - W_{S_i}| \geq \sqrt{\beta}$, no sample is taken at such time t, as depicted in Fig. 3. In other words, the threshold-based control is disabled during $[S_i, S_i+Y_i)$ and is reactivated at time S_i+Y_i. This is a key difference from previous studies on threshold policies [1]–[10].

4. *Threshold policy in time variation* [11]–[13]: The sampling times are given by

$$S_{i+1} = \inf\left\{t \geq S_i + Y_i : t - S_i \geq \beta\right\}. \tag{5}$$

The optimal sampling problem for minimizing the MMSE subject to a sampling frequency constraint is formulated as

$$\min_{\pi \in \Pi} \ \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (W_t - \hat{W}_t)^2 dt\right] \tag{6}$$

$$\text{s.t.} \ \liminf_{n\to\infty} \frac{1}{n}\mathbb{E}[S_n] \geq \frac{1}{f_{\max}}. \tag{7}$$

Problem (6) is a constrained continuous-time Markov decision problem with a continuous state space. Somewhat to our surprise, we were able to exactly solve (6):

Theorem 1. *There exists $\beta \geq 0$ such that the sampling policy* (4) *is optimal to* (6)*, and the optimal β is determined by solving*[3]

$$\mathbb{E}[\max(\beta, W_Y^2)] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}\right), \tag{8}$$

where Y is a random variable with the same distribution as Y_i. The optimal value of (6) *is then given by*

$$\mathsf{mmse}_{\text{opt}} \triangleq \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{6\mathbb{E}[\max(\beta, W_Y^2)]} + \mathbb{E}[Y]. \tag{9}$$

Proof. See Section IV. □

The optimal policy in (4) and (8) is called the "MMSE-optimal" policy. Note that one can use the bisection method or other one-dimensional search method to solve (8) with quite low complexity. Interestingly, this optimal policy does not suffer from the "curse of dimensionality" issue encountered in many Markov decision problems.

Notice that the feasible policies in Π can use the complete history of the Wiener process $(W_t : t \in [0, S_{i+1}])$ to determine S_{i+1}. However, the MMSE-optimal policy in (4) and (8) only requires recent knowledge of the Wiener process $(W_t - W_{S_i} : t \in [S_i + Y_i, S_{i+1}])$ to determine S_{i+1}.

Moreover, according to (8), the threshold $\sqrt{\beta}$ is determined by the maximum sampling frequency $f_{\max}$ and the distribution of the signal variation W_Y during the channel delay Y. It is worth noting that W_Y *is a random variable that tightly couples the source process and the channel delay*. This is different from the traditional wisdom of information theory where source coding and channel coding can be treated separately.

[3]If $\beta \to 0$, the last terms in (8) and (13) are determined by L'Hôpital's rule.

A. Signal-Independent Sampling and the Age-of-Information

Let $\Pi_{\text{sig-independent}} \subset \Pi$ denote the set of signal-independent sampling policies, defined as

$$\Pi_{\text{sig-independent}} = \{\pi \in \Pi : \pi \text{ is independent of } W_t, t \geq 0\}.$$

For each $\pi \in \Pi_{\text{sig-independent}}$, the MMSE (6) can be written as (see Appendix A for its proof)

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Delta(t)dt\right], \tag{10}$$

where

$$\Delta(t) = t - S_i, \ t \in [D_i, D_{i+1}), \ i = 0, 1, 2, \ldots, \tag{11}$$

is the *age-of-information* [14], that is, the time difference between the generation time of the freshest received sample and the current time t. If the policy space in (6) is restricted from Π to $\Pi_{\text{sig-independent}}$, (6) reduces to the following age-of-information optimization problem [11], [12]:

$$\min_{\pi \in \Pi_{\text{sig-independent}}} \ \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Delta(t)dt\right] \tag{12}$$

$$\text{s.t.} \ \liminf_{n\to\infty} \frac{1}{n}\mathbb{E}[S_n] \geq \frac{1}{f_{\max}}.$$

Problem (6) is significantly more challenging than (12), because in (6) the sampler needs to make decisions based on the evolution of the signal process W_t, which is not required in (12). More powerful techniques than those in [11], [12] are developed in Section IV to solve (6).

Theorem 2. *[11], [12] There exists $\beta \geq 0$ such that the sampling policy* (5) *is optimal to* (12)*, and the optimal β is determined by solving*

$$\mathbb{E}[\max(\beta, Y)] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[\max(\beta^2, Y^2)]}{2\beta}\right). \tag{13}$$

The optimal value of (12) *is then given by*

$$\mathsf{mmse}_{\text{age-opt}} \triangleq \frac{\mathbb{E}[\max(\beta^2, Y^2)]}{2\mathbb{E}[\max(\beta, Y)]} + \mathbb{E}[Y]. \tag{14}$$

The sampling policy in (5) and (13) is referred to as the "age-optimal" policy. Because $\Pi_{\text{sig-independent}} \subset \Pi$,

$$\mathsf{mmse}_{\text{opt}} \leq \mathsf{mmse}_{\text{age-opt}}. \tag{15}$$

In the following, the asymptotics of the MMSE-optimal and age-optimal sampling policies at low/high channel delay or low/high sampling frequencies are studied.

B. Low Channel Delay or Low Sampling Frequency

Let

$$Y_i = dX_i \tag{16}$$

represent the scaling of the channel delay Y_i with d, where $d \geq 0$ and the X_i's are *i.i.d.* positive random variables. If $d \to 0$ or $f_{\max} \to 0$, we can obtain from (8) that (see Appendix B for its proof)

$$\beta = \frac{1}{f_{\max}} + o\left(\frac{1}{f_{\max}}\right), \tag{17}$$

where $f(x) = o(g(x))$ as $x \to a$ means that $\lim_{x\to a} f(x)/g(x) = 0$. Hence, the MMSE-optimal policy in (4) and (8) becomes

$$S_{i+1} = \inf\left\{t \geq S_i : |W_t - W_{S_i}| \geq \sqrt{\frac{1}{f_{\max}}}\right\}, \tag{18}$$

and as shown in Appendix B, the optimal value of (6) becomes

$$\mathsf{mmse}_{\text{opt}} = \frac{1}{6f_{\max}} + o\left(\frac{1}{f_{\max}}\right). \tag{19}$$

The sampling policy (18) was also obtained in [7] for the case that $Y_i = 0$ for all i.

If $d \to 0$ or $f_{\max} \to 0$, one can show that the age-optimal policy in (5) and (13) becomes uniform sampling (2) with $\beta = 1/f_{\max} + o(1/f_{\max})$, and the optimal value of (12) is $\mathsf{mmse}_{\text{age-opt}} = 1/(2f_{\max}) + o(1/f_{\max})$. Therefore,

$$\lim_{d\to 0} \frac{\mathsf{mmse}_{\text{opt}}}{\mathsf{mmse}_{\text{age-opt}}} = \lim_{f_{\max}\to 0} \frac{\mathsf{mmse}_{\text{opt}}}{\mathsf{mmse}_{\text{age-opt}}} = \frac{1}{3}. \tag{20}$$

C. High Channel Delay or Unbounded Sampling Frequency

If $d \to \infty$ or $f_{\max} \to \infty$, as shown in Appendix C, the MMSE-optimal policy for solving (6) is given by (4) where β is determined by solving

$$2\beta\mathbb{E}[\max(\beta, W_Y^2)] = \mathbb{E}[\max(\beta^2, W_Y^4)]. \tag{21}$$

Similarly, if $d \to \infty$ or $f_{\max} \to \infty$, the age-optimal policy for solving (12) is given by (5) where β is determined by solving

$$2\beta\mathbb{E}[\max(\beta, Y)] = \mathbb{E}[\max(\beta^2, Y^2)]. \tag{22}$$

In these limits, the ratio between $\mathsf{mmse}_{\text{opt}}$ and $\mathsf{mmse}_{\text{age-opt}}$ depends on the distribution of Y.

When the sampling frequency is unbounded, i.e., $f_{\max} = \infty$, one logically reasonable policy is the zero-wait policy in (3) [11]–[14]. This zero-wait policy achieves the maximum throughput and the minimum queueing delay of the channel. Surprisingly, this zero-wait policy *does not always* minimize the age-of-information in (12) and *almost never* minimizes the MMSE in (6), as stated below:

Theorem 3. *If $f_{\max} = \infty$, the zero-wait policy is optimal for solving* (6) *if and only if $Y = 0$ with probability one.*

Proof. See Appendix D. □

Theorem 4. *[12] If $f_{\max} = \infty$, the zero-wait policy is optimal for solving* (12) *if and only if*

$$\mathbb{E}[Y^2] \leq 2 \operatorname{ess\,inf} Y \ \mathbb{E}[Y], \tag{23}$$

where $\operatorname{ess\,inf} Y = \sup\{y \in [0,\infty) : \Pr[Y < y] = 0\}$.

Proof. See Appendix D. □

Note that the condition in Theorem 4 is weaker than that of Theorem 5 in [12].

IV. Proof of the Main Result

We establish Theorem 1 in four steps: First, we employ the strong Markov property of the Wiener process to simplify the optimal sampling problem (6). Second, we study the Lagrangian dual problem of the simplified problem, and decompose the Lagrangian dual problem into a series of *mutually independent* per-sample control problems. Each of these per-sample control problems is a continuous-time Markov decision problem. Third, we utilize optimal stopping theory [25] to solve the per-sample control problems. Finally, we show that the Lagrangian duality gap is zero. By this, the original problem (6) is solved. The details are as follows.

A. Problem Simplification

We first provide a lemma that is crucial for simplifying (6).

Lemma 1. *In the optimal sampling problem* (6) *for minimizing the MMSE of the Wiener process, it is suboptimal to take a new sample before the previous sample is delivered.*

Proof. See Appendix E. □

In recent studies on age-of-information [11], [12], Lemma 1 was intuitive and hence was used without a proof: If a sample is taken when the channel is busy, it needs to wait in the queue until its transmission starts, and becomes stale while waiting. A better method is to wait until the channel becomes idle, and then generate a new sample, as stated in Lemma 1. However, this lemma is not intuitive in the MMSE minimization problem (6): The proof of Lemma 1 relies on the strong Markov property of Wiener process, which may not hold for other signal processes.

By Lemma 1, we only need to consider a sub-class of sampling policies $\Pi_1 \subset \Pi$ such that each sample is generated and submitted to the channel after the previous sample is delivered, i.e.,

$$\Pi_1 = \{\pi \in \Pi : S_i = G_i \geq D_{i-1} \text{ for all } i\}. \tag{24}$$

This completely eliminates the waiting time wasted in the queue, and hence the queue is always kept empty. The *information* that is available for determining S_i includes the history of signal values $(W_t : t \in [0, S_i])$ and the channel delay $(Y_1, \ldots, Y_{i-1})$ of previous samples.[4] To characterize this statement precisely, let us define the σ-fields $\mathcal{F}_t = \sigma(W_s : s \in [0,t])$ and $\mathcal{F}_t^+ = \cap_{s>t}\mathcal{F}_s$. Then, $\{\mathcal{F}_t^+, t \geq 0\}$ is the filtration (i.e., a non-decreasing and right-continuous family of σ-fields) of the Wiener process W_t. Given the transmission durations $(Y_1, \ldots, Y_{i-1})$ of previous samples, S_i is a *stopping time* with respect to the filtration $\{\mathcal{F}_t^+, t \geq 0\}$ of the Wiener process W_t, that is

$$[\{S_i \leq t\} | Y_1, \ldots, Y_{i-1}] \in \mathcal{F}_t^+. \tag{25}$$

[4]Note that the generation times $(S_1, \ldots, S_{i-1})$ of previous samples are also included in this information.

Then, the policy space Π_1 can be alternatively expressed as

$$\begin{aligned}\Pi_1 =\{&S_i : [\{S_i \leq t\}|Y_1,\ldots,Y_{i-1}] \in \mathcal{F}_t^+,\\ &S_i = G_i \geq D_{i-1} \text{ for all } i,\\ &T_i = S_{i+1}-S_i \text{ is a regenerative process}\}.\end{aligned} \quad (26)$$

Let $Z_i = S_{i+1}-D_i \geq 0$ represent the *waiting time* between the delivery time D_i of sample i and the generation time S_{i+1} of sample $i+1$. Then, $S_i = Z_0 + \sum_{j=1}^{i-1}(Y_j+Z_j)$ and $D_i = \sum_{j=0}^{i-1}(Z_j+Y_{j+1})$. If $(Y_1,Y_2,\ldots)$ is given, $(S_0,S_1,\ldots)$ is uniquely determined by $(Z_0,Z_1,\ldots)$. Hence, one can also use $\pi = (Z_0,Z_1,\ldots)$ to represent a sampling policy.

Because T_i is a regenerative process, by following the renewal theory in [26] and [22, Section 6.1], one can show that $\frac{1}{n}\mathbb{E}[S_n]$ is a convergent sequence and

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T (W_t-\hat{W}_t)^2dt\right]\\ =&\lim_{n\to\infty}\frac{\mathbb{E}\left[\int_0^{D_n}(W_t-\hat{W}_t)^2dt\right]}{\mathbb{E}[D_n]}\\ =&\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt\right]}{\sum_{i=0}^{n-1}\mathbb{E}[Y_i+Z_i]},\end{aligned}$$

where in the last step we have used $\mathbb{E}[D_n] = \mathbb{E}[\sum_{i=0}^{n-1}(Z_i+Y_{i+1})] = \mathbb{E}[\sum_{i=0}^{n-1}(Y_i+Z_i)]$. Hence, (6) can be rewritten as the following Markov decision problem:

$$\mathsf{mmse}_{\text{opt}} \triangleq \min_{\pi\in\Pi_1}\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt\right]}{\sum_{i=0}^{n-1}\mathbb{E}[Y_i+Z_i]} \quad (27)$$

$$\text{s.t.}\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i+Z_i] \geq \frac{1}{f_{\max}}, \quad (28)$$

where $\mathsf{mmse}_{\text{opt}}$ is the optimal value of (27).

In order to solve (27), let us consider the following Markov decision problem with a parameter $c \geq 0$:

$$p(c)\triangleq\min_{\pi\in\Pi_1}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt - c(Y_i+Z_i)\right] \quad (29)$$

$$\text{s.t.}\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i+Z_i] \geq \frac{1}{f_{\max}},$$

where $p(c)$ is the optimum value of (29).

Lemma 2. *The following assertions are true:*

(a). $\mathsf{mmse}_{\text{opt}} \gtreqqless c$ *if and only if* $p(c) \gtreqqless 0$.

(b). *If* $p(c)=0$*, the solutions to* (27) *and* (29) *are identical.*

Proof. See Appendix F. □

Hence, the solution to (27) can be obtained by solving (29) and seeking a $c_{\text{opt}} \geq 0$ such that

$$p(c_{\text{opt}}) = 0. \quad (30)$$

B. Lagrangian Dual Problem of (29)

Although (29) is a continuous-time Markov decision problem with a continuous state space (not a convex optimization problem), it is possible to use the Lagrangian dual approach to solve (29) and show that it admits no duality gap.

Define the following Lagrangian

$$\begin{aligned}&L(\pi;\lambda,c)\\ =&\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt-(c+\lambda)(Y_i+Z_i)\right]\\ &+\frac{\lambda}{f_{\max}}.\end{aligned} \quad (31)$$

Let

$$g(\lambda,c) \triangleq \inf_{\pi\in\Pi_1} L(\pi;\lambda,c). \quad (32)$$

Then, the Lagrangian dual problem of (29) is defined by

$$d(c) \triangleq \max_{\lambda\geq 0} g(\lambda,c), \quad (33)$$

where $d(c)$ is the optimum value of (33). Weak duality [27], [28] implies that $d(c) \leq p(c)$. In Section IV-D, we will establish strong duality, i.e., $d(c_{\text{opt}}) = p(c_{\text{opt}}) = 0$, at the optimal choice of $c = c_{\text{opt}}$.

In the sequel, we solve (32). By considering the sufficient statistics of the Markov decision problem (32), we obtain

Lemma 3. *For any* $\lambda \geq 0$*, there exists an optimal solution* $(Z_0,Z_1,\ldots)$ *to* (32) *in which* Z_i *is independent of* $(W_t, t\in[0,W_{S_i}])$ *for all* $i=1,2,\ldots$

Proof. See Appendix G. □

Using the stopping times and martingale theory of the Wiener process, we obtain the following lemma:

Lemma 4. *Let* $\tau \geq 0$ *be a stopping time of the Wiener process* W_t *with* $\mathbb{E}[\tau^2]<\infty$*, then*

$$\mathbb{E}\left[\int_0^\tau W_t^2dt\right] = \frac{1}{6}\mathbb{E}\left[W_\tau^4\right]. \quad (34)$$

Proof. See Appendix H. □

If Z_i is independent of $(W_t, t\in[0,W_{S_i}])$, by using Lemma 4, we can show that for every $i=1,2,\ldots$,

$$\begin{aligned}&\mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt\right]\\ =&\frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^4\right]+\mathbb{E}[Y_i+Z_i]\,\mathbb{E}[Y_i],\end{aligned} \quad (35)$$

which is proven in Appendix I.

Define the σ-fields $\mathcal{F}_t^s = \sigma(W_{s+v}-W_s : v\in[0,t])$ and $\mathcal{F}_t^{s+} = \cap_{v>t}\mathcal{F}_v^s$, as well as the filtration $\{\mathcal{F}_t^{s+}, t\geq 0\}$ of the time-shifted Wiener process $\{W_{s+t}-W_s, t\in[0,\infty)\}$. Define $\mathfrak{M}_s$ as the set of square-integrable stopping times of $\{W_{s+t}-W_s, t\in[0,\infty)\}$, i.e.,

$$\mathfrak{M}_s = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^{s+}, \mathbb{E}\left[\tau^2\right] < \infty\}.$$

By using (35) and considering the sufficient statistics of the Markov decision problem (32), we obtain

Theorem 5. *An optimal solution* $(Z_0, Z_1, \ldots)$ *to* (32) *satisfies*

$$Z_i = \min_{\tau \in \mathfrak{M}_{S_i+Y_i}} \mathbb{E}\left[\frac{1}{2}(W_{S_i+Y_i+\tau} - W_{S_i})^4 - \beta(Y_i+\tau)\middle| W_{S_i+Y_i} - W_{S_i}, Y_i\right], \quad (36)$$

where β *is given by*

$$\beta = 3(c + \lambda - \mathbb{E}[Y]). \quad (37)$$

Proof. See Appendix J. □

C. Per-Sample Optimal Stopping Solution to (36)

We use optimal stopping theory [25] to solve (36). Let us first pose (36) in the language of optimal stopping. A continuous-time two-dimensional Markov chain X_t on a probability space $(\mathbb{R}^2, \mathcal{F}, \mathbb{P})$ is defined as follows: Given the initial state $X_0 = x = (s, b)$, the state X_t at time t is

$$X_t = (s + t, b + W_t), \quad (38)$$

where $\{W_t, t \geq 0\}$ is a standard Wiener process. Define $\mathbb{P}_x(A) = \mathbb{P}(A|X_0 = x)$ and $\mathbb{E}_x Z = \mathbb{E}(Z|X_0 = x)$, respectively, as the conditional probability of event A and the conditional expectation of random variable Z for given initial state $X_0 = x$. Define the σ-fields $\mathcal{F}_t^X = \sigma(X_v : v \in [0, t])$ and $\mathcal{F}_t^{X+} = \cap_{v>t} \mathcal{F}_v^X$, as well as the filtration $\{\mathcal{F}_t^{X+}, t \geq 0\}$ of the Markov chain X_t. A random variable $\tau : \mathbb{R}^2 \to [0, \infty)$ is said to be a *stopping time* of X_t if $\{\tau \leq t\} \in \mathcal{F}_t^{X+}$ for all $t \geq 0$. Let $\mathfrak{M}$ be the set of square-integrable stopping times of X_t, i.e.,

$$\mathfrak{M} = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^{X+}, \mathbb{E}\left[\tau^2\right] < \infty\}.$$

Our goal is to solve the following optimal stopping problem:

$$\sup_{\tau \in \mathfrak{M}} \mathbb{E}_x g(X_\tau), \quad (39)$$

where the function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$g(s, b) = \beta s - \frac{1}{2} b^4 \quad (40)$$

with parameter $\beta \geq 0$, and x is the initial state of the Markov chain $X(t)$. Notice that (39) becomes (36) if the initial state is $x = (Y_i, W_{S_i+Y_i} - W_{S_i})$, and W_t is replaced by $W_{S_i+Y_i+t} - W_{S_i}$.

Theorem 6. *For all initial state* $(s, b) \in \mathbb{R}^2$ *and* $\beta \geq 0$*, an optimal stopping time for solving* (39) *is*

$$\tau^* = \inf\left\{t \geq 0 : |b + W_t| \geq \sqrt{\beta}\right\}. \quad (41)$$

In order to prove Theorem 6, let us define the function $u(x) = \mathbb{E}_x g(X_{\tau^*})$ and establish some properties of $u(x)$.

Lemma 5. $u(x) \geq g(x)$ *for all* $x \in \mathbb{R}^2$*, and*

$$u(s, b) = \begin{cases} \beta s - \frac{1}{2} b^4, & \text{if } b^2 \geq \beta; \\ \beta s + \frac{1}{2}\beta^2 - \beta b^2, & \text{if } b^2 < \beta. \end{cases} \quad (42)$$

Proof. See Appendix K. □

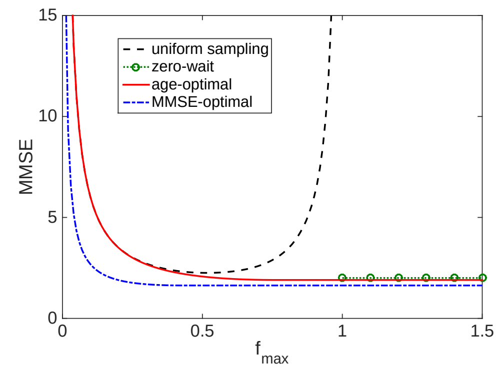


Fig. 4: MMSE vs. $f_{\max}$ tradeoff for *i.i.d.* exponential channel delay.

A function $f(x)$ is said to be *excessive* for the process X_t if [25]

$$\mathbb{E}_x f(X_t) \leq f(x), \text{ for all } t \geq 0, \ x \in \mathbb{R}^2. \quad (43)$$

By using the Itô-Tanaka-Meyer formula in stochastic calculus, we can obtain

Lemma 6. *The function* $u(x)$ *is excessive for the process* X_t*.*

Proof. See Appendix L. □

Now, we are ready to prove Theorem 6.

Proof of Theorem 6. In Lemma 5 and Lemma 6, we have shown that $u(x) = \mathbb{E}_x g(X_{\tau^*})$ is an excessive function and $u(x) \geq g(x)$. In addition, it is known that $\mathbb{P}_x(\tau^* < \infty) = 1$ for all $x \in \mathbb{R}^2$ [29, Theorem 8.5.3]. These conditions, together with the Corollary to Theorem 1 in [25, Section 3.3.1], imply that τ^* is an optimal stopping time of (39). This completes the proof. □

An immediate consequence of Theorem 6 is

Corollary 1. *An optimal solution to* (36) *is*

$$Z_i = \begin{cases} \inf\left\{t \geq 0 : |W_{S_i+Y_i+t} - W_{S_i}| \geq \sqrt{\beta}\right\}, & \text{if } \beta \geq 0; \\ 0, & \text{if } \beta < 0. \end{cases} \quad (44)$$

D. Zero Duality Gap between (29) *and* (33) *at* $c = c_{opt}$

Theorem 7. *The following assertions are true:*

(a). *The duality gap between* (29) *and* (33) *is zero such that* $d(c) = p(c)$.

(b). *A common optimal solution to* (6)*,* (27)*, and* (29) *with* $c = c_{\text{opt}}$*, is given by* (4) *and* (8).

Proof Sketch of Theorem 7. We first use Theorem 5 and Corollary 1 to find a geometric multiplier [27] for the primal problem (29). Hence, the duality gap between (29) and (33) is zero, because otherwise there exists no geometric multiplier [27, Section 6.1-6.2]. By this, part (a) is proven. Next, we consider the case of $c = c_{\text{opt}}$ and use Lemma 2 to prove part (b). See Appendix M for the details. □

Hence, Theorem 1 follows from Theorem 7.

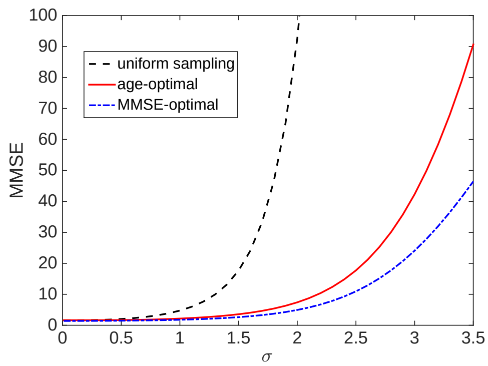


Fig. 5: MMSE vs. the scale parameter σ of *i.i.d.* log-normal channel delay for $f_{\max} = 0.8$.

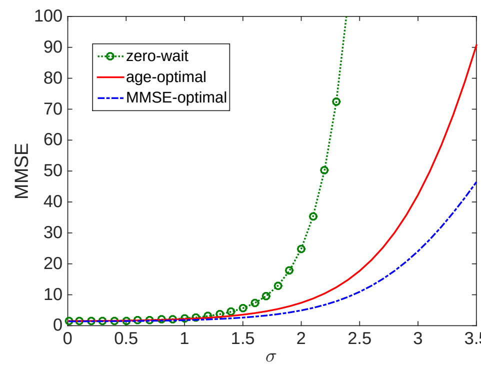


Fig. 6: MMSE vs. the scale parameter σ of *i.i.d.* log-normal channel delay for $f_{\max} = 1.5$.

V. Numerical Results

In this section, we evaluate the estimation performance achieved by the following four sampling policies:

1. *Uniform sampling:* The policy in (2) with $\beta = f_{\max}$.
2. *Zero-wait sampling [11]–[14]:* The sampling policy in (3), which is feasible when $f_{\max} \geq \mathbb{E}[Y_i]$.
3. *Age-optimal sampling [11], [12]:* The sampling policy in (5) and (13), which is the optimal solution to (12).
4. *MMSE-optimal sampling:* The sampling policy in (4) and (8), which is the optimal solution to (6).

Let $\mathsf{mmse}_{\text{uniform}}$, $\mathsf{mmse}_{\text{zero-wait}}$, $\mathsf{mmse}_{\text{age-opt}}$, and $\mathsf{mmse}_{\text{opt}}$, be the MMSEs of uniform sampling, zero-wait sampling, age-optimal sampling, MMSE-optimal sampling, respectively. According to (15), as well as the facts that uniform sampling is feasible for (12) and zero-wait sampling is feasible for (12) when $f_{\max} \geq \mathbb{E}[Y_i]$, we can obtain

$$\mathsf{mmse}_{\text{opt}} \leq \mathsf{mmse}_{\text{age-opt}} \leq \mathsf{mmse}_{\text{uniform}},$$
$$\mathsf{mmse}_{\text{opt}} \leq \mathsf{mmse}_{\text{age-opt}} \leq \mathsf{mmse}_{\text{zero-wait}}, \text{ when } f_{\max} \geq \mathbb{E}[Y_i],$$

which fit with our numerical results below.

Figure 4 depicts the tradeoff between MMSE and $f_{\max}$ for *i.i.d.* exponential channel delay with mean $\mathbb{E}[Y_i] = 1/\mu = 1$. Hence, the maximum throughput of the channel is $\mu = 1$. In this setting, $\mathsf{mmse}_{\text{uniform}}$ is characterized by eq. (25) of [14], which was obtained using a D/M/1 queueing model. For small values of $f_{\max}$, age-optimal sampling is similar with uniform sampling, and hence $\mathsf{mmse}_{\text{age-opt}}$ and $\mathsf{mmse}_{\text{uniform}}$ are of similar values. However, as $f_{\max}$ approaches the maximum throughput 1, $\mathsf{mmse}_{\text{uniform}}$ increases to infinite. This is because the queue length in uniform sampling is large at high sampling frequencies, and the samples become stale during their long waiting times in the queue. On the other hand, $\mathsf{mmse}_{\text{opt}}$ and $\mathsf{mmse}_{\text{age-opt}}$ decrease with respect to $f_{\max}$. The reason is that the set of feasible policies satisfying the constraints in (6) and (12) becomes larger as $f_{\max}$ grows, and hence the optimal values of (6) and (12) are decreasing in $f_{\max}$. Moreover, the gap between $\mathsf{mmse}_{\text{opt}}$ and $\mathsf{mmse}_{\text{age-opt}}$ is large for small values of $f_{\max}$. The ratio $\mathsf{mmse}_{\text{opt}}/\mathsf{mmse}_{\text{age-opt}}$ tends to $1/3$ as $f_{\max} \to 0$, which is in accordance with (20). As we expected, $\mathsf{mmse}_{\text{zero-wait}}$ is larger than $\mathsf{mmse}_{\text{opt}}$ and $\mathsf{mmse}_{\text{age-opt}}$ when $f_{\max} \geq 1$.

Figure 5 and Figure 6 illustrate the MMSE of *i.i.d.* log-normal channel delay for $f_{\max} = 0.8$ and $f_{\max} = 1.5$, respectively, where $Y_i = e^{\sigma X_i}/\mathbb{E}[e^{\sigma X_i}]$, $\sigma > 0$ is the scale parameter of log-normal distribution, and $(X_1, X_2, \ldots)$ are *i.i.d.* Gaussian random variables with zero mean and unit variance. Because $\mathbb{E}[Y_i] = 1$, the maximum throughput of the channel is 1. In Fig. 5, since $f_{\max} < 1$, zero-wait sampling is not feasible and hence is not plotted. As the scale parameter σ grows, the tail of the log-normal distribution becomes heavier and heavier. We observe that $\mathsf{mmse}_{\text{uniform}}$ grows quickly with respect to σ, much faster than $\mathsf{mmse}_{\text{opt}}$ and $\mathsf{mmse}_{\text{age-opt}}$. In addition, the gap between $\mathsf{mmse}_{\text{opt}}$ and $\mathsf{mmse}_{\text{age-opt}}$ increases as σ grows. In Fig. 6, because $f_{\max} > 1$, $\mathsf{mmse}_{\text{uniform}}$ is infinite and hence is not plotted. We can find that $\mathsf{mmse}_{\text{zero-wait}}$ grows quickly with respect to σ and is much larger than $\mathsf{mmse}_{\text{opt}}$ and $\mathsf{mmse}_{\text{age-opt}}$.

VI. Conclusion

In this paper, we have investigated optimal sampling and remote estimation of the Wiener process over a channel with random delay. The optimal sampling policy for minimizing the mean square estimation error subject to a sampling frequency constraint has been obtained. We prove that the optimal sampling policy is a threshold policy, and find the optimal threshold. Analytical and numerical comparisons with several important sampling policies, including age-optimal sampling, zero-wait sampling, and classic uniform sampling, have been provided. The results in this paper generalize recent research on ago-of-information by adding a signal model, and can be also considered a contribution to the rich literature on remote estimation by adding a channel that consists of a queue with random delay.

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Appendix A
Proof of (10)

If π is independent of $\{W_t, t \in [0,\infty)\}$, the S_i's and D_i's are independent of $\{W_t, t \in [0,\infty)\}$. Hence,

$$
\begin{aligned}
&\mathbb{E}\left\{\int_{D_i}^{D_{i+1}} (W_t - \hat{W}_t)^2 dt\right\} \\
\overset{(a)}{=}&\mathbb{E}\left\{\mathbb{E}\left\{\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt \middle| S_i, D_i, D_{i+1}\right\}\right\} \\
\overset{(b)}{=}&\mathbb{E}\left\{\int_{D_i}^{D_{i+1}} \mathbb{E}\left\{(W_t - W_{S_i})^2 | S_i, D_i, D_{i+1}\right\} dt\right\} \\
\overset{(c)}{=}&\mathbb{E}\left\{\int_{D_i}^{D_{i+1}} \mathbb{E}\left[(W_t - W_{S_i})^2\right] dt\right\} \\
\overset{(d)}{=}&\mathbb{E}\left\{\int_{D_i}^{D_{i+1}} (t - S_i) dt\right\} \\
\overset{(e)}{=}&\mathbb{E}\left\{\int_{D_i}^{D_{i+1}} \Delta(t) dt\right\},
\end{aligned}
$$

where step (a) is due to the law of iterated expectations, step (b) is due to Fubini's theorem, step (c) is because S_i, D_i, D_{i+1} are independent of the Wiener process, step (d) is due to Wald's identity $\mathbb{E}[W_T^2] = T$ [30, Theorem 2.48] and the strong Markov property of the Wiener process [30, Theorem 2.16], and step (e) is due to (11). By this, (10) is proven.

Appendix B
Proofs of (17) and (19)

If $f_{\max} \to 0$, (8) tells us that

$$\mathbb{E}[\max(\beta, W_Y^2)] = \frac{1}{f_{\max}},$$

which implies

$$\beta \le \frac{1}{f_{\max}} \le \beta + \mathbb{E}[W_Y^2] = \beta + \mathbb{E}[Y].$$

Hence,

$$\frac{1}{f_{\max}} - \mathbb{E}[Y] \le \beta \le \frac{1}{f_{\max}}.$$

If $f_{\max} \to 0$, (17) follows. Because Y is independent of the Wiener process, using the law of iterated expectations and the Gaussian distribution of the Wiener process, we can obtain $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2]$ and $\mathbb{E}[W_Y^2] = 3\mathbb{E}[Y]$. Hence,

$$
\begin{aligned}
\beta \le \mathbb{E}[\max(\beta, W_Y^2)] \le \beta + \mathbb{E}[W_Y^2] = \beta + \mathbb{E}[Y], \\
\beta^2 \le \mathbb{E}[\max(\beta^2, W_Y^4)] \le \beta^2 + \mathbb{E}[W_Y^4] = \beta^2 + 3\mathbb{E}[Y^2].
\end{aligned}
$$

Therefore,

$$\frac{\beta^2}{\beta + \mathbb{E}[Y]} \le \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{\mathbb{E}[\max(\beta, W_Y^2)]} \le \frac{\beta^2 + 3\mathbb{E}[Y^2]}{\beta}. \tag{45}$$

By combining (9), (17), and (45), (19) follows in the case of $f_{\max} \to 0$.

If $d \to 0$, then $Y \to 0$ and $W_Y \to 0$ with probability one. Hence, $\mathbb{E}[\max(\beta, W_Y^2)] \to \beta$ and $\mathbb{E}[\max(\beta^2, W_Y^4)] \to \beta^2$. Substituting these into (8) and (45), yields

$$\lim_{d\to 0} \beta = \frac{1}{f_{\max}}, \quad \lim_{d\to 0} \left\{ \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{6\mathbb{E}[\max(\beta, W_Y^2)]} + \mathbb{E}[Y] \right\} = \frac{1}{6 f_{\max}}.$$

By this, (17) and (19) are proven in the case of $d \to 0$. This completes the proof.

Appendix C
Proof of (21)

If $f_{\max} \to \infty$, the sampling frequency constraint in (6) can be removed. By (8), the optimal β is determined by (21).

If $d \to \infty$, let us consider the equation

$$\mathbb{E}[\max(\beta, W_Y^2)] = \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}. \tag{46}$$

If Y grows by a times, then β and $\mathbb{E}[\max(\beta, W_Y^2)]$ in (46) both should grow by a times, and $\mathbb{E}[\max(\beta^2, W_Y^4)]$ in (46) should grow by a^2 times. Hence, if $d \to \infty$, it holds in (8) that

$$\frac{1}{f_{\max}} \leq \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta} \tag{47}$$

and the solution to (8) is given by (21). This completes the proof.

Appendix D
Proofs of Theorems 3 and 4

Proof of Theorem 3. The zero-wait policy can be expressed as (4) with $\beta = 0$. Because Y is independent of the Wiener process, using the law of iterated expectations and the Gaussian distribution of the Wiener process, we can obtain $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2]$. According to (21), $\beta = 0$ if and only if $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2] = 0$ which is equivalent to $Y = 0$ with probability one. This completes the proof. □

Proof of Theorem 4. In the one direction, the zero-wait policy can be expressed as (5) with $\beta \leq \operatorname{ess\,inf} Y$. If the zero-wait policy is optimal, then the solution to (22) must satisfy $\beta \leq \operatorname{ess\,inf} Y$, which further implies $\beta \leq Y$ with probability one. From this, we can get

$$2\operatorname{ess\,inf} Y \mathbb{E}[Y] \geq 2\beta\mathbb{E}[Y] = \mathbb{E}[Y^2], \tag{48}$$

By this, (23) follows.

In the other direction, if (23) holds, we will show that the zero-wait policy is age-optimal by considering the following two cases.

Case 1: $\mathbb{E}[Y] > 0$. By choosing

$$\beta = \frac{\mathbb{E}[Y^2]}{2\mathbb{E}[Y]}, \tag{49}$$

we can get $\beta \leq \operatorname{ess\,inf} Y$ and hence

$$\beta \leq Y \tag{50}$$

with probability one. According to (49) and (50), such a β is the solution to (22). Hence, the zero-wait policy expressed by (5) with $\beta \leq \operatorname{ess\,inf} Y$ is the age-optimal policy.

Case 2: $\mathbb{E}[Y] = 0$ and hence $Y = 0$ with probability one. In this case, $\beta = 0$ is the solution to (22). Hence, the zero-wait policy expressed by (5) with $\beta = 0$ is the age-optimal policy.

Combining these two cases, the proof is completed. □

Appendix E
Proof of Lemma 1

Suppose that in policy π, sample i is generated when the channel is busy sending another sample, and hence sample i needs to wait for some time before submitted to the channel, i.e., $S_i < G_i$. Let us consider a *virtual* sampling policy $\pi' = \{S_0, \ldots, S_{i-1}, G_i, S_{i+1}, \ldots\}$. We call policy π' a virtual policy because the generation time of sample i in policy π' is $S'_i = G_i$ and it may happen that $S'_i > S_{i+1}$. However, this will not affect our proof below. We will show that the MMSE of policy π' is smaller than that of policy $\pi = \{S_0, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots\}$.

Note that the Wiener process $\{W_t : t \in [0, \infty)\}$ does not change according to the sampling policy, and the sample delivery times $\{D_1, D_2, \ldots\}$ remain the same in policy π and policy π'. Hence, the only difference between policies π and π' is that *the generation time of sample i is postponed from S_i to G_i*. The MMSE estimator of policy π is given by (1) and the MMSE estimator of policy π' is given by

$$\hat{W}_t = \begin{cases} 0, & t \in [0, D_1); \\ W_{G_i}, & t \in [D_i, D_{i+1}); \\ W_{S_j}, & t \in [D_j, D_{j+1}),\ j \neq i, j \geq 1. \end{cases} \tag{51}$$

Because $S_i \leq G_i \leq D_i \leq D_{i+1}$, by the strong Markov property of the Wiener process [30, Theorem 2.16], $\int_{D_i}^{D_{i+1}} 2[W_t - W_{G_i}]dt$ and $W_{G_i} - W_{S_i}$ are mutually independent. Hence,

$$\begin{aligned} &\mathbb{E}\left\{ \int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt \right\} \\ =&\mathbb{E}\left\{ \int_{D_i}^{D_{i+1}} (W_t - W_{G_i})^2 + (W_{G_i} - W_{S_i})^2 dt \right\} \\ &+ \mathbb{E}\left\{ \int_{D_i}^{D_{i+1}} 2(W_t - W_{G_i})(W_{G_i} - W_{S_i}) dt \right\} \\ =&\mathbb{E}\left\{ \int_{D_i}^{D_{i+1}} (W_t - W_{G_i})^2 + (W_{G_i} - W_{S_i})^2 dt \right\} \\ &+ \mathbb{E}\left\{ \int_{D_i}^{D_{i+1}} 2(W_t - W_{G_i}) dt \right\} \mathbb{E}[W_{G_i} - W_{S_i}]. \end{aligned}$$

Note that the channel is busy whenever there exist some generated samples that are not delivered to the estimator. Hence, during the time interval $[S_i, G_i]$, the channel is busy sending some samples generated before S_i in policy π. Because $\mathbb{E}[Y_j^2] < \infty$, we can get $\mathbb{E}[Y_j] < \infty$ and

$$\mathbb{E}[G_i - S_i] \leq \mathbb{E}\left[\sum_{j=1}^{i-1} Y_j \right] < \infty.$$

By Wald's identity [30, Theorem 2.44], we have $\mathbb{E}[W_{G_i} - W_{S_i}] = 0$ and hence

$$\mathbb{E}\left\{\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt\right\} \geq \mathbb{E}\left\{\int_{D_i}^{D_{i+1}} (W_t - W_{G_i})^2 dt\right\}. \tag{52}$$

Therefore, the MMSE of policy π' is smaller than that of policy π.

By repeating the above arguments for all samples i satisfying $S_i < G_i$, one can show that policy $\pi'' = \{S_0, G_1, \ldots, G_{i-1}, G_i, G_{i+1}, \ldots\}$ is better than policy $\pi = \{S_0, S_1, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots\}$. This completes the proof.

Appendix F
Proof of Lemma 2

Part (a) is proven in two steps:

Step 1: We will prove that $\mathsf{mmse}_{\text{opt}} \leq c$ if and only if $p(c) \leq 0$.

If $\mathsf{mmse}_{\text{opt}} \leq c$, then there exists a policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ that is feasible for both (27) and (29), which satisfies

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt\right]}{\sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right]} \leq c. \tag{53}$$

Hence,

$$\lim_{n\to\infty} \frac{\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right]}{\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right]} \leq 0. \tag{54}$$

Because the inter-sampling times $T_i = Y_i + Z_i$ are regenerative, the renewal theory [26] tells us that the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right]$ exists and is positive. By this, we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right] \leq 0. \tag{55}$$

Therefore, $p(c) \leq 0$.

On the reverse direction, if $p(c) \leq 0$, then there exists a policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ that is feasible for both (27) and (29), which satisfies (55). From (55), we can derive (54) and (53). Hence, $\mathsf{mmse}_{\text{opt}} \leq c$. By this, we have proven that $\mathsf{mmse}_{\text{opt}} \leq c$ if and only if $p(c) \leq 0$.

Step 2: We needs to prove that $\mathsf{mmse}_{\text{opt}} < c$ if and only if $p(c) < 0$. This statement can be proven by using the arguments in *Step 1*, in which "$\leq$" should be replaced by "$<$". Finally, from the statement of *Step 1*, it immediately follows that $\mathsf{mmse}_{\text{opt}} > c$ if and only if $p(c) > 0$. This completes the proof of part (a).

Part (b): We first show that each optimal solution to (27) is an optimal solution to (29). By the claim of part (a), $p(c) = 0$ is equivalent to $\mathsf{mmse}_{\text{opt}} = c$. Suppose that policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ is an optimal solution to (27). Then, $\mathsf{mmse}_{\pi} = \mathsf{mmse}_{\text{opt}} = c$. Applying this in the arguments of (53)-(55), we can show that policy π satisfies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right] = 0.$$

This and $p(c) = 0$ imply that policy π is an optimal solution to (29).

Similarly, we can prove that each optimal solution to (29) is an optimal solution to (27). By this, part (b) is proven.

Appendix G
Proof of Lemma 3

Because the Y_i's are *i.i.d.*, Z_i is independent of $Y_{i+1}, Y_{i+2}, \ldots$, and the strong Markov property of the Wiener process [30, Theorem 2.16], in the Lagrangian $L(\pi; \lambda)$ the term related to Z_i is

$$\mathbb{E}\left[\int_{S_i+Y_i}^{S_i+Y_i+Z_i+Y_{i+1}} (W_t - W_{S_i})^2 dt - (c+\lambda)(Y_i + Z_i)\right], \tag{56}$$

which is determined by the control decision Z_i and the recent information of the system $\mathcal{I}_i = (Y_i, (W_{S_i+t} - W_{S_i}, t \geq 0))$. According to [31, p. 252] and [32, Chapter 6], $\mathcal{I}_i$ is the *sufficient statistic* for determining Z_i in (32). Therefore, there exists an optimal policy $(Z_0, Z_1, \ldots)$ in which Z_i is determined based on only $\mathcal{I}_i$, which is independent of $(W_t : t \in [0, S_i])$. This completes the proof.

Appendix H
Proof of Lemma 4

According to Theorem 2.51 and Exercise 2.15 of [30], $W_t^4 - 6\int_0^t W_s^2 ds$ and $W_t^4 - 6tW_t^2 + 3t^2$ are two martingales of the Wiener process $\{W_t, t \in [0, \infty)\}$. Hence, $\int_0^t W_s^2 ds - tW_t^2 + t^2/2$ is also a martingale of the Wiener process.

Because the minimum of two stopping times is a stopping time and constant times are stopping times [29], it follows that $t \wedge \tau$ is a bounded stopping time for every $t \in [0, \infty)$, where $x \wedge y = \min[x, y]$. Then, it follows from Theorem 8.5.1 of [29] that for every $t \in [0, \infty)$

$$\mathbb{E}\left[\int_0^{t\wedge\tau} W_s^2 ds\right] = \frac{1}{6}\mathbb{E}\left[W_{t\wedge\tau}^4\right] \tag{57}$$

$$= \mathbb{E}\left[(t\wedge\tau)W_{t\wedge\tau}^2 - \frac{1}{2}(t\wedge\tau)^2\right]. \tag{58}$$

Notice that $\int_0^{t\wedge\tau} W_s^2 ds$ is positive and increasing with respect to t. By applying the monotone convergence theorem [29, Theorem 1.5.5], we can obtain

$$\lim_{t\to\infty} \mathbb{E}\left[\int_0^{t\wedge\tau} W_s^2 ds\right] = \mathbb{E}\left[\int_0^{\tau} W_s^2 ds\right].$$

The remaining task is to show that

$$\lim_{t\to\infty} \mathbb{E}\left[W_{t\wedge\tau}^4\right] = \mathbb{E}\left[W_{\tau}^4\right]. \tag{59}$$

Towards this goal, we combine (57) and (58), and apply Cauchy-Schwarz inequality to get

$$\begin{aligned}&\mathbb{E}\left[W_{t\wedge\tau}^4\right]\\=&\mathbb{E}\left[6(t\wedge\tau)W_{t\wedge\tau}^2-3(t\wedge\tau)^2\right]\\\leq&6\sqrt{\mathbb{E}\left[(t\wedge\tau)^2\right]\mathbb{E}\left[W_{t\wedge\tau}^4\right]}-3\mathbb{E}\left[(t\wedge\tau)^2\right].\end{aligned}$$

Let $x=\sqrt{\mathbb{E}\left[W_{t\wedge\tau}^4\right]/\mathbb{E}\left[(t\wedge\tau)^2\right]}$, then $x^2-6x+3\leq 0$. By the roots and properties of quadratic functions, we obtain $3-\sqrt{6}\leq x\leq 3+\sqrt{6}$ and hence

$$\mathbb{E}\left[W_{t\wedge\tau}^4\right]\leq(3+\sqrt{6})^2\mathbb{E}\left[(t\wedge\tau)^2\right]\leq(3+\sqrt{6})^2\mathbb{E}\left[\tau^2\right]<\infty.$$

Then, we use Fatou's lemma [29, Theorem 1.5.4] to derive

$$\begin{aligned}&\mathbb{E}\left[W_{\tau}^4\right]\\=&\mathbb{E}\left[\lim_{t\to\infty}W_{t\wedge\tau}^4\right]\\\leq&\liminf_{t\to\infty}\mathbb{E}\left[W_{t\wedge\tau}^4\right]\\\leq&(3+\sqrt{6})^2\mathbb{E}\left[\tau^2\right]<\infty.\end{aligned}\tag{60}$$

Further, by (60) and Doob's maximal inequality [30, Theorem 12.30] and [29, Theorem 5.4.3],

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,\infty)}W_{t\wedge\tau}^4\right]\\=&\mathbb{E}\left[\sup_{t\in[0,\tau]}W_{t}^4\right]\\\leq&\left(\frac{4}{3}\right)^4\mathbb{E}\left[W_{\tau}^4\right]<\infty.\end{aligned}\tag{61}$$

Because $W_{t\wedge\tau}^4\leq\sup_{t\in[0,\infty)}W_{t\wedge\tau}^4$ and $\sup_{t\in[0,\infty)}W_{t\wedge\tau}^4$ is integrable, (59) follows from dominated convergence theorem [29, Theorem 1.5.6]. This completes the proof.

APPENDIX I
PROOF OF (35)

By using (26) and the condition that Z_i is independent of $(W_t, t\in[0,W_{S_i}])$, we obtain that for given Y_i and Y_{i+1}, Y_i and $Y_i+Z_i+Y_{i+1}$ are stopping times of the time-shifted Wiener process $\{W_{S_i+t}-W_{S_i}, t\geq 0\}$. Hence,

$$\begin{aligned}&=\mathbb{E}\left\{\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt\right\}\\&=\mathbb{E}\left\{\int_{Y_i}^{Y_i+Z_i+Y_{i+1}}(W_{S_i+t}-W_{S_i})^2dt\right\}\\&\overset{(a)}{=}\mathbb{E}\left\{\mathbb{E}\left\{\int_{Y_i}^{Y_i+Z_i+Y_{i+1}}(W_{S_i+t}-W_{S_i})^2dt\middle|Y_i,Y_{i+1}\right\}\right\}\\&\overset{(b)}{=}\frac{1}{6}\mathbb{E}\left\{\mathbb{E}\left\{(W_{S_i+Y_i+Z_i+Y_{i+1}}-W_{S_i})^4\middle|Y_i,Y_{i+1}\right\}\right\}\\&\quad-\frac{1}{6}\mathbb{E}\left\{\mathbb{E}\left\{(W_{S_i+Y_i}-W_{S_i})^4\middle|Y_i,Y_{i+1}\right\}\right\}\\&\overset{(c)}{=}\frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}}-W_{S_i})^4\right]-\frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i}-W_{S_i})^4\right],\end{aligned}\tag{62}$$

where step (a) and step (c) are due to the law of iterated expectations, and step (b) is due to Lemma 4. Because $S_{i+1}=S_i+Y_i+Z_i$, we have

$$\begin{aligned}&\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}}-W_{S_i})^4\right]\\=&\mathbb{E}\left\{\left[(W_{S_i+Y_i+Z_i}-W_{S_i})+(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})\right]^4\right\}\\=&\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^4\right]\\&+4\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^3(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})\right]\\&+6\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^2(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^2\right]\\&+4\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^3\right]\\&+\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^4\right]\\=&\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^4\right]\\&+4\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^3\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})\right]\\&+6\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^2\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^2\right]\\&+4\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^3\right]\\&+\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^4\right],\end{aligned}$$

where in the last equation we have used the fact that Y_{i+1} is independent of Y_i and Z_i, and the strong Markov property of the Wiener process [30, Theorem 2.16]. Because

$$\begin{aligned}&\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^3|Y_{i+1}\right]\\=&\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})|Y_{i+1}\right]=0,\end{aligned}$$

by the law of iterated expectations, we have

$$\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^3\right]=\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})\right]=0.$$

In addition, Wald's identity tells us that $\mathbb{E}\left[W_\tau^2\right]=\mathbb{E}\left[\tau\right]$ for any stopping time τ with $\mathbb{E}\left[\tau\right]<\infty$. Hence,

$$\begin{aligned}&\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}}-W_{S_i})^4\right]\\=&\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^4\right]+6\mathbb{E}\left[Y_i+Z_i\right]\mathbb{E}\left[Y_{i+1}\right]\\&+\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^4\right].\end{aligned}\tag{63}$$

Finally, because $(W_{S_i+t}-W_{S_i})$ and $(W_{S_{i+1}+t}-W_{S_{i+1}})$ are both Wiener processes, and the Y_i's are *i.i.d.*,

$$\mathbb{E}\left[(W_{S_i+Y_i}-W_{S_i})^4\right]=\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^4\right].\tag{64}$$

Combining (62)-(64), yields (35).

Appendix J
Proof of Theorem 5

By (35), (56) can be rewritten as

$$\begin{aligned}&\mathbb{E}\left[\int_{S_i+Y_i}^{S_i+Y_i+Z_i+Y_{i+1}}(W_t-W_{S_i})^2dt-(c+\lambda)(Y_i+Z_i)\right]\\&=\frac{1}{6}(W_{S_i+Y_i+Z_i}-W_{S_i})^4-\frac{\beta}{3}(Y_i+Z_i)\\&=\frac{1}{6}[(W_{S_i+Y_i}-W_{S_i})+(W_{S_i+Y_i+Z_i}-W_{S_i+Y_i})]^4-\frac{\beta}{3}(Y_i+Z_i).\end{aligned} \tag{65}$$

Because the Y_i's are *i.i.d.* and the strong Markov property of the Wiener process [30, Theorem 2.16], the term in (65) is determined by the control decision Z_i and the information $\mathcal{I}_i' = (W_{S_i+Y_i} - W_{S_i}, Y_i, (W_{S_i+Y_i+t} - W_{S_i+Y_i},\ t \geq 0))$. According to [31, p. 252] and [32, Chapter 6], $\mathcal{I}_i'$ is the *sufficient statistic* for determining the waiting time Z_i in (32). Therefore, there exists an optimal policy $(Z_0, Z_1, \ldots)$ in which Z_i is determined based on only $\mathcal{I}_i'$. By this, (32) is decomposed into a sequence of per-sample control problems (36). In addition, because the Y_i's are *i.i.d.* and the strong Markov property of the Wiener process, the Z_i's in this optimal policy are *i.i.d.* Similarly, the $(W_{S_i+Y_i+Z_i} - W_{S_i})$'s in this optimal policy are *i.i.d.*

Appendix K
Proof of Lemma 5

Case 1: If $b^2 \geq \beta$, then (41) tells us that

$$\tau^* = 0 \tag{66}$$

and

$$u(x) = \mathbb{E}[g(X_0)|X_0 = x] = g(x) = \beta s - \frac{1}{2}b^4. \tag{67}$$

Case 2: If $b^2 < \beta$, then $\tau^* > 0$ and $(b+W_{\tau^*})^2 = \beta$. Invoking Theorem 8.5.5 in [29], yields

$$\mathbb{E}_x\tau^* = -(\sqrt{\beta} - b)(-\sqrt{\beta} - b) = \beta - b^2. \tag{68}$$

Using this, we can obtain

$$\begin{aligned}u(x) &= \mathbb{E}_x g(X(\tau^*))\\&= \beta(s + \mathbb{E}_x\tau^*) - \frac{1}{2}\mathbb{E}_x\left[(b + W_{\tau^*})^4\right]\\&= \beta(s + \beta - b^2) - \frac{1}{2}\beta^2\\&= \beta s + \frac{1}{2}\beta^2 - b^2\beta.\end{aligned} \tag{69}$$

Hence, in Case 2,

$$u(x) - g(x) = \frac{1}{2}\beta^2 - b^2\beta + \frac{1}{2}b^4 = \frac{1}{2}(b^2 - \beta)^2 \geq 0.$$

By combining these two cases, Lemma 5 is proven.

Appendix L
Proof of Lemma 6

The function $u(s, b)$ is continuous differentiable in (s, b). In addition, $\frac{\partial^2}{\partial^2 b}u(s, b)$ is continuous everywhere but at $b = \pm\sqrt{\beta}$. By the Itô-Tanaka-Meyer formula [30, Theorem 7.14 and Corollary 7.35], we obtain that almost surely

$$\begin{aligned}&u(s+t, b+W_t) - u(s, b)\\&= \int_0^t \frac{\partial}{\partial b}u(s+r, b+W_r)dW_r\\&\quad + \int_0^t \frac{\partial}{\partial s}u(s+r, b+W_r)dr\\&\quad + \frac{1}{2}\int_{-\infty}^{\infty} L^a(t)\frac{\partial^2}{\partial b^2}u(s+r, b+a)da,\end{aligned} \tag{70}$$

where $L^a(t)$ is the local time that the Wiener process spends at the level a, i.e.,

$$L^a(t) = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t 1_{\{|W_s - a| \leq \epsilon\}}ds, \tag{71}$$

and 1_A is the indicator function of event A. By the property of local times of the Wiener process [30, Theorem 6.18], we obtain that almost surely

$$\begin{aligned}&u(s+t, b+W_t) - u(s, b)\\&= \int_0^t \frac{\partial}{\partial b}u(s+r, b+W_r)dW_r\\&\quad + \int_0^t \frac{\partial}{\partial s}u(s+r, b+W_r)dr\\&\quad + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial b^2}u(s+r, b+W_r)dr.\end{aligned} \tag{72}$$

Because

$$\frac{\partial}{\partial b}u(s, b) = \begin{cases} -2b^3, & \text{if } b^2 \geq \beta;\\ -2\beta b, & \text{if } b^2 < \beta,\end{cases}$$

we can obtain that for all $t \geq 0$ and all $x = (s, b) \in \mathbb{R}^2$

$$\mathbb{E}_x\left\{\int_0^t \left[\frac{\partial}{\partial b}u(s+r, b+W_r)\right]^2 dr\right\} < \infty. \tag{73}$$

This and Thoerem 7.11 of [30] imply that $\int_0^t \frac{\partial}{\partial b}u(s+r, b+W_r)dW_r$ is a martingale and

$$\mathbb{E}_x\left[\int_0^t \frac{\partial}{\partial b}u(s+r, b+W_r)dW_r\right] = 0,\ \forall\ t \geq 0. \tag{74}$$

By combining (38), (72), and (74), we get

$$\mathbb{E}_x\left[u(X_t)\right] - u(x) = \mathbb{E}_x\left\{\int_0^t \left[\frac{\partial}{\partial s}u(X_r) + \frac{1}{2}\frac{\partial^2}{\partial b^2}u(X_r)\right]dr\right\}. \tag{75}$$

It is easy to compute that if $b^2 > \beta$,

$$\frac{\partial}{\partial s}u(s, b) + \frac{1}{2}\frac{\partial^2}{\partial b^2}u(s, b) = \beta - 3b^2 \leq 0;$$

and if $b^2 < \beta$,

$$\frac{\partial}{\partial s}u(s,b) + \frac{1}{2}\frac{\partial^2}{\partial b^2}u(s,b) = \beta - \beta = 0.$$

Hence,

$$\frac{\partial}{\partial s}u(s,b) + \frac{1}{2}\frac{\partial^2}{\partial b^2}u(s,b) \le 0 \tag{76}$$

for all $(s,b) \in \mathbb{R}^2$ except for $b = \pm\sqrt{\beta}$. Since the Lebesgue measure of those r for which $b + W_r = \pm\sqrt{\beta}$ is zero, we get from (75) and (76) that $\mathbb{E}_x[u(X_t)] \le u(x)$ for all $x \in \mathbb{R}^2$ and $t \ge 0$. This completes the proof.

Appendix M
Proof of Theorem 7

Theorem 7 is proven in three steps:

Step 1: We will show that the duality gap between (29) *and* (33) *is zero, i.e.,* $d(c) = p(c)$.

To that end, we needs to find $\pi^\star = (Z_0, Z_1, \ldots)$ and $\lambda^\star$ that satisfy the following conditions:

$$\pi^\star \in \Pi, \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i] - \frac{1}{f_{\max}} \ge 0, \tag{77}$$

$$\lambda^\star \ge 0, \tag{78}$$

$$L(\pi^\star; \lambda^\star, c_{\text{opt}}) = \inf_{\pi\in\Pi_1} L(\pi; \lambda^\star, c_{\text{opt}}), \tag{79}$$

$$\lambda^\star\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i] - \frac{1}{f_{\max}}\right\} = 0. \tag{80}$$

According to Theorem 5 and Corollary 1, the solution $\pi^\star$ to (79) is given by (44) where $\beta = 3(c + \lambda^\star - \mathbb{E}[Y])$. In addition, as shown in the proof of Theorem 5, the Z_i's in policy $\pi^\star$ are *i.i.d.* From (77), (78), and (80), $\lambda^\star$ is determined by considering two cases: If $\lambda^\star > 0$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i] = \mathbb{E}[Y_i + Z_i] = \frac{1}{f_{\max}}. \tag{81}$$

If $\lambda^\star = 0$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i] = \mathbb{E}[Y_i + Z_i] \ge \frac{1}{f_{\max}}. \tag{82}$$

Hence, such $\pi^\star$ and $\lambda^\star$ satisfy (77)-(80). By [27, Prop. 6.2.5], $\pi^\star$ is an optimal solution to the primal problem (29) and $\lambda^\star$ is a geometric multiplier [27] for the primal problem (29). In addition, the duality gap between (29) and (33) is zero, because otherwise there exists no geometric multiplier [27, Section 6.1-6.2].

Step 2: We will show that a common optimal solution to (6), (27)*, and* (29) *with* $c = c_{\text{opt}}$*, is given by* (4) *where* $\beta \ge 0$ *is determined by solving*

$$\mathbb{E}[Y_i + Z_i] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4]}{2\beta}\right), \tag{83}$$

where the last term is determined by L'Hôpital's rule if $\beta \to 0$.

We consider the case that $c = c_{\text{opt}}$. In *Step 1*, we have shown that policy $\pi^\star$ in (44) with $\beta = 3(c_{\text{opt}} + \lambda^\star - \mathbb{E}[Y])$ is an optimal solution to (29). According to the definition of c_{opt} in (30), $p(c_{\text{opt}}) = 0$. By Lemma 2(b), this policy $\pi^\star$ is also an optimal solution to (27). In addition, $p(c_{\text{opt}}) = 0$ and Lemma 2(a) imply $\mathsf{mmse}_{\text{opt}} = c_{\text{opt}}$. Substituting policy $\pi^\star$ and (35) into (27), yields

$$\begin{aligned} c_{\text{opt}} &= \lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4 + (Y_i + Z_i)\mathbb{E}[Y]\right]}{6\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i]} \\ &= \frac{\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]}{6\mathbb{E}[Y_i + Z_i]} + \mathbb{E}[Y], \end{aligned} \tag{84}$$

where in the last equation we have used that the Z_i's are *i.i.d.* and the $(W_{S_i+Y_i+Z_i} - W_{S_i})$'s are *i.i.d.*, which were shown in the proof of Theorem 5. According to (84), $c_{\text{opt}} \ge \mathbb{E}[Y]$. Hence, $\beta = 3(c_{\text{opt}} + \lambda^\star - \mathbb{E}[Y]) \ge 0$, in which case policy $\pi^\star$ in (44) is exactly (4).

The value of β can be obtained by considering the following two cases:

Case 1: If $\lambda > 0$, then (84) and (81) imply that

$$\mathbb{E}[Y_i + Z_i] = \frac{1}{f_{\max}}, \tag{85}$$

$$\beta > 3(c_{\text{opt}} - \mathbb{E}[Y]) = \frac{\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]}{2\mathbb{E}[Y_i + Z_i]}. \tag{86}$$

Case 2: If $\lambda = 0$, then (84) and (82) imply that

$$\mathbb{E}[Y_i + Z_i] \ge \frac{1}{f_{\max}}, \tag{87}$$

$$\beta = 3(c_{\text{opt}} - \mathbb{E}[Y]) = \frac{\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]}{2\mathbb{E}[Y_i + Z_i]}. \tag{88}$$

Combining (85)-(88), (83) follows. By (84), the optimal value of (27) is given by

$$\mathsf{mmse}_{\text{opt}} = \frac{\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4]}{6\mathbb{E}[Y_i + Z_i]} + \mathbb{E}[Y]. \tag{89}$$

Step 3: We will show that the expectations in (83) *and* (89) *are given by*

$$\mathbb{E}[Y_i + Z_i] = \mathbb{E}[\max(\beta, W_Y^2)], \tag{90}$$

$$\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4] = \mathbb{E}[\max(\beta^2, W_Y^4)]. \tag{91}$$

According to (44) with $\beta \ge 0$, we have

$$\begin{aligned} &W_{S_i+Y_i+Z_i} - W_{S_i} \\ &= \begin{cases} W_{S_i+Y_i} - W_{S_i}, & \text{if } |W_{S_i+Y_i} - W_{S_i}| \ge \sqrt{\beta}; \\ \sqrt{\beta}, & \text{if } |W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}. \end{cases} \end{aligned}$$

Hence,

$$\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4] = \mathbb{E}[\max(\beta^2, (W_{S_i+Y_i} - W_{S_i})^4)]. \tag{92}$$

In addition, from (66) and (68) we know that if $|W_{S_i+Y_i} - W_{S_i}| \ge \sqrt{\beta}$

$$\mathbb{E}[Z_i|Y_i] = 0;$$

otherwise,

$$\mathbb{E}[Z_i|Y_i] = \beta - (W_{S_i+Y_i} - W_{S_i})^2.$$

Hence,

$$\mathbb{E}[Z_i|Y_i] = \max[\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0].$$

Using the law of iterated expectations, the strong Markov property of the Wiener process, and Wald's identity $\mathbb{E}[(W_{S_i+Y_i} - W_{S_i})^2] = \mathbb{E}[Y_i]$, yields

$$\begin{aligned}
&\mathbb{E}[Z_i + Y_i] \\
=&\mathbb{E}[\mathbb{E}[Z_i|Y_i] + Y_i] \\
=&\mathbb{E}[\max(\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0) + Y_i] \\
=&\mathbb{E}[\max(\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0) + (W_{S_i+Y_i} - W_{S_i})^2] \\
=&\mathbb{E}[\max(\beta, (W_{S_i+Y_i} - W_{S_i})^2)]. \qquad (93)
\end{aligned}$$

Finally, because W_t and $W_{S_i+t} - W_{S_i}$ are of the same distribution, (90) and (91) follow from (93) and (92), respectively. This completes the proof.