

Magnetic Symmetries and Vortices In Chern-Simons Theories

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Abstract

We study the locality properties of the vortex operators in compact $U(1)$ Maxwell-Chern-Simons and $SU(N)$ Yang-Mills-Chern-Simons theories in 2+1 dimensions. We find that these theories do admit local vortex operators and thus in the UV regularized versions should contain stable magnetic vortices. In the continuum limit however the energy of these vortex excitations generically is logarithmically UV divergent. Nevertheless the classical analysis shows that at small values of CS coefficient κ the vortices become light. It is conceivable that they in fact become massless and condense due to quantum effects below some small κ . If this happens the magnetic symmetry breaks spontaneously and the theory is confining.

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1 Introduction.

Recently there has been proposed [1, 2] a very general argument (reviewed below) connecting the realization of magnetic Z_N symmetry, and the vacuum expectation values of the spatial 't Hooft loop V and spatial Wilson loop W in $SU(N)$ gauge theory. The motivation for these arguments is the study of confinement/deconfinement phase transitions in such theories. In 2+1 dimensions the role of the spatial 't Hooft loop operator is played by the magnetic vortex operator $V(x)$ which creates a magnetic vortex at x . The magnetic vortex operator is a canonical operator that acts on the physical Hilbert space of theory, unlike the Polyakov loop which is also used to study the phase structure of gauge theories, but which is not a canonical operator [1, 2]. An important part of the aforementioned argument concerns the *locality* of this magnetic vortex operator. In this note we study the locality properties of the magnetic vortex operator in 2+1 dimensional gauge theories when a Chern-Simons term is included for the gauge field. We consider first the case of compact $U(1)$ and then $SU(N)$. Including the Chern-Simons term does not alter the classical compact gauge symmetries of the system, but we might expect new behavior because the Chern-Simons term generates massive gauge degrees of freedom at the perturbative level [3].

Several related issues have been studied previously. In the absence of the Chern-Simons term, compact QED in 2+1 dimensions is confining due to monopoles [4], while $SU(N)$ Yang-Mills in 2+1 dimensions is confining due to the condensation of Z_N monopoles [5]. The existence of similar monopoles with a Chern-Simons term included has been analyzed for compact $U(1)$ in [6, 7, 8], and for $SU(N)$ in [9, 10, 11]. The general picture emerging from these studies is that confinement of electric charge is destroyed by the inclusion of the Chern-Simons term due to the binding of monopole-antimonopole pairs. However, these instanton-based approaches involve complex instantons, whose physical interpretation is acknowledged to be incompletely understood [11]. We thus feel it will be helpful to analyze these models in a different but complementary manner. Compact Maxwell-Chern-Simons theories have also been studied on the lattice in [12], also revealing

bound monopole-antimonopole pairs. The strong coupling limit of SU(N) Yang-Mills-Chern-Simons theories, and in particular the Polyakov loop, have been analyzed in [13] using the connection to topological field theories. And a Hamiltonian analysis [14] of SU(N) Yang-Mills-Chern-Simons theories indicates that the Wilson loop is expected not to have an area law behavior. Cornwall has argued [15] that there is a phase transition at a finite value of the Chern-Simons coefficient due to an interplay of perturbative and non-perturbative mass generation effects. As further motivation, much is known about the existence and properties of magnetic vortices in Chern-Simons theories coupled to matter fields [16, 17]. Here, instead, we want to focus on the question of vortices in the theory with just gauge fields, and no additional matter fields.

The key objects in this discussion are the spatial Wilson loop operator (along the spatial boundary of the system):

$$W = \text{Tr} \mathcal{P} \exp \left\{ i \oint_{C \rightarrow \infty} dx_i A_i \right\} \quad (1)$$

and the magnetic vortex creation operator $V(x)$, whose precise definition will be given below for the U(1) and SU(N) cases [see (11) and (41)]. In the SU(N) case, W generates the discrete Z_N global symmetry, while $V(x)$ can be viewed as a canonical local order parameter for this symmetry [1, 2]. Together, W and V satisfy the 't Hooft algebra [5]:

$$W V(x) W^\dagger = e^{\frac{2\pi i}{N}} V(x) \quad (2)$$

The general argument developed in [1, 2] for SU(N) Yang-Mills, without a Chern-Simons term, can be briefly summarized as follows.

First, one establishes that $V(x)$ is a local canonical order parameter that maps physical states to physical states, and so can be used to distinguish the phases of the theory in a gauge invariant manner. Given this, consider the theory at zero temperature. The spatial Wilson loop operator (the generator of the magnetic Z_N symmetry) acts on the ground state by transforming the Z_N -non invariant fields. There are two natural possibilities:

(i) the ground state is Z_N invariant, so that $\langle V \rangle = 0$. The operator W only has an effect near the boundary, so W has a perimeter law behavior. In this case, we expect vortex states in the spectrum, to carry the unbroken Z_N symmetry.

(ii) the ground state breaks Z_N , so that $\langle V \rangle \neq 0$. Now W acts nontrivially in the bulk so that W has an area law behavior. In this case we expect no vortices in the spectrum.

At higher temperatures, these correspondences no longer hold. Even in an unbroken phase, the thermal ensemble involves states with nontrivial Z_N charges, so that W acts significantly in the bulk and so generically has an area law behavior even though the symmetry is restored.

In this note, we ask how much of this argument is modified by the inclusion of a Chern-Simons term. We will show that the Wilson loop still generates the appropriate discrete symmetry. But Yang-Mills-Chern-Simons (YMCS) theories are completely massive, and so the Wilson loop is expected to have a perimeter law behavior [14]. By the previous argument it should then follow that the magnetic symmetry is unbroken and the spectrum should contain vortex states. There is one way to avoid this conclusion. The relation between the VEV of W and V only holds if V is a local operator. The question of locality of V has not been studied in YMCS theories. Thus it is possible that V is nonlocal, in which case the magnetic symmetry does not actually have a local order parameter. If that is the case the symmetry can be unbroken, W may have perimeter law and there could still be no magnetic vortex states. We note that a similar situation occurs in YM theories with fundamental matter[20]. This is the question we want to address in this note. We want to examine more carefully the question of locality of the vortex operator in YMCS and understand whether the spectrum contains magnetic vortices.

In Section 2 we start our discussion by considering a simpler theory - compact electrodynamics with CS term. Conceptual questions here are similar but the technical side is much simpler. Our results are somewhat surprising. We find that generically the theory does indeed contain local vortex operators and a global discrete magnetic symmetry which is unbroken. Nevertheless in the continuum limit there are no magnetic vortices in the spectrum. The reason is that the energy of such a vortex is logarithmically UV divergent. We find however that with a particular scaling of the CS coefficient (logarithmically vanishing when UV cutoff is removed) the energy of the vortex becomes finite. This suggests

that the theory may indeed have a phase with finite energy vortex states and vanishing photon mass. In Section 3 we extend our discussion to non-Abelian YMCS theories. We find that here again the local gauge invariant vortex field exists. The situation in the continuum limit appears to be similar to the Abelian case with a possible vortex phase at small values of the CS coefficient. Section 4 contains a short summary of our results.

2 Compact QED with the Chern Simons term.

The Lagrangian of Abelian Chern Simons theory in the formal continuum limit is

$$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \quad (3)$$

The gauge coupling g^2 has dimension of mass and κ is dimensionless. Equations of motion read as $\partial_\nu F^{\mu\nu} = \kappa g^2 \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$. The mass of the gauge particle is $M = \kappa g^2$. The canonical structure of the theory is simplest in the Hamiltonian gauge, $A_0 = 0$.

The Hamiltonian is

$$H = \frac{1}{2g^2}(E_i^2 + B^2) \quad (4)$$

with canonical momenta related to the time derivatives of the fields by $\Pi^i = -\frac{1}{g^2}\dot{A}_i + \frac{\kappa}{2}\epsilon^{ij}A_j$.

The gauge fields and canonical momenta form the canonical algebra, and the algebra involving the electric fields is

$$[E_i(\vec{x}), E_j(\vec{y})] = -i\kappa g^4 \epsilon_{ij} \delta^2(\vec{x} - \vec{y}), \quad [A_i(\vec{x}), E_j(\vec{y})] = -ig^2 \delta_{ij} \delta^2(\vec{x} - \vec{y}) \quad (5)$$

The Gauss law,

$$\partial_i E^i - \kappa g^2 \epsilon^{ij} \partial_i A_j = 0 \quad (6)$$

on a given spatial slice Σ which we take to be the plane, generates time independent local gauge transformations. The elements of the local gauge group near a point $x_0 \in \Sigma$ take the form

$$U(x_0) = \exp\left\{\frac{1}{g^2}i \int_{\Sigma} d^2x \lambda(x, x_0)(\partial_i E^i - g^2 \kappa \epsilon^{ij} \partial_i A_j)\right\} \quad (7)$$

such that $UAU^{-1} = A + d\lambda$. In the non-compact theory $\lambda \in \mathbb{R}$ should be a single-valued function on Σ such that the eigenvalue of the operator (7) on physical states is unity. Singular λ 's correspond to transformations which are in general nontrivial on the physical states.

Our interest however is in the compact theory. This means that magnetic vortices of flux 2π must be physically unobservable. As discussed in [21] this amounts to further restricting the physical Hilbert space to states which are trivial under the action of the vortex operator.

In other words certain large gauge transformations must act on the physical states trivially in the compact theory as opposed to the non-compact one. The compact gauge group therefore includes these singular gauge transformations in addition to the regular ones, which form the gauge group in the noncompact theory. Consider a multi-valued angle function $\theta(x, x_0)$ which is singular at one point and has a discontinuity along a straight curve $C(x_0)$ that starts at the point x_0 and goes to infinity. The operator of the gauge transformation with this singular gauge function creates a magnetic vortex. Its explicit form (after partial integration and dropping a boundary term owing to the fact that all gauge invariant fields decay at infinity) is

$$\tilde{V}(x_0) = \exp\left\{-\frac{1}{g^2}i \int_{\Sigma} d^2x \tilde{\partial}_i \theta(x - x_0) \left(E^i - g^2 \kappa \epsilon^{ij} A_j\right)\right\} \quad (8)$$

We have defined

$$\tilde{\partial}_i \theta(x - x_0) = \partial_i \theta(x - x_0) - 2\pi \epsilon_{ij} c(x)_j \delta(x - C(x_0)) = \frac{\epsilon_{ij} (x - x_0)_j}{(x - x_0)^2} \quad (9)$$

$c(x)_j$ is a unit vector tangent to the curve $C(x, x_0)$.

We may want to include \tilde{V} into the compact gauge group. However to be part of the gauge group, it must commute (at least weakly) with other elements of the group. One can check explicitly that $\tilde{V}(x, x_0)$ does not commute with the elements of the noncompact group. To rectify this situation we define, following [21] a slightly modified operator

$$V(C, x_0) = \exp\left\{\frac{2\pi i}{g^2} \epsilon_{ij} \int d^2x c(x)_j \delta(x - C(x, x_0)) E_i(x)\right\} \quad (10)$$

This operator is merely a “collection” of the electric fields which are perpendicular to the curve $C(x_0)$. More explicitly one can write it in the following form

$$V(C, x_0) = \exp \frac{2\pi i}{g^2} \epsilon_{ij} \int_C dl_i E_j(x) \quad (11)$$

Gauge invariance of this operator, $[V, U] = 0$, follows immediately. We also need to check the commutativity of $V(x)$ with $V(y)$. Defining the volume form $v = \frac{1}{2} \epsilon_{ij} dx^i \wedge dx^j$ on Σ we have the following commutation rules,

$$V(C_0)V(C_1) = V(C_1)V(C_0)\exp \left\{ i8\pi^2 \kappa L(C_0, C_1) \right\} \quad (12)$$

Where $L(C_1, C_2) = \int_{\Sigma} v \delta^2(C_0 - C_1)$. It is clear that if the curves cross each other then $L = \pm 1$ and if they are parallel $L = 0$. In order that V be a Lorentz scalar the commutator should not depend on the curves C_1 and C_2 To guarantee this we need to set

$$4\pi\kappa = k \in \mathbb{Z} \quad (13)$$

We find therefore that the requirement of compactness quantizes the coefficient of the Chern Simons term very much like in the non-Abelian theory .

To see that (11) creates magnetic vortices of integer strength it is enough to consider the commutator

$$[B(x), V^m(x_0)] = 2\pi m \delta^2(x - x_0) V^m(x_0), \quad m \in \mathbb{Z}. \quad (14)$$

Therefore $V^m(x_0)$ creates a point-like magnetic vortex of vorticity m and magnetic flux $2\pi m$. Being gauge invariant this operator also creates an electric charge

$$Q = \frac{1}{g^2} \int_{\Sigma} d^2x \partial_i E^i = \frac{k}{4\pi} \int_{\Sigma} d^2x B = \frac{mk}{2} \quad (15)$$

Since V has to be included in the gauge group, the magnetic flux and the electric charge created by it must be unobservable. Therefore the Hamiltonian of the theory must commute with V . The noncompact Hamiltonian eq.(4) does not quite do the job. It should be modified but in such a way that in the continuum limit the same form is recovered for smooth fields. The modified Hamiltonian that satisfies these conditions has been suggested

in [21]. Since the UV structure is important for our considerations, it is most usefully presented in the lattice notations

$$H_B = \frac{1}{a^4 g^2 n^2} \sum_x (1 - \text{Re } e^{i n a^2 B(x)}), \quad H_E = \frac{m^2 g^2}{4\pi^2 a^2} \sum_x (1 - \text{Re } e^{i 2\pi \frac{a}{m g^2} \epsilon_{ij} \hat{n}_j E_i(x)}), \quad (16)$$

a is the lattice spacing and $m, n \in \mathbb{Z}$ and \hat{n}_j is the unit vector parallel to the link. The normalization of the electric and magnetic terms is such that in the naive continuum limit $a \rightarrow 0$ they reduce to B^2 and E^2 respectively.

Using the Gauss' law one can see that if $2n = k$, the magnetic part H_B becomes a combination of the vortex operators V . Therefore without loss of generality we assume $2n < k$. For $m = 1$ the electric part of the Hamiltonian is also a sum of a fundamental vortex and anti-vortex, and we take $m > 1$.

Now that we have the formulation of the compact CS QED we can ask about the locality properties of vortex operators. The operator V we have considered so far is of no interest of itself, since it is trivial on all physical states. We thus have to look at the operators which create magnetic flux smaller than 2π

$$V_p(C, x_0) = \exp \frac{2p\pi i}{g^2} \epsilon_{ij} \int_C dl_i E_j(x), \quad p \in \mathbb{Q} \quad (17)$$

Here p is a rational number $p \in (0, 1)$. The question we are asking is, are there such values of p for which V_p is a gauge invariant local operator. The gauge invariance with respect to the noncompact gauge group is straightforward, since V_p only depends on the electric field, and the electric field itself is gauge invariant. However V_p should also commute with the "fundamental" vortex V , since V is part of the gauge group. Therefore we have

$$[V, V_p] = 0 \quad \implies kp = l \in \mathbb{Z} \quad \text{and } l < k. \quad (18)$$

This condition is already informative. For example we see that there are no non-trivial vortices in $k = 1$ theory. The condition of locality requires that $V_p(x)$ commute with each other at different points x and y . This commutator also should be independent of the contour C in the definition eq.(17).

$$[V_p(x), V_p(y)] = 0 \quad \implies kp^2 = r \in \mathbb{Z} \quad (19)$$

Both equations (18) and (19) have to be satisfied for the existence of non-trivial local vortices. Whether it is possible or not to satisfy these equations clearly depends on the CS coefficient k . For example there are no solutions for $k = 2$ and $k = 3$ theories. For $k = 4$ we can choose $l = 2$ and this gives a vortex of vorticity $p = 1/2$. In general one can solve the constraints in the following way. Writing CS coefficient in terms of its prime factors, $k = q_1 q_2 q_3 \dots q_m$, where all q_i are not necessarily different, one has the following two conditions to satisfy

$$q_1 q_2 q_3 \dots q_m p = l, \quad q_1 q_2 q_3 \dots q_m p^2 = r. \quad (20)$$

The first condition is solved if p divides k which means, without loss of generality, we have

$$p = \frac{1}{q_1 q_2 q_3 \dots q_i}, \quad \text{where } i < m \quad (21)$$

Using this in the second condition one can see that the most general form of k which allows vortices will be

$$\text{If } k = t^2 z \implies p = \frac{1}{t}, \quad t \geq 2 \text{ and } t, z \in \mathbb{Z} \quad (22)$$

For example if k is a prime number there are no solutions. Generically it is easier to find a solution at large values of k .

The above relations also show that should a solution exist, there is always a vortex of minimal vorticity. All the other local vortices are simple powers of this minimal vortex. For example for $k = 36$, the above conditions give three solutions (and their integer multiples), $p = (1/2, 1/3, 1/6)$. Obviously “ $p = 1/6$ ” is the minimal vortex. The minimal value of $p = \frac{1}{w}$ determines the global magnetic symmetry group of the theory as Z_w .

One last requirement that V_p should satisfy, is locality with respect to the energy density eq.(16). In obvious notation

$$[h_E(x), V_p(y)] = 0, \quad x \neq y \implies \frac{kp}{m} \in \mathbb{Z} \quad (23)$$

This can always be satisfied by choosing $m = k$. To satisfy the other condition

$$[h_B(x), V_p(y)] = 0 \implies \frac{np}{2} \in \mathbb{Z} \quad (24)$$

we can take $n = 2$. Certainly one can define other Hamiltonians which will be compatible with the above conditions. We see therefore that for those values of k for which vortex operators are local with respect to each other we can always choose the Hamiltonian such that they are also local relative to the Hamiltonian density.

Thus we conclude that for many values of k local physical vortex operators exist. They are order parameters for a global Z_w magnetic symmetry. The value of w is determined by k through the solution of the equations for minimal p . Thus the argument described in the introduction applies and, at least in the lattice theory there are vortex states. Calculating their energy in the lattice theory is not a simple matter. However the interesting question is whether these states survive in the continuum limit. That is to say, whether their energy stays finite as the lattice spacing approaches zero.

We note that the continuum limit of the theory (16) is a slightly delicate matter. One must certainly take the limit $a \rightarrow 0$. However since we expect in the continuum the scaling B and E to be $ga^{-3/2}$, to guarantee the expandability of the electric exponential in eq.(16) we should take $gka^{1/2} \gg 1$. For the expandability of the magnetic exponential we should keep $ga^{1/2} \ll 1$. These two conditions are compatible if $k \gg 1$. With this scaling the mass of the photon can be also kept finite in the continuum limit. One should keep in mind that this scaling of the couplings is sufficient to get the continuum limit, but it may not be necessary. In particular it could certainly happen that at finite k the scaling of B and E changes close to the cutoff and the continuum limit still exists.

In the continuum limit for smooth configurations of the fields the theory is described by the Lagrangian eq.(3). However while solving continuum equations we may sometimes encounter field configurations with fast variations. For these configurations it is important to take into account the compactness of the theory. In particular consider the electric field created by the "minimal" vortex operator $V_{1/w}$.

$$[V(x), E_i(y)] = E_i(y) + e_i(x, y), \quad e_i(x, y) = \frac{1}{w} g^4 \kappa \hat{n}(y)_i \delta(y - C(x, y)) \quad (25)$$

where $\hat{n}(y)$ is the vector tangential the curve C at the point y . Since the operator $V(x)$ is local, its only observable action in the compact theory is at the point x . However if we

just calculate the energy using the naive Hamiltonian eq.(4) we find infrared divergence proportional to the length of the curve C . Clearly if faced with this type of configurations in continuum calculations we should subtract this infrared divergence by hand. Rather than do this we find it convenient to think about it in the following way. Let us split the general electric field configuration into a smooth piece and a piece that contains arbitrary number of strings of the type of eq.(25)

$$E^i = E_{smooth}^i + e^i \quad (26)$$

and subtract the contribution of e_i in the Hamiltonian. The only remnant of e_i then is in the Gauss' law, since e_i of eq.(25) corresponds to a pointlike charge $\frac{k}{2w}$ at the point x . Thus the smooth field E_{smooth}^i satisfies not the naive Gauss' law, but rather a modified one

$$\partial_i E_{smooth}^i \Big|_{\text{mod } \frac{k}{2w} \delta^2(x)} - \kappa g^2 \epsilon^{ij} \partial_i A_j = 0 \quad (27)$$

In other words we can work entirely in terms of E_{smooth} if we remember that we may allow Gauss' law to be violated by the presence of pointlike charges of charge $\kappa g^2/w$. The appearance of w in this way is the only remnant of the compactness of the theory. In the following we will work in terms of the smooth fields but will drop the subscript *smooth* for brevity.

With this caveat in mind, to determine the energy of the magnetic vortex in the continuum limit we now should solve the continuum equations of motion. For a minimum vorticity solution $1/w$ following [23, 24] one can take the time independent symmetric ansatz,

$$A_i(r) = \epsilon_{ij} \frac{x_j}{r^2} [g(r) - \frac{1}{w}] \quad A_0(r) = h(r) \quad (28)$$

The equations of motion read

$$g''(r) - \frac{1}{r} g'(r) - r M h'(r) = 0, \quad (29)$$

$$h''(r) + \frac{1}{r} h'(r) - \frac{M}{r} g'(r) = 0, \quad (30)$$

where $M = g^2\kappa$. We are looking for the solutions with vorticity $1/w$. The magnetic field is $B = -\frac{1}{r}g'(r)$ so we impose $g(0) = 1/w$ and $g(\infty) = 0$ and we also demand that the fields decay exponentially at infinity. The solution under these conditions can be found as

$$g(r) = \frac{Mr}{w}K_1(Mr), \quad B(r) = \frac{M^2}{w}K_0(Mr) \quad (31)$$

$$h(r) = -\frac{M}{w}K_0(Mr) \quad E^i = -\frac{x^i M^2}{wr}K_1(Mr), \quad (32)$$

where K_0, K_1 are the Bessel functions. The energy of this vortex follows as

$$\mathcal{E} = \frac{\pi}{g^2} \int_0^\infty r dr \left\{ \left(\frac{dh}{dr} \right)^2 + \frac{1}{r^2} \left(\frac{dg}{dr} \right)^2 \right\} \quad (33)$$

$$\mathcal{E} = \frac{\pi\kappa}{w^2} M \left(-\gamma_E + \ln 2 - \ln \frac{M}{\Lambda} \right) \quad (34)$$

where Λ is the ultraviolet cutoff scale.

This result warrants several comments. First, we see that the energy of the vortex is IR finite. This is closely related to the fact that the locality of the operator $V_{1/w}$ allowed us to "violate" the Gauss' law. Looking at the electric and magnetic fields in eqn (32) we indeed see that the naive Gauss' law is violated precisely by the amount allowed by eq.(27). Without this deficit in the Gauss law any solution would have infrared divergent energy. This is because any nonvanishing magnetic flux Φ would require a long range electric field $E_i \propto g^2\Phi\kappa\frac{x_i}{x^2}$ to satisfy the Gauss' law. The Coulomb energy of this field is logarithmically divergent in IR. The absence of this long range piece in the field and the associated IR divergence in the energy is what distinguishes the compact and the noncompact theory.

Second, the energy is UV divergent. Thus the magnetic vortices do not survive in the continuum limit as finite energy excitations. This of course does not contradict our original argument. The Z_w symmetry is unbroken in the vacuum and excitations carrying the quantum numbers of this symmetry are very heavy. The symmetry therefore seems completely irrelevant for the low energy dynamics. The curious thing though is that although the energy of the vortex is large, it is still much lower than the natural ultraviolet

scale Λ . The vortices therefore are not genuine ultraviolet objects in the lattice theory, but rather occupy an intermediate scale between the UV scale Λ and the IR scale M . In fact bringing M down to zero makes vortices light. At $M = \mu \ln^{-1} \frac{\Lambda}{\mu}$ the energy of the vortex is finite⁴. This behavior in fact is very reminiscent of vortices in the Higgs phase of the Abelian Higgs model. The mass of such a vortex at weak coupling is very large; $M \propto M_v^2/g^2 \ln M_H/M_v$, where M_v is the mass of the massive photon and M_H is the mass of the Higgs particle. However as the photon mass decreases, that is as the theory approaches the phase transition line, vortices become light. On the phase transition itself they in fact become massless and condense in the Coulomb phase. It is not unlikely that similar phenomenon occurs in our model. As κ decreases at fixed g^2 the photon becomes lighter, and the mass of the vortex also decreases. It could happen that at some value of κ the vortices actually become massless and drive a phase transition into a phase with broken magnetic symmetry. A transition of precisely such type was conjectured to happen in the lattice model at $k = 8$ in [25] and was seen in the variational calculation of [21]. Of course, within the naive continuum limit we consider here we are unable to see such behavior. However the fact that the vortices become light within the validity of the naive continuum limit is quite suggestive in this respect. It is interesting to note that if the vortices indeed condense, the magnetic Z_w symmetry is spontaneously broken. By virtue of the argument given in the introduction this means that the low k phase is confining.

3 Non-Abelian theories.

We now want to extend our analysis to non-Abelian theories. We will study the $SU(N)$ gauge theory with the the Lagrangian

$$\mathcal{L} = \frac{1}{2g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} - \kappa \epsilon^{\mu\nu\lambda} \text{tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) \quad (35)$$

⁴The reader may wonder why we are not bothered by the factor κ in eq.(34). After all we saw that in the continuum limit naturally $\kappa \rightarrow \infty$. The point is that the flux of the minimal vortex generically scales as $w^2 \propto k$. For example if $k = x^2$ with some integer x , then clearly the minimal solution of the eq.(22) corresponds to $w = x$. Thus k/w^2 is finite in the continuum limit.

CS coefficient has the well known quantization, $4\pi\kappa = k$ [3]. We denote $A_\mu = A_\mu^a T^a$ and $\text{tr}[T^a, T^b] = -\frac{1}{2}\delta^{ab}$. The Lie algebra generators obey $[T^a, T^b] = f^{abc}T^c$. The field strength and the covariant derivative are $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, $D_\mu = \partial_\mu + [A_\mu, \cdot]$. The classical equations of motion follow as

$$D_\mu F^{\mu\nu} - \frac{1}{2}g^2\kappa\epsilon^{\nu\lambda\mu}F_{\lambda\mu} = 0 \quad (36)$$

The canonical structure of this theory is similar to the Abelian case. In the $A_0^a = 0$ gauge we have [17]

$$\Pi_i^a = -\frac{1}{g^2}E_i^a + \frac{\kappa}{2}\epsilon^{ij}A_j^a, \quad \text{where} \quad E_i^a = \dot{A}_i^a. \quad (37)$$

The Hamiltonian is

$$H = \frac{1}{2g^2}\{(E_i^a)^2 + (B^a)^2\}, \quad \text{where} \quad B^a = \frac{1}{2}\epsilon_{ij}F_{ij}^a. \quad (38)$$

The canonical algebra is

$$[E_i^a(\vec{x}), E_j^b(\vec{y})] = -i\delta^{ab}\kappa g^4\epsilon_{ij}\delta^2(\vec{x} - \vec{y}), \quad [A_i^a(\vec{x}), E_j^b(\vec{y})] = -ig^2\delta^{ab}\delta_{ij}\delta^2(\vec{x} - \vec{y}) \quad (39)$$

In terms of the momenta the Gauss law is

$$(D_i\Pi_i)^a = -\frac{\kappa}{2}\epsilon^{ij}\partial_i A_j^a \quad (40)$$

In the non Abelian YMCS theory, the large Wilson loop still commutes with the Hamiltonian. This is obvious in the Hamiltonian formalism, since the commutation relation between the vector potential and the chromoelectric field is unaffected by the presence of the CS term. The form of the Hamiltonian in terms of A_i and E_i is also the same as without the CS term. Since W is a function of A_i only, its commutator with the Hamiltonian is exactly the same as in the theory without CS. Therefore the fundamental Wilson loop still generates a symmetry.

Again our question is whether the theory admits local vortex operators. In the non-Abelian theory our choices are more limited than in compact QED. In the SU(N) Yang

Mills theory, the only candidates for local operators are those that create quantized flux [5, 18]. The vortex operator in YM theory is [18, 19]:

$$V(x) = \exp\left\{\frac{4\pi i}{g^2 N} \int_C dy^i \epsilon_{ij} \text{Tr}(Y E_i(y))\right\} \quad (41)$$

where the hyper charge generator Y is defined as

$$Y = \text{diag}(1, 1, \dots, -(N-1)) \quad (42)$$

and the electric field is taken in the matrix notation $E_i = T^a E_i^a$. It can be proven that in SU(N) YM theory, this operator despite its nonlocal and gauge non-invariant appearance is in fact a local, gauge invariant, Lorentz scalar field [18, 19]. The way it was constructed there was to require that it satisfies the 'tHooft algebra [5] with the fundamental Wilson loop

$$V^\dagger(x)W(C)V(x) = \exp\left\{\frac{2\pi i}{N}n(x, C)\right\}W(C) \quad (43)$$

with $n(x, C)$ being the linking number on the plane between the point x and the closed curve C .

We claim that the operator in (41) is also the appropriate vortex operator when a Chern-Simons term is included for the gauge field. The commutation relation (43) is still satisfied by the expression (41). However an additional requirement was that V be gauge invariant. Here we should be more specific what we mean by that. The expression (41) is not explicitly gauge invariant since it depends on the chromoelectric field in the hyper charge direction. However for $\kappa = 0$ it has been proven that the matrix elements of V between physical states (those that satisfy Gauss' law) and non-physical states (non-singlet under gauge transformations) vanish. This means that when we calculate matrix elements of any number of operators V between gauge invariant states, only intermediate states from the physical sector contribute and so the form (41) can be safely used even though it is not explicitly gauge invariant. Let us briefly recap the proof[2]. The wave functional of any physical state in a theory without Chern Simons term depends only on

gauge invariant characteristics of the vector potential, i.e. only on the values of Wilson loops over all possible contours.

$$\Psi[A_i] = \Psi[\{W(C)\}] \quad (44)$$

Consider the action on such a state of the operator $V(x)$ and its gauge transform $V_\Omega(x)$.

$$\begin{aligned} V|\Psi\rangle &= \Psi_V[A_i] = \Psi[\{VW(C)V^\dagger\}] \\ V_\Omega|\Psi\rangle &= \Psi_V^\Omega[A_i] = \Psi[\{V_\Omega W(C)V_\Omega^\dagger\}] \end{aligned} \quad (45)$$

The action of $V(x)$ and $V_\Omega(x)$ on the Wilson loop is identical - they both multiply it by the center group phase (which stays unaffected by Ω) if x is inside C and do nothing otherwise.

Therefore

$$V|\Psi\rangle = V_\Omega|\Psi\rangle \quad (46)$$

for any physical state Ψ . Thus we have

$$\Omega V|\Psi\rangle = \Omega V_\Omega^\dagger|\Psi\rangle = V|\Psi\rangle \quad (47)$$

where the first equality follows from the fact that a physical state is invariant under action of any gauge transformation Ω and the second equality follows from eq.(46). But this equation is nothing but the statement that the state $V|\Psi\rangle$ is physical, i.e. invariant under any nonsingular gauge transformation. Thus we have proved that V transforms a physical state into another physical state.

In the Chern-Simons theory the vortex operator should also be gauge invariant. We thus have to check that it transforms a physical state into another physical state. The difference with the pure YM theory is that the wave function of a gauge invariant state does not depend only on the Wilson loops. The physical wave function should satisfy the following equation

$$i(D_i \frac{\delta}{\delta A_i})^a \Psi[A] = \frac{\kappa}{2} \epsilon_{ij} \partial_i A_j^a \Psi[A] \quad (48)$$

The general form of Ψ has been determined in [14] in terms of certain nonlinear variables. For our purposes we find it more convenient to work directly in terms of the vector potentials A_i . Let us take Ψ in the form

$$\Psi = \exp\{-iS\} \quad (49)$$

Then the eikonal S satisfies a linear inhomogeneous equation

$$D_i^{ab} \frac{\delta}{\delta A_i^b} S[A] = \frac{\kappa}{2} \epsilon_{ij} \partial_i A_j^a \quad (50)$$

The solution of the homogeneous equation is indeed any functional that depends on Wilson loops $S_0[W]$. We can find a particular solution of the inhomogeneous equation using the following argument. $S[A]$ must be a functional whose change under a standard gauge transformation of vector potentials $\delta A_i = D_i \lambda$ is proportional to $\frac{\kappa}{2} \int d^2x \epsilon_{ij} \partial_i A_j^a \lambda^a$. Such a functional can be represented as a Chern Simons action on a space with a boundary. Let us introduce an additional coordinate $\tau \in [-\infty, 1]$ and functions of three coordinates $A_i(x, \tau)$ so that at the boundary $\tau = 1$, the value of these functions is equal to the value of the vector potentials in our theory $A_i(x, \tau = 1) = A_i(x)$. Let us write the Chern Simons term (in the Weyl gauge) on this manifold

$$S_{CS} = \int_{-\infty}^1 d\tau \int d^2x \epsilon_{ij} A_i^a(x, \tau) \dot{A}_j^a(x, \tau) \quad (51)$$

Under the τ independent gauge transformation this action changes by a boundary term

$$\delta S_{CS} = - \int d^2x \epsilon_{ij} \lambda^a(x) \partial_i A_j^a(x, \tau = 1) = - \int d^2x \epsilon_{ij} \lambda^a(x) \partial_i A_j^a(x) \quad (52)$$

which is precisely of the form required to satisfy eq.(50). A particular solution of eq.(50) is therefore

$$S_p = -\frac{\kappa}{2} S_{CS} = -\frac{\kappa}{2} \int_{-\infty}^1 d\tau \int d^2x \epsilon_{ij} A_i^a(x, \tau) \dot{A}_j^a(x, \tau) \quad (53)$$

The introduction of the extra coordinate τ and the expression eq.(51) is not at all unnatural. One should view this extra coordinate as parameterizing a curve in the field space. With this interpretation we have

$$d\tau \dot{A}_j(x, \tau) = \delta A_j \quad (54)$$

and

$$\int_{-\infty}^1 d\tau \int d^2x \epsilon_{ij} A^i(x, \tau) \dot{A}_j(x, \tau) = \int_{\mathcal{C}} \delta A_i \epsilon_{ij} A_j \quad (55)$$

where the line integral is taken over the trajectory \mathcal{C} in the field space which ends at the point $\{A_i(x)\}$.

We have thus determined the general form of the wave function of a physical state in the YMCS theory to be

$$\Psi[A] = \exp\left\{i\frac{\kappa}{2} \int_{-\infty}^1 d\tau \int d^2x \epsilon_{ij} A^i(x, \tau) \dot{A}_j(x, \tau)\right\} \Psi_0[W] \quad (56)$$

Now it is straightforward to see how the vortex operator acts on it. Under the action of the vortex operator

$$V(x) A_i^a(y, \tau) V^\dagger(x) = A_i(x, \tau) + \frac{4\pi}{N} \text{Tr} Y T^a \epsilon_{ij} \int dz_j \delta^2(z - C(x, y)) \quad (57)$$

Remembering that $E_i^a = -g^2 \Pi_i^a + g^2 \kappa \epsilon_{ij} A_j^a$, we see that the change in the phase factor in the wave functional is exactly cancelled by the A -dependent term in the vortex operator eq.(41).

$$V(x) e^{-iS_p} V^\dagger(x) = e^{-iS_p}. \quad (58)$$

Thus

$$V\Psi[A] = \exp\left\{i\frac{\kappa}{2} \int_{-\infty}^1 d\tau \int d^2x \epsilon_{ij} A^i(x, \tau) \dot{A}_j(x, \tau)\right\} \Psi_0[V^\dagger W V] \quad (59)$$

Clearly a gauge transformed vortex operator V_Ω has exactly the same action on the wave functional $\Psi[A]$,

$$V\Psi = V_\Omega\Psi \quad (60)$$

which establishes gauge invariance of V in the same sense as in the YM theory.

We now can check the locality of the operator V by calculating straightforwardly the relevant commutation relation. A simple calculation gives $[V(x), V(y)] = 0$. Thus the operators are local with respect to each other. When considering the locality with

respect to the Hamiltonian density we are faced with the same ambiguity as in the Abelian theory. The electric part of naive continuum Hamiltonian is not local relative to V , since E_i is shifted by the action of V along the curve C . Just like in the Abelian case one should consider a properly regularized version of H in order to be able to draw a definite conclusion. In the non-Abelian case such a regularized Hamiltonian is not available. However in the Abelian case we saw that there is quite a lot of flexibility in defining such a regularized version. In particular we saw that whenever the vortex operators were local with respect to each other, we were always able to define the local Hamiltonian density. We expect that this situation persists in the non-Abelian theory too.

The situation in the continuum limit is again similar to the Abelian case. There are no finite energy solutions of the non-Abelian equations of motion which have finite vorticity. The only way to find such solutions would be again to relax the Gauss' law constraint by allowing point like charges which correspond to the singular chromoelectric field created by V . However again those IR finite configurations will have UV logarithmically divergent energy. In fact taking Abelian ansatz the YM equations of motion reduce to those we considered in the previous section and thus lead to the same energy dependence on the UV cutoff. Strictly speaking this conclusion is only valid for large enough value of k , since for small k quantum corrections to this classical analysis may be large. Thus again it is possible that at small k the theory is in a different phase as suggested in [15].

4 Summary.

To summarize, we have studied the question of locality of the vortex creation operator in compact Chern- Simons theories. We have found that compact CS QED does admit local vortex operators for many values of the CS coefficient k . The energy of the vortex excitations however generically is logarithmically UV divergent in the continuum limit. With a particular scaling of the CS coefficient these vortices become light and might condense at small values of k . Our results for non-Abelian CSYM theory are similar. Local vortex operators exist, but the particles that carry vorticity are heavy in the continuum limit.

These results are broadly compatible with suggestions made in the literature that at low values of the Chern Simons coefficient the YMCS theory might undergo a phase transition. If this happens it is very likely that this other phase has a broken magnetic symmetry and is therefore confining. This is a very interesting possibility which seems worthwhile exploring by numerical lattice methods.

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