Asymptotic equivalence of differential equations and asymptotically almost periodic solutions

M. U. Akhmet[1]*, M.A. Tleubergenova[2] and A. Zafer[3]

[1] Department of Mathematics and Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey, marat@metu.edu.tr
[2] Department of Mathematics, K. Zhubanov Aktobe State Pedagogical University, 463000 Aktobe, pr. Moldagulovoy, 34, Kazakhstan,

[3] Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey, zafer@metu.edu.tr

Keywords: *Asymptotic equivalence; Biasymptotic equivalence; Asymptotically almost periodic solution; Biasymptotically almost periodic solution*

2000 Mathematics Subject Classification: 34A30, 34C27, 34C41, 34D10.

*Corresponding author. M.U. Akhmet is previously known as M. U. Akhmetov.

Proposed running head: Asymptotic equivalence, almost periodic solutions

Abstract

In this paper we establish asymptotic (biasymptotic) equivalence between spaces of solutions of a given linear homogeneous system and a perturbed system. The perturbations are of either linear or weakly linear characters. Existence of a homeomorphism between subspaces of almost periodic and asymptotically (biasymptotically) almost periodic solutions is also obtained.

1 Introduction and Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ be sets of all natural and real numbers, respectively. Denote by $||\cdot||$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$, and by $C(\mathbb{X}, \mathbb{Y})$ the space of all continuous functions defined on $\mathbb{X}$ with values in $\mathbb{Y}$.

We shall recall the definitions of almost periodic and asymptotically almost periodic functions, see [1, 5, 7] for more details.

A number $\tau \in \mathbb{R}$ is called an $\epsilon-$ translation number of a function $f \in C(\mathbb{R}, \mathbb{R}^n)$ if $||f(t+\tau) - f(t)|| < \epsilon$ for all $t \in \mathbb{R}$. A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is called almost periodic if for a given $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists a relatively dense set of $\epsilon-$ translation numbers of f. The set of all almost periodic functions is denoted by $\mathcal{AP}(\mathbb{R})$.

A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is called asymptotically almost periodic if there is a function $g \in \mathcal{AP}(R)$ and a function $\phi \in C(\mathbb{R}, \mathbb{R}^n)$ with $\lim_{t\to\infty} \phi(t) = 0$ such that $f(t) = g(t) + \phi(t)$.

The basic definition of an almost periodic function given by H. Bohr has been modified by several authors [1, 5, 7, 11, 21, 27]. Below we introduce a new notion with regard to almost periodic functions.

Definition 1.1 *A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is called biasymptotically almost periodic if $f(t) = g(t) + \phi(t)$ for some $g \in \mathcal{AP}(\mathbb{R})$ and $\phi \in C(\mathbb{R}, \mathbb{R}^n)$ with* $\lim_{t\to\pm\infty} \phi(t) = 0$.

Note that every biasymptotically almost periodic function is a pseudo almost periodic function [27], but not conversely.

In this paper we are concerned with the linear system

$$y' = [A(t) + B(t)]y, \tag{1}$$

which may be viewed as a perturbation of

$$x' = A(t)x, \tag{2}$$

where $x, y \in \mathbb{R}^n$, and $A, B \in C(\mathbb{R}, \mathbb{R}^{n\times n})$.

Moreover we consider the quasilinear systems of the form

$$y' = Cy + f(t, y) \tag{3}$$

and the corresponding homogeneous linear system

$$x' = Cx, \tag{4}$$

where $x, y \in \mathrm{R}^n$, $C \in \mathbb{R}^{n\times n}$, and $f(t, x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ such that

$$f(t, 0) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Definition 1.2 ([2, 16]) *A homeomorphism between solutions $x(t)$ and $y(t)$ is called an asymptotic equivalence if $x(t) - y(t) \to 0$ as $t \to \infty$.*

Definition 1.3 *A homeomorphism between solutions $x(t)$ and $y(t)$ is called an biasymptotic equivalence if $x(t) - y(t) \to 0$ as $t \to \pm\infty$.*

Our main objective is to investigate the problem of asymptotic equivalence of systems and to prove the existence of asymptotically and biasymptotically almost periodic solutions of (1) and (3) .

The classical theorem of Levinson [14] states that if the trivial solution of (2) is uniformly stable, $A(t) \equiv A$, and

$$\int_0^\infty || B(t) || \; dt < \infty, \tag{5}$$

then (1) and (2) are asymptotically equivalent. In the case when A is not a constant matrix, Wintner [28] proved that the above conclusion remains valid if all solutions of (2) are bounded, (5) is satisfied, and

$$\liminf_{t\to\infty} \int_0^t \mathrm{Trace}[B(s)]\, ds > -\infty.$$

Later, Yakubovič [25] considered (3) and obtained asymptotic equivalence of (4) and (3), see [16, 25] for details. After the pioneering works of Levinson, Wintner, and Yakubovič, the problem of asymptotic equivalence of differential systems including linear, nonlinear, and functional equations has been investigated by many authors; see e.g. [2, 12–20], and the references cited therein. Two interesting articles in this direction which also motivate our study here in this paper were written by M. Ráb [18, 19]. In fact, the main result in [19] is an improvement of the earlier one in [18], which we have employed in our work.

Asymptotically almost periodic functions were introduced by Fréchet [8, 9]. The existence of this type of solutions was investigated by A.M. Fink [7] (Theorem 9.5) for the first time. For more results on the existence of asymptotically almost periodic solutions of different types of equations we refer to [12, 13, 15, 17, 24, 26] and the references cited therein. In this work we exploit the idea of A.M. Fink to obtain the existence of asymptotically almost periodic solutions of linear and quasilinear systems. Moreover, we prove a theorem about biasymptotic equivalence of linear systems and a theorem on the existence of biasymptotically almost periodic solutions. Apparently the notions of biasymptotic equivalence and a biasymptotically almost periodic function are introduced for the first time in this paper.

The paper is organized as follows. In the next section, we prove a main lemma of Ráb and obtain sufficient conditions concerning the asymptotic equivalence of (1) and (2), and the existence of a family of asymptotically almost periodic solutions of the system (1). The third section is devoted to the problem of the asymptotic equivalence of systems (4) and (3) and the problem of existence of asymptotically almost periodic solutions of the system (3). The last section concerns with biasymptotic equivalence problem and the existence of biasymptotically almost periodic solutions of (1). In addition, examples are given to illustrate the results.

2 Asymptotic equivalence of linear systems and asymptotically almost periodic solutions

Let $X(t)$, $X(0) = I$, be a fundamental matrix solution of (2). Setting $y = X(t)u$, we easily see from (1) that

$$u' = P(t)u, \tag{6}$$

where $P(t) = X^{-1}(t)B(t)X(t)$.

Assume that

(C_1) $\int_0^\infty ||P(t)||dt < \infty.$

The following lemma has been obtained Ráb [18], [19], for which we include a proof for convenience.

LEMMA 2.1 *If* (C_1) *is valid then the matrix differential equation*

$$\Psi' = P(t)(\Psi + I) \tag{7}$$

has a solution $\Psi(t)$ *which satisfies* $\Psi(t) \to 0$ *as* $t \to \infty$.

Proof. Construct a sequence of $n \times n$ matrices $\{\Psi_k\}$ defined on $R_+ = [0, \infty)$ as follows:

$$\Psi_0(t) = I, \quad \Psi_k(t) = -\int_t^\infty P(s)\Psi_{k-1}(s)\, ds \ \text{ for } k = 1, 2, \ldots.$$

Fix ϵ, $0 < \epsilon < 1$. In view of (C_1) there exists a $t_1 > 0$ such that

$$\int_t^\infty ||P(s)|| ds < \epsilon \quad \text{for all } t > t_1.$$

It follows that $||\Psi_k(t)|| < \epsilon^k, \quad k \in N$, and consequently the series $\sum_{k=1}^\infty \Psi_k(t)$ is convergent uniformly for $t \in [t_1, \infty)$. Letting $\Psi(t) = \sum_{k=1}^\infty \Psi_k(t)$, one can easily check that Ψ satisfies

$$\Psi(t) = -\int_t^\infty P(s)[I + \Psi(s)] ds \tag{8}$$

and hence it is a solution (7). From (8) it also follows that $\Psi \to 0$ as $t \to \infty$, which completes that proof.

We may assume that

(C_2) $\lim_{t\to\infty} X(t)\Psi(t) = 0$.

THEOREM 2.1 *Suppose that conditions* (C_1) *and* (C_2) *hold. Then (1) and (2) are asymptotically equivalent.*

Proof. Let t be sufficiently large, $t \geq t_1$ say. In view of (8) we see that the function $u(t) = [I + \Psi(t)]c$, $c \in \mathbb{R}^n$, is a solution of (6) defined on $[t_1, \infty)$ and hence

$$y(t) = X(t)[I + \Psi(t)]c \tag{9}$$

is a solution of (1) .

Since $\Psi(t) \to 0$ as $t \to \infty$, there exists a $t_2 > t_1$ such that $I + \Psi(t_2)$ is nonsingular. Let $x^0 = X(t_2)c$ and $y^0 = X(t_2)(I + \Psi(t_2))c$. Denote by $y(t, c) = y(t, t_2, x^0)$ and $x(t, c) = x(t, t_2, y^0)$ the solutions of (1) and (2) satisfying $x(t_2) =$

x^0 and $y(t_2) = y^0$, respectively. Now, because of the existence and uniqueness of solutions of linear differential equations and the fact that $I+\Psi(t_2)$ is nonsingular, the relation

$$y^0 = X(t_2)[I + \Psi(t_2)]X^{-1}(t_2)x^0$$

defines an isomorphism between solutions $x(t)$ of (2) and $y(t)$ of (1) such that

$$y(t) = x(t) + X(t)\Psi(t)c$$

for $t > t_1$. The last equality, in view of (C_2), completes the proof.

COROLLARY 2.1 *Suppose that the system (2) has a $k-$parameter ($k \leq n$) family σ_2 of almost periodic solutions, and that the conditions* (C_1), (C_2) *are satisfied. Then there exists a $k-$parameter family σ_1 of asymptotic almost periodic solutions of (1), and σ_1 is isomorphic σ_2.*

EXAMPLE 2.1 Consider the systems

$$x'' - 2(t+1)^{-2}x = 0 \tag{10}$$

and

$$y'' - [2(t+1)^{-2} + b(t)]y = 0, \tag{11}$$

where $b(t)$ is a continuous function defined on R_+. We assume that there exist real numbers $K_1 > 0$ and $\alpha > 0$ such that

$$\mid b(t) \mid < K_1 e^{-\alpha t} \quad \text{for all } t \in R_+. \tag{12}$$

Notice that (10) has solutions $x_1(t) = (t+1)^2$ and $x_2(t) = (t+1)^{-1}$.

If we transform the above second order equations into systems of the form (1) and (2) we identify that

$$A = \begin{bmatrix} 0 & 1 \\ 2/(t+1)^2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ b(t) & 0 \end{bmatrix}.$$

It is easy to see that for a given $\epsilon > 0$ there exists $K > 0$ such that

$$\| P(t) \| \leq K e^{(-\alpha+\epsilon)t} \quad \text{for all } t \in R_+. \tag{13}$$

Fix ϵ so that $\beta := \alpha + \epsilon < 0$. Then (6) is satisfied, i.e.,

$$\int_0^\infty ||P(t)||\, dt < \infty,$$

and

$$\| \Psi(t) \| \leq e^{K|\beta|^{-1} e^{\beta t}} - 1. \tag{14}$$

Moreover, using (13) and (14) one can show that $X(t)\Psi(t) \to 0$ as $t \to \infty$, i.e., (C_2) holds.

Since the conditions of Theorem 2.1 are fulfilled, we may conclude that (10) and (11) are asymptotically equivalent whenever (12) holds.

EXAMPLE 2.2 Let $b(t)$ be a continuous function such that $|b(t)| \leq K_1 e^{-\alpha t}$ for all $t \in R_+$ for some $\alpha > 0$, $K_1 > 0$, and $C \in \mathbb{R}^{5\times 5}$. Consider

$$y' = (A + B(t))y, \tag{15}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi & 0 \\ 0 & 0 & -\pi & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix},$$

$B(t) = b(t)C$, and $\beta > 0$ satisfies $\alpha - 2\beta > 0$.

The associated equation $x' = Ax$ has a fundamental matrix

$$X(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 & 0 \\ -\sin t & \cos t & 0 & 0 & 0 \\ 0 & 0 & \cos \pi t & \sin \pi t & 0 \\ 0 & 0 & \sin \pi t & \cos \pi t & 0 \\ 0 & 0 & 0 & 0 & e^{\beta t} \end{pmatrix}.$$

The equality

$$P(t) = X^{-1}(t)B(t)X(t) = b(t)X^{-1}(t)CX(t)$$

implies that there exists a $K > 0$ such that $||P(t)|| \leq Ke^{-(\alpha-\beta)t}$ for all $t \in R_+$. Therefore, (C_1) is valid. We also see that

$$||\Psi(t)|| \leq \sum_{k=1}^{\infty} \frac{(Ke^{-(\alpha-\beta)t})^k}{(\alpha-\beta)^k k!} = 1 - e^{K(\alpha-\beta)^{-1}e^{-(\alpha-\beta)t}},$$

and hence $X(t)\Psi(t) \to 0$ as $t \to \infty$. In view of Corollary 2.1 we conclude that system (15) has a $4-$ parameter family of asymptotically almost periodic solutions. More precisely they are asymptotically "quasiperiodic" solutions and every such solution has a torus as the $\omega-$ limit set.

3 Asymptotic equivalence of linear and quasilinear systems

Let $\alpha = \min_j \Re\lambda_j$ and $\beta = \max_j \Re\lambda_j$, where $\Re\lambda_j$ denotes the real part of the eigenvalue λ_j of the matrix C. Let m_α and m_β be the maximum of degrees of elementary divisors of C corresponding to eigenvalues with real part equal to α and β, respectively. Clearly, there exist constants κ_1, κ_2 such that

$$||e^{Ct}|| \leq \kappa_1 t^{m_\beta - 1} e^{\beta t} \quad \text{and } ||e^{-Ct}|| \leq \kappa_2 t^{m_\alpha - 1} e^{-\alpha t}$$

for all $t \in R_+ = [0, \infty)$.

The following conditions are to be assumed:

(C_3) $||f(t, x_1) - f(t, x_2)|| \leq \eta(t)||x_1 - x_2||$ for all $(t, x_1), (t, x_2) \in R_+ \times R^n$, and for some nonnegative function $\eta(t)$ defined on R_+;

(C_4) $L := \int_0^\infty t^{m_\beta + m_\alpha - 2} e^{(\beta-\alpha)t} \eta(t)\, dt < \infty.$

LEMMA 3.1 *If* (C_3) *and* (C_4) *are valid, then every solution of*

$$u' = e^{-Ct} f(t, e^{Ct} u) \tag{16}$$

is bounded on R_+ *and for each solution* u *of (16) there exists a constant vector* $c_u \in \mathbb{R}^n$ *such that* $u(t) \to c_u$ *as* $t \to \infty$.

Proof. Let $u(t) = u(t, t_0, u_0)$ denote the solution of (16) satisfying $u(t_0) = u_0$, $t_0 \geq 0$. It is clear that

$$u(t) = u_0 + \int_{t_0}^{t} e^{-Cs} f(s, e^{Cs}u(s))\, ds, \quad t \geq t_0.$$

By using (C_3) and $f(t, 0) = 0$, we see that

$$|u(t)| \leq |u_0| + k_1 \int_{t_0}^{t} s^{m_\beta + m_\alpha - 2} e^{(\beta-\alpha)s} \eta(s) |u(s)|\, ds, \quad t \geq t_0$$

for some $k_1 > 0$. In view of (C_4) and Gronwall's inequality, we have

$$|u(t)| \leq |u_0|\, e^{k_1 \int_{t_0}^{t} s^{m_\beta + m_\alpha - 2} e^{(\beta-\alpha)s} \eta(s)\, ds} \leq |u_0|\, e^{k_1 L} < \infty, \quad t \geq t_0.$$

Let $M_0 = \max\{|u(t)| : t \in [0, t_0]\}$ and $M = \max\{M_0, |u_0|\, e^{k_1 L}\}$. Then we have $|u(t)| \leq M$ for all $t \in R_+$.

To prove the second part of the theorem, we first note that

$$\left| \int_{t_0}^{t} e^{-Cs} f(s, e^{Cs}u(s))\, ds \right| \leq M k_1 \int_{0}^{\infty} t^{m_\beta + m_\alpha - 2} e^{(\beta-\alpha)t} \eta(t) dt < \infty.$$

So we may define

$$c_u = u_0 + \int_{t_0}^{\infty} e^{-Cs} f(s, e^{Cs}u(s))\, ds.$$

It follows that

$$u(t) = c_u - \int_{t}^{\infty} e^{-Cs} f(s, e^{Cs}u(s)) ds,$$

which completes the proof.

The following lemma can be easily justified by a direct substitution.

LEMMA 3.2 *If $y(t)$ is a solution of (3), then there is a solution $u(t)$ of (16) such that*

$$y(t) = e^{Ct} u(t). \tag{17}$$

Conversely, if $u(t)$ is a solution of (16) then $y(t)$ in (17) is a solution of (3).

THEOREM 3.1 *If conditions* (C_3) *and* (C_4) *are satisfied, then every solution* $y(t)$ *of (3) possesses an asymptotic representation of the form*

$$y(t) = e^{C\,t}[c + o(1)],$$

where $c \in \mathbb{R}^n$ *is a constant vector and for a solution* $u(t)$ *of (16),*

$$o(1) = -\int_t^\infty e^{-Cs} f(s, e^{Cs}u(s))\, ds.$$

Proof. The proof follows from Lemma 3.1 and Lemma 3.2.

THEOREM 3.2 *Assume that* (C_3) *and* (C_4) *are fulfilled, and*

(C_5) $\lim_{t\to\infty} \int_t^\infty (s-t)^{m_\alpha - 1} s^{m_\beta - 1} e^{\alpha(t-s)} e^{\beta s} \eta(s) ds = 0.$

Then (3) and (4) are asymptotically equivalent.

Proof. In view of Lemma 3.1 we see that

$$\begin{aligned} y(t) &= e^{Ct}[c_u - \int_t^\infty e^{-Cs} f(s, e^{Cs}u(s)) ds] \\ &= x(t) - \int_t^\infty e^{C(t-s)} f(s, e^{Cs}u(s)) ds, \end{aligned}$$

where $x(t) = e^{Ct}c_u$ is a solution of (4) and $u(t) = u(t, t_0, u_0)$ is a solution of (16). It is clear that a given u_0 results in a one-to-one correspondence between $x(t)$ and $y(t)$. In view of (C_5), we also see that $x(t) - y(t) \to 0$ as $t \to \infty$, which completes the proof of the theorem.

In [25], Yakubovich proved that if

$$\lim_{t\to\infty} \int_t^\infty s^{m_\beta + m_\alpha - 2} e^{\beta s} \eta(s) ds = 0 \qquad (18)$$

then (3) and (4) are asymptotically equivalent. It is clear that if $\alpha > 0$ then condition (C_5) is weaker than (18).

The following assertion is a simple corollary of the Theorem 3.2

Corollary 3.1 *Suppose that conditions* (C_3)*,* (C_4)*,* (C_5) *hold, and that system (4) has a* $k-$ *parameter* ($k \leq n$) *family* γ_1 *of almost periodic solutions. Then (3) admits a* $k-$*parameter family* γ_2 *of asymptotically almost periodic solutions, and* γ_1 *is homeomorphic* γ_2*.*

4 Biasymptotic equivalence of linear systems and biasymptotically almost periodic solutions

With regard to systems (1) and (2) we shall make use of the following conditions:

(C_6) $A(-t) = -A(t)$ for all $t \in \mathbb{R}$.

(C_7) $B(-t) = B(t)$ for all $t \in \mathbb{R}$.

We will rely on the following two lemmas. The first lemma is almost trivial.

Lemma 4.1 *If* (C_6) *is satisfied then* $X(-t) = X(t)$ *for all* $t \in \mathbb{R}$*, and if in addition* (C_7) *holds then* $P(-t) = -P(t)$ *for all* $t \in \mathbb{R}$*.*

Lemma 4.2 *Assume that conditions* (C_1)*,* (C_6)*,* (C_7) *are valid. Then (7) has a solution* $\Psi(t)$ *which satisfies* $\Psi(-t) = \Psi(t)$ *for all* $t \in \mathbb{R}$ *and* $\Psi \to 0$ *as* $t \to \infty$*.*

Proof. By Lemma 2.1 there exists a solution $\Psi_+(t)$ of (7) which is defined for $t \geq t_1$ and satisfies $\Psi_+ \to 0$ as $t \to \infty$.

Using $P(-t) = -P(t)$ we see that

$$\int_{-\infty}^{-t_1} ||P(s)||\, ds = \int_{t_1}^{\infty} ||P(s)||\, ds < \epsilon.$$

We may define a sequence of $n \times n$ matrices $\{\tilde{\Psi}_k\}$ for $t \in (-\infty, 0]$ as follows:

$$\tilde{\Psi}_0(t) = I, \quad \tilde{\Psi}_k = \int_{-\infty}^{t} P(s)\tilde{\Psi}_{k-1}(s)\, ds \ \text{ for } k = 1, 2, \ldots.$$

As in the proof of Lemma 2.1, the matrix function $\Psi_-(t) = \sum_{k=1}^{\infty} \tilde{\Psi}_k(t)$ satisfies

$$\Psi(t) = \int_{-\infty}^{t} P(s)(I + \Psi(s))ds$$

and hence becomes a solution of (7) for $t \le -t_1$.

On the other hand, since $\Psi_0(-t) = \tilde{\Psi}_0(t) = I$ we have

$$\begin{aligned} \Psi_1(-t) &= -\int_{-t}^{\infty} P(s)\Psi_0(s)ds = \int_{t}^{-\infty} P(-s)\Psi_0(-s)ds \\ &= \int_{-\infty}^{t} P(s)\tilde{\Psi}_0(s)ds = \tilde{\Psi}_1(t). \end{aligned}$$

It follows by induction that $\Psi_k(-t) = \tilde{\Psi}_k(t)$ for all $k = 0, 1, 2, \ldots$ and for all $t \le -t_1$. Hence $\Psi_+(-t) = \Psi_-(t)$ for all $t \le -t_1$. Continuing Ψ_+ and Ψ_- as solutions of (7), one can obtain that $\Psi_+(-t) = \Psi_-(t)$ for all $t \le 0$.

Define

$$\Psi(t) = \begin{cases} \Psi_+ & \text{if } t \ge 0, \\ \Psi_- & \text{if } t < 0. \end{cases}$$

Clearly $\Psi(t)$ is a solution of (7) satisfying $\Psi(-t) = \Psi(t)$ for all $t \in \mathbb{R}$ and $\Psi \to 0$ as $t \to \infty$. This completes the proof.

The following results are analogous to Theorem 2.1 and Corollary 2.1.

THEOREM 4.1 *Suppose that* (C_1), (C_2), (C_6), (C_7) *are valid. Then (1) and (2) are asymptotically biequivalent.*

COROLLARY 4.1 *Suppose that* (C_1), (C_2), (C_6), (C_7) *are valid, and that (2) has a* $k-$ *parameter* ($k \le n$) *family* ν_1 *of almost periodic solutions. Then (1) admits a* $k-$*parameter family* ν_2 *of biasymptotically almost periodic solutions, and* ν_1 *is isomorphic* ν_2.

EXAMPLE 4.1 Consider the system

$$y' = (A(t) + B(t))y, \tag{19}$$

where

$$A(t) = \begin{pmatrix} \sin \pi t & 0 \\ 0 & \sin \sqrt{5}t \end{pmatrix},$$

$B(t) = \cos t\, e^{-\alpha|t|}\, C$ with $\alpha > 0$ a real number and $C \in \mathbb{R}^{2\times 2}$.

Applying Corollary 4.1 one can conclude that every solution of (19) is bi-asymptotically quasiperiodic.

References

[1] L. Amerio and G. Prouse, Almost periodic functions and functional equations, Van Nostrand Reinohold Company, New York, 1961.

[2] L. Cesari, Asymptotic behavior and stability problems in ordinary differential equations, Springer-Verlag, Berlin, Heidelberg, New York, 1963.

[3] J.M.Bebernes and N.X. Vinh, On the asymptotic behavior of linear differential equations, Amer. Math. Monthly. 72 (1965) 285–287.

[4] S. K. Choi, Y. H. Goo, and N. J. Koo, Asymptotic equivalence between two linear differential systems, Ann. differential Equations, 13 (1997) 44-52.

[5] C. Corduneanu, Almost periodic functions, Interscience Publishers, New York, 1961.

[6] R.B. Evans, Asymptotic equivalence of linear functional differential equations, J. Math. Anal. Applic., 51 (1975) 223-228.

[7] A.M. Fink, Almost periodic differential quations, Lecture notes in mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[8] M. Fréchet, Sur le théoréme ergodique de Birkhoff, (French) C. R. Acad. Sci. Paris, 213 (1941) 607-609.

[9] M. Fréchet, Les fonctions asymptotiquement presque-periodiques continues, (French) C. R. Acad. Sci. Paris, 213 (1941) 520-522.

[10] J.R. Haddock, T. Krisztin, and J. H. Wu, Asymptotic equivalence of neutral and infinite retarded differential equations, Nonlinear Anal., 14 (1990) 369-377.

[11] A. Halanay and D. Wexler Qualitative theory of impulsive systems, Edit. Acad. RPR, Bucuresti, 1968 (Romanian).

[12] Hernández M., Eduardo; Pelicer, Mauricio L., Asymptotically almost periodic and almost periodic solutions for partial neutral differential equations. Appl. Math. Lett., 18 (2005), 1265-1272.

[13] Hernández M., Eduardo; Pelicer, Mauricio L.; Dos Santos, Jos P. C. Asymptotically almost periodic and almost periodic solutions for a class of evolution equations. Electron. J. Differential Equations 2004, No. 61, 15 pp.

[14] N. Levinson, The asymptotic nature of solutions of linear systems of differential equations, Duke Math. J., 15 (1948) 111-126.

[15] T. Morozan, Asymptotic almost periodic solutions for Riccati equations of stochastic control, Stud. Cerc. Mat., 46 (1994) 603-612.

[16] V.V. Nemytskii and V.V. Stepanov, Qualitative theory of Differential Equations, Princeton University Press, Princeton, New Jersey, 1966.

[17] V. F. Puljaev and Z. B. Caljuk, Asymptotically almost periodic solutions of a Volterra integral equation, (Russian) Mathematical analysis, Kuban. Gos. Univ. Naucn. Trudy, 180 (1974) 127-131.

[18] M. Ráb, Über lineare perturbationen eines systems von linearen differentialgleichungen, Czech. Math. J., 83 (1958) 222-229.

[19] M. Ráb, Note sur les formules asymptotiques pour les solutions d'un systéme d'équations différentielles linéaires, Czech. Math. J., 83 (1958) 222-229.

[20] S. Saito, Asymptotic equivalence of quasilinear ordinary differential systems, Math. Japan, 37 (1992) 503-513.

[21] A.M. Samoilenko and N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[22] G. Sansone, R. Conti, Non-linear Differential Equations, McMillan, New York, 1964.

[23] M. Švec, Asymptotic relationship between solutions of two systems of differential equations, Czech. Math. J., 29 (1974) 44-58.

[24] C. Varsan, Stochastic differential equations and asymptotic almost periodic solutions, Rev. Roumaine Math. Pures Appl., 35 (1990) 485-493.

[25] V.A. Yakubovich, On the asymptotic behavior of systems of differential equations, Mat. Sbornik, 28 (1951) 217-240 .

[26] Rong, Yuan, Existence of almost periodic solutions of functional-differential equations of neutral type. J. Math. Anal. Appl., 165 (1992) 524-538.

[27] C. Zhang, Pseudo almost periodic solutions of some differential equations. J. Math. Anal. and Appl., 181 (1994) 62-76.

[28] A. Wintner, Linear variations of constants, Amer. Jour. Math., 68 (1946) 185-213.