

Deterministic Transformations of Multipartite Entangled States with Tensor Rank 2

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Abstract

Transformations involving only local operations assisted with classical communication are investigated for multipartite entangled pure states having tensor rank 2. All necessary and sufficient conditions for the possibility of deterministically converting truly multipartite, rank-2 states into each other are given. Furthermore, a chain of local operations that successfully achieves the transformation has been identified for all allowed transformations. The identified chains have two nice features: (1) each party needs to carry out at most one local operation and (2) all of these local operations are also deterministic transformations by themselves. Finally, it is found that there are disjoint classes of states, all of which can be identified by a single real parameter, which remain invariant under deterministic transformations.

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I. INTRODUCTION

Entanglement is a physical resource that enables one to carry out classically impossible tasks such as teleportation[1] and dense coding[2]. As such tasks require only special entangled states to be used in their implementation, the transformation of entanglement has become a major problem that has been widely studied. Entanglement purification[3] is an example of such transformations where mixed states, which are necessarily produced by noisy quantum communication channels when the entangled particles are distributed to distant parties, are converted into pure states. The main problem in entanglement transformations is to understand the conditions and the necessary protocols of the conversion process of a given state by local quantum operations assisted with classical communication (LOCC) to another desired state.

The transformations between pure bipartite-entangled states have been understood best due to the existence of the Schmidt decomposition of such states. The first important result on such transformations is the pioneering work of Bennett *et al.*[4] on the asymptotic transformations where multiple copies of the same state are needed to be converted, which establishes the entropy of entanglement as the sole currency of conversion. The next major step was taken by Lo and Popescu[5] who laid much of the groundwork for the transformation of single copies of pure bipartite states. Subsequently, Nielsen[6] discovered the rules of deterministic transformations where a simple connection between the entanglement transformations and the mathematical theory of majorization is established. Based on these developments, the conditions for probabilistic transformations have also been determined[7, 8].

In contrast to the bipartite case, Schmidt decomposition is not available for multipartite entanglement between three or more particles[9], and hence not much is known about the transformations of such states. Some general results about the transformations of multipartite pure states are given in Ref. 10. Apart from this, all known transformation rules are obtained for a restricted class of states. For example, it has been shown that if the given and desired states in question have a Schmidt decomposition, then the transformation rules for bipartite states can be directly applied[11]. There are also works focusing on states that lack a Schmidt decomposition. Namely, the probabilistic distillation of the tripartite GHZ state[12] and some aspects of the deterministic transformation between GHZ class states[13] of three qubits have been studied. A systematic treatment of transformations of this kind

of state is the subject of this article.

First, consider p particles distributed to p distant persons (parties) where $p \geq 3$ and let $|\psi\rangle$ be a state of these particles. This state can be written as a sum of product states as

$$|\psi\rangle = \sum_{i=1}^r \left| \varphi_i^{(1)} \otimes \varphi_i^{(2)} \otimes \cdots \otimes \varphi_i^{(p)} \right\rangle \quad (1)$$

where $\left\{ \left| \varphi_i^{(k)} \right\rangle \right\}_{i=1}^r$ are vectors in the state space of the k th particle (but these r vectors are not necessarily orthogonal to each other). The minimum possible value of the number of terms r in that expression is called the *tensor rank* of the state $|\psi\rangle$. The tensor rank is sometimes also called the *Schmidt rank*, and its base-2 logarithm gives the *Schmidt measure* of the multipartite states[14, 15].

The main subject of this article is states with tensor rank 2, which will be simply called as rank 2 states. As the matrix rank of the reduced density matrices for each party is at most 2, these states can always be considered as states of p qubits. They can be obtained by LOCC with non-zero probability from the generalized GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|0, 0, \dots, 0\rangle + |1, 1, \dots, 1\rangle) \quad . \quad (2)$$

But note that they also contain states that cannot be used to distill the GHZ state, i.e., states where some parties are unentangled from the rest, even some bipartite entangled states are also included in the set of rank 2 states.

The rank 2 states can be considered as the simplest type of entangled states. Since any local operation on such states produces either a rank 2 or a product state, the analysis of their transformations should be somewhat simpler than the ones on other multipartite states. Since they also lack a Schmidt decomposition, it will be interesting to investigate their transformations. The purpose of this article is twofold. First, it proposes a simple parametrization of rank 2 states. This parametrization essentially does the same job that the Schmidt decomposition does for the analysis of bipartite states: it simplifies the identification of states that can be converted to each other by local unitaries, an essential task in the analysis of transformations. It also simplifies the analysis and parametrization of local operations. Second purpose of this article is to completely describe the deterministic transformations between truly multipartite rank 2 states. It is hoped that, the results obtained in this article will shed light on future studies on transformations between more complicated states.

The organization of the article is as follows. In section II, a parametrization of the rank 2 states is given and the equivalence relation under the local unitaries is described. After that, using the established parametrization of states, local quantum operations that can be carried out by each party are described and two possible parametrizations of these operations are proposed. In section III, the necessary and sufficient conditions for the possibility of deterministic transformations between two given multipartite states are obtained. Finally, a brief conclusion is given in IV.

II. THE DESCRIPTION OF THE STATES AND THE LOCAL OPERATIONS

A. The parametrization of states with ranks 1 and 2

By definition, any rank-2 state $|\psi\rangle$ can be expressed in the form

$$|\psi\rangle = \frac{1}{\sqrt{N}} (|\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_p\rangle + z |\beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_p\rangle) \quad (3)$$

where $|\alpha_k\rangle$ and $|\beta_k\rangle$ are *normalized* states in the Hilbert space \mathcal{H}_k of the particle possessed by the party- k , where their relative phases are adjusted suitably such that they have a real, non-negative inner product $c_k = \langle \alpha_k | \beta_k \rangle$ (i.e., $c_k \geq 0$), and z is a complex number. Here N is a normalization factor. The overall phase of the state $|\psi\rangle$ can be eliminated by absorbing it into the overall phase of a pair $\{|\alpha_k\rangle, |\beta_k\rangle\}$ for one of the parties.

Apart from the vectors $|\alpha_k\rangle$ and $|\beta_k\rangle$, this state depends on one complex parameter z , and p real parameters c_1, \dots, c_p , which will simply be called as the *cosines*, belonging to the closed interval $[0, 1]$. The collection of these parameters will be denoted by $\lambda = (z; c_1, c_2, \dots, c_p)$. The $(p+1)$ -tuple λ will be considered as a point in a space Λ which is essentially $\mathbb{C} \times [0, 1]^p$. However, there are a few adjustments to be made before defining the space Λ precisely. First, if $|\psi\rangle$ is a rank-2 state, then the complex number z has to be non-zero. However, treating the product states (i.e., rank-1 states) by the same parametrization has some advantages. For this reason, $z = 0$ values are also included as possible values of this parameter. Moreover, the value $z = \infty$ should also be included as a possible value for this parameter, where $|\psi\rangle$ is again a product state. In other words, the parameter z can be chosen from the extended complex numbers $\mathbb{C}' = \mathbb{C} \cup \{\infty\}$. Apart from this, note that the point $(z = -1; c_1 = 1, \dots, c_p = 1)$ cannot possibly be identified with a state. As a result, this point is excluded from Λ . Hence,

the parameter space Λ is defined as $\Lambda = \mathbb{C}' \times [0, 1]^p \setminus \{(-1; 1, 1, \dots, 1)\}$. Consequently, any rank-2 or rank-1 state can be expressed by using a point λ in so-defined space Λ .

It is possible to distinguish three types of states that can be represented as a point in Λ . The following list describes these types and gives the necessary rules for understanding the type of the state a point $\lambda = (z; c_1, \dots, c_p)$ represents.

- (1) The product states, which are included into Λ for completeness due to the fact that some local operations produce them. The point λ corresponds to a product state if and only if either (i) $z = 0$, or (ii) $z = \infty$, or (iii) the cosines of at least $p - 1$ parties are 1.
- (2) Bipartite entangled states. A point λ is a bipartite entangled state if and only if $z \neq 0, \infty$ and *exactly* $p - 2$ of the cosines are 1 and the remaining two cosines are strictly less than 1. If λ is a bipartite state between parties k_1 and k_2 , then we should have $c_{k_1}, c_{k_2} < 1$. For all of these states, the cosines can be chosen in a multitude of different ways, proving that the current parametrization is rather inconvenient for these types of states.
- (3) The rest of the states, i.e., those that are neither product nor bipartite entangled, will be called *truly multipartite* states. The point λ corresponds to a truly multipartite state if and only if $z \neq 0, \infty$ and at least three of the cosines are strictly less than 1.

Note that the concurrence[16] can be used as a measure of the entanglement of a given party k with all the other parties. When the state $|\psi\rangle$ in Eq. (3) is considered as an entangled state between the party- k and the rest of the parties, the related concurrence can be computed as,

$$\mathcal{C}_k = 2\sqrt{\det \rho^{(k)}} \quad (4)$$

where $\rho^{(k)}$ is the reduced density matrix for the party k , which is defined by $\rho^{(k)} = \text{tr}_{1,2,\dots,k-1,k+1,\dots,p} |\psi\rangle \langle \psi|$. It is straightforward to compute these concurrences for the state in Eq. (3) as

$$\mathcal{C}_k = \frac{2|z| \sqrt{1 - c_k^2} \sqrt{1 - (c_1 \cdots c_{k-1} c_{k+1} \cdots c_p)^2}}{N} . \quad (5)$$

Note that for $z \neq 0, \infty$, the concurrence \mathcal{C}_k can be non-zero (and hence the party- k is entangled with the others) if and only if $c_k < 1$ and there is another cosine which is less than 1. Using this, all of the rules listed above can be justified easily.

B. Local Unitary Equivalence

Next, a description of local unitary (LU) equivalence between states that are expressed by using the parametrization given above must be provided. Two states $|\psi\rangle$ and $|\phi\rangle$ are called LU-equivalent, if there are local unitary operators V_k on each local Hilbert space \mathcal{H}_k such that $|\psi\rangle = (V_1 \otimes \cdots \otimes V_p) |\phi\rangle$. Obviously, LU-equivalent states can be converted into each other by LOCC, with necessary local quantum operations being the indicated unitaries. The opposite is also true. If two states $|\psi\rangle$ and $|\phi\rangle$ are LOCC-convertible into each other, then $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent[10].

It is obvious that any two states described by the same parameter $\lambda = (z; c_1, \dots, c_p)$ are LU-equivalent. Such states differ only in the pairs of states $\{|\alpha_k\rangle, |\beta_k\rangle\}$ which have a common inner product c_k , and therefore it is possible find a unitary operator that converts such pairs into similar pairs. Hence, the point $\lambda = (z; c_1, \dots, c_p)$ denotes a collection states which are all LU-equivalent to the following representative state

$$|\Phi(\lambda)\rangle = \frac{1}{\sqrt{N(\lambda)}} (|0 \otimes 0 \otimes \cdots \otimes 0\rangle + z |w_{c_1} \otimes w_{c_2} \otimes \cdots \otimes w_{c_p}\rangle) \quad (6)$$

where $|w_{c_k}\rangle = c_k |0\rangle + \sqrt{1 - c_k^2} |1\rangle$ and

$$N(\lambda) = 1 + |z|^2 + c_1 c_2 \cdots c_p (z + z^*) \quad . \quad (7)$$

Apart from this, it is possible to express the *same* state by using two different points of Λ , say λ and λ' . Expressed in a different but equivalent way: there might be different points λ and λ' of Λ such that the representative states $|\Phi(\lambda)\rangle$ and $|\Phi(\lambda')\rangle$ are LU-equivalent. The points λ and λ' will be called (LU) equivalent if this happens and that relation will be denoted as $\lambda \sim \lambda'$.

For any given point $\lambda = (z; c_1, \dots, c_p)$, let us define its *conjugate* point by $\hat{\lambda} = (1/z; c_1, \dots, c_p)$, i.e., point obtained by inverting the z -parameter only. It can be seen easily that $\lambda \sim \hat{\lambda}$. It turns out that, if none of the cosines of λ vanish, then the equivalence class of λ is formed by the pair of points $\{\lambda, \hat{\lambda}\}$. Precise criteria for deciding whether two given points of Λ are equivalent are given below.

Let $\lambda = (z; c_1, c_2, \dots, c_p)$ and $\lambda' = (z'; c'_1, c'_2, \dots, c'_p)$. The rules of equivalence depend on the types of the states as follows.

- (1) If λ is product state, then $\lambda \sim \lambda'$ if and only if λ' is a product state.

- (2) If λ is a bipartite entangled state between party- k_1 and party- k_2 , then $\lambda \sim \lambda'$ if and only if λ' is also a bipartite state between the same parties and they have the same concurrences, in other words

$$\frac{|z| \sqrt{1 - c_{k_1}^2} \sqrt{1 - c_{k_2}^2}}{N(\lambda)} = \frac{|z'| \sqrt{1 - c_{k_1}'^2} \sqrt{1 - c_{k_2}'^2}}{N(\lambda')}. \quad (8)$$

- (3) If λ is truly multipartite then $\lambda \sim \lambda'$ if and only if (a) the corresponding cosines are identical (i.e., $c_k = c_k'$ for all k) and (b) the following condition holds for z depending on whether there is a vanishing cosine or not:

- (i) when no cosines vanish: either $z' = z$ or $z' = 1/z$,
- (ii) when there is a vanishing cosine: either $|z'| = |z|$ or $|z'| = 1/|z|$.

The statements for the product and bipartite states are straightforward. The last rule follows simply from the following theorem.

Theorem 1. Let $|\psi\rangle$ be a state of p particles ($p \geq 2$) which is expressed as

$$|\psi\rangle = \sum_{i=1}^r \left| \varphi_i^{(1)} \otimes \varphi_i^{(2)} \otimes \cdots \otimes \varphi_i^{(p)} \right\rangle \quad (9)$$

where, for each k , $F^{(k)} = \left\{ \left| \varphi_1^{(k)} \right\rangle, \left| \varphi_2^{(k)} \right\rangle, \dots, \left| \varphi_r^{(k)} \right\rangle \right\}$ is a set of r non-zero, possibly unnormalized vectors from the Hilbert space \mathcal{H}_k . Let m denote the number of parties k for which the set $F^{(k)}$ is linearly independent. Then the following statements hold.

(a) If $m \geq 1$, then the set of r vectors $G = \left\{ \left| \varphi_i^{(1)} \otimes \varphi_i^{(2)} \otimes \cdots \otimes \varphi_i^{(p)} \right\rangle \right\}_{i=1}^r$ is also linearly independent.

(b) If $F^{(\ell)}$ is linearly independent, then for all $k \neq \ell$, $\text{supp } \rho^{(k)} = \text{span } F^{(k)}$.

(c) If $m \geq 2$, then $|\psi\rangle$ has tensor rank r .

(d) If $m \geq 3$, then the expression of $|\psi\rangle$ as a superposition of r product states is unique.

In other words, if $\left| \beta_i^{(k)} \right\rangle$ are vectors such that

$$|\psi\rangle = \sum_{i=1}^r \left| \beta_i^{(1)} \otimes \beta_i^{(2)} \otimes \cdots \otimes \beta_i^{(p)} \right\rangle \quad (10)$$

then there is a permutation Q of r objects such that for any $i = 1, \dots, r$ we have

$$\left| \beta_i^{(1)} \otimes \beta_i^{(2)} \otimes \cdots \otimes \beta_i^{(p)} \right\rangle = \left| \varphi_{Q_i}^{(1)} \otimes \varphi_{Q_i}^{(2)} \otimes \cdots \otimes \varphi_{Q_i}^{(p)} \right\rangle. \quad (11)$$

Proof: To simplify the proof, it can be assumed that the parties are relabelled so that the first m sets of vectors, i.e., $F^{(1)}, \dots, F^{(m)}$, are linearly independent. This assumption is employed in all of the cases treated below.

For (a), suppose that there are numbers a_1, a_2, \dots, a_r such that

$$\sum_{i=1}^r a_i \left| \varphi_i^{(1)} \otimes \varphi_i^{(2)} \otimes \dots \otimes \varphi_i^{(p)} \right\rangle = 0 \quad . \quad (12)$$

Let $|\Theta\rangle = |\Theta\rangle_{2\dots p}$ be an arbitrary vector in the Hilbert space of all parties except the 1st.

The inner product with this state gives

$$\sum_{i=1}^r \left| \varphi_i^{(1)} \right\rangle \left(a_i \left\langle \Theta \left| \varphi_i^{(2)} \otimes \dots \otimes \varphi_i^{(p)} \right\rangle \right) = 0 \quad . \quad (13)$$

Due to linear independence of $F^{(1)}$, we have $a_i \left\langle \Theta \left| \varphi_i^{(2)} \otimes \dots \otimes \varphi_i^{(p)} \right\rangle = 0$ for all i . As $|\Theta\rangle$ is arbitrary and each $\left| \varphi_i^{(k)} \right\rangle$ is non-zero, it necessarily follows that $a_i = 0$. This shows that the set of vectors G is linearly independent.

For (b), suppose that $\ell = 1$ and $k = 2$ without loss of generality. The reduced density matrix for the second party is

$$\rho^{(2)} = \sum_{i,j=1}^r S_{ji} \left| \varphi_i^{(2)} \right\rangle \left\langle \varphi_j^{(2)} \right| \quad , \quad (14)$$

where S is the overlap matrix given by $S_{ji} = \langle \chi_j | \chi_i \rangle$ and $|\chi_i\rangle = \left| \varphi_i^{(1)} \otimes \varphi_i^{(3)} \otimes \dots \otimes \varphi_i^{(p)} \right\rangle$. Using the result in (a), it can be seen that the set of r -vectors $\{|\chi_i\rangle\}_{i=1}^r$ is also linearly independent. Therefore, the overlap matrix S is strictly positive definite.

From the expression of $\rho^{(2)}$, it is obvious that the support of $\rho^{(2)}$ is included in $\text{span } F^{(2)}$. To show that these two subspaces are identical, let us assume the contrary. Let $|\varphi'\rangle$ be a non-zero vector in $\text{span } F^{(2)}$ but orthogonal to the support of $\rho^{(2)}$. Then, at least one of $b_i = \left\langle \varphi_i^{(2)} \left| \varphi' \right\rangle$ is non-zero and therefore $\langle \varphi' | \rho^{(2)} | \varphi' \rangle = \sum_{ij} S_{ji} b_j^* b_i > 0$, which is a contradiction. This then shows that the two subspaces are identical, i.e., $\text{supp } \rho^{(2)} = \text{span } F^{(2)}$.

For (c), note that both $F^{(1)}$ and $F^{(2)}$ are linearly independent and hence by (b), $\text{supp } \rho^{(k)} = \text{span } F^{(k)}$ for all parties k . In particular, $\rho^{(1)}$ has matrix rank r . This shows that $|\psi\rangle$ cannot be written as a sum of product states with less than r terms. Hence, the tensor rank of $|\psi\rangle$ is r .

Finally, consider the statement in part (d). Let $B^{(k)} = \left\{ \left| \beta_1^{(k)} \right\rangle, \left| \beta_2^{(k)} \right\rangle, \dots, \left| \beta_r^{(k)} \right\rangle \right\}$. Since the tensor rank of $|\psi\rangle$ is r , all of the vectors $\left| \beta_i^{(k)} \right\rangle$ are non-zero. Moreover, since the

matrix rank of $\rho^{(k)}$ is r for $k = 1, 2, 3$, the sets $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ are linearly independent. In short, the prerequisite conditions for the theorem and part (d) are satisfied for the new vectors $|\beta_i^{(k)}\rangle$ as well. Hence we have $\text{span } B^{(k)} = \text{span } F^{(k)}$.

Since $\text{span } B^{(1)} = \text{span } F^{(1)}$ and $B^{(1)}$ is also linearly independent, there is an $r \times r$ invertible matrix Z such that

$$|\beta_i^{(1)}\rangle = \sum_{j=1}^r Z_{ji} |\varphi_j^{(1)}\rangle \quad . \quad (15)$$

Inserting this into the expansions of $|\psi\rangle$ we get

$$\sum_j |\varphi_j^{(1)} \otimes \varphi_j^{(2)} \otimes \cdots \otimes \varphi_j^{(p)}\rangle = \sum_{ij} Z_{ji} |\varphi_j^{(1)} \otimes \beta_i^{(2)} \otimes \cdots \otimes \beta_i^{(p)}\rangle \quad . \quad (16)$$

Using the linear independence of $F^{(1)}$, we get

$$|\varphi_j^{(2)} \otimes \cdots \otimes \varphi_j^{(p)}\rangle = \sum_{i=1}^r Z_{ji} |\beta_i^{(2)} \otimes \cdots \otimes \beta_i^{(p)}\rangle \quad (17)$$

which must hold true for all r . In here, a rank-1 state (product state) is expanded as a sum of r product states. Note that $B^{(2)}$ and $B^{(3)}$ are both linearly independent and therefore part (c) of the current theorem can be applied to this expression. It then directly follows that only one number in the sequence $Z_{j1}, Z_{j2}, \dots, Z_{jr}$ can be non-zero (otherwise we get a contradiction for the tensor rank of the state on the left-hand side). As Z is a square matrix, each row and each column contains only one non-zero entry.

Let Q be the permutation that gives the index of the non-zero entry for a given column. In other words, $Z_{ji} \neq 0$ only for $j = Q_i$. Then, we have

$$|\beta_i^{(1)}\rangle = Z_{Q_i i} |\varphi_{Q_i}^{(1)}\rangle \quad , \quad (18)$$

$$|\varphi_{Q_i}^{(2)} \otimes \cdots \otimes \varphi_{Q_i}^{(p)}\rangle = Z_{Q_i i} |\beta_i^{(2)} \otimes \cdots \otimes \beta_i^{(p)}\rangle \quad , \quad (19)$$

$$\implies |\beta_i^{(1)} \otimes \beta_i^{(2)} \otimes \cdots \otimes \beta_i^{(p)}\rangle = |\varphi_{Q_i}^{(1)} \otimes \varphi_{Q_i}^{(2)} \otimes \cdots \otimes \varphi_{Q_i}^{(p)}\rangle \quad , \quad (20)$$

which is what is aimed to be proved. \square

At this point, let us briefly investigate the implication of the theorem for the rank-2 states. Let $|\psi\rangle$ be the state defined by Eq. (3) and suppose that $z \neq 0, \infty$. In this case, the conditions in the theorem are satisfied with $r = 2$. Here, the statement that the set $F^{(k)} = \{|\alpha_k\rangle, |\beta_k\rangle\}$ is linearly independent is equivalent to $c_k < 1$. Hence, part (c) is the

case for truly multipartite states. As a result, the expansion of $|\psi\rangle$ in Eq. (3) is unique. The most one can do in here is to exchange the places of the two terms, which essentially changes λ to $\hat{\lambda}$. Consequently, the cosines of individual parties do not change. The rest of the rule follows trivially.

C. Local Operations by a Single Party

A local quantum operation (measurement) applied by a single party- k can be described by the general measurement formalism, i.e., there is a set of possible outcomes and for each outcome ℓ , there is an associated measurement operator M_ℓ on \mathcal{H}_k , all of which satisfy the probability-sum condition

$$\sum_{\ell} M_{\ell}^{\dagger} M_{\ell} = \mathbf{1}_k \quad . \quad (21)$$

If the state before the operation is $|\psi\rangle$, then the outcome ℓ occurs with probability $p_{\ell} = \langle \psi | (M_{\ell}^{\dagger} M_{\ell}) \otimes \mathbf{1}'_k | \psi \rangle$ and the state collapses to $|\psi^{(\ell)}\rangle \propto (M_{\ell} \otimes \mathbf{1}'_k) |\psi\rangle$ (up to normalization), where $\mathbf{1}'_k$ denotes the identity operator acting on all parties except the k th one. The main purpose of this section is to provide two alternative descriptions of the general measurement formalism that are more suitable to work with when using the Λ space for state parametrization.

1. First parametrization of local operations

Suppose that the initial state $|\psi\rangle$ is given as in Eq. (3) and consider a local operation described by M_{ℓ} . Define four real parameters for each outcome ℓ as follows

$$A_{\ell} = \|M_{\ell} |\alpha_k\rangle\| \quad , \quad (22)$$

$$B_{\ell} = \|M_{\ell} |\beta_k\rangle\| \quad , \quad (23)$$

$$C_{\ell} e^{i\gamma_{\ell}} = \frac{1}{A_{\ell} B_{\ell}} \langle \alpha_k | M_{\ell}^{\dagger} M_{\ell} | \beta_k \rangle \quad , \quad (24)$$

where the norm is defined as $\|\phi\rangle\| = \sqrt{\langle \phi | \phi \rangle}$. Here C_{ℓ} is taken to be a non-negative real number and the phase γ_{ℓ} is defined accordingly. It follows that $C_{\ell} \leq 1$ by Schwarz inequality. The parameters A_{ℓ} and B_{ℓ} are also necessarily non-negative. If the state before the operation corresponds to the point $\lambda = (z; c_1, \dots, c_p)$, then the collapsed state for the

outcome ℓ corresponds to the point $\lambda^{(\ell)} = (z^{(\ell)}; c_1, \dots, c_{k-1}, C_\ell, c_{k+1}, \dots, c_p)$ where

$$z^{(\ell)} = z \frac{B_\ell e^{i\gamma_\ell}}{A_\ell} \quad , \quad (25)$$

and the probability of that outcome is given by

$$p_\ell = A_\ell^2 \frac{N(\lambda^{(\ell)})}{N(\lambda)} \quad (26)$$

$$= \frac{A_\ell^2 + |z|^2 B_\ell^2 + A_\ell B_\ell c_1 \cdots c_{k-1} C_\ell c_{k+1} \cdots c_p (ze^{i\gamma_\ell} + z^* e^{-i\gamma_\ell})}{N(\lambda)} \quad . \quad (27)$$

Hence, when describing the effect of a local operation by a single party, the values of four real parameters for each outcome are needed: A_ℓ , B_ℓ , C_ℓ and γ_ℓ . Obviously, possible values of these parameters are restricted by the probability-sum condition (21) and the Schwarz inequality, which read

$$\sum_{\ell=1}^n A_\ell^2 = \sum_{\ell=1}^n B_\ell^2 = 1 \quad , \quad (28)$$

$$\sum_{\ell=1}^n A_\ell B_\ell C_\ell e^{i\gamma_\ell} = c_k \quad , \quad (29)$$

$$C_\ell \leq 1 \quad , \quad (30)$$

when party- k is carrying out the operation.

It appears that these are the only restrictions on these parameters. In other words, if a set of non-negative numbers A_ℓ , B_ℓ , C_ℓ and angles γ_ℓ satisfy Eqs. (28-30), then it is possible to construct measurement operators M_ℓ on the space \mathcal{H}_k which would produce the same set of parameters. For showing this, consider only the case where $c_k < 1$ (otherwise, party k is unentangled with the remaining parties and what she does has no effect on the state). Let $\{|\alpha_k^\perp\rangle, |\beta_k^\perp\rangle\}$ be the dual basis in \mathcal{H}_k , which satisfy

$$\langle \alpha_k^\perp | \alpha_k \rangle = \langle \beta_k^\perp | \beta_k \rangle = 1 \quad , \quad (31)$$

$$\langle \alpha_k^\perp | \beta_k \rangle = \langle \beta_k^\perp | \alpha_k \rangle = 0 \quad . \quad (32)$$

The associated vectors are simply given by

$$|\alpha_k^\perp\rangle = \frac{1}{1 - c_k^2} (|\alpha_k\rangle - c_k |\beta_k\rangle) \quad , \quad (33)$$

$$|\beta_k^\perp\rangle = \frac{1}{1 - c_k^2} (-c_k |\alpha_k\rangle + |\beta_k\rangle) \quad . \quad (34)$$

Next, define a new set of operators P_ℓ on \mathcal{H}_k as

$$P_\ell = A_\ell^2 |\alpha_k^\perp\rangle \langle \alpha_k^\perp| + B_\ell^2 |\beta_k^\perp\rangle \langle \beta_k^\perp| + (A_\ell B_\ell C_\ell e^{i\gamma_\ell} |\alpha_k^\perp\rangle \langle \beta_k^\perp| + h.c.) \quad . \quad (35)$$

It is straightforward to show that P_ℓ is positive semidefinite (where the inequality $C_\ell \leq 1$ is employed) and $\sum_\ell P_\ell = \mathbb{1}_k$ (where the remaining restrictions, (28) and (29), are employed). In short, the set of operators $\{P_\ell\}$ forms a positive-operator valued measure (POVM). The measurement operators can be simply defined as $M_\ell = \sqrt{P_\ell}$. It is then easy to check that the same parameters are produced by these measurement operators. This completes the proof that one only needs to satisfy the conditions (28-30) when employing the parametrization of local operations by party- k .

There are a number of remarks that should be made about the parametrization of local measurements described above. First, notice that this parametrization *depends on the initial parameter point* λ through the appearance of the cosine c_k in (29). Second, if an alternative, LU-equivalent point is used for the initial point, then the parametrization of the operation changes accordingly, even though it is the same operation on the same state. For example, if $\hat{\lambda}$ is used instead of λ , then the parameters of the operation change as $A_\ell \leftrightarrow B_\ell$, $\gamma_\ell \rightarrow -\gamma_\ell$ and C_ℓ remains same so that $\lambda^{(\ell)} \rightarrow \widehat{\lambda}^{(\ell)}$. Third, local operations do not change the cosines of the other parties, i.e., if party- k is carrying out the operation, then $c_{k'}$ remains the same for all $k' \neq k$. The only changes are in the cosine of party- k (i.e., c_k becomes C_ℓ now) and the parameter z .

Fourth, any two different outcomes that produce the same final point (e.g., outcomes $\ell_1 \neq \ell_2$ with $\lambda^{(\ell_1)} = \lambda^{(\ell_2)}$) can be combined to a single outcome by choosing a new set of parameters. Hence, when analyzing local operations, it can be supposed without loss of generality that different outcomes correspond to different final points of Λ . However, different outcomes corresponding to different but LU-equivalent points of Λ (e.g., outcomes $\ell_1 \neq \ell_2$ with $\lambda^{(\ell_1)} \neq \lambda^{(\ell_2)}$ but $\lambda^{(\ell_1)} \sim \lambda^{(\ell_2)}$) cannot in general be combined to a single outcome by a simple redefinition of local operation parameters.

Finally, the special outcomes where either $A_\ell = 0$ or $B_\ell = 0$ produces a product state where $z^{(\ell)}$ is either 0 or ∞ . The outcomes for which $A_\ell = B_\ell = 0$ can be simply discarded from consideration because they will always have zero probability of occurrence.

A nice application of the first parametrization is the following theorem which essentially

expresses the idea that any non-unitary local operation produces an outcome which is closer to the product states.

Theorem 2. Consider a local operation by party- k on a state corresponding to point $\lambda = (z; c_1, \dots, c_p)$. Then,

- (a) There is at least one outcome ℓ for which $C_\ell \geq c_k$.
- (b) If $c_k \neq 0$ and the local operation is not a set of random unitary transformations, then there is at least one outcome ℓ for which $C_\ell > c_k$.
- (c) If $|z| \geq 1$, there is at least one outcome m for which $|z^{(m)}| \geq |z|$.

Proof: The statement in (a) holds trivially for the special case $c_k = 0$; it also holds when the local operation is a unitary transformation or a set of random unitary transformations which never change the state parameters λ . Consequently, proving (b) also proves (a). To prove (b), assume the contrary, i.e., suppose that $c_k > 0$ and the final cosines do not exceed c_k for all outcomes ($C_\ell \leq c_k$ for all ℓ). Then,

$$c_k = \sum_{\ell} A_{\ell} B_{\ell} C_{\ell} e^{i\gamma_{\ell}} \leq \sum_{\ell} A_{\ell} B_{\ell} C_{\ell} \quad (36)$$

$$\leq c_k \sum_{\ell} A_{\ell} B_{\ell} \leq c_k \sqrt{\left(\sum_{\ell} A_{\ell}^2\right)\left(\sum_{\ell} B_{\ell}^2\right)} = c_k \quad (37)$$

and, as a result, all inequalities must be equalities. Namely, we should have $\gamma_{\ell} = 0$ when $A_{\ell} B_{\ell} C_{\ell} \neq 0$, $C_{\ell} = c_k$ when $A_{\ell} B_{\ell} \neq 0$ and $A_{\ell} = B_{\ell}$ from the Schwarz inequality. The last relation rules out the product-state producing outcomes (which are the cases where either $A_{\ell} = 0$ or $B_{\ell} = 0$, but not both). Hence we have $C_{\ell} = c_k$, $\gamma_{\ell} = 0$ and $z^{(\ell)} = z$ for all ℓ . This means that all outcomes are identical and the state has not changed. In other words, k th party has made a local unitary transformation for all outcomes ℓ , which is contrary to the assumption.

For (c), again assume the contrary and suppose $|z^{(\ell)}| < |z|$ for all outcomes ℓ . This implies that $B_{\ell} < A_{\ell}$. However,

$$1 = \sum_{\ell} B_{\ell}^2 < \sum_{\ell} A_{\ell}^2 = 1 \quad , \quad (38)$$

which is a contradiction. Therefore the statement in (c) holds. \square

2. Second parametrization of local operations

For the local operation described above, the restrictions (28,29) imply the following identities

$$\sum_{\ell} p_{\ell} \frac{1}{N(\lambda^{(\ell)})} = \frac{1}{N(\lambda)} \quad , \quad (39)$$

$$\sum_{\ell} p_{\ell} \frac{|z^{(\ell)}|^2}{N(\lambda^{(\ell)})} = \frac{|z|^2}{N(\lambda)} \quad , \quad (40)$$

$$\sum_{\ell} p_{\ell} \frac{z^{(\ell)} C_{\ell}}{N(\lambda^{(\ell)})} = \frac{z c_k}{N(\lambda)} \quad . \quad (41)$$

An important feature of these relations is that they are expressed entirely in terms of two real parameters (the probability p_{ℓ} and the final cosine C_{ℓ}) and one complex parameter ($z^{(\ell)}$), which are all one needs for describing the effect of the local operation. More importantly, these relations form a basis for an alternative parametrization of the local operation by party- k . In other words, if a set of outcomes are given and for each outcome ℓ , two real numbers, p_{ℓ} and C_{ℓ} , and one complex number, $z^{(\ell)}$, are given such that they satisfy: (i) $\{p_{\ell}\}$ are probabilities, (ii) $0 \leq C_{\ell} \leq 1$ and (iii) the relations (39) and (41) are satisfied, then it is possible to construct a local operation for party- k such that the final point $\lambda^{(\ell)} = (z^{(\ell)}; c_1, \dots, c_{k-1}, C_{\ell}, c_{k+1}, \dots, c_p)$ is obtained with probability p_{ℓ} . To prove this, we simply define

$$A_{\ell} = \sqrt{\frac{p_{\ell} N(\lambda)}{N(\lambda^{(\ell)})}} \quad (42)$$

$$B_{\ell} = \frac{|z^{(\ell)}|}{|z|} \sqrt{\frac{p_{\ell} N(\lambda)}{N(\lambda^{(\ell)})}} \quad (43)$$

$$\gamma_{\ell} = \arg(z^{(\ell)}/z) \quad (44)$$

and check that Eq. (28-30) are satisfied. (The special cases $z^{(\ell)} = 0, \infty$ or $z = 0, \infty$ can be handled by a limiting procedure without any problem.)

The second parametrization shares many features with the first. For example, it depends on the initial point λ , different outcomes corresponding to same point in Λ can be combined into a single outcome, etc. However, the second parametrization is more convenient, and thus more useful because of the direct appearance of the parameters of the final states in the conditions (39-41).

III. DETERMINISTIC TRANSFORMATIONS OF STATES BY MANY PARTIES

Consider the general LOCC transformation by successive local operations carried out by many parties on rank 2 states. Using classical communication, the parties can coordinate their actions depending on the outcomes of previous measurements. For the case of deterministic transformations, it is required that all of the possible final states are LU equivalent to each other. In this section, deterministic transformation of an initial state λ to a final state λ' is considered. The main problem is the determination of the necessary and sufficient conditions for the possibility of such a transformation and the design of a chain of local measurements when the transformation is possible.

The special case where λ corresponds to a bipartite state falls into the scope of Nielsen's theorem[6] and is not needed to be considered in here. Moreover, the special case where the final state λ' is a bipartite state appears to be a complicated problem due to the presence of an enormous number of points in the LU-equivalence class of all bipartite states in Λ . Hence, in the rest of this section, it will be assumed that both of the initial and the final states are truly multipartite.

Because of the LU-equivalences $\lambda \sim \hat{\lambda}$ and $\lambda' \sim \hat{\lambda}'$, there is some freedom in the choice of the initial and final points. Whenever convenient, this freedom will be used to choose λ and λ' so that their z parameters have modulus greater than or equal to 1. The following necessary conditions of deterministic transformations can be expressed in a simple way when such a choice is made.

Corollary: Let $\lambda = (z; c_1, \dots, c_p)$ and $\lambda' = (z'; c'_1, \dots, c'_p)$ be truly multipartite such that $|z| \geq 1$ and $|z'| \geq 1$. If λ can be transformed into λ' by LOCC, then $|z'| \geq |z|$ and $c'_k \geq c_k$ for all k .

This statement follows straightforwardly from theorem 2. The set of sufficient conditions, however, are expressed differently depending on whether the initial and the final states have a vanishing cosine or not. A separate analysis has to be given in each special case, which can be found in the following three subsections.

A. Transformations into states with a vanishing cosine

If λ to λ' conversion is possible and λ' has vanishing cosines, then λ should also have vanishing cosines for the same parties. Hence, this case deals with transformations between states with vanishing cosines. In this case, it turns out that the necessary conditions given in the corollary above are also sufficient.

Theorem 3. Let $\lambda = (z; c_1, \dots, c_p)$ and $\lambda' = (z'; c'_1, \dots, c'_p)$ be truly multipartite, $|z| \geq 1$, $|z'| \geq 1$, and λ' has a vanishing cosine parameter. Then λ can be LOCC converted into λ' if and only if $|z'| \geq |z|$ and $c'_k \geq c_k$ for all parties k .

Proof: Necessity is obvious from the corollary. For proving the sufficiency of the conditions, it must be shown that any desired parameter (one of the cosines or the z parameter) can be increased without changing the others. In order to simplify the proof, suppose that the first party has a vanishing cosine in λ' and hence $c'_1 = c_1 = 0$. Suppose also that both z and z' are positive real numbers.

First, note that the first party can increase the z parameter without changing any of the cosines. In other words, the state $\lambda = (z; 0, c_2, \dots, c_p)$ can be converted to $(z'; 0, c_2, \dots, c_p)$ for any real number z' with $z' > z$. In terms of the first parametrization of local measurements, this can be achieved by a two outcome measurement, having the following parameters

$$A_1 = \sqrt{\frac{z'^2 z^2 - 1}{z'^4 - 1}} \quad , \quad (45)$$

$$A_2 = z' \sqrt{\frac{z'^2 - z^2}{z'^4 - 1}} \quad , \quad (46)$$

$$B_1 = \frac{z'}{z} \sqrt{\frac{z'^2 z^2 - 1}{z'^4 - 1}} \quad , \quad (47)$$

$$B_2 = \frac{1}{z} \sqrt{\frac{z'^2 - z^2}{z'^4 - 1}} \quad , \quad (48)$$

$$C_1 = C_2 = \gamma_1 = \gamma_2 = 0 \quad . \quad (49)$$

It can be easily seen that the conditions (28-30) are satisfied by these parameters and both of the final points are LU-equivalent to $(z'; 0, c_2, \dots, c_p)$.

Second, any party other than the first can increase their own cosine parameter to any desired value without changing any other parameter. For this purpose, consider k th party with $k \neq 1$ whose initial and final cosines satisfying $c'_k > c_k \geq 0$. The k th party can carry out the following two-outcome measurement, which is expressed in the first parametrization

as,

$$A_1 = A_2 = B_1 = B_2 = \frac{1}{\sqrt{2}} \quad , \quad (50)$$

$$C_1 = C_2 = c'_k \quad , \quad (51)$$

$$\gamma_1 = -\gamma_2 = \arccos \frac{c_k}{c'_k} \quad . \quad (52)$$

It is straightforward to check that the conditions (28-30) are satisfied. If the initial state point is $(z; 0, c_2, \dots, c_{k-1}, c_k, c_{k+1}, \dots, c_p)$, then the final point for both outcomes is LU-equivalent to $(z; 0, c_2, \dots, c_{k-1}, c'_k, c_{k+1}, \dots, c_p)$.

For transforming λ into λ' , the first party increases only the modulus of the z parameter while the rest of the parties increase their cosines. Note that all of these local operations are deterministic. Moreover, they can be carried out in any order without changing the operation parameters. \square

B. Transformations from states with non-zero cosines

Next case that will be dealt with is transformations between states without vanishing cosines. For such transformations, the following representation of the complex z -parameters of the points of Λ turns out to be very useful. As a result, a brief explanation of that representation is necessary at this point. Let z be a complex number having the polar decomposition $z = \exp(\rho + i\theta)$. Two real valued functions $n = n(z)$ and $s = s(z)$ are defined as

$$n = \frac{\cos \theta}{\cosh \rho} = \frac{2 \operatorname{Re} z}{|z|^2 + 1} \quad , \quad (53)$$

$$s = \frac{\sin \theta}{\sinh \rho} = \frac{2 \operatorname{Im} z}{|z|^2 - 1} \quad . \quad (54)$$

Note that n takes on values in the closed interval $[-1, 1]$ while s takes on values in the closed interval $[-\infty, +\infty]$. In particular, s has the value $\pm\infty$ on the unit circle $|z| = 1$. At the special points $z = \pm 1$ of the unit circle, however, s does not have a definite value or limit. Fortunately, these two points are the only places where n reaches its boundary values, namely $n = +1$ only at $z = 1$. Similarly, $n = -1$ only at $z = -1$.

The correspondence between z and the pair (n, s) is two-to-one for all points on the complex plane except $z = \pm 1$. First note that, if z is replaced with $1/z$, then both of these

two functions do not change: $n(1/z) = n(z)$ and $s(1/z) = s(z)$. The opposite is also true, i.e., if $n(z) = n(z')$ and $s(z) = s(z')$ then we either have $z = z'$ or $z = 1/z'$. Hence, the pairs of values (n, s) are identical for conjugate points of Λ .

Now, consider the transformation of λ into λ' where all of the cosines of λ are non-zero. In that case, all cosines of λ' should be non-zero as well if a transformation is possible. Since both states are considered to be truly multipartite, both of them have at most two LU-equivalent points in Λ . The following theorem gives the necessary and sufficient conditions for the LOCC transformation between such states.

Theorem 4. Let $\lambda = (z; c_1, \dots, c_p)$ and $\lambda' = (z'; c'_1, \dots, c'_p)$ correspond to truly multipartite states and all cosines of both states are non-zero. Let (n, s) and (n', s') denote the values of the n and s functions of the z parameters of λ and λ' respectively. It is possible to transform λ into λ' by LOCC if and only if

(a) $c'_k \geq c_k$ for all parties k , and

(b) the following equality is satisfied

$$\frac{n'}{n} = \frac{s'}{s} = \frac{c_1 c_2 \dots c_p}{c'_1 c'_2 \dots c'_p} . \quad (55)$$

Proof: First we show necessity. If λ can be converted into λ' , then part (a) follows from the corollary. The relation in (b) can be obtained from the extension of the relations (39-41) into the whole protocol as follows: Let L denote the chain of outcomes of all local operations carried out by all parties until the protocol is terminated and let p_L denote the joint probability of occurrence of that outcome. After carefully following the second parametrization of all successive local operations, one reaches to a final parameter point $\lambda^{(L)} = (z^{(L)}; c_1^{(L)}, \dots, c_p^{(L)})$ at the end of the protocol for the outcome L . Let $c(\lambda^{(L)})$ denote the product of all cosines of this state, i.e., $c(\lambda^{(L)}) = c_1^{(L)} \dots c_p^{(L)}$. Now, the successive use of relations (39-41) immediately lead to

$$\sum_L p_L \frac{1}{N(\lambda^{(L)})} = \frac{1}{N(\lambda)} , \quad (56)$$

$$\sum_L p_L \frac{|z^{(L)}|^2}{N(\lambda^{(L)})} = \frac{|z|^2}{N(\lambda)} , \quad (57)$$

$$\sum_L p_L \frac{z^{(L)} c(\lambda^{(L)})}{N(\lambda^{(L)})} = \frac{z c(\lambda)}{N(\lambda)} . \quad (58)$$

All of these relations are valid for all probabilistic transformations as well. However, for the current deterministic transformation, all final states can be either $\lambda^{(L)} = \lambda'$ or $\lambda^{(L)} = \hat{\lambda}'$. Hence, all terms within the summation can be collected into just two terms with total probabilities p and $q = (1 - p)$ respectively. Using, $N(\hat{\lambda}') = N(\lambda')/|z'|^2$, the relations above can be expressed as

$$\frac{p + q |z'|^2}{N(\lambda')} = \frac{1}{N(\lambda)} \quad , \quad (59)$$

$$\frac{p |z'|^2 + q}{N(\lambda')} = \frac{|z|^2}{N(\lambda)} \quad , \quad (60)$$

$$\frac{pz' + qz'^*}{N(\lambda')} c(\lambda') = \frac{zc(\lambda)}{N(\lambda)} \quad . \quad (61)$$

(Note that these equations are valid for the cases $z' = \pm 1$ as well, for which $\hat{\lambda}' = \lambda'$ and there should only be a single term. For these special cases, the equation above holds for all possible probabilities p .) These equations are equivalent with the following four equations

$$\frac{N(\lambda')}{N(\lambda)} = \frac{|z'|^2 + 1}{|z|^2 + 1} \quad , \quad (62)$$

$$(p - q) \frac{|z'|^2 - 1}{|z'|^2 + 1} = \frac{|z|^2 - 1}{|z|^2 + 1} \quad , \quad (63)$$

$$\frac{\text{Re } z'}{|z'|^2 + 1} c(\lambda') = \frac{\text{Re } z}{|z|^2 + 1} c(\lambda) \quad , \quad (64)$$

$$(p - q) \frac{\text{Im } z'}{|z'|^2 + 1} c(\lambda') = \frac{\text{Im } z}{|z|^2 + 1} c(\lambda) \quad . \quad (65)$$

Expressing the last three relations in terms of n and s , we get the desired relation. This completes the proof of necessity. (Again, note that for the special cases $z' = \pm 1$, the equations do not depend on the precise value of $p - q$.)

For proving sufficiency of the conditions, it will be argued that the parties consecutively make a deterministic transformation by a local operation to bring the initial state to some desired final state. Hence, there will be a chain of points λ_k for $k = 0, 1, \dots, p$,

$$\lambda_0 = \lambda \rightarrow \lambda_1 \rightarrow \lambda_2 \rightarrow \dots \rightarrow \lambda_p = \lambda'$$

where party- k takes the k th turn to change the state point from λ_{k-1} into λ_k . Here, the

intermediate points are given as

$$\begin{aligned}
\lambda_0 &= (z_0; c_1, c_2, c_3, \dots, c_p) \quad , \\
\lambda_1 &= (z_1; c'_1, c_2, c_3, \dots, c_p) \quad , \\
\lambda_2 &= (z_2; c'_1, c'_2, c_3, \dots, c_p) \quad , \\
&\dots \quad \dots \\
\lambda_p &= (z_p; c'_1, c'_2, c'_3, \dots, c'_p) \quad ,
\end{aligned}$$

where $z_0 = z$ and $z_p = z'$. The k th party essentially increases her cosine from c_k to c'_k while this change is associated by a definite change in the value of the z -parameter from z_{k-1} to z_k (see Fig. 1). The intermediate values z_k of that parameter should be found from the condition (b) of the theorem, e.g., $n(z_k) = n(z_{k-1})c_k/c'_k$ etc. Hence, what is left is the proof that, for all k , the transformation from λ_{k-1} to λ_k by the k th party can be carried out. Obviously, only the cases for which $c'_k > c_k$ are needed to be considered.

Unfortunately there are special values of the z -parameter that need a separate treatment. Note that, due to the proved necessity of the condition (b), the special values of 0 and ∞ for n or s cannot change in these deterministic transformations. Specifically, these correspond to (a) the real axis, $\text{Im } z = 0$, where $s = 0$; (b) the imaginary axis, $\text{Re } z = 0$, where $n = 0$; (c) and the unit circle, $|z| = 1$ where $s = \pm\infty$. These are curves that are invariant under deterministic LOCC transformations. Hence, their intersections, specifically $z = \pm 1, \pm i$ need special attention.

Now, consider the local operation done by the k th party to transform the state point from λ_{k-1} to λ_k . A deterministic transformation with two outcomes \pm will do the job, which is designed, depending on the special cases, as follows.

Case (I) *When z_{k-1} and z_k are not on the unit circle, i.e., with $z_m = \exp(\rho_m + i\theta_m)$, both ρ_{k-1} and ρ_k are strictly positive:* In the second parametrization of local operations, the parameters of transformation are

$$p_{\pm} = \frac{1}{2} \left(1 \pm \frac{\tanh \rho_{k-1}}{\tanh \rho_k} \right) \quad , \quad (66)$$

$$C_{\pm} = c'_k \quad , \quad (67)$$

$$z^{(\pm)} = (z_k)^{\pm 1} \quad . \quad (68)$$

Now, it is straightforward, but tedious, to check that p_{\pm} are probabilities and the

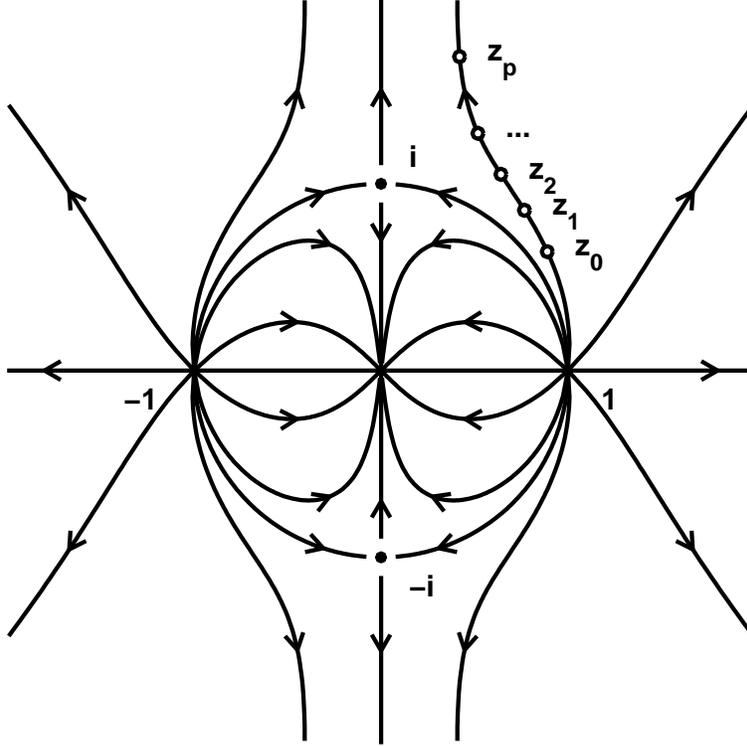


FIG. 1: The curves in the complex plane showing the z -parameter values that are left invariant by deterministic LOCC transformations (i.e., curves for which $n/s = \text{const.}$) and the direction of the shift of the z -parameter under these transformations. The successive z -parameter values for the conversion protocol are also shown. Here, the local operation of party- k changes that parameter from z_{k-1} to z_k .

parameters given above satisfy the conditions (39) and (41). Finally, it is trivial to see that the final state is LU-equivalent to λ_k .

Case (II) *Either z_{k-1} or z_k are on the unit circle (in which case both points must be on the unit circle and therefore $\rho_{k-1} = \rho_k = 0$), but both points are different from $\pm 1, \pm i$ (i.e., θ_{k-1} and θ_k are not an integer multiple of $\pi/2$):* In this case, the parametrization above in Case (I) is valid, except that the probabilities should now be expressed in terms of the polar angles as

$$p_{\pm} = \frac{1}{2} \left(1 \pm \frac{\tan \theta_{k-1}}{\tan \theta_k} \right) . \quad (69)$$

Here too, it is straightforward to check that this local operation describes the needed transformation.

Case (III) *Either $z_{k-1} = \pm i$ or $z_k = \pm i$* : Since the complex numbers $\pm i$ have (n, s) parametrization given by $(n, s) = (0, \pm\infty)$, by part (b) of the current theorem, this point cannot be changed by deterministic LOCC transformations. Hence both of the z parameters should be $\pm i$. Moreover, as z parameters of points in Λ , i and $-i$ correspond to LU-equivalent points. Hence, take $z_{k-1} = z_k = i$ without loss of generality. The main idea is that, even though the z parameter does not change, the k th party can increase her cosine for this special case. The parameters of the transformation are given as

$$p_{\pm} = \frac{1}{2} \left(1 \pm \frac{c_k}{c'_k} \right) \quad , \quad (70)$$

$$C_{\pm} = c'_k \quad , \quad (71)$$

$$z^{(\pm)} = \pm i \quad . \quad (72)$$

Case (IV) *When $z_k = \pm 1$* : That final point has an extreme n value of ± 1 . Hence, by part (b) of the current theorem, the only way this final point is reached is that $z_{k-1} = z_k$ and $c'_k = c_k$. In other words $\lambda_{k-1} = \lambda_k$ and there is no need for a transformation.

Case (V) *When $z_{k-1} = \pm 1$* : Note that by part (b) of the current theorem, the point z_k must satisfy $n(z_k) = \pm(c_k/c'_k)$. However, due to the fact that $s(z_{k-1})$ does not have a definite value, $s(z_k)$ is arbitrary. Hence, suppose that z_k is any number on the complex plane such that

$$n(z_k) = \frac{2\text{Re } z_k}{|z_k|^2 + 1} = \pm \frac{c_k}{c'_k} \quad . \quad (73)$$

Then, the parameters of the local measurement by the k th party in second parametrization are given by

$$p_{\pm} = \frac{1}{2} \quad , \quad (74)$$

$$C_{\pm} = c'_k \quad , \quad (75)$$

$$z^{(\pm)} = (z_k)^{\pm 1} \quad . \quad (76)$$

Once it is observed that

$$N(\lambda_k) = 1 + |z_k|^2 + c'_1 \cdots c'_k c_{k+1} \cdots c_p (2\text{Re } z_k) \quad , \quad (77)$$

$$= (1 + |z_k|^2) \left(1 + c'_1 \cdots c'_{k-1} c_k c_{k+1} \cdots c_p n(z_k) \frac{c'_k}{c_k} \right) \quad , \quad (78)$$

$$= (1 + |z_k|^2) \frac{N(\lambda_{k-1})}{2} \quad (79)$$

it becomes straightforward to verify that the relations (39) and (41) are satisfied and the desired final state is produced. \square

C. Transformations from states with vanishing cosines to those without any vanishing cosine

The only remaining special case that is not yet dealt with is the transformation of states with vanishing cosines to states having only non-zero cosines. For these transformations to be possible, it appears that the z -parameters of both the initial and the final states must satisfy the following, state-independent restrictions.

Theorem 5. Let $\lambda = (z; c_1, \dots, c_p)$ and $\lambda' = (z'; c'_1, \dots, c'_p)$ be truly multipartite states such that λ has a vanishing cosine and λ' has no vanishing cosines. It is possible to transform λ to λ' by LOCC if and only if

(a) $c'_k \geq c_k$ for all k ,

(b) $|z| = 1$.

(c) z' is purely imaginary.

Proof: For proving the necessity of the conditions, the relations (62-65) can be used. Inserting $c(\lambda) = 0$ and $c(\lambda') \neq 0$ into the last two equations we get $\text{Re } z' = 0$ and $p = q$. This then leads to $|z| = 1$. Part (a) follows from the corollary again.

For proving sufficiency, first suppose that the first party has vanishing cosine for the initial state, i.e., $c_1 = 0$. In theorem 3, it is shown that all parties except the first can increase their cosines to any desired value without changing anything else. Hence, the initial state $\lambda = (z; 0, c_2, \dots, c_p)$ can be transformed into $\tilde{\lambda} = (z; 0, c'_2, \dots, c'_p)$. At this point, 1st party can do a single measurement and change the state from $\tilde{\lambda}$ into λ' . The needed measurement has two outcomes with the following parameters in the second parametrization

$$p_{\pm} = \frac{1}{2} \quad , \quad (80)$$

$$C_{\pm} = c'_1 \quad , \quad (81)$$

$$z^{(\pm)} = (z')^{\pm 1} \quad . \quad (82)$$

It can be checked that the relations (39) and (41) are satisfied and the measurement produces the desired final state. \square

D. Invariants of deterministic LOCC transformations

Having found all of the necessary and sufficient conditions for deterministic LOCC transformations of truly multipartite states into each other, a few remarks can be made about some interesting features of these transformations. The most important of these is the existence of some invariants. Many of these can be derived from the particular relation in condition (b) of theorem 4. For example, the phase angle of $(z - z^{-1})$ modulo π is an invariant for states having only non-zero cosines.

Although the deterministic transitions from a given state λ are allowed only to a restricted set of states, it might be useful to consider also the sets of states that can be transformed into λ . With this in mind, it can be seen that the set of truly multipartite states can be partitioned into various disjoint sets, between which no deterministic transformations are possible. An invariant that is capable of finely describing such a partition is given by

$$\xi(\lambda) = c_1 c_2 \cdots c_p \frac{z + z^*}{1 + |z|^2} \quad (83)$$

$$= c_1 \cdots c_p n(z) = \frac{N(\lambda)}{1 + |z|^2} - 1 \quad , \quad (84)$$

for $\lambda = (z; c_1, \dots, c_p)$. The existence of such an invariant has been first discussed by Spedalieri[13]. The invariance of this quantity follows directly from Eq. (62) which is valid for all deterministic transformations into truly multipartite states. Note that $\xi(\lambda)$ has the same value for all LU-equivalent truly multipartite points of Λ . Note also that $\xi(\lambda)$ takes on values in the open interval $(-1, 1)$.

For any ξ with $-1 < \xi < 1$, define M_ξ to be the set of all truly multipartite λ for which $\xi(\lambda) = \xi$. Hence, any element of M_ξ can only be deterministically converted *into* or *from* some element of the same set. This is the finest partition of the rank 2 states having that property. Any state point λ (or its whole LU-equivalence class) in these sets will be called as an *ancestor* if it cannot be obtained deterministically from a different state. In that respect, the ancestors can be thought as *the most entangled* states, meaning that there are no other candidates for being even more entangled. All states represented in M_ξ can be obtained from one of the ancestors but it appears that some of these sets contain different LU-inequivalent ancestors.

The transformations within the set M_0 are treated in all three of the theorems 3, 4 and 5. This set contains the states that have either a vanishing cosine or a purely imaginary

z parameter. All states in this set can be obtained from a single ancestor, the GHZ state $\lambda_{\text{GHZ}} = (1; 0, 0, \dots, 0)$, in other words states which are LU-equivalent to Eq. (2). An interesting special subset of M_0 is formed from states with non-zero cosines having a z -parameter equal to $\pm i$, i.e., the set L_i , which is defined as

$$L_i = \{(z; c_1, c_2, \dots, c_p) : c_1 \cdots c_p \neq 0, z = \pm i\} \quad (85)$$

Note that L_i is also invariant under deterministic LOCC. If a state in L_i is transformed deterministically to a truly multipartite state, then the final state must be in L_i as well. As a result, only the cosines of the state can be increased in a deterministic transformation. It is not possible to change the z parameter to any value other than $\pm i$.

The transformations within the sets M_ξ for which $\xi \neq 0$ are covered in theorem 4. These sets contain infinitely many, LU-inequivalent ancestors. The ancestors are those members of M_ξ that have a z parameter equal to $+1$ (if $\xi > 0$) or -1 (if $\xi < 0$). Note that a given ancestor can generate through deterministic LOCC transformations only a subset of the points in M_ξ . Moreover, all non-ancestor states can be generated from different ancestors. Hence, the partial order in M_ξ induced by the LOCC-convertibility relationship is very non-trivial.

IV. CONCLUSIONS

All of the necessary and sufficient conditions for the possibility of converting truly-multipartite rank-2 states into each other are given. The main theorems are listed under three different headings depending on whether the states have vanishing cosines or not. It is found that the multipartite states can be partitioned into disjoint subsets, which are defined by a single continuous parameter, in such a way that all deterministic transformations are allowed within each subset only. The ancestor states, which can be considered as the most entangled states for deterministic transformations, are identified. They are either the GHZ state or the states that correspond to λ for which the z -parameter is ± 1 and all cosines are non-zero.

For allowed transformations, a specific protocol for converting the initial state to the final one is also proposed. In all of the special cases investigated, the whole transformation can be divided into a series of p successive steps. Each step is associated with one of the

parties and is a deterministic transformation by itself. In each step, the associated party carries out a local generalized measurement; subsequently informs all other parties about the outcome by classical communication; and finally, all parties apply an appropriate local unitary transformation. Depending on the initial and final states, some of the parties may not need to do carry out any measurement in their steps but every party makes *at most one* such measurement. Moreover, it can be shown that the precise ordering of the parties can be changed arbitrarily; but the individual steps of local operations may depend on the ordering.

These protocols have sufficient simplicity so that a comparison to transformations of bipartite states can be made. Any deterministic or probabilistic transformation between bipartite states can be carried out with a protocol where (i) one of the parties carries out a single general measurement, (ii) sends the outcome to the other party by one-way classical communication, (iii) and the other party subsequently carries out a local unitary transformation[5]. It does not matter which party, the 1st or the 2nd, does the local measurement.

For the transformations of truly multipartite states, it is obvious that, in general all parties must carry out a non-trivial local measurement. For example, if the k th cosines of initial and final states are different, then party- k must carry out a local measurement because there is no other way for changing that parameter. Associated with this necessity, each party must be able to send classical information to all of the other parties for the generic case, i.e., multi-way classical communication is also needed. Hence, keeping these restrictions in mind, it can be argued that the protocols proposed in this article, which require *at most one local operation for each party*, are the simplest possible protocols. This feature is also present in the protocol used in Ref. 12 for the distillation of the three-partite GHZ state with maximum probability. An interesting question is this: for which kind of multipartite entanglement transformations it is sufficient to carry out the conversion by at most one local operation for each party?

There are still unsolved problems in the LOCC transformation of rank-2 states. One of those is the transformation into bipartite entangled states and the other is the probabilistic transformations. Hopefully, the approach taken in this article, i.e., particular Λ representation of the states and the associated description of local operations will prove useful in the solution of these problems as well.

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