HIGHER DIMENSIONAL METRICS OF COLLIDING GRAVITATIONAL PLANE WAVES

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We give a higher even dimensional extension of vacuum colliding gravitational plane waves with the combinations of collinear and non-collinear polarized four-dimensional metric. The singularity structure of space-time depends on the parameters of the solution.

I. INTRODUCTION

One of the main fields of interest in general relativity is the collision of gravitational plane waves. The structure of the field equations, physical and geometrical interpretations, and various solution-generating techniques have been described in [1]. Khan-Penrose [2] and Szekeres [3] have found exact solutions of the vacuum Einstein equations describing the collision of impulsive and shock plane waves with collinear polarizations. Nutku-Halil [4] generalized the Khan-Penrose metric to the case of non-collinear polarizations. Later several authors have studied exact solutions of the Einstein-vacuum and Einstein-Maxwell equations, describing the collision of the gravitational and electromagnetic plane waves. In general relativity, various techniques are known for generating different solutions of vacuum and electrovacuum Einstein field equations [5]. In this context recently various new solution-generating techniques have been given for vacuum and electrovacuum cases [6, 7]. In the low energy limit of string theory the colliding gravitational plane waves and some exact solutions are given in [8, 9]. Also a formulation of the colliding gravitational plane waves in metric-affine theories is given in [10]. In connection with string theory colliding gravitational plane waves were studied in [11]- [13].

Recently, motivated by the results obtained in [14, 15], we showed that starting from a Ricci flat metric of a four-dimensional geometry admitting two Killing vector fields it is possible to generate a whole class $2N = 2 + 2n$-dimensional Ricci flat metrics [16]. As an explicit example we constructed higher even dimensional metrics of colliding gravitational waves from the corresponding four dimensional vacuum Szekeres metrics.

In this paper we give a full construction of higher even dimensional colliding gravitational plane waves with the combinations of collinear and non-collinear polarized four-dimensional metrics . In particular, we show that there is no higher even dimensional solution for the Nutku-Halil solution. The singularity structure of this higher dimensional solutions is also examined by using the curvature invariant.

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II. HIGHER DIMENSIONAL COLLIDING GRAVITATIONAL PLANE WAVE GEOMETRIES

In [16] we have studied some Ricci flat geometries with an arbitrary signature. We presented a procedure to construct solutions to some higher even dimensional Ricci flat metrics. According to our theorem stated in [16], if the metric functions $\mathcal{U}(x^a)$, $h_{o\,i}(x^a)$, and $\mathcal{M}_i(x^a)$, for each $i = 1, 2, \cdots, n$, form a solution to the four dimensional vacuum field equations for the metric

$$ds^2 = e^{-\mathcal{M}_i}\, \eta_{ab}\, dx^a\, dx^b + e^{\mathcal{U}}\, (h_{0\,i})_{ab}\, dy^a\, dy^b, \qquad (1)$$

where η_{ab} is the metric of the flat 2-geometry with an arbitrary signature (0 or ± 2), then the metric of the $2N = 2 + 2n$ dimensional geometry defined below

$$ds^2 = e^{-M}\, \eta_{ab}\, dx^a\, dx^b + \sum_{i=1}^{n} \epsilon_i\, e^{u_i}\, (h_{0\,i})_{ab}\, dy_i^a\, dy_i^b, \quad (2)$$

solves the vacuum equations, where $\epsilon_i = \pm 1$, $M = \bar{M} + \tilde{M}$, $\tilde{M} = \sum_{i=1}^{n} \mathcal{M}_i$. $\bar{M}$ solves

$$\frac{1}{2}(\nabla_\eta^2 \bar{M})\, \eta_{ab} + (n-1)\,\mathcal{U}_{,ab} - \frac{1}{2}[\bar{M}_{,a}\,\mathcal{U}_{,b} + \bar{M}_{,b}\,\mathcal{U}_{,a} - \bar{M}_{,d}\,\mathcal{U}^{d}_{,}\,\eta_{ab}] - \frac{1}{2}\sum_{i=1}^{n} \partial_a\, u_i \partial_b\, u_i + \frac{1}{2} n \partial_a\, \mathcal{U} \partial_b\, \mathcal{U} = 0, \quad (3)$$

and $\mathcal{U}$ and u_i solve the following equations, respectively:

$$\partial_a[\eta^{ab}\, e^{\mathcal{U}} \partial_b \mathcal{U}] = 0, \qquad (4)$$

$$\nabla_\eta^2\, u_i + \eta^{ab}\, \mathcal{U}_{,a}\, u_{i,b} = 0. \qquad (5)$$

Here the local coordinates of the $(2n+2)$ dimensional geometry are given by $x^\alpha = (x^a, y_1^a, y_2^a, \cdots, y_n^a)$. Given any four-dimensional metric of colliding vacuum gravitational plane wave geometry we have their extensions to higher dimensions for arbitrary $2N$ without solving further differential equations. In particular, taking $u_i = m_i\mathcal{U}$, where $m_i (i = 1, 2, ..., n)$ are real constants satisfying

$$\sum_{i=1}^{n} m_i = 1, \qquad \sum_{i=1}^{n} (m_i)^2 = m^2, \tag{6}$$

and the signature of flat-space metric with null coordinates is

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x^1 = u,\, x^2 = v,$$

then the solutions to the Eqs. (3) and (4) are found to be

$$e^{-M} = (f_u\, g_v)^{-n+1} (f+g)^{\frac{m^2+n-2}{2}} e^{-\sum_{i=1}^{n} \mathcal{M}_i}, \tag{7}$$

$$e^{\mathcal{U}} = f(u) + g(v), \tag{8}$$

where f and g are arbitrary functions of their arguments and Eq. (4) is automatically satisfied as a result of Eq. (8). Therefore, the above exact solutions describe the collision of gravitational waves for arbitrary $n > 1$.

In [16] we found a family of solutions when the four dimensional metrics are the Szekeres metrics [3] (collinear four dimensional metrics). In the next section we generalize this solution by adding non-collinear metrics to the Szekeres metrics. We also show that, in our formalism, there is no higher dimensional metric constructed by the non-collinear four dimensional metric alone.

III. HIGHER DIMENSIONAL VACUUM SOLUTIONS

We take the following metric as the metric describing a plane wave geometry in $2N$ dimensions.

$$\begin{aligned} ds^2 &= 2e^{-M}\, du dv \\ &+ \sum_{j=1}^{n_a} (f+g)^{m_j} \frac{|(1-E_j)\, dx_j + i\,(1+E_j)\, dy_j)|^2}{1 - E_j\, \bar{E}_j} \\ &+ \sum_{j=n_a+1}^{n} (f+g)^{m_j} (e^{V_j}\, dx_j^2 + e^{-V_j}\, dy_j^2), \end{aligned} \tag{9}$$

where the complex functions E_i (non-collinear case) and the real functions V_i (collinear case) satisfy the Ernst and Euler-Poisson-Darboux equations, respectively,

$$\begin{aligned} &(1 - E_i\bar{E}_i) \left[2(f+g)\, E_{i,\,fg} + E_{i,\,f} + E_{i,\,g}\right] \\ &= -4(f+g)\bar{E}_i\, E_{i,\,f}\, E_{i,\,g}, \qquad i = 1, 2, ..., n_a, \end{aligned} \tag{10}$$

$$2(f+g)V_{i,\,fg} = -V_{i,\,f} - V_{i,\,g}, \qquad i = 1, 2, ..., n - n_a, \tag{11}$$

with the following solutions:

$$E_i = e^{i\alpha_i}(\frac{1}{2} - f)^{1/2}(\frac{1}{2} + g)^{1/2} + e^{i\beta_i}(\frac{1}{2} + f)^{1/2}(\frac{1}{2} - g)^{1/2}, \tag{12}$$

$$V_i = -2k_i\, \tanh^{-1}(\frac{\frac{1}{2} - f}{\frac{1}{2} + g})^{\frac{1}{2}} - 2\ell_i\, \tanh^{-1}(\frac{\frac{1}{2} - g}{\frac{1}{2} + f})^{\frac{1}{2}}, \tag{13}$$

where α_i, β_i, k_i, and ℓ_i are arbitrary constants and there is no sum on i in Eq. (10). The metric function M given in (9) is

$$\begin{aligned} e^{-M} &= (f_u\, g_v)^{-n+1} (f+g)^{\frac{m^2+n-2}{2}} \\ &\times e^{-\sum_{i=1}^{n_a} \mathcal{M}_i^{(1)}} e^{-\sum_{i=n_a+1}^{n} \mathcal{M}_i^{(2)}}, \end{aligned} \tag{14}$$

where the metric functions $\mathcal{M}_i^{(1)}$ and $\mathcal{M}_i^{(2)}$ are, respectively,

$$e^{-\mathcal{M}_i^{(1)}} = \frac{f_u g_v[-\gamma_i^2(f+g)^2 + 2(\gamma_i^2 - 4)(1+4fg)fg + \frac{\gamma_i^2}{4} - 1]}{(f+g)[(1+4fg) + 2\gamma_i(\frac{1}{4} - f^2)^{1/2}(\frac{1}{4} - g^2)^{1/2}](\frac{1}{4} - f^2)^{1/2}(\frac{1}{4} - g^2)^{1/2} d_i}, \tag{15}$$

$$e^{-\mathcal{M}_i^{(2)}} = \frac{c_i f_u g_v (f+g)^{\frac{\tau_i}{2}}}{(\frac{1}{2} - f)^{k_i^2/2}(\frac{1}{2} - g)^{\ell_i^2/2}(\frac{1}{2} + f)^{\ell_i^2/2}(\frac{1}{2} + g)^{k_i^2/2}\, [(\frac{1}{2} - f)^{1/2}(\frac{1}{2} - g)^{1/2} + (\frac{1}{2} + f)^{1/2}(\frac{1}{2} + g)^{1/2}]^{2k_i\ell_i}}, \tag{16}$$

where $\gamma_i = 2\cos(\alpha_i - \beta_i)$, $\tau_i = k_i^2 + \ell_i^2 + 2k_i\ell_i - 1$, and d_i, c_i are constants.

The metric function e^{-M} must be continuous across the null boundaries. To make it so we define

$$k^2 \equiv \sum_{i=1}^{n-n_a} k_i^2, \quad \ell^2 \equiv \sum_{i=1}^{n-n_a} \ell_i^2, \quad s \equiv \sum_{i=1}^{n-n_a} k_i\ell_i, \tag{17}$$

and we assume that the functions f and g take the form

$$f = \frac{1}{2} - (e_1\, u)^{n_1}, \quad g = \frac{1}{2} - (e_2 v)^{n_2}, \tag{18}$$

where $e_1,\, e_2,\, n_1 \geq 2,\; n_2 \geq 2$ are real constants. Then the metric function e^{-M} is continuous across the boundaries if

$$k^2 + n_a = 2(1 - \frac{1}{n_1}), \qquad \ell^2 + n_a = 2(1 - \frac{1}{n_2}) \tag{19}$$

with

$$1 \leq k^2 < 2, \quad 1 \leq \ell^2 < 2. \tag{20}$$

Therefore, the metric function e^{-M} reads

$$e^{-M} = \frac{(f+g)^{\frac{\sigma}{2}}}{(\frac{1}{2}+f)^{(n_a+\ell^2)/2}(\frac{1}{2}+g)^{(n_a+k^2)/2}}[(\frac{1}{(\frac{1}{2}-f)^{1/2}(\frac{1}{2}-g)^{1/2}+(\frac{1}{2}+f)^{1/2}(\frac{1}{2}+g)^{1/2}})^{2s}]\; e^{-\Gamma}, \tag{21}$$

where

$$e^{-\Gamma} = \Pi_{i=1}^{n_a} \frac{\gamma_i^2(f+g)^2 - 2(\gamma_i^2-4)(1+4fg)fg - \frac{\gamma_i^2}{4} + 1}{(1+4fg) + 2\gamma_i(\frac{1}{4}-f^2)^{1/2}(\frac{1}{4}-g^2)^{1/2}}$$

and $\sigma = m^2 + n - n_a + k^2 + \ell^2 + 2s - 3$. We may set

$$-2e_1e_2n_1n_2\Pi_{i=1}^{n-n_a} c_i = \Pi_{i=1}^{n_a} d_i. \tag{22}$$

When we consider only the non-collinear case where $n_a = n$ and $k^2 = \ell^2 = s = 0$, conditions in Eq. (19) imply $n_1 = n_2$. Then Eqs. (19) and (20) reduce to

$$n = 2(1 - \frac{1}{n_1}), \tag{23}$$

where $n_1 \geq 2$ and n is a positive integer. The only possible solution is $n = 1,\; n_1 = 2$ which corresponds to the Nutku-Halil solution. Hence if we take all E_i as the Nutku-Halil metric functions it is not possible to find an appropriate metric function M which is continuous across the null boundaries for $n > 1$. This is the reason why one has to take the higher dimensional metric as the combinations of collinear and non-collinear polarizations.

A. Singularity structure

We now discuss the nature of the space-time singularity. That is, we study the behavior of the metric function M as $f+g$ tends to zero. For this purpose, using the result obtained in [16] for the curvature invariant

$$I = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}, \tag{24}$$

as

$$I \sim e^{2M} \frac{(f_u g_v)^2}{(f+g)^4}, \tag{25}$$

we find

$$I \sim (f_u\, g_v)^2\, (f+g)^{-\mu}, \tag{26}$$

as $f + g \to 0$. Here $\mu = k^2 + \ell^2 + m^2 + 2s + 4n_a + 2$. For the four dimensional case ($n = 1$) $n_a = 0$ with $k = \ell = s = 1$ and $m^2 = 1$ this corresponds to the singularity structure of the Khan-Penrose solution. $n_a = 1$, $k^2 = \ell^2 = s = 0$ and $m^2 = 1$ corresponds to the Nutku-Halil solution. It is known that both solutions have the same singularity structures. $n_a = 0$, with $k = k_1$, $\ell = \ell_1$, and $m^2 = 1$ corresponds to the singularity structure of the Szekeres solution. The singularity structure in the higher dimensional spacetimes can be made weaker or stronger than the four dimensional cases by choosing the constants m_i, k_i, and ℓ_i properly.

IV. CONCLUSION

In this work we gave a higher even dimensional generalization of vacuum colliding gravitational plane waves with the combinations of collinear and non-collinear polarizations. We also discussed the singularity structure of the corresponding spacetimes, and showed that the strength of the singularity depends on arbitrary parameters m_i. We also showed that it is not possible to construct the higher dimensional metric by non-collinear four dimensional metric functions E_i (10) alone. Einstein's equations and continuity conditions force us to superpose the collinear and non-collinear metric functions.

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