DESIGN AND CONTROL OF A 2 DOF STABILIZER

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ABSTRACT

DESIGN AND CONTROL OF A 2 DOF STABILIZER

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Majority of mission critical airborne, marine or land platforms and vehicles are equipped with highly accurate cameras, infra-red imagers, sighting systems, communication antennas and gun systems. While these systems function, the platforms they are attached to are in constant movement. The vibration they are subject to may cause temporary communication interruption, low image quality or other functional degradations. One of the most proven methods to eliminate these problems is utilizing gimbal stabilization. In this study, an experimental set-up to test and verify different control methods to satisfy stabilization performance of an inertial stabilization system, has been designed and manufactured. A very detailed mathematical modelling for inverse and forward kinematics of the system and dynamic analysis involving nonlinear dynamic effects such as static and dynamic mass unbalance and friction has been obtained by using Lagrange methods. Then, based on the equation of motion derived previously, the mathematical foundation for different control strategies has been developed. Finally, the theoretical background developed has been tested and verified using the experimental set-up. The results of the experiments are compared and performances of the different control methods such as PID control, LQR

control and their robust and adaptive versions are compared. The results of this study will be used to design a family of stabilizers to be used with Mast systems and hence this study is supported by Figes Milmast Inc.

Keywords: Mathematical Modeling, Gimbal, Stabilization, Control, Sensor Management, Robotics

ÖZ

2 SERBESTLİK DERECELİ STABİLİZATÖR TASARIMI VE KONTROLÜ

Kılıç, Zafer
Yüksek Lisans, Makina Mühendisliği Bölümü
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Kritik öneme sahip operasyonlara katılan kara, hava ve deniz araçlarının büyük çoğunluğu yüksek hassasiyetli kameralar, kızılötesi sistemler, nişangâh ve silah sistemleri ve haberleşme ekipmanlarıyla donatılmıştır. Bu sistemler hareketli platformlar üzerine bütünlenmektedir. Bu sistemler çalışır haldeyken maruz kaldıkları titreşim geçici haberleşme kesintilerine, düşük kalitede görüntülere ve başka fonksiyonel arızalara neden olmaktadır. Bunu önlemenin en etkin yollarından birisi kendisini kanıtlamış stabilizatör sistemlerinin kullanılmasıdır. Bu çalışmada, ataletli bir sistemin stabilizasyon performansını test etmek ve doğrulamak için bir deney düzeneği tasarlanmış ve bütünlenmiştir. Sistemin ileri ve geri kinematik denklemleri ve dinamik analizinin türetilmesi için kütle dengesizliği ve sürtünme gibi lineer olmayan dinamik etkileri de içeren detaylı bir matematik modeli Lagrange denklemleri kullanılarak oluşturulmuştur. Bunun ardından hareket denklemi temel alınarak farklı kontrol metodlarının matematik modelleri türetilmiştir. Son olarak ise bu modeller deney düzeneğinde test edilmiş ve PID, LQR ve bu kontrolcülerin robust ve adaptif versiyonları gibi birbirinden farklı kontrolcülerin performansları karşılaştırılmıştır. Bu çalışma, Mast sistemlerinde kullanılmak üzere geliştirilecek stabilizatör ürün ailesinin tasarımı için bilgi

kümesi oluşturmuş olacaktır.

Anahtar Kelimeler: Matematiksel Modelleme, Gimbal, Stabilizasyon, Kontrol, Sensor Uygulamaları, Robotik

To my family

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LIST OF ABBREVIATIONS

ALE	Acceleration loop equation
COTS	Common of the shelf
DOF	Degree of freedom
DAQ	Data acquisition card
IMU	Inertial measurement unit
ISP	Inertially stabilized platform
LCE	Loop closure equation
LOS	Line of sight
LQG	Linear quadratic gaussian
LQR	Linear quadratic regulator
LQRD	Linear quadratic regulator discrete
N-E	Newton-Euler
PID	Proportional integral derivative
P & ID	Piping and instrumentation diagram
QFT	Quantitative feedback theory
UAV	Unmanned aerial vehicle
VLE	Velocity loop equation
$\hat{C}^{(a,b)}$	Transformation matrix from Body-a to Body-b
$\check{J}_k$	Inertia tensor of Body-k with respect to its own center of mass
$\vec{F}_{ab}$	Force vector applied from of Body-a to Body-b
$\vec{M}_{ab}$	Moment vector applied from of Body-a to Body-b
$\vec{r}_k$	Position vector from ground to center of mass of Body-k
$\vec{v}_k$	Velocity vector from ground to center of mass of Body-k
$\vec{a}_k$	Acceleration vector from ground to center of mass of Body-k

$\vec{r}_{ab}$	Position vector from Body-a to Body-b
$\vec{w}_k$	Angular velocity vector of Body-k with respect to ground
$\vec{w}_{a/b}$	Angular velocity vector of Body-a with respect to Body-a
$\vec{\alpha}_k$	Angular acceleration vector of Body-k with respect to ground
$\overline{u}_k$	Unit vector k
$\overline{u}_k^{(a/b)}$	Unit vector k of Body-a with respect to Body-b
$s\theta$	sine function of variable θ
$c\theta$	cosine function of variable θ
θ_{131}	Payload angle actuated by hydraulic cylinder 1
θ_{132}	Payload angle actuated by hydraulic cylinder 2
θ_5	Local platform angle of the outer gimbal
θ_6	Local platform angle of the inner gimbal
θ_5^*	Global platform angle of the outer gimbal
θ_6^*	Global platform angle of the inner gimbal
T_5	Torque applied by the servo motor of the outer gimbal
T_6	Torque applied by the servo motor of the inner gimbal

CHAPTER 1

INTRODUCTION

Inertially stabilized platforms (ISPs) perform an outstanding job in stabilization and pointing and they are used in conjunction with a broad array of sensors, cameras, telescopes, and weapon systems. Their history dates back to 100 years ago and since then they have been an essential part of a variety range of application areas such as surveillance, target tracking, missile guidance, gun-turret control, communications, astronomical telescopes, and hand-held cameras [7].

The application platforms include almost every type of moving vehicle, from satellites to submarines, and are even used on some handheld and ground-mounted devices. One of the renown worldwide applications is the Hubble telescope, which actually as a whole is a gyro-stabilized ISP designed to point at distant stars and galaxies to within a few milliarcsec and hold the optical axis steady to a fraction of this angle to avoid blurring the magnified image [7].

Another common application area, although as not remarkable as the Hubble telescope, is UAV mounted targeting camera. This instrument has to endure through climate perturbations and UAV movements. In order to prevent blurry or out of frame images which makes rendering the precise acquisition of a target nearly impossible, utilization of stabilization is inevitable. Therefore, it is evident that an accurate dynamic stabilization to control and cancel motion and vibration is required for correct operation of any types of dynamic platform mission critical instrumentation.

As J.M. Hilkert (2008) stated “ISP electromechanical configurations are as diverse as the applications for which they are designed.” A typical architecture of an ISP consists of an assembly of structure, bearings, and motors called a gimbal to which a gyroscope, or a set of gyroscopes, is mounted [7]. The purpose of the gyros is to send a command to gimbal motors to counteract the movement it has sensed. The placement of the payload or the sensor depends on the application. In some applications it is more convenient to mount them directly on the gimbal assembly; whereas in other applications such as optic applications the mirrors or other optical elements are mounted on the gimbal and the sensor is fixed to the vehicle. Generally, at least two orthogonal gimbals are required to stabilize and point. Majority of the applications usually provide more degrees of freedom to achieve better isolation from the host vehicle [7].

In our study, the experimental setup comprises of two different gimbals assembled together. Both of these gimbals have two degrees of freedom on tilt and roll axes. The outer gimbal is used to simulate movement of a military vehicle, based on the data collected from a real life operation. The inner gimbal, on the other hand, is used to stabilize the platform by using the feedback from different instruments and devices. The experimental setup has been designed to be versatile such that the center of mass and inertia of the platform to be stabilized can be adjusted by using portable weights. By this way, the performance of the controllers are verified when the system is subjected mass unbalance. Moreover, it enables different feedbacks such as coordinates obtained by resistive touchpad, an accelerometer which will be used as a gyro, incline meters, encoders and linear transducers to be used for control purposes. All these feedbacks are required at different stages of the study. The procedure for this study has an evolutionary nature that involves experiments with relatively simpler tasks such as ball on a plate in the initial steps which later to be followed by harder experiments and tasks with different control methods ultimately ending up with a precisely stabilized platform that can enable laser tracking in real time.

1.1 Performance Concepts

What we expect from an ISP is to prevent the payload sensor from rotating in inertial space. However, there are some cases that might regard this attempt insufficient. The sensor system may need to be controlled in a specific manner in order to obtain a clear image of a certain target. It requires the image to be held steady within the sensor FOV without concern for the aim-point. Moreover, in applications involving a beam such as laser range finder which should be directed to a target; it may be crucial to hold the target exactly at the center of the FOV, while very small amounts of high-frequency rotation or jitter might be inconsequential. There are also some applications which requires control of both the jitter and the aim point.

Mass stabilization is the main approach used by design engineers; which simply refers to the Newton's first and second laws of motion, but applied to rotational motion. Newton's first law asserts that a body does not accelerate with respect to an inertial frame unless a torque is applied whereas Newton's second law states that the angular acceleration of a homogeneous body under a net torque is inversely is proportional moment of inertia of the body, J.

To put it in a very simple way, all you have to do is to ensure that the applied torque turns to be zero in order to prevent the object from rotating with respect to inertial space. Nevertheless, in practical this is a very difficult task to accomplish considering all the torque disturbances from various sources. Another aspect of interest is to achieve a rotational motion in response to an input command by means of a controller. This is where the design engineers utilize instruments such as displacement gyros which are mounted on the object in order to measure the inertial rotation about the axes which are intended to be controlled and stabilized. The gyro, which provides feedback, therefore is an essential component of the closed loop servo system. By this way, the system can be controlled by means of external command inputs and it becomes resistant to all kinds of disturbances [2].

1.2 Structural Consideration

One of the most challenging aspects of stabilization design is related to the structural interaction and integrity of the system. When undermined, it can easily dominate the performance. Thus, one has to carefully give attention to this detail in the design phase. Three critical components that play an important role in this matter are the structural characteristics of the payload, the gimbal axes and the support mechanism. These factors combined with the structure of the design determine the structural mode of the system which usually refers to the shape and the particular frequency of resonance. Remind that the shape and frequency of the modes is a function of structural stiffness, damping and mass distribution and it gives us clue about how the whole structure will behave when subjected to load in a particular direction again under that specific frequency. To sum up, the amplitude of the response depends on the amplitude and frequency of the vibration input that triggers the modes to be responded [2].

1.3 Motivation and Problem Definition

Gimbal systems have a wide application area concerning the angular stabilization of camera, gun or telescope in an inertial coordinate system. Land platforms are one of the common application areas and mast systems integrated to these platforms require stabilized platforms. Mast or tower systems are fixed to the base of the vehicle or directly to the earth but that do not mean that these systems are completely stable. In an extreme example, the top of the Eiffel tower can sway six to seven meters in the wind. It is common and self-proven practice to benefit from gyro stabilization based on the distance and size of the targets you are tracking, as well as the height and stability of the tower for mast and tower applications.

Figes Milmast Inc. has been supplying several types and configurations of Mast systems to defence industry for several years. They have a wide range of these systems compatible with the majority of the armoured vehicles of defence industry and there has been a demand for inertial stabilizers to be supplied with their mast products. After getting in touch with me for consultancy for designing a family of stabilizers, we had an agreement to executive this project as a joint study of both academy and industry. The know-how obtained from this thesis study will be an initiative for the design of family of these products.

For this reason, we specified the purpose and objectives of the thesis study. In order to satisfy our success criteria, real time experimental data is required. Therefore, a mast product of Figes supplied to Roketsan which has a 8 meter of elongated height and 1,8 meter of closed height has been deployed to one of the military vehicles while it is on a mission. During the mission the deflection angles at the top of the mast are measured with respect to the vehicle base and are tabulated. These tabulated angle values will act as an input for the outer gimbal in order to test the performance of our stabilization. In Figure 1.1 below is an illustration of a mast product assembled to the armoured vehicle.

The aim of this study is to develop a prototype with 2 gimbal axis with 15-20 mrad stabilization error and 5 kg payload.

Figure 1.1: Illustration of the mast on an armoured vehicle, retrieved from [5]

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1.4 The Outline of the Thesis

The contents of the chapters are as follows:

Chapter 2 provides a literature survey of mathematical modelling of gimbal systems and different control methods employed for stabilization purposes.

In Chapter 3, mechanical design details of the system are explained. Moreover, the infrastructure for experimental set-up is explained in terms of mechanical, electronics and software aspects.

Chapter 4 explains the mathematical modelling of the system. The mathematical modelling includes derivation of ball and beam equations, inverse and forward kinematics of the disturber, derivation of Newton-Euler and Lagrange equations for stabilizer.

In chapter 5, controller design methods have been discussed, simulation and experiment Simulink models have been explained and experiment results have been presented.

Finally, chapter 6 in addition to the discussion of the experimental results; problems encountered and solution methods are discussed. Future work and conclusions are presented.

CHAPTER 2

LITERATURE SURVEY

The Gimbal stabilization systems are closed loop control systems mounted on a ground or aerial vehicle in military or civil applications for keeping line of sight (LOS) of a camera pointing towards a moving or fixed target and reject the disturbances due to maneuvers of the vehicle. Mechanical stabilization is necessary to avoid blur and jitter in the image. Two axis gimbal systems are the most common configuration in both literature and practice. In two axis gimbal systems, the camera collects an image fixed about the roll axis and the gimbal system is electromechanically controlled for giving LOS direction through consecutive rotations around pan and tilt directions. Since the gimbal system is mounted on a vehicle, the disturbances are generally due to sudden changes in orientation of the vehicle or unabsorbed shocks induced through the body of the vehicle, or gimbal mass imbalance. The common method for controlling gimbal systems is PID control. PID control systems become a de facto standard in practice since the availability of control hardware and software present in today's quadcopter and aerial surveillance industries. PID systems are also easy to implement and tune. The inner-outer loop control systems are more common in military and professional applications. The more advanced methods generally utilize a PID controller in the inner loop and adaptive or robust disturbance rejection controller on the outer loop [12].

In Figure 2.1, a basic stabilization mechanism for an axis of a gimbal is shown. All disturbances collectively contribute and they show up in the system as torque disturbances; which in turn alter the inertia of the system.

The plant model is the base of a control system. The performance of the gimbal system is heavily impaired due to a poor plant model. When we consider the two-axis

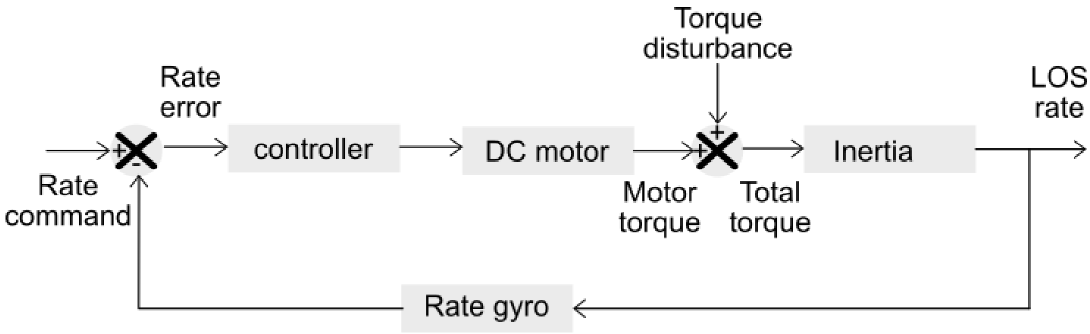

Figure 2.1: Single axis gimbal stabilization system, retrieved from [5]

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gimbal system, it is obviously a multivariable system. The simplest approach to control the system is to assume that the control of the two joints is decoupled. This is called decentralized control. The two joints of the gimbal systems are usually considered as separate systems in controller design for lightweight or slow gimbal systems. In this approximation the coupling effects are considered as disturbances. However, for high inertia or high speed gimbal systems, there are a significant coupling between these two joints. In [1], the cross-coupling between the joints are investigated. This is referred to as centralized control. Centralized control structures take advantages of knowledge of the dynamics to compensate for coupling effects. Often the coupling effects are reduced by adding a compensating torque based on the knowledge the dynamics to compensate for coupling effects. According to the study, the mass imbalance induce inertial cross-coupling and it is one of the three sources of disturbance. The other sources are listed as the base (vehicle) rotations, and gimbal system imperfections.

If high-response rates are desired, the nonlinear control systems can be considered. In [6], a robust inverse dynamics controller along with adaptive control is investigated. The robust inverse dynamics controller is used to provide rejection of external disturbances. However, in the paper it is stated that this controller is sensitive to unmodeled dynamics of the gimbal system and may cause chattering. To mitigate this effect an adaptive controller is devised.

The system identification refers to construction of a mathematical model from noisy data by experiments. The mathematical model (plant model) is required for building advanced control systems. Time domain and frequency domain simulations, stability analysis, controller design, and predicting the performance of a controller all requires a mathematical model. In [6], system identification of a gimbal system in frequency domain is considered. As also mentioned in the study, the experiment design was the most important part since least possible bias and minimum variance were desired. In the study, parametric and non-parametric modelling is described where modelling is based on experiment's inputs and outputs. Non-parametric modelling requires experiments at various frequencies and there is no relation between frequencies. In parametric modelling, system is described with finite number of parameters. Noise model is estimated using a nonparametric modelling scheme. On the other hand, experiment data is fitted to a parametric system model.

The modelling of a robotic (or a gimbal) system requires mathematically tedious analytical closed form kinematic and dynamic relationships. There are commercial symbolic manipulation engines for tackling these relationships. As an example, in [9], tracking radar gimbal control is considered. The forward and inverse kinematics and dynamics are obtained using MapleSim's symbolic engine. However the availability of the symbolic engine license in research facilities dictates the software. For detailed information on how to model and control a system in MATLAB/Simulink we refer the reader to [10]. In [3], mechanical modelling and control example of a 4 axis gyroscope platform is given. Simulink/SimMechanics is used in their study.

A nonlinear system's (all mechanical systems are inherently nonlinear) stability criterion does not determine a unique controller. Each stabilizing controller is a possible controller for the target system. Among these possibilities, optimal control seeks for the best behaving one. However, optimality of mathematically defined control system depends on the modelling accuracy on the real system. Since the target mechanical system cannot be modelled exactly, there is always a discrepancy between model and the actual system. For this reason, optimal control system may not work as intended on the actual system. To deal with system uncertainty, robust control scheme can be considered. The main advantage of a robust control system is its stabilizing behaviour under uncertainty and surfacing of an unmodelled aspect of the system, etc.

In [2], techniques utilized in UAV (Unmanned Aerial Vehicle) control are listed as Linear Quadratic Regulator (LQR), Linear Quadratic Gaussian (LQG), H_2, H_∞, Quantitative Feedback Theory (QFT), advanced self tuning techniques for PID, Neural Networks and Adaptive Control. These techniques are not extensively studied for gimbal control in literature. For this reason, in their study, controller design and implementation for the PID controller and its associated auto-tuning algorithm, H_∞ control, QFT and adaptive control schemes are investigated.

In [13], gimbal trajectory planner and gimbal motion controller subsystems are developed based on mathematical models of their gimbal system in MATLAB/Simulink. The trajectory planner transforms the input signal to a reference signal relative a vehicle fixed reference frame. Two motion controllers are developed; namely a PID controller and a linear quadratic controller (LQR).

In [14], robust inverse dynamics control and sliding mode control schemes are utilized in the inner control loop. The sliding mode controller applies a set valued control signal that causes the system to jump from one continuous regime to another based on the current position on the state space. To stabilize the LOS in case of disturbances, a high level (outer loop) controller is implemented. Both direct and indirect LOS stabilization are investigated. In the direct LOS stabilization, the angular rate sensors are mounted on LOS axis. On the other hand, in the indirect stabilization, the rate sensors are mounted on the base of the gimbal. Since, the mechanical structure of the current device in this thesis allows application of rate sensor in inner gimbal, the direct LOS stabilization method, explained in [2], is useful.

In [11], fuzzy controller is implemented on 3 DOF stabilizing platform. A decentralized control scheme is considered, resulting in 3 separate controllers for 3 axes. Fuzzy logic is especially suitable for systems with high uncertainty.

CHAPTER 3

EXPERIMENTAL SETUP

3.1 Mechanical Design of the Set-up

The experimental set-up is constructed to be formed from two main assemblies. First one is called the disturber and is located underneath the second main assembly which is called the stabilizer. Disturber is responsible for generating the desired motion of the stabilizer assembly. As the name suggests, it disturbs the stabilizer so that we can evaluate its performance. Meanwhile, stabilizer is expected to control the platform orientation with respect to an inertial frame by means of generating necessary motor torques.

We have come up with two different concept designs for this study. In the first one, all of the motion actuators are selected to be servo motors. In the second concept design, disturber actuators are replaced with hydraulic cylinders. The stabilizer module remains unchanged and the only difference between these two versions is how we actuate the disturber motions.

Moreover, the designs enable different feedbacks such as coordinates obtained by resistive touchpad, angles obtained from inclinometers, encoders or gyroscopes for both concept designs and actuator strokes from linear transducers in the second concept design to be used for control purposes.

When cost and time schedule analysis is done, we have seen that the second concept design will be cheaper and at the same time it will take less time to assemble the set-up. Therefore, the second concept design has been selected to be best concept design and we have decided to move on with this design. However, in this report we will mention about the first design version too; so that evolution of the study can be understood. More importantly, it is also beneficial to explain this design to a some extend; because some of the analysis, calculations and deductions we have made as a result of this design are adapted to second concept design.

3.1.1 First Concept Design

In this design, the experimental set-up comprises of two different gimbals assembled together. Both of these gimbals have two degrees of freedom on tilt and roll axes. The outer gimbal is used to simulate movement of a military vehicle, based on the data collected from a real life operation. The inner gimbal, on the other hand, is used to stabilize the platform by using the feedback from different instruments and devices.

Both versions of the experimental set-up has been designed to be versatile such that the center of mass and inertia of the platform to be stabilized can be adjusted by using portable weights. By this way, the performance of the controllers are verified when the system is subjected mass unbalance.

The isometric views of the experimental set-up are given in Figure 3.1. Moreover, the exploded view of the assembly is presented in Figure 3.2.

Figure 3.1: Isometric views of the first version experimental set-up

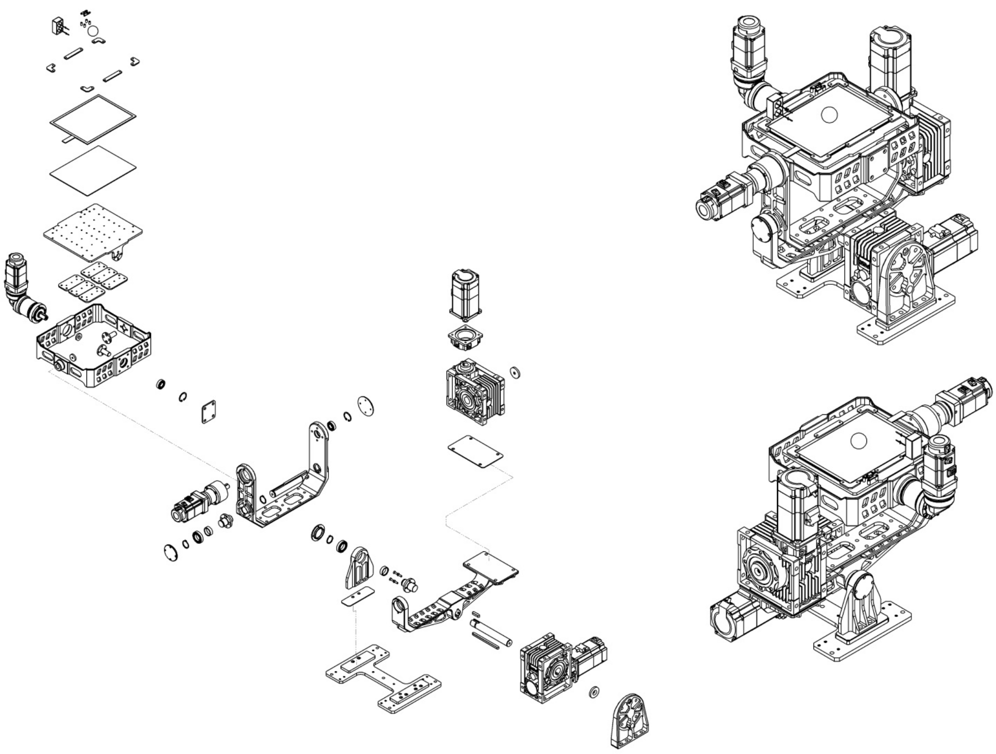


Figure 3.2: Exploded view of the first version of the experimental set-up

At this point some section views of the design will be presented and design details will be explained. In Figure 3.3, degrees of freedoms and locations of the sections are shown.

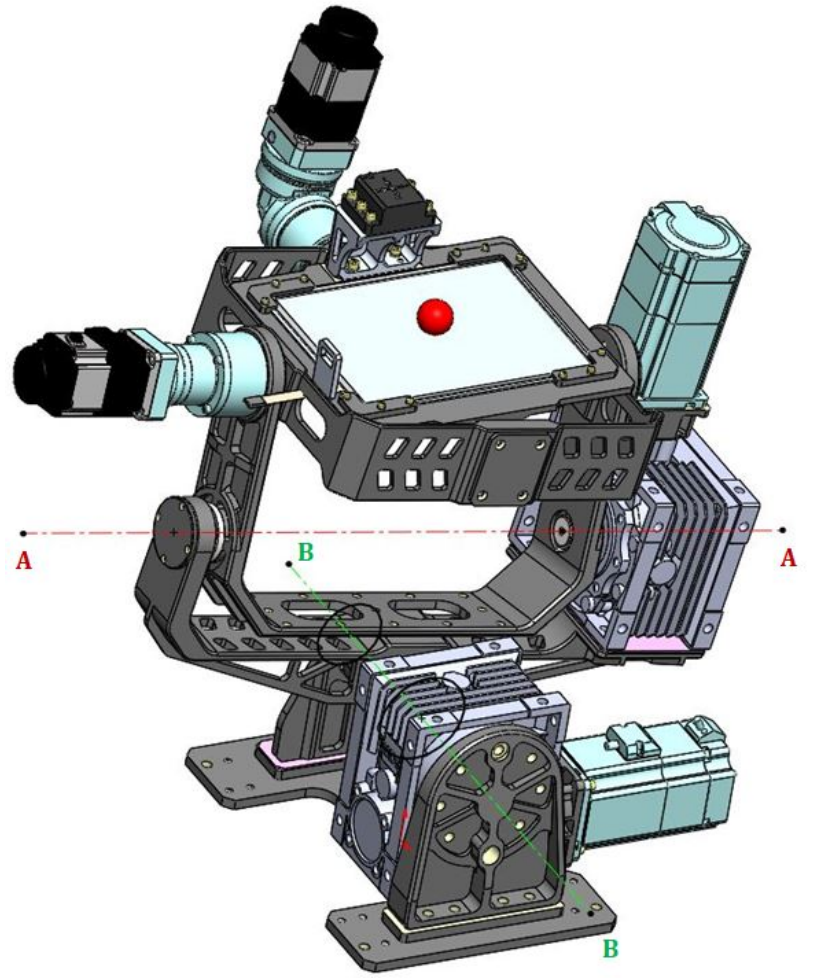


Figure 3.3: Representation of the degree of freedoms and section locations

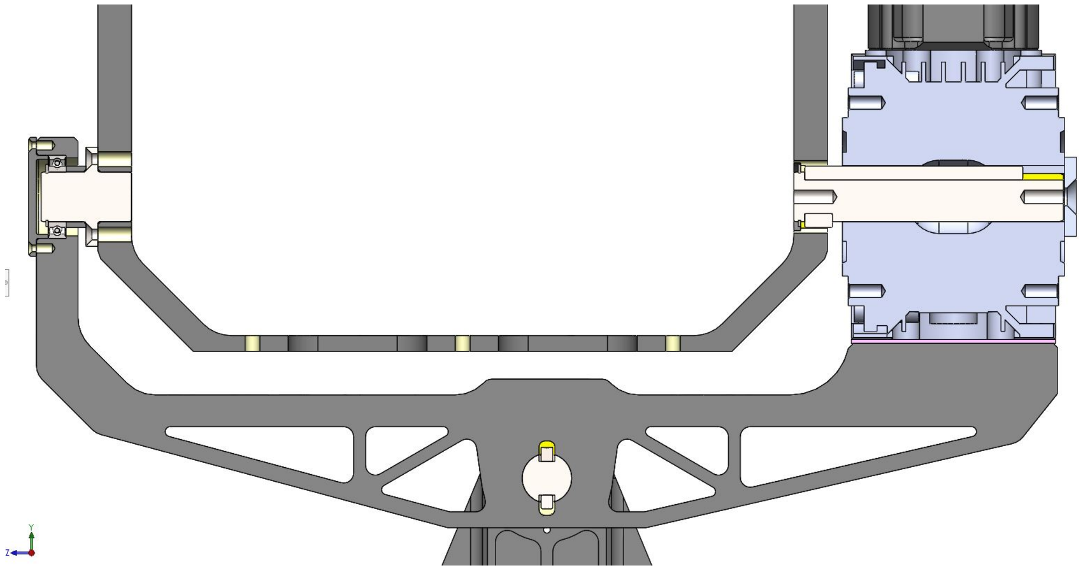


Figure 3.4: Section A-A of first version of disturber

In Figure 3.4, on the right gearbox and the power transmitting shaft can be seen. the gearbox is a worm gear type so that it changes the direction of the servo motor axis perpendicularly and is able to generate high transmission ratio. The change of motion of the motor by the gearbox is a must feature needed for compact design. Moreover, the shaft has a key for power transmission purposes. Its axial movement is restricted by means of a retaining ring on the left and a cap on the right.

On the left counterpart, a bearing and its shaft are located. Shaft is assembled to the inner body by means of countersunk screws and moves together with the inner body. Bearing has a locating bearing arrangement which means that it dictates the axial position of the remaining parts. In order to achieve that arrangement it has been fixed from a cap on the left of the outer ring which is immobile by the way, and on the left from a machined resting surface of the outer axis main body. Inner ring, which is mobile, is restrained by means of a retaining ring on the right and a bushing on the right. Note that bushing is located between the inner axis body and the right inner surface of the bearing. In order to use the locating bearing arrangement, the bearing has to be able to withstand both radial and axial loads. The bearing type used in this design is a deep groove type so it can handle both types of loads.

On the right, you can see that a pink shim has been inserted to the design. Although all the parts have been designed to be convenient for 5 axis machining and continuous features such as left and right shaft holes are instructed to be machined in a single operation; due to the uncertainty in the gearbox dimensions; there is always a risk that both left and right shaft holes may not be concentric. This shim is meant to be final machined after critical dimensions of the delivered gearbox and other machined parts are precision measured by CMM. Therefore, it will give just the required thickness so that both these opposing shaft holes will be concentric.

This arrangement has been used in all of the servo motor actuated degrees of freedom in both concept designs. Each module takes its reference from the outer surrounding module and generates motion accordingly. When all these motions are combined, an articulated complex motion patterns can be achieved.

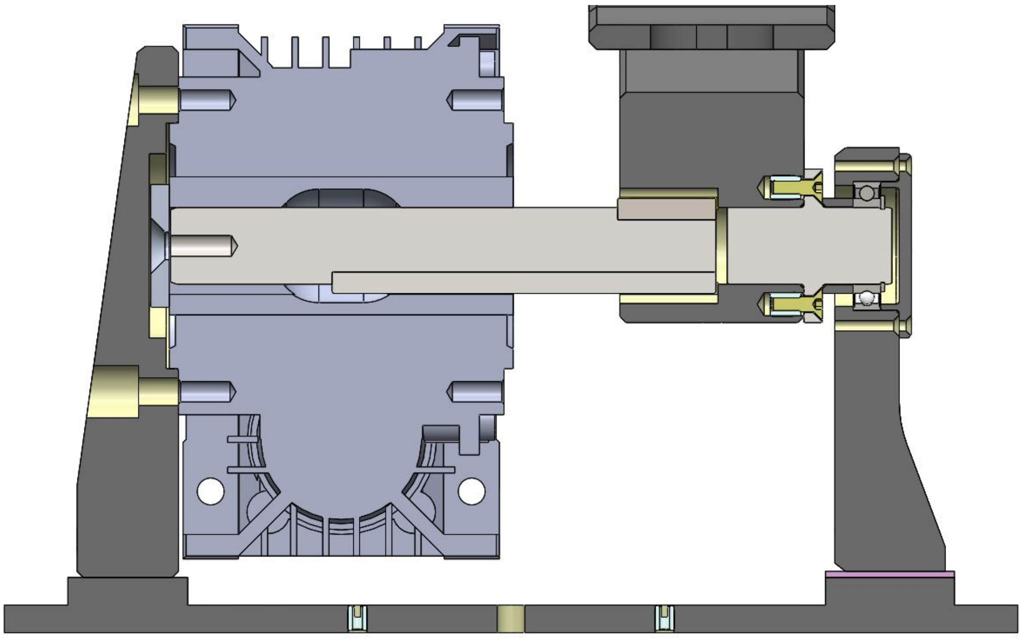


Figure 3.5: Section B-B of first version of disturber

The section given in 3.5 shows the most outer axis. For all other degree of freedoms the motion actuating mechanisms take reference from the nearest outer axis. Here, as you can see the reference is taken from the ground as this is the very first degree of freedom of the machine. This time you can see the bearing on the right and power transmitting machine elements on the right. All the design considerations are identical as mentioned in the explanation of section view given in 3.4.

3.1.2 Second Concept Design

As it has been mentioned before due to time and cost parameter; this design has been selected to be the best concept design. As it has been decided to be the final design; it has been manufactured and assembled. In figures 3.6 & 3.7 you can see the isometric view and exploded view, respectively.

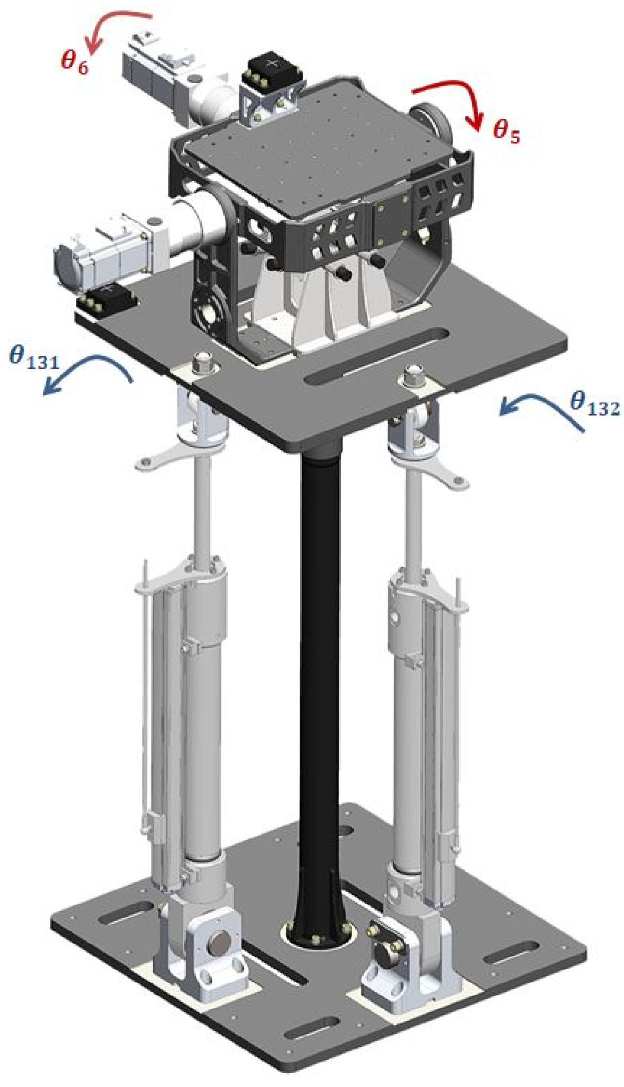


Figure 3.6: Representation of the payload and platform angles

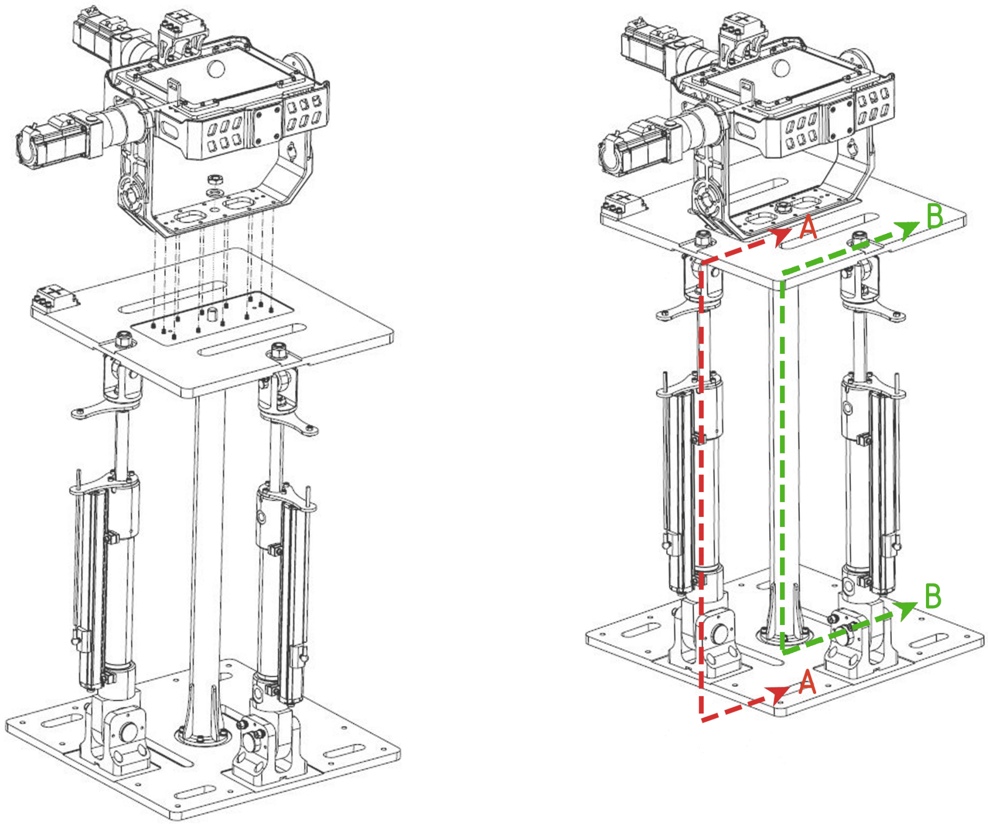


Figure 3.7: Exploded view of the final design of experimental set-up

3.1.3 Mechanical Design of the Disturber

The disturber is a spatial mechanism consisting of revolute, prismatic and ball joints. The representation of the mechanism is given in 3.8.

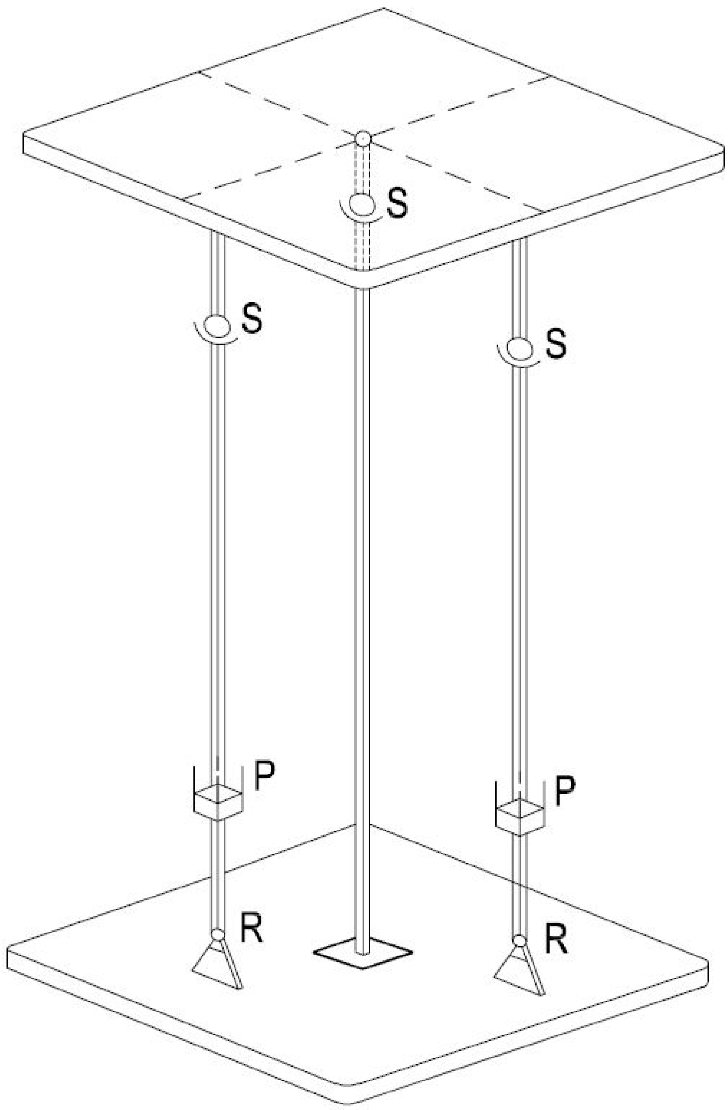


Figure 3.8: Schematic representation of the disturber mechanism

Similar to planar mechanisms, loop closure equations which define the kinematic relationships of the mechanism can be written for spatial mechanisms too. A number of different mathematical forms can be used for this purpose including vectors, dual numbers, quaternions and transformation matrices.

However, special constraints have been added to the design so that the two payload angles θ_{131} & θ_{132} have become independent of each other. Moreover, two actuators and ball joint support in the middle of the payload base plate are included in the design to satisfy symmetry conditions. Therefore, all in all, the spherical mechanism can be represented by two planar mechanisms. In this configuration, each actuator is responsible for a single DOF and the motion of neither actuators affect the other actuator's motion. In other words, the motions of the actuators are independent from each other and no cross coupling occurs.

In Figure 3.9, exploded view of the disturber mechanism can be seen.

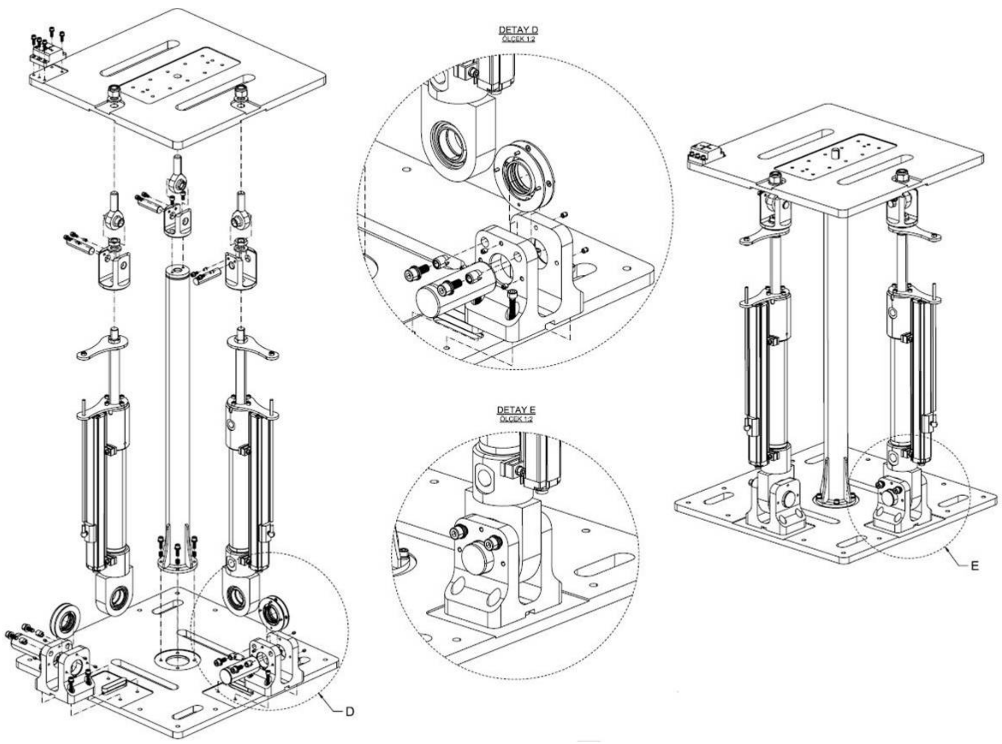


Figure 3.9: Exploded view of the disturber

As it can be seen from 3.8 & 3.9; there are three ball joints connecting the two hydraulic cylinders and middle support to the base plate of the payload. Both actuators are connected to the ground by means of revolute joints and actuators themselves are prismatic joints. Preliminary design sketch that has been used to create the concept design and check the motions of the mechanism is given in Figure 3.10.

In figures 3.11 & 3.12 critical section views are presented to explain the design details.

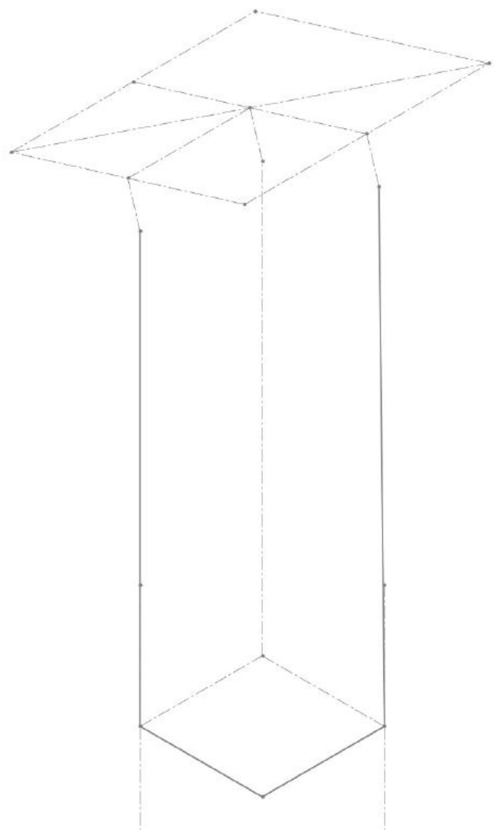


Figure 3.10: Concept design sketch of the disturber mechanism

In figures 3.11 & 3.12 critical section views are presented to explain the design details. Also, section planes are shown in Figure 3.8.

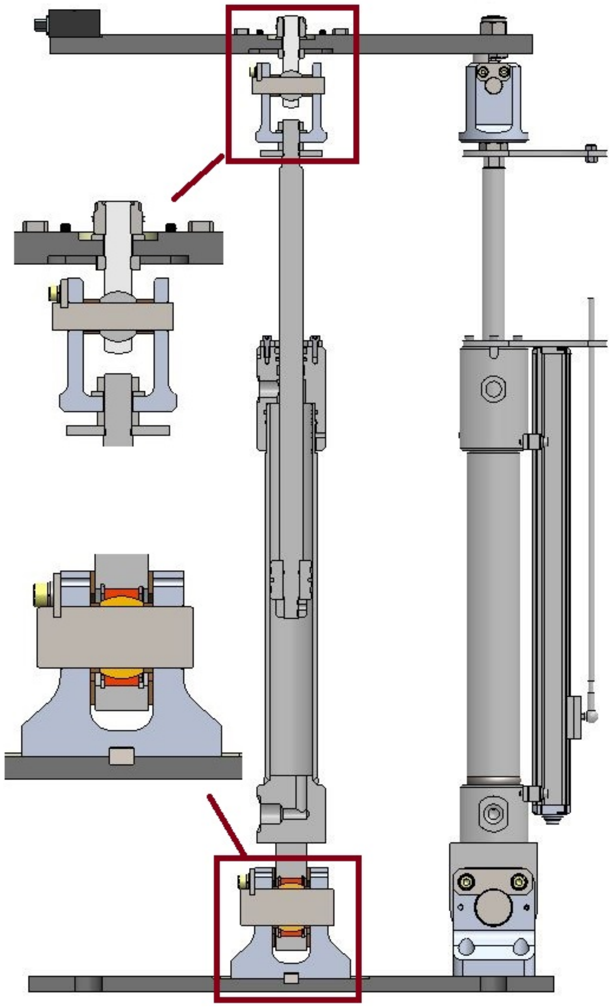


Figure 3.11: Section A-A of second version of disturber

When we look at the bottom detail, we can see that a ball joint has been located in the bottom mechanical interface of the system. The reason for this choice is that this bottom interface, the U body and the ball joint inside it, is a COTS item readily available and easy to assemble. However, according to kinematic synthesis of the mechanism, what we need here is a revolute joint. The ball joint is converted to a revolute joint by eliminating its extra rotation degree of freedoms. In order to accomplish that bronze ring has been inserted both on the top and bottom sides of the ball joint. By this way, lateral axial movement of the hydraulic cylinder is prevented and at the same time extra rotation degree of freedom is restricted. Moreover, bronze rings have a low friction coefficient so the axial movement of the hydraulic cylinder in this joint has been very smooth. One more important point about the bronze rings is that a joint without clearance can be achieved by giving right tolerances and ability of bronze material to shrink.

Similarly again in the bottom detail, we can see that the rotation of the shaft which works with the ball joint of the hydraulic cylinder has been fixed by means of a lock plate. More explanations about the lock plate is given in the paragraph underneath the Figure 3.12.

As it can be seen a key has been located between the housing which connects the ground and the hydraulic cylinder and the ground plate. The key aligns the hydraulic cylinder assembly so that the assembly process becomes much smoother and misalignment issues are prevented. Moreover, it takes the moment and shear forces which are very undesirable for the screws.

When we look at the top detail, we can see that jam nut technique has been used in the threaded connections to prevent loosening of the nuts. An extra precaution we have taken about these connections is to utilize the Loctite adhesives that is used to lock threaded connections. Again bronze rings, ball joint and lock plate with the shaft method has been used in the top joint. The bottom nut between the platform plate and the top joint is a jam nut but at the same time it has been used for calibration purposes. All of the ball joints have a jam nut at the bottom of the plate and by turning these the platform can be levelled by means of water gage.

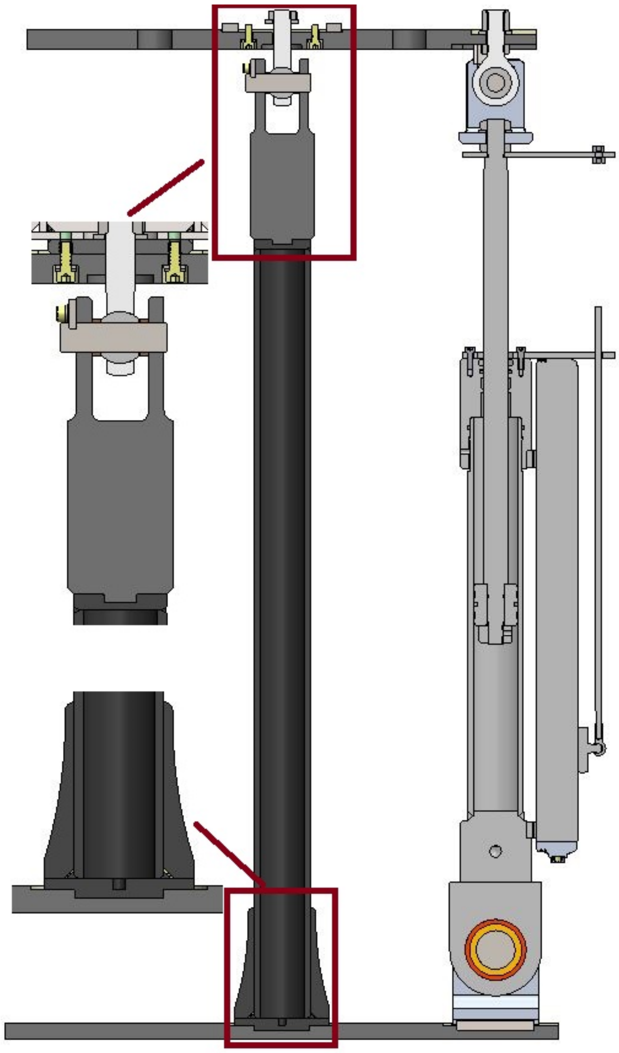


Figure 3.12: Section B-B of second version of disturber

If we look at the detailed view of the bottom part of the middle support, it can be seen that the middle support has been aligned to the ground plate by means of a precision machined circular feature. This design detail enables reference of the ground plate to be transferred to the middle support. Therefore, the assembly process is much easier and tolerance accumulation which will prevent assembling more parts on top of each other, is avoided by this way. Furthermore, by this way the screws are not subjected to shear forces which they are not durable enough. This feature takes all moments subjected to this area so that the screws are not subjected to shear forces. This design logic has been implemented through every design detail. Again similarly, you can see that the top support has been aligned with the long middle support by using the same logic.

When we look at the top detail, we can see that the axial movement of the ball joint is prevented by two bronze rings. In order to ease the rotation of the ball joint, the shaft inside the ball joint is fixed. In order to prevent the rotation of the shaft which the ball joint rotates with respect to, a lock plate has been inserted. As it can be seen on the detailed view located on the left; it is a rectangular plate and it is inserted into a rectangular groove feature in the shaft. When these planar surfaces touch each other the shaft cannot rotate since this lock plate is also fixed to a fixed body by the screws. Also, again due to screws of the lock plate, the axial movement of the shaft is also restricted.

Lastly, we will talk about the design of the hydraulic cylinder. The hydraulic cylinder has been designed for moderate loads. The working principle is very simple. It has a mobile part called piston. The main function of the piston is to separate the pressure zones inside the barrel. The piston is machined with grooves to fit elastomer or metal seals and bearing elements. The direction and the velocity of the hydraulic cylinder is decided according to the pressure difference on the upper and lower surfaces of the piston. According to the sign of this pressure difference it retracts or extends and according to the magnitude of the pressure difference, it retracts or extends slower or faster.

Other elements are cylinder barrel, cylinder base, cylinder head, piston rod, seal gland and seals. The main function of the cylinder body is to hold cylinder pressure. The cylinder barrel we have used has been made from honed tube.

Cylinder base, which has been welded, is used to enclose the pressure chamber at one end. Similarly, the mechanical interface which includes a ball joint inside has been assembled by pins to the cylinder base and then welded.

The main function of the head is to enclose the pressure chamber from the other end. The head contains an integrated rod sealing arrangement so that it can be combined with a seal gland. The head is connected to the body by means of threading. O-rings are used in between head and barrel.

The piston rod we have used is a hard chrome-plated piece of cold-rolled steel (C1045) which is attached to the piston and extends from the cylinder through the rod-end head. The piston rod connects the hydraulic actuator to the machine component doing the work. This connection is in the form of a machine thread. Jam nut type assembly method, which uses double nuts also called jam nut to prevent loosening, has been used to secure the upper ball joint assembly.

Last component is called the seal gland. The cylinder head is fitted with seals to prevent the pressurized oil from leaking past the interface between the rod and the head. This area is called the seal gland. The advantage of a seal gland is that it can be easy removed and seal replacement can be easily done.

In Figure 3.13, these parts are shown in section view of the hydraulic cylinder.

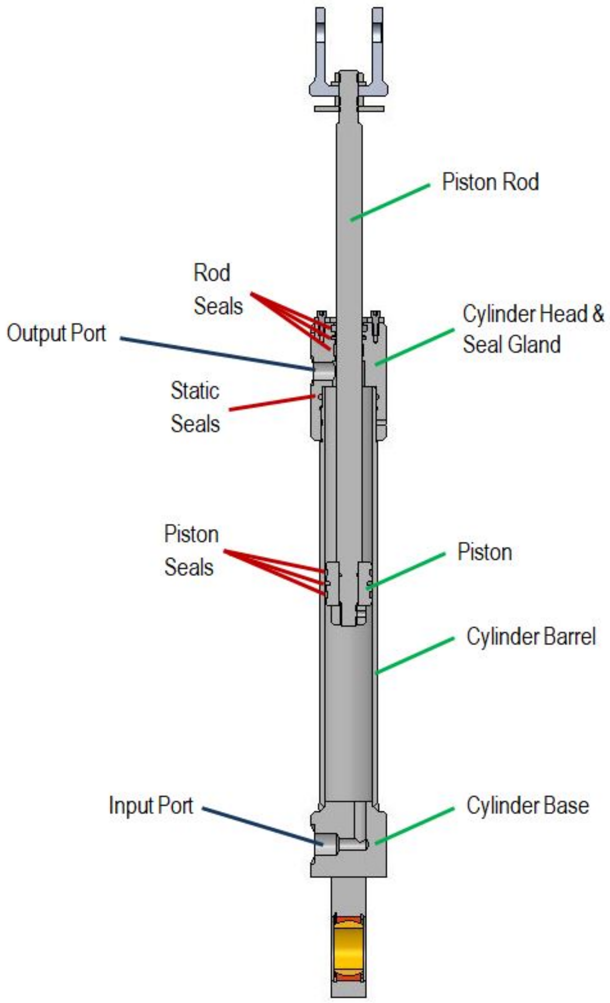
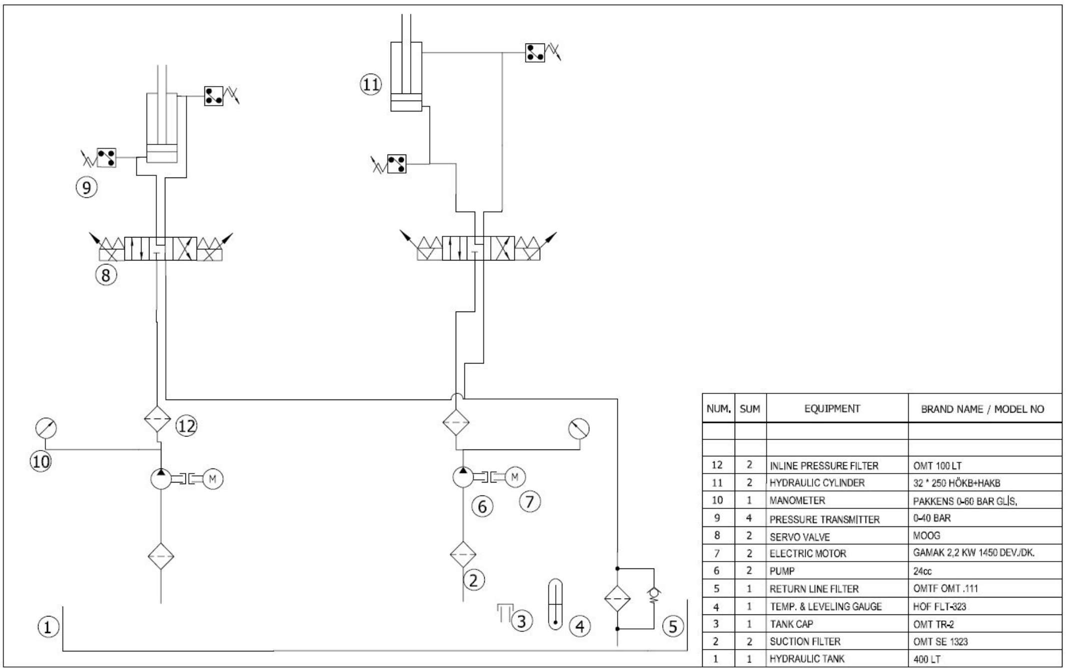


Figure 3.13: Sectional view of the hydraulic cylinder

3.1.4 Hydraulic Design of the Disturber

In Figure 3.14 P & ID drawing of hydraulics system is given.



NUM.	SUM	EQUIPMENT	BRAND NAME / MODEL NO
12	2	INLINE PRESSURE FILTER	OMT 100 LT
11	2	HYDRAULIC CYLINDER	32 * 250 HÖKB+HAKB
10	1	MANOMETER	PAKKENS 0-60 BAR GL|S.
9	4	PRESSURE TRANSMITTER	0-40 BAR
8	2	SERVO VALVE	MOOG
7	2	ELECTRIC MOTOR	GAMAK 2,2 KW 1450 DEV./DK.
6	2	PUMP	24cc
5	1	RETURN LINE FILTER	OMTF OMT .111
4	1	TEMP. & LEVELING GAUGE	HOF FLT-323
3	1	TANK CAP	OMT TR-2
2	2	SUCTION FILTER	OMT SE 1323
1	1	HYDRAULIC TANK	400 LT

Figure 3.14: P & ID drawing of the hydraulic system

The system has been designed to consist of two identical subsystems for controlling both actuators independently. The electrical motor drives the pump which transfers the oil from the tank to the valves. In order to protect the equipment from the dirt; a suction filter has been placed in the pipeline before it reaches to the pump. Then after the pump, it is filtered again by an in-line type filter. Next, oil arrives at the valve. The valve is a 4/3 servo valve; which has an ability to precisely adjust the valve opening; therefore controlling the flowrate. It has been assembled with a plate and pressure regulators. This configuration simplifies the system since it packs essential equipments all together. The pressure regulator is required to release the pressure when it reaches a critical value by discharging excessive oil back to the tank. This limit is adjustable in the models we have used.

On the left block, pressure line is connected to the lower compartment of the hydraulic cylinder which causes an increase in the cylinder stroke. Similarly, on the right block, the pressure line is connected to the upper compartment of the hydraulic cylinder which results in a decrease in the cylinder stroke. As we have mentioned before, it is not an on-off type mechanism; indeed by adjusting the valve opening precisely the hydraulic cylinder velocity can be controlled. On the neutral configuration, when the valve compartment is on the middle, the pressure line is blocked; which results in a pressure increase in the pressure line. Although pressure relief valves are included in the design, this is still not a good practice. Therefore, two precautions have been taken in order to prevent this. First one is avoiding stall configuration of the actuators and constantly moving them. This has been implemented in the motion profiles. However, there is no guarantee that all the motion profiles will always demand constant movement. Therefore, as a secondary precaution, compensators have been installed to prevent the pressure build-up, protect the equipment and prevent accidents. Unlike pressure regulators, compensators are installed into pumps and reduce or stop the flowrate when pressure is sensed on the pump outlet port.

The last element of the hydraulic line is the hydraulic cylinder. Explanation about design of the hydraulic cylinder can be found in section 3.1.3. In Figure 3.15, real life assembled hydraulic components are shown. Moreover, in Table 3.1, bill of materials can be found.

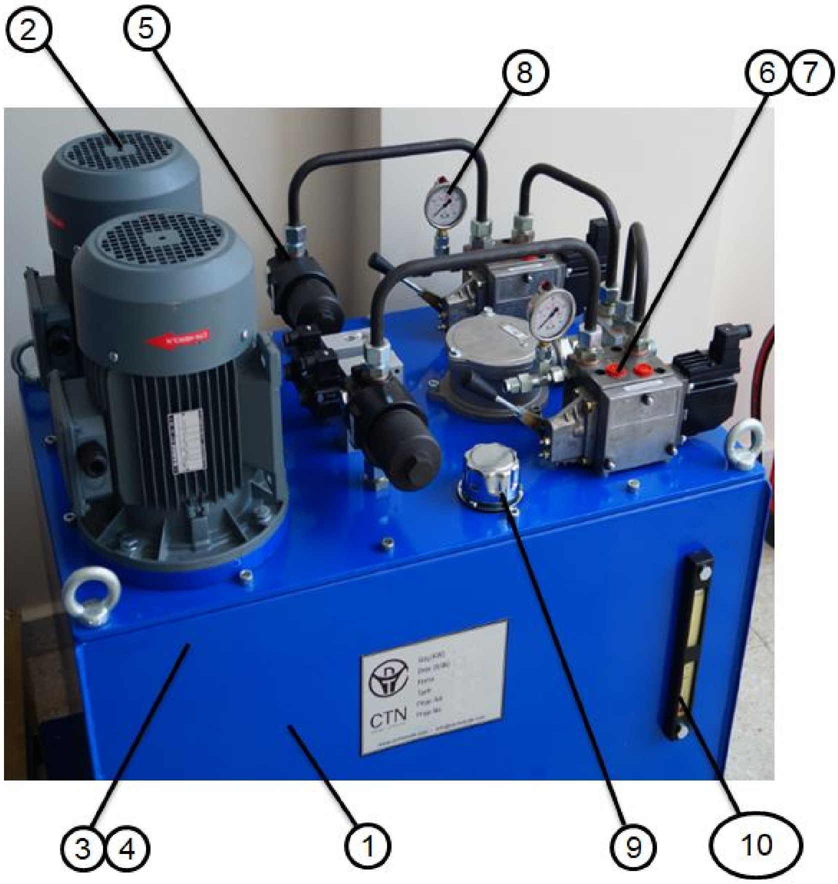


Figure 3.15: Photo of the hydraulic components

Table 3.1: Bill of materials of the hydraulic components

Item Number	Definition
1	Hydraulic Tank (400 litres)
2	Electric Motor
3	Pump
4	Suction Filter
5	Pressure Line Filter
6	Servo Valve
7	Plate & Pressure Regulator
8	Manometer
9	End cover
10	Level & Temperature Gauge

3.1.5 Mechanical Design of the Stabilizer

The isometric and exploded view of the stabilizer is given in figures 3.16 & 3.17, respetively.

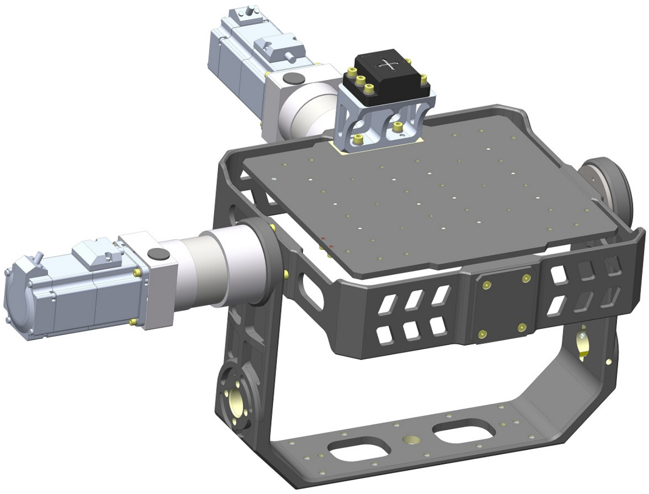


Figure 3.16: Isometric view of the stabilizer

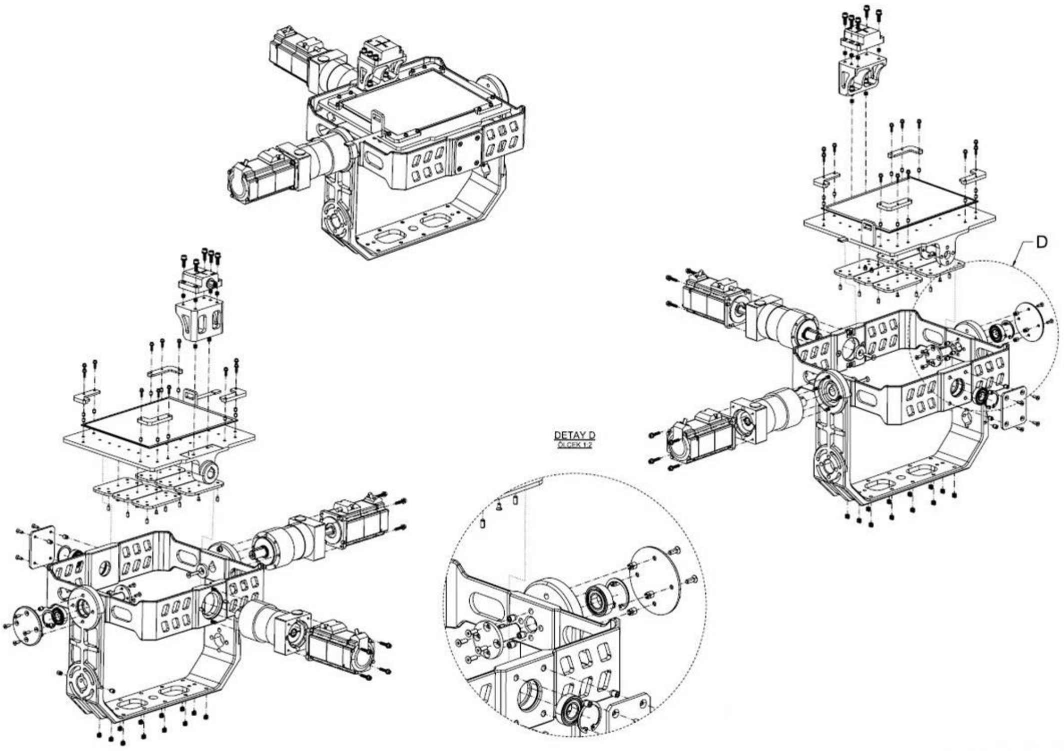


Figure 3.17: Exploded view of the stabilizer

In order to explain design details critical sectional views are given in 3.18 & 3.19

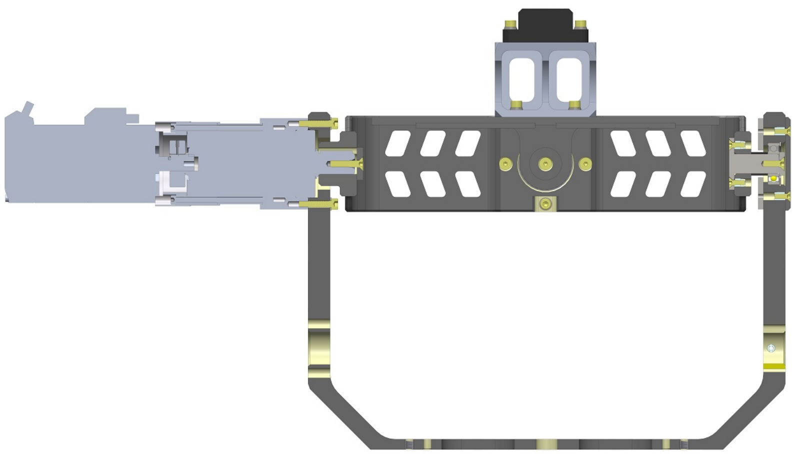


Figure 3.18: Section A-A of stabilizer

On the left, it can be seen that outer axis is actuated by means of a servo motor coupled with a gearbox. The connection between the servo motor and the gearbox is maintained by a clamping hub. Moreover, the gearbox and servo motor is connected to Body-5, which is in the shape of U. On the other hand, a key is used for power transmission between the gearbox and the body. Similar to the other bearing arrangements, which have been explained in detail in section 3.1.1; a locating bearing arrangement has been used on the concentric opposing hole feature. This arrangement can be seen on the right of Figure 3.18.

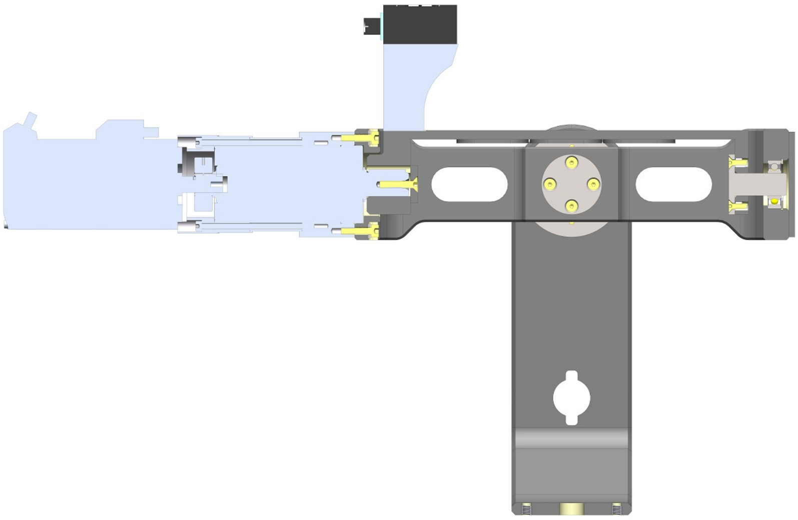


Figure 3.19: Section B-B of stabilizer

Similar to the outer axis arrangement, a servo motor and gearbox is located on the left whereas a locating bearing arrangement is used on the right. This time, servo motor and gearbox is connected to Body-6 which is in the shape of a rectangular frame.

3.1.6 Mechanical Design of the Shock Absorber

Shock absorber consists of several spring mechanisms to absorb the energy of the gimbals. It restricts the maximum angle freedom of the gimbals in order to prevent damage to the system. It has been added to design later on, since it has been observed that there can be some accidents even though all of the actuators are limited by the software. Exploded view of the shock absorber can be seen in Figure 3.20.

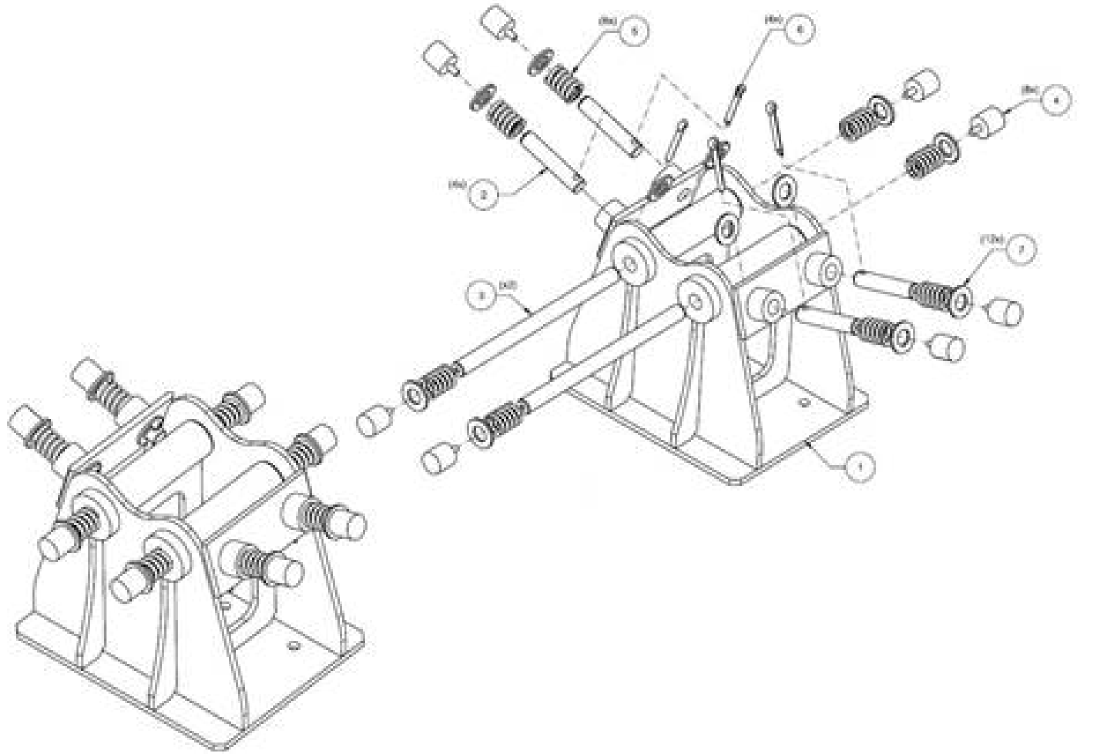


Figure 3.20: Exploded view of the shock absorber

3.2 Electronics & Software Infrastructure

For compact design criteria, all the electronic hardware has been assembled in a rack cabinet. This cabinet includes a PC and its Simulink Real Time Target PC, Humusoft MF 634 Data Acquisition card, two Omron R88D-GT02H servo motor drivers, an Ardunio UNO, miscellaneous electronic equipment for electric motors and servo valves in the hydraulic system. Ardunio UNO has been used firstly for touchpad; and then later for BMI-088 gyroscope. In addition to these, there are two electric boxes for contact breakers and fuses. These two boxes are located next to the hydraulic tank. These two boxes can be seen in Figure 3.21 The front and rear views and isometric view of the rack cabinet are given figures 3.22, 3.23 & 3.24, respectively.

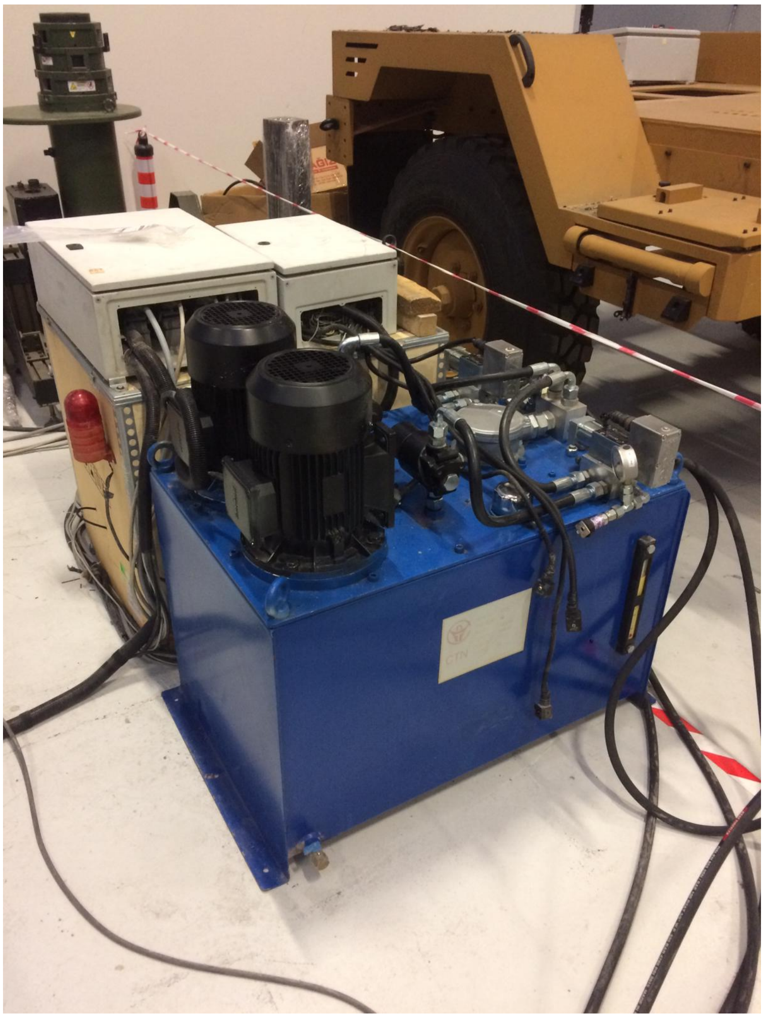


Figure 3.21: Photo of the electrical fuse boxes next to hydraulic tank

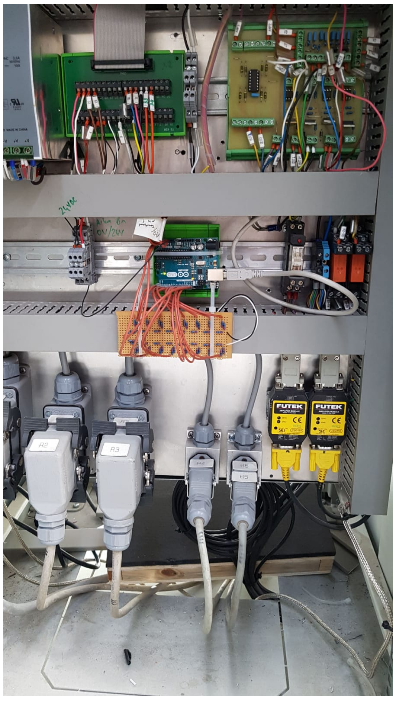


Figure 3.22: Front view of the rack cabinet

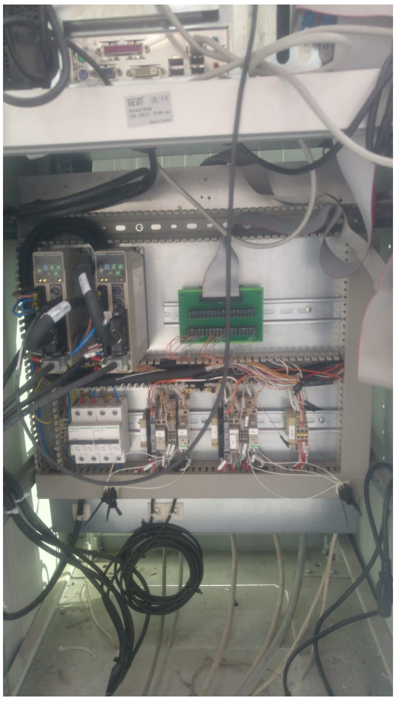


Figure 3.23: Rear view of the rack cabinet

Figure 3.24: Photo of the rack cabinet

Inputs and outputs of the DAQ have been arranged as follows:

AD0 Incline meter 1 – theta 132	1	20	DA0 Servo valve 1
AD1 Incline meter 1 – theta 131	2	21	DA1 Servo valve 2
AD2 Linear transducer 1	3	22	DA2 Servo motor 1
AD3 Linear transducer 2	4	23	DA3 Servo motor 2
AD4 Incline meter 2 – theta 5	5	24	DA4
AD5 Incline meter 2 – theta 6	6	25	DA5
AD6	7	26	
AD7	8	27	
AGND	9	28	+5V
DA6	10	29	GND
DA7	11	30	DOUT0 Valve 1 motor on/off
DIN0	12	31	DOUT1 Valve 2 motor on/off
DIN1	13	32	DOUT2 Alarm on/off
DIN2	14	33	DOUT3 Servo motor 1 on/off
DIN3	15	34	DOUT4 Servo motor 2 on/off
DIN4	16	35	DOUT5
DIN5	17	36	DOUT6
DIN6	18	37	DOUT7
DIN7	19		

IRC0A+		1	20	IRC3A+
IRC0A-		2	21	IRC3A-
IRC0B+	Encoder 1	3	22	IRC3B+
IRC0B-		4	23	IRC3B-
IRC0I+		5	24	IRC3I+
IRC0I-		6	25	IRC3I-
IRC1A+		7	26	TRIG
IRC1A-		8	27	
IRC1B+	Encoder 2	9	28	+5V
IRC1B-		10	29	GND
IRC1I+		11	30	T0IN
IRC1I-		12	31	T0OUT
IRC2A+		13	32	T1IN
IRC2A-		14	33	T1OUT
IRC2B+		15	34	T2IN
IRC2B-		16	35	T2OUT
IRC2I+		17	36	T3IN
IRC2I-		18	37	T3OUT
GND		19		

Figure 3.25: DAQ signal allocation list

These pin connections on the DAQ can be seen in Figure 3.26.

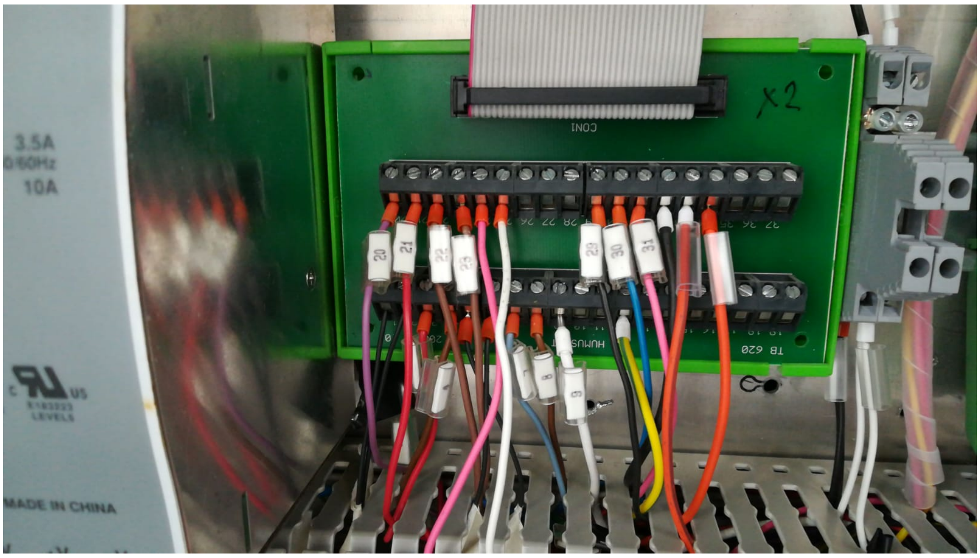


Figure 3.26: Photo of the DAQ pin connection

3.3 Overview of the Set-up

All in all, real life photos of the physical set-up can be found in figures. Note that, first photo represents to earlier times of the set-up whereas the latter one represents its current condition.

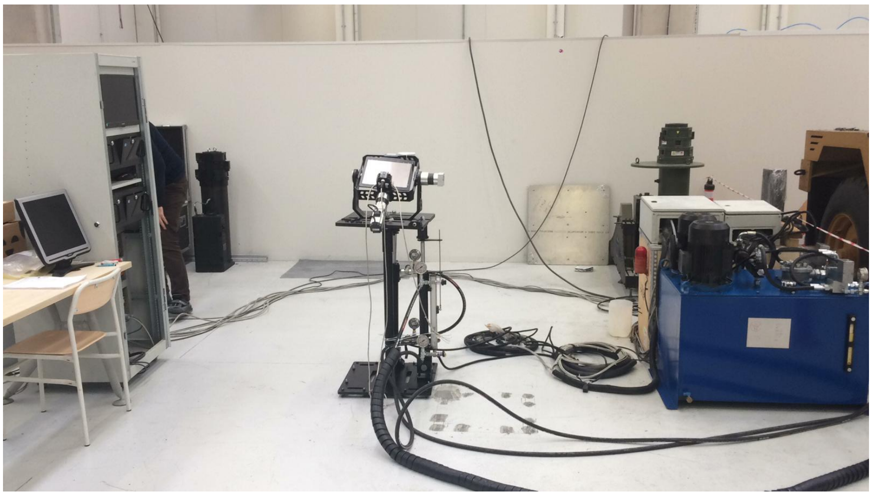


Figure 3.27: Photo of the physical set-up from earlier stages of the study

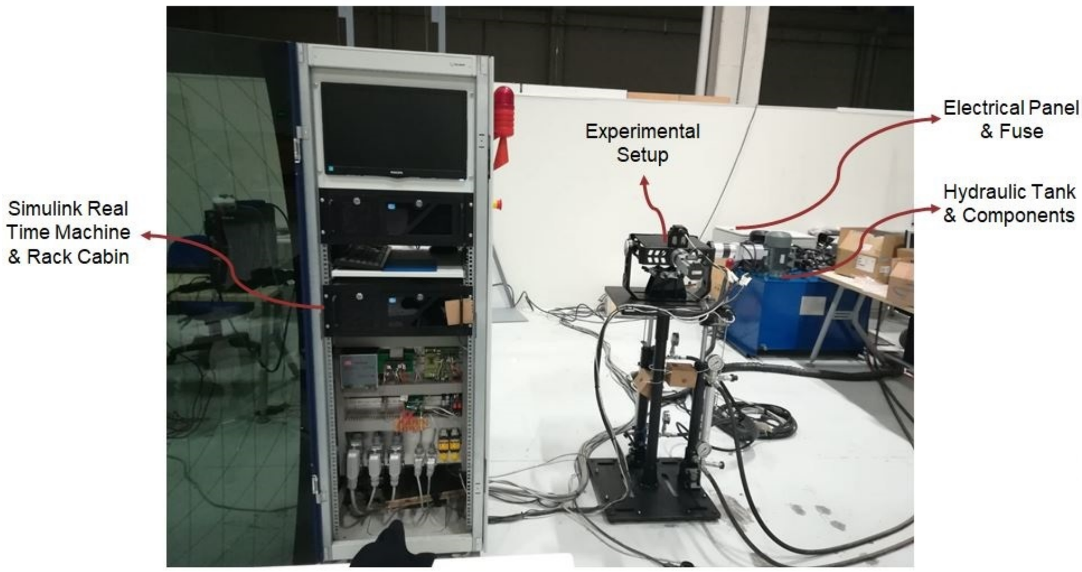


Figure 3.28: Photo of the current physical set-up

CHAPTER 4

MATHEMATICAL MODELLING

4.1 Ball and Plate

4.1.1 Adaptation of the Problem to Experimental Set-up

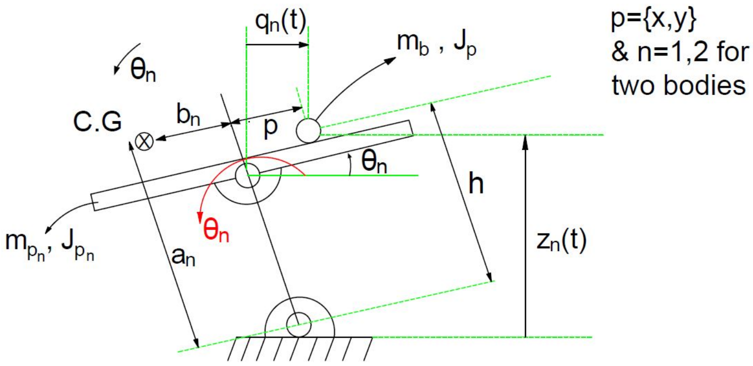


Figure 4.1: Representation of ball and beam adaptation

The kinetic energy of the plate is:

$$K_p = \frac{1}{2} J_{pn} \dot{\theta}_n^2 + \frac{1}{2} m_{pn} V_{pn}^2 \tag{4.1}$$

Similarly, the kinetic energy of the ball is:

$$K_b = \frac{1}{2} J_b \dot{\theta}_b^2 + \frac{1}{2} m_b V_b^2 \tag{4.2}$$

The total kinetic energy of the system is:

$$K_t = K_p + K_b \tag{4.3}$$

$$K_t = \frac{1}{2} J_{pn} \dot{\theta}_n^2 + \frac{1}{2} m_{pn} V_{pn}^2 + \frac{1}{2} J_b \dot{\theta}_b^2 + \frac{1}{2} m_b V_b^2 \tag{4.4}$$

where

$$V_{pn}^2 = (a_n^2 + b_n^2)\dot{\theta}_n^2 \tag{4.5}$$

$$\dot{\theta}_b^2 = \frac{\dot{p}^2}{r^2} \tag{4.6}$$

$$V_b^2 = \dot{p}^2 + (p^2 + h^2)\dot{\theta}_n^2 \tag{4.7}$$

Substitution of (4.5)- (4.7) into (4.4) yields the following result:

$$K_t = \frac{1}{2}[J_{pn} + m_{pn}\underbrace{(a_n^2 + b_n^2)}_{r_n^2}) + m_b(p^2 + h^2)]\dot{\theta}_n^2 + \frac{1}{2}(\frac{J_p}{r^2} + m_b)\dot{p}^2 \tag{4.8}$$

The potential energy of the people is:

$$U_p = m_{pn}g(b_n sin\theta_n + a_n cos\theta_n) \tag{4.9}$$

Similarly, the potential energy of the ball is:

$$U_b = m_b g(psin\theta_n + hcos\theta_n) \tag{4.10}$$

The total potential energy of the system is:

$$U_t = U_p + U_b \tag{4.11}$$

$$U_t = m_{pn}g(b_n sin\theta_n + a_n cos\theta_n) + m_b g(psin\theta_n + hcos\theta_n) \tag{4.12}$$

The Lagrange of the system is:

$$L = K - U \tag{4.13}$$

$$L = \frac{1}{2}[J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p^2 + h^2)]\dot{\theta}_n^2 + \frac{1}{2}(\frac{J_p}{r^2} + m_b)\dot{p}^2 \tag{4.14}$$
$$- m_b g(psin\theta_n + hcos\theta_n) - m_{pn}g(b_n sin\theta_n + a_n cos\theta_n)$$

The first Lagrange equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) - \frac{\partial L}{\partial p} = 0 \tag{4.15}$$

The terms of (4.15) can be obtained as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) = \left(\frac{J_b}{r^2} + m_b\right)\ddot{p} \tag{4.16}$$

$$\frac{\partial L}{\partial p} = m_b p\dot{\theta}_n^2 - m_b g sin\theta_n \tag{4.17}$$

Substitution of (4.16) & (4.17) into (4.15) yields the Lagrange equation:

$$L_1 = \left(\frac{J_b}{r^2} + m_b\right)\ddot{p} - m_b p \dot{\theta}_n^2 + m_b g sin\theta_n = 0 \tag{4.18}$$

Similarly, the second Lagrange equation is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_n}\right) - \frac{\partial L}{\partial \theta_n} = T_n \tag{4.19}$$

The terms of (4.19) can be obtained as follows:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_n}\right) &= \frac{d}{dt}[J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p^2 + b^2)]\dot{\theta}_n \\ &= \big(J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p^2 + b^2)\big) + 2m_b p\dot{p}\dot{\theta}_n \quad (4.20) \\ \frac{\partial L}{\partial \theta_n} &= -m_b g(pcos\theta_n - hsin\theta_n) - m_{pn} g(b_n cos\theta_n - a_n sin\theta_n) \quad (4.21)\end{aligned}$$

Substitution of (4.20) & (4.21) into (4.19) yields the second Lagrange equation:

$$\begin{aligned}L_2 = \big(J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p^2 + b^2)\big)\ddot{\theta}_n + 2m_b p\dot{p}\dot{\theta}_n + m_b g(pcos\theta_n - hsin\theta_n) \\ + m_{pn} g(b_n cos\theta_n - a_n sin\theta_n) = T_n \quad (4.22)\end{aligned}$$

4.1.1.1 Nonlinear and Linear State-Variable Representation

The equations we derived can be written in state variable representation. The state vector can be defined as follows:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} p(t) \\ \dot{p}(t) \\ \theta_n(t) \\ \dot{\theta}_n(t) \end{bmatrix} \tag{4.23}$$

These variables are the minimum set of variables required to determine the future response of the system. For that purpose, we also need the input and the current state. By using the equation of motion, the derivatives of state variables can be obtained. From (4.18),

$$\ddot{p} = \frac{-m_b g sin\theta_n + m_b p \dot{\theta}_n^2}{\left(\frac{J_b}{r^2} + m_b\right)} \Rightarrow \dot{x}_2 = \frac{m_b x_1 x_4^2 - m_b g sin(x_3)}{\left(\frac{J_b}{r^2} + m_b\right)}$$

From (4.22),

$$\ddot{\theta}_n = \frac{T_n - 2m_b p\dot{p}\dot{\theta}_n - m_b g(pcos\theta_n - hsin\theta_n) - m_{pn} g(pcos\theta_n - a_n sin\theta_n)}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p^2 + h^2)} \quad (4.24)$$

$$\dot{x}_4 = \frac{T_n - 2m_b x_1 x_2 x_4 - m_b g\big(x_1 cos(x_3) - hsin(x_3)\big) - m_{pn} g\big(b_n cos(x_3) - a_n sin(x_3)\big)}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(x_1^2 + h^2)} \quad (4.25)$$

$$\dot{x}_1 = x_2 \quad (4.26)$$

$$\dot{x}_3 = x_4 \quad (4.27)$$

Therefore,

$$f(x,t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{m_b x_1 x_4^2 - m_b g sin(x_3)}{\left(\dfrac{J_b}{r_2} + m_b\right)} \\ x_4 \\ \dfrac{T_n - 2m_b x_1 x_2 x_4 - m_b g\big(x_1 cos(x_3) - hsin(x_3)\big) - m_{pn} g\big(b_n cos(x_3) - a_n sin(x_3)\big)}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(x_1^2 + h^2)} \end{bmatrix} \quad (4.28)$$

The state space equations can be defined as follows:

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t) \quad (4.29)$$

where,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (4.30)$$

Before the system is actuated, the platform rests around an equilibrium point. At that point, the state variables are defined as follows:

$$x_0 = \begin{bmatrix} p_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4.31)$$

The torque required to maintain this state can be found from (4.25);

$$T_n - 2m_b x_1 x_2 x_4 - m_b g\big(x_1 cos(x_3) - hsin(x_3)\big) - m_{pn} g\big(b_n cos(x_3) - a_n sin(x_3)\big) = 0 \quad (4.32)$$

When we evaluate (4.32) around operating point,

$$T_0 = \cancel{2m_b x_1 x_2 x_4} + m_b g\,\big(\underbrace{x_1 cos(x_3)}_{1} - \cancel{hsin(x_3)}\big) + m_{pn} g\,\big(\underbrace{b_n cos(x_3)}_{1} - \cancel{a_n sin(x_3)}\big) = 0 \tag{4.33}$$

$$T_0 = m_b g p_0 + m_{pn} g b_n \tag{4.34}$$

In order to design a controller; we need to linearize the system. Thus, we need to obtain the Jacobian matrix which is obtained by expanding the function by Taylor series; neglecting the higher order terms and evaluation around the operating point. Hence,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} \cancel{\partial\dot{x}_1/\partial x_1} & \partial\dot{x}_1/\partial x_2 & \cancel{\partial\dot{x}_1/\partial x_3} & \cancel{\partial\dot{x}_1/\partial x_4} \\ \partial\dot{x}_2/\partial x_1 & \cancel{\partial\dot{x}_2/\partial x_2} & \partial\dot{x}_2/\partial x_3 & \partial\dot{x}_2/\partial x_4 \\ \cancel{\partial\dot{x}_3/\partial x_1} & \cancel{\partial\dot{x}_3/\partial x_2} & \cancel{\partial\dot{x}_3/\partial x_3} & \partial\dot{x}_3/\partial x_4 \\ \partial\dot{x}_4/\partial x_1 & \partial\dot{x}_4/\partial x_2 & \partial\dot{x}_4/\partial x_3 & \partial\dot{x}_4/\partial x_4 \end{bmatrix}}_{Jacobian\ matrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \cancel{\partial\dot{x}_1/\partial T_n} \\ \cancel{\partial\dot{x}_2/\partial T_n} \\ \cancel{\partial\dot{x}_3/\partial T_n} \\ \partial\dot{x}_4/\partial T_n \end{bmatrix} T_n \tag{4.35}$$

Evaluating the terms in Jacobian matrix:

$$\frac{\partial\dot{x}_1}{\partial x_1} = \frac{\partial\dot{x}_1}{\partial x_3} = \frac{\partial\dot{x}_1}{\partial x_4} = \frac{\partial\dot{x}_2}{\partial x_2} = \frac{\partial\dot{x}_3}{\partial x_1} = \frac{\partial\dot{x}_3}{\partial x_2} = \frac{\partial\dot{x}_3}{\partial x_3} = 0 \tag{4.36}$$

$$\frac{\partial\dot{x}_1}{\partial x_2} = \frac{\partial\dot{x}_3}{\partial x_4} = 1; \qquad \frac{\partial\dot{x}_2}{\partial x_1} = \frac{m_b x_4^2}{J_b/r^2 + m_b}; \qquad \frac{\partial\dot{x}_2}{\partial x_3} = \frac{-m_b g cos(x_3)}{J_b/r^2 + m_b} \tag{4.37}$$

Continue with the remaining longer terms:

$$\frac{\partial\dot{x}_2}{\partial\dot{x}_4} = \frac{2m_b x_1 x_4}{J_b/r^2 + m_b} \tag{4.38}$$

$$\frac{\partial\dot{x}_4}{\partial x_1} = \frac{-2m_b x_2 x_4 - m_b g cos(x_3)\big(J_{pn} + m_{pn}\big(a_n^2 + b_n^2\big) + m_b(x_1^2 + h^2)\big)}{\big(J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(x_1^2 + h^2)\big)^2} - \frac{2m_b x_1\Big(T_n - 2m_b x_1 x_2 x_4 - m_b g\big(x_1 cos(x_3) - hsin(x_3)\big) - m_{pn} g\big(b_n cos(x_3) - a_n sin(x_3)\big)\Big)}{\big(J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(x_1^2 + h^2)\big)^2} \tag{4.39}$$

$$\frac{\partial\dot{x}_4}{\partial x_2} = \frac{-2m_b x_4}{\big(J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(x_1^2 + h^2)\big)^2} \tag{4.40}$$

$$\frac{\partial\dot{x}_4}{\partial x_3} = \frac{m_b g\big(x_1 sin(x_3) + hcos(x_3)\big) + m_{pn} g\big(b_n sin(x_3) + a_n cos(x_3)\big)}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(x_1^2 + h^2)} \tag{4.41}$$

$$\frac{\partial\dot{x}_4}{\partial x_4} = \frac{-2m_b x_1 x_2}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(x_1^2 + h^2)} \tag{4.42}$$

Similarly,

$$\frac{\partial \dot{x}_1}{\partial T_n} = \frac{\partial \dot{x}_2}{\partial T_n} = \frac{\partial \dot{x}_3}{\partial T_n} = 0 \tag{4.43}$$

$$\frac{\partial \dot{x}_4}{\partial T_n} = \frac{1}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(x_1^2 + h^2)} \tag{4.44}$$

When we evaluate (4.36)-(4.44) around operating point, we'll obtain the following equation:

$$\left(\frac{\partial \dot{x}_2}{\partial x_1}\right)\bigg|_{x_2} = 0 \quad \left(\frac{\partial \dot{x}_2}{\partial x_3}\right)\bigg|_{x_0} = \frac{-m_b g}{J_b/r^2 + m_b} \qquad \left(\frac{\partial \dot{x}_2}{\partial x_4}\right)\bigg|_{x_0} = 0 \tag{4.45}$$

$$\left(\frac{\partial \dot{x}_4}{\partial x_2}\right)\bigg|_{x_0} = 0 \quad \left(\frac{\partial \dot{x}_4}{\partial x_3}\right)\bigg|_{x_0} = \frac{m_b g h + m_{pn} g a_n}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p_0^2 + h^2)} \quad \left(\frac{\partial \dot{x}_4}{\partial x_4}\right)\bigg|_{x_0} = 0 \tag{4.46}$$

$$\begin{aligned} \left(\frac{\partial \dot{x}_4}{\partial x_1}\right)\bigg|_{x_0} &= \frac{-m_b g}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p_0^2 + h^2)} \\ &\quad - \frac{2 m_b p_0 (T_0 - m_b g p_0) - m_{pn} g b_n}{(J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p_0^2 + h^2))^2} \end{aligned} \tag{4.47}$$

$$\frac{\partial \dot{x}_4}{\partial T_n} = \frac{1}{J_{pn} + m_{pn}(a_n^2 + b_n^2)^2 + m_b(p_0^2 + h^2)} \tag{4.48}$$

Rewriting (5.6);

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-m_b g}{(J_b/r_2 + m_b)} & 0 \\ 0 & 0 & 0 & 1 \\ (\partial x_4/\partial x_1)\big|_{x_0} & 0 & \dfrac{m_b g h + m_{pn} g a_n}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p_0^2 + h^2)} & 0 \end{bmatrix}}_{A = \partial f/\partial x(x_0,\, T_0)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p_0^2 + h^2)} \end{bmatrix}}_{B = \partial f/\partial T_n(x_0,\, T_0)} T_n \end{aligned} \tag{4.49}$$

where,

$$\frac{\partial x_4}{\partial x_1}\bigg|_{x_0, T_0} = \frac{-m_b g}{J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p_0^2 + h^2)}$$

$$-\frac{2m_b p_0(+m_b g p_0 + m_{pn} g b_n - m_b g p_0) - m_{pn} g b_n}{(J_{pn} + m_{pn}(a_n^2 + b_n^2) + m_b(p_0^2 + h^2))^2} \tag{4.50}$$

4.2 Kinematics of Disturber

4.2.1 Forward Kinematics of Second Actuator

4.2.1.1 Position Analysis

When viewed from the front plane, the mechanism can be represented as in Figure 4.2.

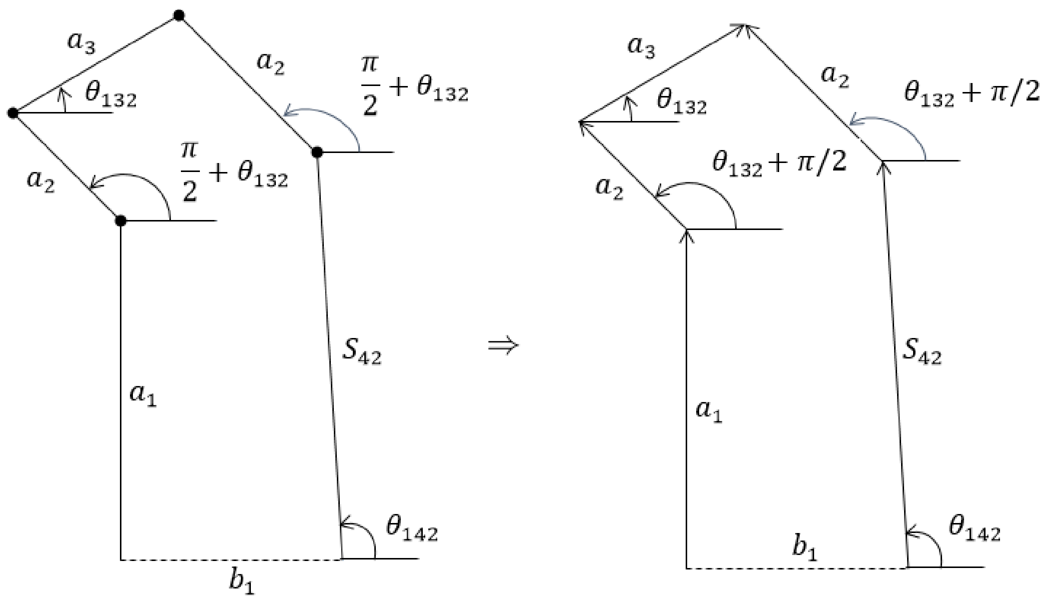


Figure 4.2: Front view of the second actuator

When we break the chain from point Q, the loop closure equation can be written as following:

$$a_1 i + a_2 e^{(\theta_{132}+\pi/2)} + a_3 e^{i\theta_{132}} = b_1 + s_{42} e^{i\theta_{142}} + a_2 e^{i(\theta_{132}+\pi/2})} \tag{4.51}$$

$$a_1 i + a_3 e^{i\theta_{13}} = b_1 + s_{42} e^{i\theta_{14}} \tag{4.52}$$

Rearrange the terms in (4.52),

$$s_{42}e^{i\theta_{142}} = a_1 i - b_1 + a_3 e^{i\theta_{132}} \tag{4.53}$$

The complex conjugate is:

$$s_{42}e^{i\theta_{142}} = -a_1 i - b_1 + a_3 e^{-i\theta_{132}} \tag{4.54}$$

Multiplying (4.53) & (4.54) side by side:

$$\begin{aligned} s_{42}^2 = & -a_1^2 i^2 - a_1 \cancel{b_1} i + a_1 a_3 i e^{i\theta_{132}} + a_1 \cancel{b_1} i + b_1^2 - a_3 b_1 e^{-i\theta_{132}} \\ & - a_1 a_3 i e^{i\theta_{132}} - a_3 b_1 e^{i\theta_{13}} + a_3^2 \end{aligned} \tag{4.55}$$

$$s_{42}^2 = a_1^2 + a_3^2 + b_1^2 + a_1 a_3 i \underbrace{\left(e^{-i\theta_{132}} - e^{-i\theta_{132}}\right)}_{K_1} - a_3 b_3 \underbrace{\left(e^{-i\theta_{132}} + e^{-i\theta_{132}}\right)}_{K_2} \tag{4.56}$$

Expanding K_1,

$$e^{-i\theta_{132}} - e^{i\theta_{132}} = \cancel{cos\theta_{132}} - isin\theta_{132} - \cancel{cos\theta_{132}} - isin\theta_{132} = -2isin\theta_{132} \tag{4.57}$$

$$\Rightarrow \; e^{-i\theta_{132}} - e^{i\theta_{132}} = 2isin\theta_{132} \tag{4.58}$$

Expanding K_2,

$$e^{-i\theta_{132}} + e^{i\theta_{132}} = cos\theta_{132} - \cancel{isin\theta_{132}} + cos\theta_{132} + \cancel{isin\theta_{132}} = 2icos\theta_{132} \tag{4.59}$$

$$\Rightarrow \; e^{-i\theta_{132}} + e^{i\theta_{132}} = 2icos\theta_{132} \tag{4.60}$$

Combining (4.56) with (4.58) & (4.60) results in:

$$s_{42}^2 = a_1^2 + a_3^2 + b_1^2 + a_1 a_3 i(-2sin\theta_{132}) - 2a_3 b_1 cos\theta_{132} \tag{4.61}$$

$$s_{42}^2 = a_1^2 + a_3^2 + b_1^2 + 2a_1 a_3 sin\theta_{132}) - 2a_3 b_1 cos\theta_{132} \tag{4.62}$$

Similar to what we have done previously in the position analysis of the first actuator; we can use half tangent expressions to linearize (4.62). Remind that these expressions were;

$$sin\theta_{132} = \frac{2tan(\frac{1}{2}\theta_{132})}{[1 + tan^2(\dfrac{1}{2}\theta_{132})]}, \qquad cos\theta_{132} = \frac{[1 - tan^2(\frac{1}{2}\theta_{132})]}{[1 + tan^2(\dfrac{1}{2}\theta_{132})]} \tag{4.63}$$

Substitution of (4.161) into (4.62) yields that,

$$s_{42}^2 = a_1^2 + a_3^2 + b_1^2 + 2a_1 a_3 \frac{2tan(\frac{1}{2}\theta_{132})}{1 + tan^2(\frac{1}{2}\theta_{132})} - 2a_3 b_1 \frac{1 - tan^2(\frac{1}{2}\theta_{132})}{1 + tan^2(\frac{1}{2}\theta_{132})} \tag{4.64}$$

$$
\begin{aligned}
s_{42}^2[1+tan^2(\frac{1}{2}\theta_{132})] &= a_1^2+a_3^2+b_1^2[1+tan^2(\frac{1}{2}\theta_{132})]+4a_1a_3tan(\frac{1}{2}\theta_{132}) \\
&- 2a_3b_1[1-tan^2(\frac{1}{2}\theta_{132})] \\
&= \overbrace{(s_{42}^2-a_1^2-a_3^2-b_1^2-2a_3b_1)}^{A}tan^2(\frac{1}{2}\theta_{132}) - \overbrace{4a_1a_3tan(\frac{1}{2}\theta_{132})}^{B} \\
&+ \overbrace{(s_{42}^2-a_1^2-a_3^2-b_1^2+2a_3b_1)}^{C} = 0
\end{aligned}
\tag{4.65}
$$

As found before in (4.164), this equation results in:

$$A\tan^2(\frac{1}{2}\theta_{132})+Btan^2(\frac{1}{2}\theta_{132})+C=0$$

where

$$A = s_4^2-a_1^2-a_3^2-b_1^2-2a_3b_1 \tag{4.66}$$

$$B = -4a_1a_3 \tag{4.67}$$

$$C = s_4^2-a_1^2-a_3^2-b_1^2+2a_3b_1 \tag{4.68}$$

Note that (4.164) is quadratic in terms of $tan(\frac{1}{2}\theta_{132})$. Therefore, generic solution of the quadratic equations can be obtained as:

$$tan(\frac{\theta_{132}}{2}) = \frac{-B\pm\sqrt{B^2-4AC}}{2A} \tag{4.69}$$

$$\Rightarrow \boxed{\theta_{132} = 2atan_2(-B\pm\sqrt{B^2-4AC}),\ 2A)} \tag{4.70}$$

The second method for obtaining θ_{132} from (4.62), is as follows. Let's rewrite (4.62) again:

$$s_{42}^2 = a_1^2-a_3^2-b_1^2+2a_1a_3sin\theta_{132}-2a_3b_1cos\theta_{132} \tag{4.71}$$

$$\frac{s_{42}^2-a_1^2-a_3^2-b_1^2}{2a_3} = a_1sin\theta_{132}-b_1cos\theta_{132} \tag{4.72}$$

$$\text{Let } a_1 = m_1sin\zeta_1 \text{ and } b_1 = m_1cos\zeta_1 \tag{4.73}$$

$$\zeta_1 = atan_2(a_1,b_1) \rightsquigarrow \text{ known, } m_1 = \sqrt{a_1^2+b_1^2} \rightsquigarrow \text{ known} \tag{4.74}$$

Substituting (4.73) into (4.72);

$$\frac{s_{42}^2-a_1^2-a_3^2-b_1^2}{2a_3} = m_1sin\zeta_1sin\theta_{132}-m_1cos\zeta_1cos\theta_{132} \tag{4.75}$$

$$\frac{s_{42}^2 - a_1^2 - a_3^2 - b_1^2}{2a_3} = m_1 \underbrace{(cos\zeta_1 cos\theta_{132} - sin\zeta_1 sin\theta_{132})}_{cos(\zeta_1+\theta_{132})} \tag{4.76}$$

$$\Rightarrow \ cos(\zeta_1 + \theta_{132}) = \underbrace{\frac{a_1^2 + a_3^2 + b_1^2 - s_4^2}{2m_1 a_3}}_{K_3} \tag{4.77}$$

$$\Rightarrow \ sin(\zeta_1 + \theta_{132}) = \sigma_1\sqrt{1 - K_3^2}, \quad \sigma_1 = \pm 1 \tag{4.78}$$

$$\Rightarrow \ \zeta + \theta_{132} = atan_2(\sigma_1\sqrt{1 - K_3^2},\ K_3) \tag{4.79}$$

$$\Rightarrow \boxed{\theta_{132} = atan_2(\sigma_1\sqrt{1 - K_3^2},\ K_3)} \tag{4.80}$$

Here, σ_1 represents the open and cross configuration of the mechanism. Since $\mp$ 16° degree limit of this mechanism all fall into open configuration of this mechanism, we can take $\sigma_1 = 1$ without any loss of generality.

It is obvious that both (4.70) & (4.80) will give us the same result. Once we have determined θ_{132}, we can proceed with finding θ_{142}. To do that, let's remind (4.53)

$$s_{42}e^{i\theta_{142}} = a_1 i - b_1 + a_3 e^{i\theta_{132}}$$

Writing imaginary and real parts of (4.53);

$$\text{Re: } s_{42}cos\theta_{142} = -b_1 + a_3 cos\theta_{132} \tag{4.81}$$

$$\text{Im: } s_{42}sin\theta_{142} = a_1 + a_3 sin\theta_{132} \tag{4.82}$$

$$tan\theta_{142} = \frac{a_1 + a_3 sin\theta_{132}}{a_3 cos\theta_{132} - b_1} \tag{4.83}$$

$$\Rightarrow \boxed{\theta_{142} = atan_2(a_1 + a_3 sin\theta_{132},\ a_3 cos\theta_{132} - b_1)} \tag{4.84}$$

4.2.1.2 Velocity Analysis

Remind that in (4.53), LCE has been found as:

$$s_{42}e^{i\theta_{142}} = a_1 i + a_3 e^{i\theta_{132}} - b_1$$

Taking the derivative:

$$\dot{s}_{42}e^{i\theta_{142}} + s_{42} i\dot{\theta}_{142} e^{i\theta_{142}} = a_3 i\dot{\theta}_{132} e^{i\theta_{132}} \tag{4.85}$$

Rewriting (4.85);

$$\dot{s}_{42}e^{i\theta_{142}} = a_3 i\dot{\theta}_{132} e^{i\theta_{132}} - s_{42} i\dot{\theta}_{142} e^{i\theta_{142}} \tag{4.86}$$

The complex conjugate is :

$$\dot{s}_{42}e^{-i\theta_{142}} = -a_3 i\dot{\theta}_{132}e^{-i\theta_{132}} + s_{42}i\dot{\theta}_{142}e^{-i\theta_{142}} \tag{4.87}$$

Writing (4.86) & (4.87) in matrix form:

$$\begin{bmatrix} a_3 ie^{i\theta_{132}} & -s_{42}ie^{i\theta_{142}} \\ \\ -a_3 ie^{-i\theta_{132}} & s_{42}ie^{-i\theta_{142}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{132} \\ \\ \dot{\theta}_{142} \end{bmatrix} = \begin{bmatrix} \dot{s}_{42}e^{i\theta_{142}} \\ \\ \dot{s}_{42}e^{-i\theta_{142}} \end{bmatrix} \tag{4.88}$$

$\dot{\theta}_{132}$ can be determined by using Cramer's rule;

$$\dot{\theta}_{132} = \frac{\begin{vmatrix} \dot{s}_{42}e^{i\theta_{142}} & -s_{42}ie^{i\theta_{142}} \\ \\ \dot{s}_{42}e^{-i\theta_{142}} & s_{42}ie^{-i\theta_{142}} \end{vmatrix}}{\begin{vmatrix} a_3 ie^{i\theta_{132}} & -s_{42}ie^{i\theta_{142}} \\ \\ -a_3 ie^{-i\theta_{132}} & s_{42}ie^{-i\theta_{142}} \end{vmatrix}} \tag{4.89}$$

The determinants of (4.89) are expanded in (4.90) & (4.91) as follows:

$$\begin{vmatrix} \dot{s}_{42}e^{i\theta_{142}} & -s_{42}ie^{i\theta_{142}} \\ \\ \dot{s}_{42}e^{-i\theta_{142}} & s_{42}ie^{-i\theta_{142}} \end{vmatrix} = s_{42}\dot{s}_{42}i + s_{42}\dot{s}_{42}i = 2is_{42}\dot{s}_{42} \tag{4.90}$$

$$\begin{vmatrix} a_3 ie^{i\theta_{132}} & -s_{42}ie^{i\theta_{142}} \\ \\ -a_3 ie^{-i\theta_{132}} & s_{42}ie^{-i\theta_{142}} \end{vmatrix} = -a_3 s_{42}e^{i(\theta_{132}-\theta_{142})} - a_3 s_{42}i^2 e^{i(\theta_{142}-\theta_{132})} \tag{4.91}$$

$$= -a_3 s_{42}e^{i(\theta_{132}-\theta_{142})} + a_3 s_{42}e^{i(\theta_{142}-\theta_{132})} \tag{4.92}$$

$$= -a_3 s_{42}\big(e^{i(\theta_{142}-\theta_{132})} - e^{i(\theta_{132}-\theta_{142})}\big) \tag{4.93}$$

$$= a_3 s_{42}\Big(\cancel{cos(\theta_{132}-\theta_{142})} - isin(\theta_{132}-\theta_{142}) \tag{4.94}$$

$$- \cancel{cos(\theta_{132}-\theta_{142})} - isin(\theta_{132}-\theta_{142})\Big) \tag{4.95}$$

$$= -2ia_3 ss_{42}sin(\theta_{132}-\theta_{142}) = 2ia_3 ss_{42}sin(\theta_{142}-\theta_{132}) \tag{4.96}$$

Thus,

$$\boxed{\dot{\theta}_{132} = \frac{2is_{42}\dot{s}_{42}}{2is_{42}a_3 sin(\theta_{142}-\theta_{132})} = \frac{\dot{s}_{42}}{a_3 sin(\theta_{142}-\theta_{132})}} \tag{4.97}$$

Similarly,

$$\dot{\theta}_{142} = \frac{\begin{vmatrix} a_3 i e^{i\theta_{132}} & \dot{s}_{42} e^{i\theta_{142}} \\ \\ -a_3 i e^{-i\theta_{132}} & \dot{s}_{42} e^{-i\theta_{142}} \end{vmatrix}}{2 i a_3 s_{42} sin(\theta_{142} - \theta_{132})} \tag{4.98}$$

Since the determinant of denominator of (4.89) has already been found in (4.96); next step is to expand the determinant of the numerator.

$$\begin{vmatrix} a_3 i e^{i\theta_{132}} & \dot{s}_{42} e^{i\theta_{142}} \\ \\ -a_3 i e^{-i\theta_{132}} & \dot{s}_{42} e^{-i\theta_{142}} \end{vmatrix} = \dot{s}_{42} a_3 i e^{i(\theta_{132}-\theta_{142})} + \dot{s}_{42} a_3 i e^{(\theta_{142}-\theta_{132})} \tag{4.99}$$

$$= \dot{s}_{42} a_3 i (e^{i(\theta_{132}-\theta_{142})} + e^{(\theta_{142}-\theta_{132})}) \tag{4.100}$$

$$= \dot{s}_{42} a_3 i \big(cos(\theta_{132} - \theta_{142}) + \cancel{isin(\theta_{132} - \theta_{142})}$$
$$+ cos(\theta_{132} - \theta_{142}) - \cancel{isin(\theta_{132} - \theta_{142})} \big) \tag{4.101}$$

$$= 2\dot{s}_{42} a_3 i cos(\theta_{132} - \theta_{142}) = 2\dot{s}_{42} a_3 i cos(\theta_{142} - \theta_{132}) \tag{4.102}$$

Therefore, $\dot{\theta}_{142}$ can be expressed as:

$$\boxed{\dot{\theta}_{142} = \frac{2 i a_3 \dot{s}_{42} cos(\theta_{142} - \theta_{132})}{2 i a_3 s_{42} sin(\theta_{142} - \theta_{132})} = \frac{\dot{s}_{42} cos(\theta_{142} - \theta_{132})}{s_{42} sin(\theta_{142} - \theta_{132})}} \tag{4.103}$$

4.2.1.3 Acceleration Analysis

Remind that in (4.86), velocity loop equation, from now on will be referred as VLE, has been found as:

$$\dot{s}_{42} e^{i\theta_{142}} = a_3 i \dot{\theta}_{132} e^{i\theta_{132}} - s_{42} i \dot{\theta}_{142} e^{i\theta_{142}} \tag{4.104}$$

In order to obtain the acceleration terms, we need to take derivative of (4.86). Hence,

$$\ddot{s}_{42} e^{i\theta_{142}} + \dot{s}_{42} i \dot{\theta}_{142} e^{i\theta_{142}} = a_3 i \ddot{\theta}_{132} e^{i\theta_{132}} - a_3 \dot{\theta}_{132}^2 e^{i\theta_{132}} - \dot{s}_{42} i \dot{\theta}_{142} e^{i\theta_{142}}$$
$$- s_{42} i \ddot{\theta}_{142} e^{i\theta_{142}} + s_{42} \dot{\theta}_{142}^2 e^{i\theta_{142}} \tag{4.105}$$

Rewriting (4.105),

$$a_3 i \ddot{\theta}_{132} e^{i\theta_{132}} - s_{42} i \ddot{\theta}_{142} e^{i\theta_{142}} = \ddot{s}_{42} e^{i\theta_{142}} + \dot{s}_{42} i \dot{\theta}_{142} e^{i\theta_{142}} + a_3 \dot{\theta}_{132}^2 e^{i\theta_{132}}$$

$$+\dot{s}_{42}i\,\dot{\theta}_{142}e^{i\theta_{142}}-s_{42}\dot{\theta}_{142}^2e^{i\theta_{142}} \tag{4.106}$$

The complex conjugate is:

$$\begin{aligned}-a_3i\ddot{\theta}_{132}e^{-i\theta_{132}}+s_{42}i\ddot{\theta}_{142}e^{-i\theta_{142}} &= \ddot{s}_{42}e^{-i\theta_{142}}-\dot{s}_{42}i\dot{\theta}_{142}e^{-i\theta_{142}}+a_3\dot{\theta}_{132}^2e^{-i\theta_{132}}\\ &-\dot{s}_{42}i\dot{\theta}_{142}e^{-i\theta_{142}}-s_{42}\dot{\theta}_{142}^2e^{-i\theta_{142}}\end{aligned} \tag{4.107}$$

Writing (4.106) & (4.107) in matrix form:

$$\begin{bmatrix} a_3ie^{i\theta_{132}} & -s_{42}ie^{i\theta_{142}} \\ -a_3ie^{-i\theta_{13}} & s_{42}ie^{-i\theta_{142}} \end{bmatrix}\begin{bmatrix}\ddot{\theta}_{132}\\ \ddot{\theta}_{142}\end{bmatrix}$$
$$=\begin{bmatrix}\ddot{s}_{42}e^{i\theta_{142}}+\dot{s}_{42}i\dot{\theta}_{142}e^{i\theta_{142}}+a_3\dot{\theta}_{132}^2e^{i\theta_{132}}+\dot{s}_{42}i\dot{\theta}_{142}e^{i\theta_{142}}-s_{42}\dot{\theta}_{142}^2e^{i\theta_{142}}\\ \ddot{s}_{42}e^{-i\theta_{142}}-\dot{s}_{42}i\dot{\theta}_{142}e^{-i\theta_{142}}+a_3\dot{\theta}_{132}^2e^{-i\theta_{132}}-\dot{s}_{42}i\dot{\theta}_{142}e^{-i\theta_{142}}-s_{42}\dot{\theta}_{142}^2e^{-i\theta_{142}}\end{bmatrix} \tag{4.108}$$

Again, as we have done in (4.89); $\ddot{\theta}_{132}$ can be determined by using Cramer's rule;

$$\ddot{\theta}_{132}=\frac{Z_1\leftarrow\begin{vmatrix}\ddot{s}_{42}e^{i\theta_{142}}+\dot{s}_{42}i\dot{\theta}_{142}e^{i\theta_{142}}+a_3\dot{\theta}_{132}^2e^{i\theta_{132}} & \\ +\dot{s}_{42}i\dot{\theta}_{142}e^{i\theta_{142}}-s_{42}\dot{\theta}_{142}^2e^{i\theta_{142}} & -s_{42}ie^{i\theta_{142}} \\ \ddot{s}_{42}e^{-i\theta_{142}}-\dot{s}_{42}i\dot{\theta}_{142}e^{-i\theta_{142}}+a_3\dot{\theta}_{132}^2e^{-i\theta_{132}} & \\ -\dot{s}_{42}i\dot{\theta}_{142}e^{-i\theta_{142}}-s_{42}\dot{\theta}_{142}^2e^{i\theta_{142}} & s_{42}ie^{-i\theta_{142}}\end{vmatrix}}{Z_2\leftarrow\begin{vmatrix}a_3ie^{i\theta_{132}} & -s_{42}ie^{i\theta_{142}}\\ -a_3ie^{-i\theta_{132}} & s_{42}ie^{-i\theta_{142}}\end{vmatrix}} \tag{4.109}$$

As it can be seen from (4.109), we can name the determinant of the numerator as Z_1, while the determinant of the denominator is called Z_2. Expanding these expressions, we'll obtain the following:

$$\begin{aligned}Z_1 &= \ddot{s}_{42}s_{42}i-\dot{s}_{42}s_{42}\dot{\theta}_{142}+a_3s_{42}i\dot{\theta}_{132}^2e^{i(\theta_{132}-\theta_{142})}-\dot{s}_{42}s_{42}\dot{\theta}_{142}-s_{42}^2i\dot{\theta}_{142}^2\\ &+\ddot{s}_{42}s_{42}i+\dot{s}_{42}s_{42}\dot{\theta}_{142}+a_3s_{42}i\dot{\theta}_{132}^2e^{i(\theta_{142}-\theta_{132})}+\dot{s}_{42}s_{42}\dot{\theta}_{142}-s_{42}^2i\dot{\theta}_{142}^2\end{aligned} \tag{4.110}$$

$$\Rightarrow\ Z_1=2\ddot{s}_{42}s_{42}i+a_3s_{42}i\dot{\theta}_{132}^2(\underbrace{e^{i(\theta_{132}-\theta_{142})}+e^{i(\theta_{142}-\theta_{132})}}_{Z_3})-2s_{42}^2i\dot{\theta}_{142}^2 \tag{4.111}$$

$$Z_3 = e^{i(\theta_{132}-\theta_{142})} + e^{i(\theta_{142}-\theta_{132})} = cos\left(\theta_{132}-\theta_{142}\right) + \cancel{isin(\theta_{132}-\theta_{142})}$$
$$+ \, cos\left(\theta_{132}-\theta_{142}\right) - \cancel{isin(\theta_{132}-\theta_{142})} \tag{4.112}$$
$$\Rightarrow \; Z_3 = 2cos\left(\theta_{132}-\theta_{142}\right) \text{ or } Z_3 = 2cos\left(\theta_{142}-\theta_{132}\right) \tag{4.113}$$

Combining (4.111) & (4.113);

$$Z_1 = 2i\ddot{s}_{42}s_{42} + 2ia_3 s_{42}\dot{\theta}_{132}^2 cos\left(\theta_{142}-\theta_{132}\right) - 2is_{42}^2\dot{\theta}_{142}^2 \tag{4.114}$$

Similarly, Z_2 can be found as;

$$Z_2 = \begin{vmatrix} a_3 i e^{i\theta_{132}} & -s_{42} i e^{i\theta_{142}} \\ -a_3 i e^{-i\theta_{132}} & s_{42} i e^{-i\theta_{142}} \end{vmatrix} = -a_3 s_{42} e^{i(\theta_{132}-\theta_{142})} + a_3 s_{42} e^{i(\theta_{142}-\theta_{132})} \tag{4.115}$$
$$= a_3 s_{42}\left(e^{i(\theta_{142}-\theta_{132})} - e^{i(\theta_{132}-\theta_{142})}\right) \tag{4.116}$$
$$= a_3 s_{42}\Big(\cancel{cos\left(\theta_{142}-\theta_{132}\right)} + isin\left(\theta_{142}-\theta_{132}\right)$$
$$- \cancel{cos\left(\theta_{142}-\theta_{132}\right)} + isin\left(\theta_{142}-\theta_{132}\right)\Big) \tag{4.117}$$
$$= 2ia_3 s_{42} sin\left(\theta_{142}-\theta_{132}\right) \tag{4.118}$$

Therefore, simultaneous solution of (4.109), (4.114) & (4.118) yields the following result;

$$\ddot{\theta}_{132} = \frac{2is_{42}\left(\ddot{s}_{42} + a_3\dot{\theta}_{132}^2 cos\left(\theta_{142}-\theta_{132}\right)\right) - s_{42}\dot{\theta}_{142}^2}{2is_{42}a_3 sin\left(\theta_{142}-\theta_{132}\right)} \tag{4.119}$$
$$\boxed{\ddot{\theta}_{132} = \frac{\ddot{s}_{42} + a_3\dot{\theta}_{132}^2 cos\left(\theta_{142}-\theta_{132}\right) - s_{42}\dot{\theta}_{142}^2}{a_3 sin\left(\theta_{142}-\theta_{132}\right)}} \tag{4.120}$$

$\ddot{\theta}_{142}$ can be determined again from Cramer's rule;

$$\ddot{\theta}_{142} = \frac{Z_4 \leftarrow \begin{vmatrix} a_3 i e^{i\theta_{132}} & \begin{matrix}\ddot{s}_{42}e^{i\theta_{142}} + \dot{s}_{42}i\dot{\theta}_{142}e^{i\theta_{142}} + a_3\dot{\theta}_{132}^2 e^{i\theta_{132}} \\ +\dot{s}_{42}i\dot{\theta}_{142}e^{i\theta_{142}} - s_{42}\dot{\theta}_{142}^2 e^{i\theta_{142}}\end{matrix} \\ -a_3 i e^{-i\theta_{132}} & \begin{matrix}\ddot{s}_{42}e^{-i\theta_{142}} - \dot{s}_{42}i\dot{\theta}_{142}e^{-i\theta_{142}} + a_3\dot{\theta}_{132}^2 e^{-i\theta_{132}} \\ -\dot{s}_{42}i\dot{\theta}_{142}e^{-i\theta_{142}} - s_{42}\dot{\theta}_{142}^2 e^{-i\theta_{142}}\end{matrix} \end{vmatrix}}{2ia_3 s_{42} sin\left(\theta_{142}-\theta_{132}\right)} \tag{4.121}$$

Since the determinant of denominator of (4.121) has already been found in (4.118); next step is to expand the determinant of the numerator. As it can be seen from

(4.121), we can name the determinant of the numerator as Z_4. Expanding this expression:

$$\begin{aligned} Z_4 &= \ddot{s}_{42} a_3 i e^{i(\theta_{132}-\theta_{142})} + \dot{s}_{42} a_3 \dot{\theta}_{142} e^{i(\theta_{132}-\theta_{142})} + a_3^2 i \dot{\theta}_{132}^2 + \dot{s}_{42} a_3 \dot{\theta}_{142} e^{i(\theta_{132}-\theta_{142})} \\ &- s_{42} a_3 i \dot{\theta}_{142}^2 e^{i(\theta_{132}-\theta_{142})} + \ddot{s}_{42} a_3 i e^{i(\theta_{142}-\theta_{132})} - \dot{s}_{42} a_3 \dot{\theta}_{142} e^{i(\theta_{142}-\theta_{132})} + a_3 i \dot{\theta}_{132}^2 \\ &- \dot{s}_{42} a_3 \dot{\theta}_{142} e^{i(\theta_{142}-\theta_{132})} - s_{42} a_3 i \dot{\theta}_{142}^2 e^{i(\theta_{142}-\theta_{132})} \end{aligned} \tag{4.122}$$

$$\begin{aligned} \Rightarrow\ Z_4 &= \ddot{s}_{42} a_3 i \left(e^{i(\theta_{132}-\theta_{142})} + e^{i(\theta_{142}-\theta_{132})}\right) + \dot{s}_{42} a_3 \dot{\theta}_{142} \left(e^{i(\theta_{132}-\theta_{142})} - e^{i(\theta_{142}-\theta_{132})}\right) \\ &+ 2a_3^2 i \dot{\theta}_{132}^2 + s_{42} a_3 \dot{\theta}_{142} \left(e^{i(\theta_{132}-\theta_{142})} + e^{i(\theta_{142}-\theta_{132})}\right) \\ &- s_{42} a_3 i \dot{\theta}_{142}^2 \left(e^{i(\theta_{132}-\theta_{142})} + e^{i(\theta_{142}-\theta_{132})}\right) \end{aligned} \tag{4.123}$$

Expanding the complex terms by using Euler formula, we'll obtain the following expressions:

$$\begin{aligned} \therefore\ e^{i(\theta_{132}-\theta_{142})} + e^{i(\theta_{142}-\theta_{132})} &= cos\left(\theta_{132}-\theta_{142}\right) + \cancel{isin\left(\theta_{132}-\theta_{142}\right)} \\ &+ cos\left(\theta_{132}-\theta_{142}\right) - \cancel{isin\left(\theta_{132}-\theta_{142}\right)} \end{aligned} \tag{4.124}$$

$$\Rightarrow\ e^{i(\theta_{132}-\theta_{142})} + e^{i(\theta_{142}-\theta_{132})} = 2cos\left(\theta_{132}-\theta_{142}\right) = 2cos\left(\theta_{142}-\theta_{132}\right) \tag{4.125}$$

$$\begin{aligned} \therefore\ e^{i(\theta_{132}-\theta_{142})} - e^{i(\theta_{142}-\theta_{132})} &= \cancel{cos\left(\theta_{132}-\theta_{142}\right)} + isin\left(\theta_{132}-\theta_{142}\right) \\ &- \cancel{cos\left(\theta_{132}-\theta_{142}\right)} + isin\left(\theta_{132}-\theta_{142}\right) \end{aligned} \tag{4.126}$$

$$\Rightarrow\ e^{i(\theta_{132}-\theta_{142})} - e^{i(\theta_{142}-\theta_{132})} = 2isin\left(\theta_{132}-\theta_{142}\right) = -2isin\left(\theta_{142}-\theta_{132}\right) \tag{4.127}$$

Insert (4.125) & (4.127) into (4.123) to obtain:

$$\begin{aligned} Z_4 &= 2ia_3 \ddot{s}_{42} cos\left(\theta_{142}-\theta_{132}\right) - 2ia_3 \dot{s}_{42} \dot{\theta}_{142} sin\left(\theta_{142}-\theta_{132}\right) + 2ia_3^2 \dot{\theta}_{132}^2 \\ &- 2ia_3 \dot{s}_{42} \dot{\theta}_{142} sin\left(\theta_{142}-\theta_{132}\right) - 2ia_3 s_{42} \dot{\theta}_{142}^2 cos\left(\theta_{142}-\theta_{132}\right) \end{aligned} \tag{4.128}$$

Therefore, $\ddot{\theta}_{142}$ can be found as follows:

$$\ddot{\theta}_{142} = \frac{\begin{aligned} 2ia_3 \Big(\ddot{s}_{42} cos\left(\theta_{142}-\theta_{132}\right) - \dot{s}_{42} \dot{\theta}_{142} sin\left(\theta_{142}-\theta_{132}\right) + a_3 \dot{\theta}_{132}^2 \\ -\dot{s}_{42} \dot{\theta}_{142} sin\left(\theta_{142}-\theta_{132}\right) - s_{42} \dot{\theta}_{142}^2 cos\left(\theta_{142}-\theta_{132}\right)\Big) \end{aligned}}{2ia_3 s_{42} sin\left(\theta_{142}-\theta_{132}\right)} \tag{4.129}$$

$$\boxed{\ddot{\theta}_{142} = \frac{\begin{aligned} \ddot{s}_{42} cos\left(\theta_{142}-\theta_{132}\right) - \dot{s}_{42} \dot{\theta}_{142} sin\left(\theta_{142}-\theta_{132}\right) + a_3 \dot{\theta}_{132}^2 \\ -\dot{s}_{42} \dot{\theta}_{142} sin\left(\theta_{142}-\theta_{132}\right) - s_{42} \dot{\theta}_{142}^2 cos\left(\theta_{142}-\theta_{132}\right) \end{aligned}}{s_{42} sin\left(\theta_{142}-\theta_{132}\right)}} \tag{4.130}$$

4.2.2 Inverse Kinematics of Second Actuator

4.2.2.1 Position Analysis

Remind that in (4.53) & (4.54), the LCE and its conjugate have been found to be:

$$s_{42}e^{i\theta_{142}} = a_1 i - b_1 + a_3 e^{i\theta_{132}}$$
$$\text{Conj: } s_{42}e^{-i\theta_{142}} = -a_1 i - b_1 + a_3 e^{-i\theta_{132}}$$

Multiply (4.53) & (4.54) side by side to obtain:

$$\begin{aligned}
s_{42}^2 &= a_1^2 - \cancel{a_1 b_1} i + a_1 a_3 i e^{-i\theta_{132}} + \cancel{a_1 b_1} i + b_1^2 - a_3 b_1 e^{-i\theta_{132}} - a_1 a_3 e^{i\theta_{132}} \\
&\quad - a_3 b_1 e^{i\theta_{132}} + a_3^2 \\
&= a_1^2 + a_3^2 + b_1^2 + a_1 a_3 i \underbrace{\left(e^{-i\theta_{132}} - e^{i\theta_{132}}\right)}_{-2isin\theta_{132}} - a_3 b_1 \underbrace{\left(e^{-i\theta_{132}} - e^{i\theta_{132}}\right)}_{2cos\theta_{132}} \\
&= a_1^2 + a_3^2 + b_1^2 + 2a_1 a_3 sin\theta_{132} - 2a_3 b_1 cos\theta_{132} \\
\Rightarrow &\boxed{s_{42} = \sqrt{a_1^2 + a_3^2 + b_1^2 + 2a_1 a_3 sin\theta 132 - 2a_3 b_1 cos\theta_{132}}}
\end{aligned} \tag{4.131}$$

4.2.2.2 Velocity Analysis

The VLE and its conjugate has been found in (4.86) & (4.87) as follows:

$$\dot{s}_{42}e^{i\theta_{142}} + s_{42} i \dot{\theta}_{142} e^{i\theta_{142}} = a_3 i \dot{\theta}_{132} e^{i\theta_{132}} \tag{4.132}$$

$$\text{Conj: } \dot{s}_{42}e^{-i\theta_{142}} - s_{42} i \dot{\theta}_{142} e^{-i\theta_{142}} = -a_3 i \dot{\theta}_{132} e^{-i\theta_{132}} \tag{4.133}$$

Writing (4.86) & (4.87) in matrix form:

$$\begin{bmatrix} e^{i\theta_{142}} & s_{42} i e^{i\theta_{142}} \\ & \\ e^{-i\theta_{142}} & -s_{42} i e^{-i\theta_{142}} \end{bmatrix} \begin{bmatrix} \dot{s}_{42} \\ \\ \dot{\theta}_{142} \end{bmatrix} = \begin{bmatrix} a_3 i e^{i\theta_{132}} \\ \\ -a_3 i e^{-i\theta_{132}} \end{bmatrix} \dot{\theta}_{132} \tag{4.134}$$

From Cramer's rule, $\dot{s}_{42}$ can be determined as follows:

$$\dot{s}_{42} = \frac{\begin{vmatrix} a_3 i e^{i\theta_{132}}\dot{\theta}_{132} & s_{42} i e^{i\theta_{142}} \\ -a_3 i e^{-i\theta_{132}}\dot{\theta}_{132} & -s_{42} i e^{-i\theta_{142}} \end{vmatrix} \rightarrow Z_5}{\begin{vmatrix} e^{i\theta_{142}} & s_{42} i e^{i\theta_{142}} \\ e^{-i\theta_{142}} & -s_{42} i e^{-i\theta_{142}} \end{vmatrix} \rightarrow Z_6} \tag{4.135}$$

The determinants of (4.135) are expanded in (4.136) & (4.137) as follows:

$$\begin{aligned} \Rightarrow \; Z_5 &= a_3 s_{42}\dot{\theta}_{132} e^{i(\theta_{132}-\theta_{142})} - a_3 s_{42}\dot{\theta}_{132} e^{i(\theta_{142}-\theta_{132})} \\ &= a_3 s_{42}\dot{\theta}_{132}\left(e^{i(\theta_{132}-\theta_{142})} - e^{i(\theta_{142}-\theta_{132})}\right) \\ &= a_3 s_{42}\dot{\theta}_{132}(\cancel{cos\,(\theta_{142}-\theta_{132})} - isin\,(\theta_{142}-\theta_{132}) \\ &\quad \cancel{-cos\,(\theta_{142}-\theta_{132})} - isin\,(\theta_{142}-\theta_{132})) \\ &= -2ia_3 s_{42}\dot{\theta}_{132} sin\,(\theta_{142}-\theta_{132}) \end{aligned} \tag{4.136}$$

$$\Rightarrow \; Z_6 = -s_{42}i - s_{42}i = -2s_{42}i \tag{4.137}$$

Combining (4.135) - (4.137);

$$\dot{s}_{42} = \frac{-2is_{42}a_3\dot{\theta}_{132} sin\,(\theta_{142}-\theta_{132})}{-2is_{42}} \tag{4.138}$$

$$\Rightarrow \boxed{\dot{s}_{42} = a_3\dot{\theta}_{132} sin\,(\theta_{142}-\theta_{132})} \tag{4.139}$$

4.2.2.3 Acceleration Analysis

In (4.106) & (4.107), the acceleration expression and its conjugate has been found as:

$$\begin{aligned} \ddot{s}_{42}e^{i\theta_{142}} + s_{42}i\ddot{\theta}_{142}e^{i\theta_{142}} &= -2i\dot{s}_{42}\dot{\theta}_{142}e^{i\theta_{142}} + s_{42}\dot{\theta}_{142}^2 e^{i\theta_{142}} \\ &\quad + a_3 i\ddot{\theta}_{132}e^{i\theta_{132}} - a_3\dot{\theta}_{132}^2 e^{i\theta_{132}} \\ \text{Conj: } \ddot{s}_{42}e^{-i\theta_{142}} - s_{42}i\ddot{\theta}_{142}e^{-i\theta_{142}} &= 2i\dot{s}_{42}\dot{\theta}_{142}e^{-i\theta_{142}} + s_{42}\dot{\theta}_{142}^2 e^{-i\theta_{142}} \\ &\quad - a_3 i\ddot{\theta}_{132}e^{-i\theta_{132}} - a_3\dot{\theta}_{132}^2 e^{-i\theta_{132}} \end{aligned}$$

Writing (4.106) & (4.107) in matrix form:

$$\begin{bmatrix} e^{i\theta_{142}} & s_{42}ie^{i\theta_{142}} \\ \\ e^{-i\theta_{142}} & -s_{42}ie^{-i\theta_{142}} \end{bmatrix} \begin{bmatrix} \ddot{s}_{42} \\ \\ \ddot{\theta}_{142} \end{bmatrix}$$

$$= \begin{bmatrix} -2i\dot{s}_{42}\dot{\theta}_{142}e^{i\theta_{142}} + s_{42}\dot{\theta}_{142}^2e^{i\theta_{142}} + a_3i\ddot{\theta}_{132}e^{i\theta_{132}} - a_3\dot{\theta}_{132}^2e^{i\theta_{132}} \\ \\ 2i\dot{s}_{42}\dot{\theta}_{142}e^{-i\theta_{142}} + s_{42}\dot{\theta}_{142}^2e^{-i\theta_{142}} - a_3i\ddot{\theta}_{132}e^{-i\theta_{132}} - a_3\dot{\theta}_{132}^2e^{-i\theta_{132}} \end{bmatrix} \tag{4.140}$$

From Cramer's rule;

$$\ddot{s}_{42} = \frac{Z_7 \rightarrow \begin{vmatrix} -2i\dot{s}_{42}\dot{\theta}_{142}e^{i\theta_{142}} + s_{42}\dot{\theta}_{142}^2e^{i\theta_{142}} + a_3i\ddot{\theta}_{132}e^{i\theta_{132}} - a_3\dot{\theta}_{132}^2e^{i\theta_{132}} & s_{42}ie^{i\theta_{142}} \\ \\ 2i\dot{s}_{42}\dot{\theta}_{142}e^{-i\theta_{142}} + s_{42}\dot{\theta}_{142}^2e^{-i\theta_{142}} - a_3i\ddot{\theta}_{132}e^{-i\theta_{132}} - a_3\dot{\theta}_{132}^2e^{-i\theta_{132}} & -s_{42}ie^{-i\theta_{142}} \end{vmatrix}}{Z_8 \rightarrow \begin{vmatrix} e^{\theta_{142}} & s_{42}ie^{i\theta_{142}} \\ \\ e^{-\theta_{142}} & -s_{42}ie^{-i\theta_{142}} \end{vmatrix}} \tag{4.141}$$

Z_7 can be expressed as follows:

$$\begin{aligned} Z_7 &= -2\dot{s}_{42}s_{42}\dot{\theta}_{142} - s_{42}^2i\dot{\theta}_{142}^2 + a_3s_{42}\ddot{\theta}_{132}e^{i(\theta_{132}-\theta_{142})} + a_3s_{42}i\dot{\theta}_{132}^2e^{i(\theta_{132}-\theta_{142})} \\ &+ 2\dot{s}_{42}s_{42}\dot{\theta}_{142} - s_{42}^2i\dot{\theta}_{142}^2 - a_3s_{42}\ddot{\theta}_{132}e^{i(\theta_{142}-\theta_{132})} + a_3s_{42}i\dot{\theta}_{132}^2e^{i(\theta_{142}-\theta_{132})} \end{aligned} \tag{4.142}$$

$$\begin{aligned} \Rightarrow Z_7 &= -2s_{42}^2i\dot{\theta}_{142}^2 + a_3s_{42}\ddot{\theta}_{132}\left(e^{i(\theta_{132}-\theta_{142})} + e^{i(\theta_{142}-\theta_{132})}\right) \\ &+ a_3s_{42}i\dot{\theta}_{132}^2\left(e^{i(\theta_{132}-\theta_{142})} + e^{i(\theta_{142}-\theta_{132})}\right) \end{aligned} \tag{4.143}$$

Expanding the complex terms by using Euler formula, we'll obtain the following expressions:

$$\begin{aligned} \therefore\ e^{i(\theta_{132}-\theta_{142})} - e^{i(\theta_{142}-\theta_{132})} &= \cancel{cos\left(\theta_{142}-\theta_{132}\right)} - isin\left(\theta_{142}-\theta_{132}\right) \\ &- \cancel{cos\left(\theta_{142}-\theta_{132}\right)} - isin\left(\theta_{142}-\theta_{132}\right) \\ \Rightarrow\ e^{i(\theta_{132}-\theta_{142})} - e^{i(\theta_{142}-\theta_{132})} &= -2isin\left(\theta_{142}-\theta_{132}\right) \end{aligned} \tag{4.144}$$

$$\begin{aligned} \therefore\ e^{i(\theta_{142}-\theta_{132})} + e^{i(\theta_{142}-\theta_{132})} &= cos\left(\theta_{142}-\theta_{132}\right) - \cancel{isin\left(\theta_{142}-\theta_{132}\right)} \\ &+ cos\left(\theta_{142}-\theta_{132}\right) + \cancel{isin\left(\theta_{142}-\theta_{132}\right)} \\ \Rightarrow\ e^{i(\theta_{142}-\theta_{132})} + e^{i(\theta_{142}-\theta_{132})} &= 2cos\left(\theta_{142}-\theta_{132}\right) \end{aligned} \tag{4.145}$$

Combine (4.144) & (4.145) with (4.143) to get;

$$Z_7 = -2s_{42}^2 i\dot{\theta}_{142}^2 - 2is_{42}a_3\ddot{\theta}_{132}sin\left(\theta_{142} - \theta_{132}\right) + 2a_3s_{42}i\dot{\theta}_{132}^2cos\left(\theta_{142} - \theta_{132}\right) \tag{4.146}$$

Similarly,

$$Z_8 = -s_{42}i - s_{42}i = -2s_{42}i \tag{4.147}$$

Simultaneous solution of (4.141), (4.146) & (4.147) yields the following result:

$$\ddot{s}_{42} = \frac{-2s_{42}i\left(s_{42}\dot{\theta}_{142}^2 + a_3\ddot{\theta}_{132}sin\left(\theta_{142} - \theta_{132}\right) - a_3\dot{\theta}_{132}^2cos\left(\theta_{142} - \theta_{132}\right)\right)}{-2s_{42}i} \tag{4.148}$$

$$\Rightarrow \boxed{\ddot{s}_{42} = s_{42}\dot{\theta}_{142}^2 + a_3\ddot{\theta}_{132}sin\left(\theta_{142} - \theta_{132}\right) - a_3\dot{\theta}_{132}^2cos\left(\theta_{142} - \theta_{132}\right)} \tag{4.149}$$

4.2.3 Forward Kinematics of First Actuator

4.2.3.1 Position Analysis

When viewed from the front plane, the mechanism can be represented as in Figure 4.3.

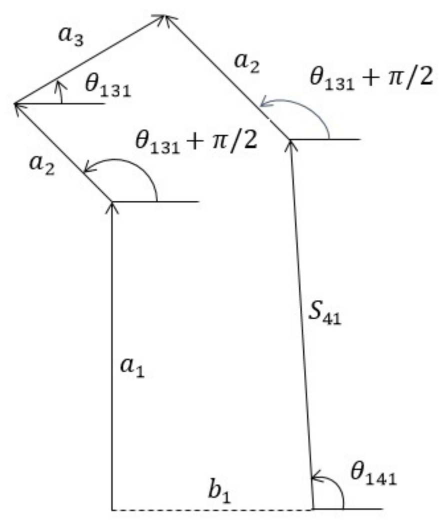


Figure 4.3: Front view of the first actuator

When we break the chain from point Q, the loop closure equation can be written as following:

$$s_{41}e^{i\theta_{141}} + \cancel{a_2 e^{i(\theta_{131}+\pi/2)}} + a_3 e^{i\theta_{131}} = b_1 + a_1 i + \cancel{a_2 e^{i(\theta_{131}+\pi/2)}} \tag{4.150}$$

$$s_{41}e^{i\theta_{141}} = a_1 i + b_1 - a_3 e^{i\theta_{131}} \tag{4.151}$$

The complex conjugate is;

$$s_{41}e^{-i\theta_{141}} = -a_1 i + b_1 - a_3 e^{-i\theta_{131}} \tag{4.152}$$

Multiply (4.151) & (4.152) side by side;

$$s_{41}^2 = a_1^2 + a_1 b_1 i - a_1 a_3 i e^{-i\theta_{131}} - a_1 b_1 i + b_1^2 - a_3 b_1 e^{-i\theta_{131}} + a_1 a_3 i e^{-i\theta_{131}} - a_3 b_1 e^{i\theta_{131}} \tag{4.153}$$

$$s_{41}^2 = a_1^2 + a_3^2 + b_1^2 + a_1 a_3 i \underbrace{\left(e^{i\theta_{131}} - e^{-i\theta_{131}}\right)}_{K_{11}} - a_3 b_1 \underbrace{\left(e^{i\theta_{131}} + e^{-i\theta_{131}}\right)}_{K_{12}} \tag{4.154}$$

Expanding K_{11};

$$K_{11} = e^{i\theta_{131}} - e^{-i\theta_{131}} = cos\left(\theta_{131}\right) + isin\left(\theta_{131}\right) - cos\left(\theta_{131}\right) + isin\left(\theta_{131}\right) \tag{4.155}$$

$$\Rightarrow \; e^{i\theta_{131}} - e^{-i\theta_{131}} = 2isin\left(\theta_{131}\right) \tag{4.156}$$

$$K_{12} = e^{i\theta_{131}} + e^{-i\theta_{131}} = cos\left(\theta_{131}\right) + isin\left(\theta_{131}\right) + cos\left(\theta_{131}\right) - isin\left(\theta_{131}\right) \tag{4.157}$$

$$\Rightarrow \; e^{i\theta_{131}} + e^{-i\theta_{131}} = 2cos\left(\theta_{131}\right) \tag{4.158}$$

Substitution of (4.156) & (4.158) into (4.154) yields that;

$$s_{41}^2 = a_1^2 + a_3^2 + b_1^2 + a_1 a_3 i\left(2isin\left(\theta_{131}\right)\right) - a_3 b_1\left(2cos\left(\theta_{131}\right)\right) \tag{4.159}$$

$$s_{41}^2 = a_1^2 + a_3^2 + b_1^2 - 2a_1 a_3 sin\left(\theta_{131}\right) - 2a_3 b_1 cos\left(\theta_{131}\right) \tag{4.160}$$

When we inspect on (4.160), we'll realize that this equation has one unknown θ_{131} since s_{41} is the input of the mechanism, which therefore will be specified by us. However, this equation is not linear. Thus, some manipulations should be done in order to obtain θ_{131}. To do this, two different methods will be applied. In the first method, the sine and cosine terms will be replaced by its half-tangent correspondents. The second method will only be applied in the analysis of second actuator in Section 4.2.1.1.

$$sin\theta_{132} = \frac{2tan(\frac{1}{2}\theta_{132})}{[1 + tan^2(\frac{1}{2}\theta_{132})]}, \qquad cos\theta_{132} = \frac{[1 - tan^2(\frac{1}{2}\theta_{132})]}{[1 + tan^2(\frac{1}{2}\theta_{132})]} \tag{4.161}$$

Substitution of (4.161) into (4.160) yields that:

$$s_{41}^2 = a_1^2 + a_3^2 + b_1^2 - 2a_1a_3\frac{2tan(\frac{1}{2}\theta_{131})}{1+tan^2(\frac{1}{2}\theta_{131})} - 2a_3b_1\frac{1-tan^2(\frac{1}{2}\theta_{131})}{1+tan^2(\frac{1}{2}\theta_{131})} \tag{4.162}$$

$$\begin{aligned} s_{41}^2\left[1+tan^2\left(\frac{1}{2}\theta_{131}\right)\right] &= a_1^2+a_3^2+b_1^2+\left(a_1^2+a_3^2+b_1^2\right)tan^2\left(\frac{1}{2}\theta_{131}\right) \\ &- 4a_1a_3tan\left(\frac{1}{2}\theta_{131}\right) - 2a_3b_1 + 2a_3b_1tan^2\left(\frac{1}{2}\theta_{131}\right) \\ &= \underbrace{\left(a_1^2+a_3^2+b_1^2-s_{41}^2+2a_3b_1\right)}_{A_1}tan^2\left(\frac{1}{2}\theta_{131}\right) \\ &\underbrace{-4a_1a_3}_{B_1}tan\left(\frac{1}{2}\theta_{131}\right) + \underbrace{a_1^2+a_3^2+b_1^2-s_{41}^2-2a_3b_1}_{C_1} = 0 \end{aligned} \tag{4.163}$$

This equation results in:

$$Atan^2\left(\frac{1}{2}\theta_{131}\right) + Btan\left(\frac{1}{2}\theta_{131}\right) + C = 0 \tag{4.164}$$

where

$$A = a_1^2+a_3^2+b_1^2-s_{41}^2+2a_3b_1 \tag{4.165}$$

$$B = -4a_1a_3 \tag{4.166}$$

$$C = a_1^2+a_3^2+b_1^2-s_{41}^2-2a_3b_1 \tag{4.167}$$

Note that (4.164) is quadratic in terms of $tan\left(\frac{1}{2}\theta_{131}\right)$. Therefore, θ_{131} can be obtained as;

$$tan\left(\frac{\theta_{131}}{2}\right) = \frac{-B \mp \sqrt{B^2-4AC}}{2A} \tag{4.168}$$

$$\Rightarrow \boxed{\theta_{131} = 2atan_2\left(-B \mp \sqrt{B^2-4AC},\ 2A\right)} \tag{4.169}$$

Once we have obtained $\theta_{131}, \theta_{141}$ can be obtained from LCE (4.151). Remind that LCE has been obtained as;

$$s_{41}e^{i\theta_{141}} = a_1i + b_1 - a_3e^{i\theta_{131}}$$

Writing the imaginary & real parts of (4.151)

$$\text{Re: } s_{41}cos\theta_{141} = b_1 - a_3cos\theta_{131} \tag{4.170}$$

$$\text{Im: } s_{41}sin\theta_{141} = a_1 - a_3sin\theta_{131} \tag{4.171}$$

Dividing (4.171) by (4.170);

$$tan\theta_{141} = \frac{a_1 - a_3 sin\theta_{131}}{b_1 - a_3 cos\theta_{131}} \tag{4.172}$$

$$\boxed{\theta_{141} = atan_2\left(a_1 - a_3 sin\theta_{131},\ b_1 - a_3 cos\theta_{131}\right)} \tag{4.173}$$

4.2.3.2 Velocity Analysis

Taking the derivative of (4.151), we'll obtain the following;

$$\dot{s}_{41}e^{i\theta_{141}} + s_{41}i\dot{\theta}_{141}e^{i\theta_{141}} = -a_3 i\dot{\theta}_{131}e^{i\theta_{131}} \tag{4.174}$$

Rewriting (4.174),

$$-\dot{s}_{41}e^{i\theta_{141}} = a_3 i\dot{\theta}_{131}e^{i\theta_{131}} + s_{41}i\dot{\theta}_{141}e^{i\theta_{141}} \tag{4.175}$$

The complex conjugate is;

$$-\dot{s}_{41}e^{-i\theta_{141}} = -a_3 i\dot{\theta}_{131}e^{-i\theta_{141}} - s_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} \tag{4.176}$$

Rewriting (4.176)

$$\dot{s}_{41}e^{-i\theta_{141}} = a_3 i\dot{\theta}_{131}e^{-i\theta_{131}} + s_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} \tag{4.177}$$

Writing (4.175)) and (4.177 in matrix form;

$$\begin{bmatrix} a_3 ie^{i\theta 131} & s_{41}ie^{i\theta_{141}} \\ a_3 ie^{-i\theta 131} & s_{41}ie^{-i\theta_{141}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{131} \\ \dot{\theta}_{141} \end{bmatrix} = \begin{bmatrix} -e^{i\theta_{141}} \\ e^{-i\theta_{141}} \end{bmatrix} \dot{s}_{41} \tag{4.178}$$

From Cramer's rule, $\dot{\theta}_{131}$ can be determined as follows:

$$\dot{\theta}_{131} = \frac{\begin{vmatrix} -\dot{s}_{41}e^{i\theta_{141}} & s_{41}ie^{i\theta_{141}} \\ \dot{s}_{41}e^{-i\theta_{141}} & s_{41}ie^{-i\theta_{141}} \end{vmatrix}}{\begin{vmatrix} a_3 ie^{i\theta_{131}} & s_{41}ie^{i\theta_{141}} \\ a_3 ie^{-i\theta_{131}} & s_{41}ie^{-i\theta_{141}} \end{vmatrix}} \tag{4.179}$$

The determinants of (4.179) are expanded in (4.180) & (4.184) as follows:

$$\begin{vmatrix} -\dot{s}_{41}e^{i\theta_{141}} & s_{41}ie^{i\theta_{141}} \\ \dot{s}_{41}e^{-i\theta_{141}} & s_{41}ie^{-i\theta_{141}} \end{vmatrix} = -\dot{s}_{41}s_{41}i - \dot{s}_{41}s_{41}i = -2\dot{s}_{41}s_{41}i \tag{4.180}$$

$$\begin{vmatrix} a_3ie^{i\theta_{131}} & s_{41}ie^{i\theta_{141}} \\ a_3ie^{-i\theta_{131}} & s_{41}ie^{-i\theta_{141}} \end{vmatrix} = -a_3s_{41}e^{i(\theta_{131}-\theta_{141})} + a_3s_{41}e^{i(\theta_{141}-\theta_{131})} \tag{4.181}$$

$$= a_3s_{41}\left(e^{i(\theta_{141}-\theta_{131})} - e^{i(\theta_{131}-\theta_{141})}\right) \tag{4.182}$$

$$= a_3s_{41}(\cancel{cos\left(\theta_{141}-\theta_{131}\right)} + isin\left(\theta_{141}-\theta_{131}\right)$$

$$\cancel{-cos\left(\theta_{141}-\theta_{131}\right)} + isin\left(\theta_{141}-\theta_{131}\right)) \tag{4.183}$$

$$= a_3s_{41}\left(2isin\left(\theta_{141}-\theta_{131}\right)\right) = 2ia_3s_{41}sin\left(\theta_{141}-\theta_{131}\right) \tag{4.184}$$

Therefore, combining (4.179) with (4.180) & (4.184);

$$\dot{\theta}_{131} = \frac{-2is_{41}\dot{s}_{41}}{2is_{41}a_3sin\left(\theta_{141}-\theta_{131}\right)} = \frac{-\dot{s}_{41}}{a_3sin\left(\theta_{141}-\theta_{131}\right)} \tag{4.185}$$

$$\boxed{\dot{\theta}_{131} = \frac{-\dot{s}_{41}}{a_3sin\left(\theta_{141}-\theta_{131}\right)}} \tag{4.186}$$

Similarly,

$$\dot{\theta}_{141} = \frac{\begin{vmatrix} a_3ie^{i\theta_{131}} & -\dot{s}_{41}e^{i\theta_{141}} \\ a_3ie^{-i\theta_{131}} & \dot{s}_{41}e^{-i\theta_{141}} \end{vmatrix}}{2ia_3s_{41}sin\left(\theta_{141}-\theta_{131}\right)} \tag{4.187}$$

Since the determinant of denominator of (4.187) has already been found in (4.184); next step is to expand the determinant of the numerator.

$$\begin{vmatrix} a_3ie^{i\theta_{131}} & -\dot{s}_{41}e^{i\theta_{141}} \\ a_3ie^{-i\theta_{131}} & \dot{s}_{41}e^{-i\theta_{141}} \end{vmatrix} = a_3\dot{s}_{41}ie^{i(\theta_{131}-\theta_{141})} + a_3\dot{s}_{41}ie^{i(\theta_{141}-\theta_{131})} \tag{4.188}$$

$$= a_3\dot{s}_{41}i\left(e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})}\right) \tag{4.189}$$

$$= a_3\dot{s}_{41}i(cos\left(\theta_{131}-\theta_{141}\right) + \cancel{isin\left(\theta_{131}-\theta_{141}\right)}$$

$$+ cos\left(\theta_{131}-\theta_{141}\right) - \cancel{isin\left(\theta_{131}-\theta_{141}\right)}) \tag{4.190}$$

$$= 2ia_3\dot{s}_{41}cos\left(\theta_{131}-\theta_{141}\right) \tag{4.191}$$

Therefore, combining (4.187) with (4.191);

$$\dot{\theta}_{141} = \frac{2ia_3\dot{s}_{41}cos\left(\theta_{141} - \theta_{131}\right)}{2ia_3 s_{41}sin\left(\theta_{141} - \theta_{131}\right)} = \frac{\dot{s}_{41}cos\left(\theta_{141} - \theta_{131}\right)}{s_{41}sin\left(\theta_{141} - \theta_{131}\right)} \tag{4.192}$$

$$\boxed{\dot{\theta}_{141} = \frac{\dot{s}_{41}cos\left(\theta_{141} - \theta_{131}\right)}{s_{41}sin\left(\theta_{141} - \theta_{131}\right)}} \tag{4.193}$$

4.2.3.3 Acceleration Analysis

Remind that in (4.174), VLE has been found as:

$$\dot{s}_{41}e^{i\theta_{141}} + s_{41}i\dot{\theta}_{141}e^{i\theta_{141}} = -a_3 i\dot{\theta}_{131}e^{i\theta_{131}}$$

In order to obtain the acceleration terms, we need to take the derivative (4.174).

$$\ddot{s}_{41}e^{i\theta_{141}} + \dot{s}_{41}i\dot{\theta}_{141}e^{i\theta_{141}} + \dot{s}_{41}i\dot{\theta}_{141}e^{i\theta_{141}} - s_{41}\dot{\theta}^2_{141}e^{i\theta_{141}} + s_{41}i\ddot{\theta}_{141}e^{i\theta_{141}} \\ = -a_3 i\ddot{\theta}_{131}e^{i\theta_{131}} + a_3\dot{\theta}^2_{131}e^{i\theta_{131}} \tag{4.194}$$

Rewriting (4.194)

$$a_3 i\ddot{\theta}_{131}e^{i\theta_{131}} + s_{41}i\ddot{\theta}_{141}e^{i\theta_{141}} = -\ddot{s}_{41}e^{i\theta_{141}} - 2\dot{s}_{41}i\dot{\theta}_{141}e^{i\theta_{141}} + s_{41}\dot{\theta}^2_{141}e^{i\theta_{141}} + a_3\dot{\theta}^2_{131}e^{i\theta_{131}} \tag{4.195}$$

The complex conjugate is:

$$-a_3 i\ddot{\theta}_{131}e^{-i\theta_{131}} - s_{41}i\ddot{\theta}_{141}e^{-i\theta_{141}} = -\ddot{s}_{41}e^{-i\theta_{141}} + 2\dot{s}_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} \\ + s_{41}\dot{\theta}^2_{141}e^{-i\theta_{141}} + a_3\dot{\theta}^2_{131}e^{-i\theta_{131}} \tag{4.196}$$

Writing (4.195) & (4.196) in matrix form:

$$\begin{bmatrix} a_3 i^{i\theta_{131}} & s_{41}ie^{i\theta_{141}} \\ -a_3 i^{-i\theta_{131}} & -s_{41}ie^{-i\theta_{141}} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_{131} \\ \ddot{\theta}_{141} \end{bmatrix} \\ = \begin{bmatrix} -\ddot{s}_{41}e^{i\theta_{141}} - 2\dot{s}_{41}i\dot{\theta}_{141}e^{i\theta_{141}} + s_{41}\dot{\theta}^2_{141}e^{i\theta_{141}} + a_3\dot{\theta}^2_{131}e^{i\theta_{131}} \\ -\ddot{s}_{41}e^{-i\theta_{141}} + 2\dot{s}_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} + s_{41}\dot{\theta}^2_{141}e^{-i\theta_{141}} + a_3\dot{\theta}^2_{131}e^{-i\theta_{131}} \end{bmatrix} \tag{4.197}$$

$\ddot{\theta}_{131}$ can be determined by using Cramer's rule as follows:

$$\ddot{\theta}_{131} = \frac{\overbrace{\begin{vmatrix} -\ddot{s}_{41}e^{i\theta_{141}} - 2\dot{s}_{41}i\dot{\theta}_{141}e^{i\theta_{141}} + s_{41}\dot{\theta}_{141}^{2}e^{i\theta_{141}} + a_3\dot{\theta}_{131}^{2}e^{i\theta_{131}} & s_{41}ie^{i\theta_{141}} \\ -\ddot{s}_{41}e^{-i\theta_{141}} + 2\dot{s}_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} + s_{41}\dot{\theta}_{141}^{2}e^{-i\theta_{141}} + a_3\dot{\theta}_{131}^{2}e^{-i\theta_{131}} & -s_{41}ie^{-i\theta_{141}} \end{vmatrix}}^{K_{13}}}{\underbrace{\begin{vmatrix} a_3ie^{i\theta_{131}} & s_{41}ie^{i\theta_{141}} \\ -a_3ie^{-i\theta_{131}} & -s_{41}ie^{-i\theta_{141}} \end{vmatrix}}_{K_{14}}} \tag{4.198}$$

As it can be seen from (4.198), we can name the determinant of the numerator as K_{13}, while the determinant of the denominator is called K_{14}. Expanding these expressions, we'll obtain the following:

$$\begin{aligned} K_{13} &= \ddot{s}_{41}s_{41}i - 2\dot{s}_{41}s_{41}\dot{\theta}_{141} - s_{41}^{2}i\dot{\theta}_{141}^{2} - a_3s_{41}i\dot{\theta}_{131}^{2}e^{i(\theta_{131}-\theta_{141})} \\ &\quad + \ddot{s}_{41}s_{41}i + 2\dot{s}_{41}s_{41}\dot{\theta}_{141} - s_{41}^{2}i\dot{\theta}_{141}^{2} - a_3s_{41}i\dot{\theta}_{131}^{2}e^{i(\theta_{141}-\theta_{131})} \end{aligned} \tag{4.199}$$

$$\Rightarrow\ K_{13} = 2\ddot{s}_{41}s_{41}i - 2s_{41}^{2}i\dot{\theta}_{141}^{2} - a_3s_{41}i\dot{\theta}_{131}^{2}\left(e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})}\right) \tag{4.200}$$

Expanding the complex terms by using Euler formula, we'll obtain the following expressions:

$$\begin{aligned} e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})} &= cos\left(\theta_{131}-\theta_{141}\right) + \cancel{isin\left(\theta_{131}-\theta_{141}\right)} \\ &\quad + cos\left(\theta_{131}-\theta_{141}\right) - \cancel{isin\left(\theta_{131}-\theta_{141}\right)} \end{aligned} \tag{4.201}$$

$$\Rightarrow\ e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})} = 2cos\left(\theta_{141}-\theta_{131}\right) \tag{4.202}$$

Insert (4.212) into (4.200) to get:

$$K_{13} = 2i\ddot{s}_{41}s_{41} - 2is_{41}^{2}\dot{\theta}_{141}^{2} - 2ia_3s_{41}\dot{\theta}_{131}^{2}cos\left(\theta_{141}-\theta_{131}\right) \tag{4.203}$$

Similarly, K_{14} can be expressed as:

$$K_{14} = a_3s_{41}e^{(\theta_{131}-\theta_{141})} - a_3s_{41}e^{(\theta_{141}-\theta_{131})} \tag{4.204}$$

$$K_{14} = a_3s_{41}\left(e^{i(\theta_{131}-\theta_{141})} - e^{i(\theta_{141}-\theta_{131})}\right) = -2ia_3s_{41}sin\left(\theta_{141}-\theta_{131}\right) \tag{4.205}$$

Combining (4.203) & (4.205) with (4.198) yields that:

$$\ddot{\theta}_{131} = \frac{2is_{41}\left(\ddot{s}_{41} - s_{41}\dot{\theta}_{141}^{2} - a_3\dot{\theta}_{131}^{2}cos\left(\theta_{141}-\theta_{131}\right)\right)}{-2ia_3s_{41}sin\left(\theta_{141}-\theta_{131}\right)}$$

$$= \frac{\left(\ddot{s}_{41} - s_{41}\dot{\theta}_{141}^2 - a_3\dot{\theta}_{131}^2 cos\left(\theta_{141} - \theta_{131}\right)\right)}{-a_3 sin\left(\theta_{141} - \theta_{131}\right)} \tag{4.206}$$

$$\Rightarrow \boxed{\ddot{\theta}_{131} = \frac{-\ddot{s}_{41} + s_{41}\dot{\theta}_{141}^2 + a_3\dot{\theta}_{131}^2 cos\left(\theta_{141} + \theta_{131}\right)}{a_3 sin\left(\theta_{141} - \theta_{131}\right)}} \tag{4.207}$$

Similarly, $\ddot{\theta}_{141}$ can also be obtained by means of Cramer's rule:

$$\ddot{\theta}_{141} = \frac{\overbrace{\begin{vmatrix} a_3 i e^{i\theta_{131}} & -\ddot{s}_{41}e^{i\theta_{141}} - 2\dot{s}_{41}i\dot{\theta}_{141}e^{i\theta_{141}} + s_{41}\dot{\theta}_{141}^2 e^{i\theta_{141}} + a_3\dot{\theta}_{131}^2 e^{i\theta_{131}} \\ & \\ -a_3 i e^{-i\theta_{131}} & -\ddot{s}_{41}e^{-i\theta_{141}} + 2\dot{s}_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} + s_{41}\dot{\theta}_{141}^2 e^{-i\theta_{141}} + a_3\dot{\theta}_{131}^2 e^{-i\theta_{131}} \end{vmatrix}}^{K_{15}}}{-2ia_3 s_{41} sin\left(\theta_{141} - \theta_{131}\right)} \tag{4.208}$$

Since the determinant of denominator of (4.208) has already been found in (4.205); next step is to expand the determinant of the numerator. As it can be seen from (4.208), we can name the determinant of the numerator as K_{15}. Expanding this expression:

$$\begin{aligned} K_{15} = &-a_3\ddot{s}_{41}ie^{i(\theta_{131}-\theta_{141})} - 2a_3\dot{s}_{41}\dot{\theta}_{141}e^{i(\theta_{131}-\theta_{141})} + a_3 s_{41}\dot{\theta}_{141}^2 e^{i(\theta_{131}-\theta_{141})} \\ &+ a_3^2 i\dot{\theta}_{131}^2 - a_3\ddot{s}_{41}ie^{i(\theta_{141}-\theta_{131})} + 2a_3\dot{s}_{41}\dot{\theta}_{141}e^{i(\theta_{141}-\theta_{131})} \\ &+ a_3 s_{41}i\dot{\theta}_{141}^2 e^{i(\theta_{131}-\theta_{141})} + a_3^2 i\dot{\theta}_{131}^2 \end{aligned} \tag{4.209}$$

$$\begin{aligned} K_{15} = &-a_3\ddot{s}_{41}i\left(e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})}\right) + 2a_3\dot{s}_{41}\dot{\theta}_{141}\left(e^{i(\theta_{141}-\theta_{131})} - e^{i(\theta_{131}-\theta_{141})}\right) \\ &+ a_3 s_{41}i\dot{\theta}_{141}^2\left(e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})}\right) + 2a_3^2 i\dot{\theta}_{131}^2 \end{aligned} \tag{4.210}$$

Expanding the complex terms by using Euler formula, we'll obtain the following expressions:

$$\begin{aligned} \therefore e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})} &= cos\left(\theta_{131} - \theta_{141}\right) + \cancel{isin\left(\theta_{131} - \theta_{141}\right)} \\ &+ cos\left(\theta_{131} - \theta_{141}\right) - \cancel{isin\left(\theta_{131} - \theta_{141}\right)} \end{aligned} \tag{4.211}$$

$$\Rightarrow e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})} = 2cos\left(\theta_{131} - \theta_{141}\right) \tag{4.212}$$

$$\begin{aligned} \therefore e^{i(\theta_{141}-\theta_{131})} - e^{i(\theta_{131}-\theta_{141})} &= \cancel{cos\left(\theta_{141} - \theta_{131}\right)} + isin\left(\theta_{141} - \theta_{131}\right) \\ &- \cancel{cos\left(\theta_{141} - \theta_{131}\right)} - isin\left(\theta_{141} - \theta_{131}\right) \end{aligned} \tag{4.213}$$

$$\Rightarrow e^{i(\theta_{141}-\theta_{131})} - e^{i(\theta_{131}-\theta_{141})} = 2isin\left(\theta_{141} - \theta_{131}\right) \tag{4.214}$$

Therefore, $\ddot{\theta}_{141}$ can be found as follows:

$$\ddot{\theta}_{141} = \frac{\begin{array}{r} -2a_3\ddot{s}_{41}c\left(\theta_{141}-\theta_{131}\right)+4a_3i\dot{s}_{41}\dot{\theta}_{141}sin\left(\theta_{141}-\theta_{131}\right) \\ +2a_3s_{41}i\dot{\theta}_{141}^2c\left(\theta_{141}-\theta_{131}\right)+2ia_3^2\dot{\theta}_{131}^2 \end{array}}{-2ia_3s_{41}sin\left(\theta_{141}-\theta_{131}\right)} \tag{4.215}$$

$$\boxed{\ddot{\theta}_{141} = \frac{\begin{array}{r} \ddot{s}_{41}cos\left(\theta_{141}-\theta_{131}\right)+2\dot{s}_{41}\dot{\theta}_{141}sin\left(\theta_{141}-\theta_{131}\right) \\ -s_{41}\dot{\theta}_{141}^2cos\left(\theta_{141}-\theta_{131}\right)-a_3\dot{\theta}_{131}^2 \end{array}}{s_{41}sin\left(\theta_{141}-\theta_{131}\right)}} \tag{4.216}$$

4.2.4 Inverse Kinematics of First Actuator

4.2.4.1 Position Analysis

The purpose of the inverse dynamics is to find s_{41} & s_{42} for desired θ_{131} & θ_{132}. Remind that the LCE and its conjugate has been found in (4.151) & (4.152).

$$s_{41}e^{i\theta_{141}} = a_1i + b_1 - a_3e^{i\theta_{131}}$$
$$\text{Conj: } s_{41}e^{-i\theta_{141}} = -a_1i + b_1 - a_3e^{-i\theta_{131}}$$

Multiply (4.151) & (4.152) side by side to obtain:

$$\begin{aligned} s_{41}^2 &= a_1^2 + a_1b_1i - a_1a_3ie^{-i\theta_{131}} - a_1b_1i + b_1^2 - a_3b_1e^{-i\theta_{131}} \\ &\quad + a_1a_3ie^{i\theta_{131}} - a_3b_1e^{i\theta_{131}} + a_3^2 \end{aligned} \tag{4.217}$$

$$\Rightarrow\ s_{41}^2 = a_1^2 + a_3^2 + b_1^2 + a_1a_3i\left(e^{i\theta_{131}} - e^{-i\theta_{131}}\right) - a_3b_1\left(e^{-i\theta_{131}} + e^{i\theta_{131}}\right) \tag{4.218}$$

Using Euler's formula:

$$e^{i\theta_{131}} - e^{-i\theta_{131}} = cos\theta_{131} + isin\theta_{131} - cos\theta_{131} + isin\theta_{131} = 2isin\theta_{131} \tag{4.219}$$

$$e^{i\theta_{131}} + e^{-i\theta_{131}} = cos\theta_{131} + isin\theta_{131} + cos\theta_{131} - isin\theta_{131} = 2cos\theta_{131} \tag{4.220}$$

Combining (4.218) - (4.220), we'll obtain the following:

$$s_{41}^2 = a_1^2 + a_3^2 + b_1^2 + a_1a_3i2is\theta_{131} - a_3b_12c\theta_{131} \tag{4.221}$$

$$s_{41}^2 = a_1^2 + a_3^2 + b_1^2 - 2a_1a_3s\theta_{131} - 2a_3b_1c\theta_{131} \tag{4.222}$$

$$\Rightarrow \boxed{s_{41} = \sqrt{a_1^2 + a_3^2 + b_1^2 - 2a_1a_3s\theta_{131} - 2a_3b_1c\theta_{131}}} \tag{4.223}$$

4.2.4.2 Velocity Analysis

The VLE and its conjugate has been found in (4.174) & (4.176) as follows:

$$\dot{s}_{41}e^{i\theta_{141}} + s_{41}i\dot{\theta}_{141}e^{i\theta_{141}} = -a_3 i\dot{\theta}_{131}e^{i\theta_{131}}$$

$$\text{Conj: } \dot{s}_{41}e^{-i\theta_{141}} - s_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} = -a_3 i\dot{\theta}_{131}e^{-i\theta_{131}}$$

Writing (4.174) & (4.176) in matrix form:

$$\begin{bmatrix} e^{i\theta_{141}} & s_{41}ie^{i\theta_{141}} \\ e^{-i\theta_{141}} & -s_{41}ie^{-i\theta_{141}} \end{bmatrix} \begin{bmatrix} \dot{s}_{41} \\ \dot{\theta}_{41} \end{bmatrix} = \begin{bmatrix} -a_3 i\dot{\theta}_{131}e^{i\theta_{131}} \\ a_3 i\dot{\theta}_{131}e^{-i\theta_{131}} \end{bmatrix} \tag{4.224}$$

$$\dot{s}_{41} = \frac{\overbrace{\begin{vmatrix} -a_3 i\dot{\theta}_{131}e^{i\theta_{131}} & s_{41}ie^{i\theta_{141}} \\ a_3 i\dot{\theta}_{131}e^{-i\theta_{131}} & -s_{41}ie^{-i\theta_{141}} \end{vmatrix}}^{K_{16}}}{\underbrace{\begin{vmatrix} e^{i\theta_{141}} & s_{41}ie^{i\theta_{141}} \\ e^{-i\theta_{141}} & -s_{41}ie^{-i\theta_{141}} \end{vmatrix}}_{K_{17}}} \tag{4.225}$$

The determinants of (4.225) are expanded in (4.226) & (4.227) as follows:

$$K_{16} = \begin{vmatrix} -a_3 i\dot{\theta}_{131}e^{i\theta_{131}} & s_{41}ie^{i\theta_{141}} \\ a_3 i\dot{\theta}_{131}e^{-i\theta_{131}} & -s_{41}ie^{-i\theta_{141}} \end{vmatrix} = -a_3 s_{41}\dot{\theta}_{131}e^{i(\theta_{131}-\theta_{141})} + a_3 s_{41}\dot{\theta}_{131}e^{i(\theta_{141}-\theta_{131})} \tag{4.226}$$

$$K_{17} = \begin{vmatrix} e^{i\theta_{141}} & s_{41}ie^{i\theta_{141}} \\ e^{-i\theta_{141}} & -s_{41}ie^{-i\theta_{141}} \end{vmatrix} = -s_{41}i - s_{41}i = -2s_{41}i \tag{4.227}$$

Rewriting (4.226);

$$\begin{aligned} K_{16} &= a_3 s_{41}\dot{\theta}_{131}\left(e^{i(\theta_{141}-\theta_{131})} - e^{i(\theta_{131}-\theta_{141})}\right) \\ &= a_3 s_{41}\dot{\theta}_{131}\left(cos\left(\theta_{131}-\theta_{141}\right) - isin\left(\theta_{131}-\theta_{141}\right)\right. \\ &\left. -cos\left(\theta_{131}-\theta_{141}\right) - isin\left(\theta_{131}-\theta_{141}\right)\right) \end{aligned}$$

$$= -2ia_3 s_{41}\dot{\theta}_{131} sin\left(\theta_{131} - \theta_{141}\right) \tag{4.228}$$

Therefore, combining (4.225), (4.227) & (4.228), $\dot{s}_{41}$ is found as:

$$\begin{aligned} \dot{s}_{41} &= \frac{-2ia_3 s_{41}\dot{\theta}_{131} s\left(\theta_{131} - \theta_{141}\right)}{-2is_{41}} \\ &= a_3\dot{\theta}_{131} sin\left(\theta_{131} - \theta_{141}\right) && (4.229) \\ \Rightarrow & \boxed{\dot{s}_{41} = -a_3\dot{\theta}_{131} sin\left(\theta_{141} - \theta_{131}\right)} && (4.230) \end{aligned}$$

4.2.4.3 Acceleration Analysis

Taking derivative of (4.174), in (4.195) the acceleration expression has been found as:

$$\ddot{s}_{41}e^{i\theta_{141}} + 2\dot{s}_{41}i\dot{\theta}_{141}e^{i\theta_{141}} + s_{41}i\ddot{\theta}_{141}e^{i\theta_{141}} - s_{41}\dot{\theta}_{141}^2 e^{i\theta_{141}} \\ = -a_3 i\ddot{\theta}_{131}e^{i\theta_{131}} + a_3\dot{\theta}_{131}^2 e^{i\theta_{131}}$$

The conjugate expression has been determined in (4.196) as;

$$\ddot{s}_{41}e^{-i\theta_{141}} - 2\dot{s}_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} + s_{41}i\ddot{\theta}_{141}e^{-i\theta_{141}} - s_{41}\dot{\theta}_{141}^2 e^{-i\theta_{141}} \\ = +a_3 i\ddot{\theta}_{131}e^{-i\theta_{131}} + a_3\dot{\theta}_{131}^2 e^{-i\theta_{131}}$$

Writing (4.195) & (4.195) in matrix form:

$$\begin{bmatrix} e^{i\theta_{141}} & s_{41}ie^{i\theta_{141}} \\ e^{-i\theta_{141}} & -s_{41}ie^{-i\theta_{141}} \end{bmatrix} \begin{bmatrix} \ddot{s}_{41} \\ \ddot{\theta}_{141} \end{bmatrix} \tag{4.231}$$

$$= \begin{bmatrix} -2\dot{s}_{41}i\dot{\theta}_{141}e^{i\theta_{141}} + s_{41}\dot{\theta}_{141}^2 e^{i\theta_{141}} - a_3 i\ddot{\theta}_{131}e^{i\theta_{131}} + a_3\dot{\theta}_{131}^2 e^{i\theta_{131}} \\ 2\dot{s}_{41}i\dot{\theta}_{141}e^{-i\theta_{141}} + s_{41}\dot{\theta}_{141}^2 e^{-i\theta_{141}} + a_3 i\ddot{\theta}_{131}e^{-i\theta_{131}} + a_3\dot{\theta}_{131}^2 e^{-i\theta_{131}} \end{bmatrix}$$

From Cramer's rule, $\ddot{s}_{41}$ can be written as:

$$\ddot{s}_{41} = \frac{K_{18} \leftarrow \begin{vmatrix} \begin{matrix} -2\dot{s}_{41} i\dot{\theta}_{141} e^{i\theta_{141}} + s_{41}\dot{\theta}_{141}^2 e^{i\theta_{141}} \\ -a_3 i\ddot{\theta}_{131} e^{i\theta_{131}} + a_3\dot{\theta}_{131}^2 e^{i\theta_{131}} \end{matrix} & s_{41} i e^{i\theta_{1}41} \\ \\ \begin{matrix} 2\dot{s}_{41} i\dot{\theta}_{141} e^{-i\theta_{141}} + s_{41}\dot{\theta}_{141}^2 e^{-i\theta_{141}} \\ +a_3 i\ddot{\theta}_{131} e^{-i\theta_{131}} + a_3\dot{\theta}_{131}^2 e^{-i\theta_{131}} \end{matrix} & -s_{41} i e^{-i\theta_{1}41} \end{vmatrix}}{K_{19} \leftarrow \begin{vmatrix} e^{i\theta_{141}} & s_{41} i e^{i\theta_{141}} \\ \\ e^{-i\theta_{141}} & -s_{41} i e^{-i\theta_{141}} \end{vmatrix}} \tag{4.232}$$

Expanding K_{18} & K_{19}:

$$\begin{aligned} K_{18} = & \cancel{-2\dot{s}_{41} s_{41}\dot{\theta}_{141}} - s_{41}^2 i\dot{\theta}_{141}^2 - a_3 s_{41}\ddot{\theta}_{131} e^{i(\theta_{131}-\theta_{141})} - a_3 s_{41} i\dot{\theta}_{131}^2 e^{i(\theta_{131}-\theta_{141})} \\ & + \cancel{2\dot{s}_{41} s_{41}\dot{\theta}_{141}} - s_{41}^2 i\dot{\theta}_{141}^2 + a_3 s_{41}\ddot{\theta}_{131} e^{i(\theta_{141}-\theta_{131})} - a_3 s_{41} i\dot{\theta}_{131}^2 e^{i(\theta_{141}-\theta_{131})} \end{aligned} \tag{4.233}$$

$$\begin{aligned} \Rightarrow \ K_{18} = & -2s_{41}^2 i\dot{\theta}_{141}^2 + a_3 s_{41}\ddot{\theta}_{131} \left(e^{i(\theta_{141}-\theta_{131})} - e^{i(\theta_{131}-\theta_{141})}\right) \\ & - a_3 s_{41} i\dot{\theta}_{131}^2 \left(e^{i(\theta_{131}-\theta_{141})} + e^{i(\theta_{141}-\theta_{131})}\right) \end{aligned} \tag{4.234}$$

Expanding the complex terms by using Euler formula, we'll obtain the following expressions:

$$\begin{aligned} \therefore e^{i(\theta_{141}-\theta_{131})} - e^{i(\theta_{131}-\theta_{141})} &= \cancel{cos\left(\theta_{141}-\theta_{131}\right)} + isin\left(\theta_{141}-\theta_{131}\right) \\ & - \cancel{cos\left(\theta_{141}-\theta_{131}\right)} + isin\left(\theta_{141}-\theta_{131}\right) \\ \Rightarrow \ e^{i(\theta_{141}-\theta_{131})} - e^{i(\theta_{131}-\theta_{141})} &= 2sin\left(\theta_{141}-\theta_{131}\right) \end{aligned} \tag{4.235}$$

$$\begin{aligned} \therefore e^{i(\theta_{131}+\theta_{141})} - e^{i(\theta_{141}-\theta_{131})} &= cos\left(\theta_{141}-\theta_{131}\right) - \cancel{isin\left(\theta_{141}-\theta_{131}\right)} \\ & + cos\left(\theta_{141}-\theta_{131}\right) + \cancel{isin\left(\theta_{141}-\theta_{131}\right)} \\ \Rightarrow \ e^{i(\theta_{131}+\theta_{141})} - e^{i(\theta_{141}-\theta_{131})} &= 2cos\left(\theta_{141}-\theta_{131}\right) \end{aligned} \tag{4.236}$$

Combining (4.234) - (4.236);

$$K_{18} = -2s_{41}^2 i\dot{\theta}_{141}^2 + 2a_3 s_{41}\ddot{\theta}_{131} sin\left(\theta_{141}-\theta_{131}\right) - 2ia_3 s_{41}\dot{\theta}_{131}^2 cos\left(\theta_{141}-\theta_{131}\right) \tag{4.237}$$

$$K_{19} = -s_{41} i - s_{41} i = -2s_{41} i \tag{4.238}$$

Simultaneous solution of (4.232), (4.237) & (4.238) yields the following result:

$$\ddot{s}_{41} = \frac{-2i\left(s_{41}^2\dot{\theta}_{141}^2 - a_3 s_{41}\ddot{\theta}_{131} sin\left(\theta_{141} - \theta_{131}\right) + a_3 s_{41}\dot{\theta}_{131}^2 cos\left(\theta_{141} - \theta_{131}\right)\right)}{-2is_{41}} \tag{4.239}$$

$$\boxed{\ddot{s}_{41} = s_{41}\dot{\theta}_{141}^2 - a_3\ddot{\theta}_{131} sin\left(\theta_{141} - \theta_{131}\right) + a_3\dot{\theta}_{131}^2 cos\left(\theta_{141} - \theta_{131}\right)} \tag{4.240}$$

4.3 Denavit - Hartenberg Convention

In Figure 4.4, a sketch that shows unit vectors of all reference frames is given. Note that same coloured unit vectors represent the unit vectors of the same reference frame.

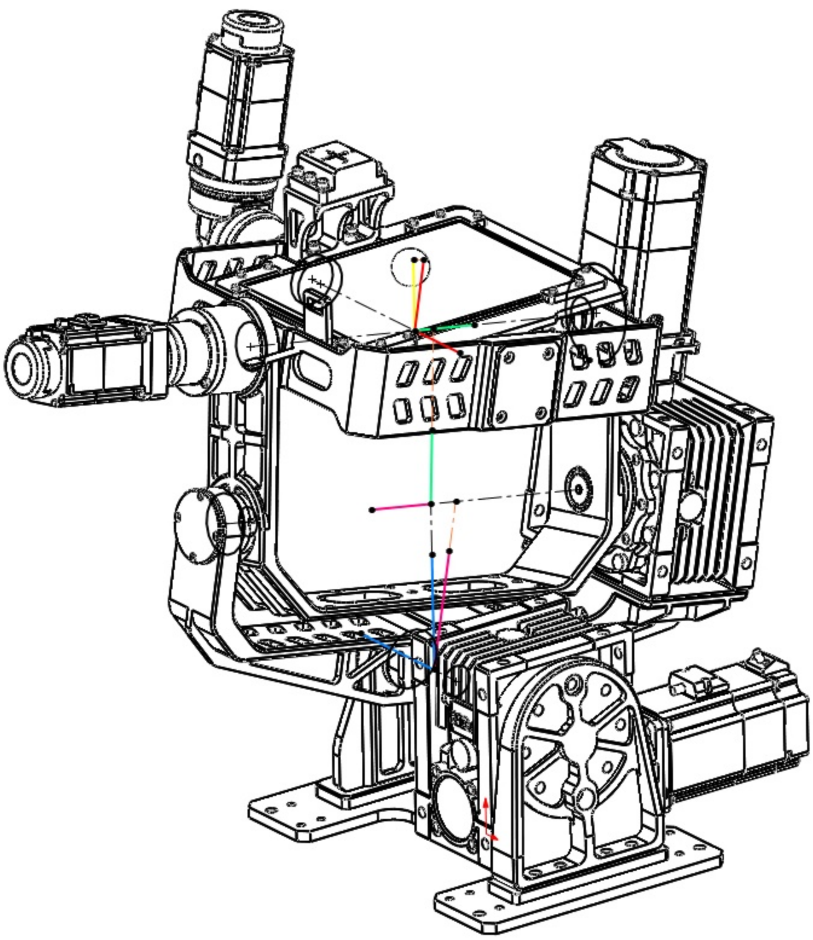


Figure 4.4: Sketch showing unit vectors of the coordinate frames

Table 4.1: D-H convention for first concept design

	a_k	α_k	s_k	θ_k
1	a_1	$\pi/2$	0	θ_1
2	a_2	π	s_2	θ_2
3	0	$-\pi/2$	s_3	θ_3
4	0	0	0	θ_4

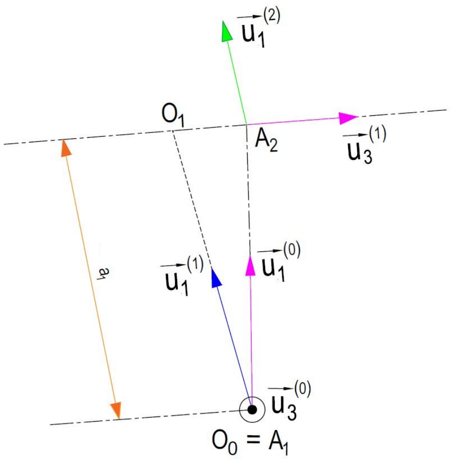


Figure 4.5: Schematic representation of parameters included in the first row of Table 4.1

A_1: intersection of $\vec{u}_3^{(0)}$ & $\vec{u}_1^{(1)}$

A_1O_1 : Distance between $\vec{u}_3^{(0)}$ & $\vec{u}_3^{(1)}$ along $\vec{u}_1^{(1)}$

O_0A_1 : Distance between $\vec{u}_1^{(0)}$ & $\vec{u}_1^{(1)}$ along $\vec{u}_3^{(0)}$

$a_1 = A_1O_1 \;\Rightarrow\; a_1 = 142$ mm

$\alpha_1 \rightarrow \measuredangle\, [\vec{u}_3^{(0)} \rightarrow \vec{u}_3^{(1)}]$ about $\vec{u}_1^{(1)} \;\Rightarrow\; \alpha_1 = \pi/2$

$s_1 = O_0A_1$ (along $\vec{u}_3^{(0)}$) $\Rightarrow\; s_1 = 0$ mm

$\theta_1 = \measuredangle\, [\vec{u}_1^{(0)} \rightarrow \vec{u}_1^{(1)}$ about $\vec{u}_3^{(0)}]$

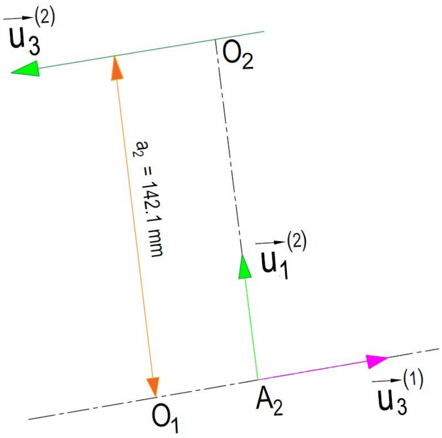


Figure 4.6: Schematic representation of parameters included in the second row of Table 4.1

A_2 : intersection of $\vec{u}_3^{(1)}$ & $\vec{u}_1^{(2)}$

A_2O_2 : Distance between $\vec{u}_3^{(1)}$ & $\vec{u}_3^{(2)}$ along $\vec{u}_1^{(2)}$

O_1A_2 : Distance between $\vec{u}_1^{(1)}$ & $\vec{u}_1^{(2)}$ along $\vec{u}_3^{(1)}$

$a_2 = A_2O_2 \;\Rightarrow\; a_2 = 142.1$ mm

$\alpha_2 \rightarrow \measuredangle\, [\vec{u}_3^{(1)} \rightarrow \vec{u}_3^{(2)}]$ about $\vec{u}_1^{(2)} \;\Rightarrow\; \alpha_2 = \pi$

$s_2 = O_1A_2$ (along $\vec{u}_3^{(1)}$) $\Rightarrow\; s_2 = 25$ mm

$\theta_2 = \measuredangle\, [\vec{u}_1^{(1)} \rightarrow \vec{u}_1^{(2)}$ about $\vec{u}_3^{(1)}]$

A_3 : intersection of $\vec{u}_3^{(2)}$ & $\vec{u}_1^{(3)}$

A_3O_3 : Distance between $\vec{u}_3^{(2)}$ & $\vec{u}_3^{(3)}$ along $\vec{u}_1^{(3)}$

O_2A_3 : Distance between $\vec{u}_1^{(2)}$ & $\vec{u}_1^{(3)}$ along $\vec{u}_3^{(2)}$

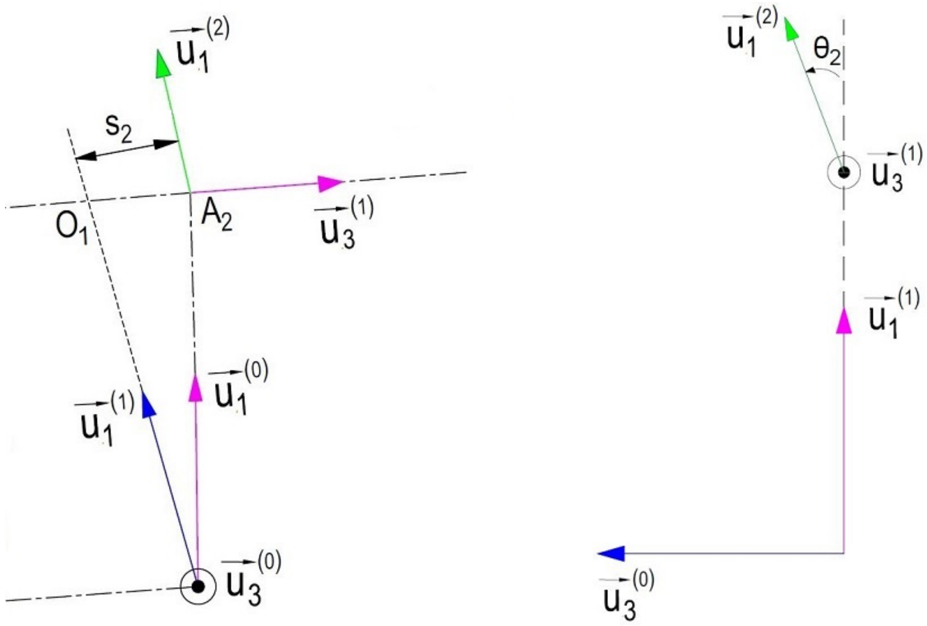


Figure 4.7: Schematic representation of additional parameters included in the second row of Table 4.1

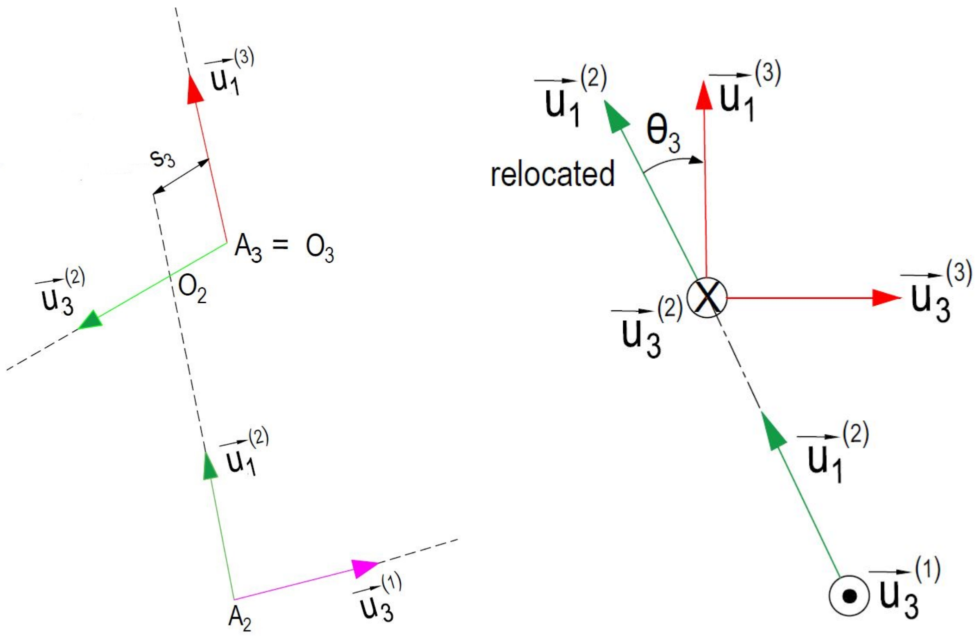


Figure 4.8: Schematic representation of parameters included in the third row of Table 4.1

$$a_3 = A_3O_3 \;\Rightarrow\; a_3 = 0 \text{ mm}$$
$$\alpha_3 \rightarrow \measuredangle\,[\vec{u}_3^{(2)} \rightarrow \vec{u}_3^{(3)}] \text{ about } \vec{u}_1^{(3)} \;\Rightarrow\; \alpha_3 = -\pi/2$$
$$s_3 = O_2A_3 \text{ (along } \vec{u}_3^{(2)}) \;\Rightarrow\; s_3 = 19 \text{ mm}$$
$$\theta_3 = \measuredangle\,[\vec{u}_1^{(2)} \rightarrow \vec{u}_1^{(3)} \text{ about } \vec{u}_3^{(2)}]$$

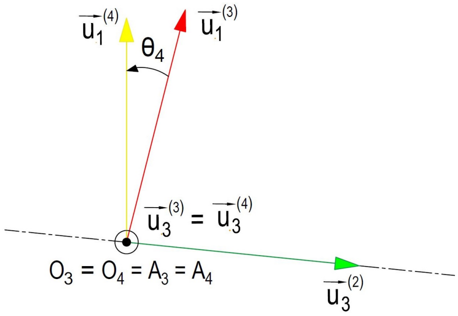


Figure 4.9: Schematic representation of parameters included in the fourth row of Table 4.1

A_4 : intersection of $\vec{u}_3^{(3)}$ & $\vec{u}_1^{(4)}$
A_4O_4 : Distance between $\vec{u}_3^{(3)}$ & $\vec{u}_3^{(4)}$ along $\vec{u}_1^{(4)}$
O_3A_4 : Distance between $\vec{u}_1^{(3)}$ & $\vec{u}_1^{(4)}$ along $\vec{u}_3^{(3)}$

$$a_4 = A_4O_4 \;\Rightarrow\; a_4 = 0 \text{ mm}$$
$$\alpha_4 \rightarrow \measuredangle\,[\vec{u}_3^{(3)} \rightarrow \vec{u}_3^{(4)}] \text{ about } \vec{u}_1^{(4)} \;\Rightarrow\; \alpha_4 = 0$$
$$s_4 = O_3A_4 \text{ (along } \vec{u}_3^{(3)}\text{)} \;\Rightarrow\; s_4 = 0 \text{ mm}$$
$$\theta_4 = \measuredangle\,[\vec{u}_1^{(3)} \rightarrow \vec{u}_1^{(4)} \text{ about } \vec{u}_3^{(3)}]$$

By using the variables tabulated in Table 4.1, we can write the transformation matrices as follows:

$$\hat{C}^{(0,1)} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\pi/2} \tag{4.241}$$
$$\hat{C}^{(1,2)} = e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2} = e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\pi} \tag{4.242}$$
$$\hat{C}^{(2,3)} = e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\alpha_3} = e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} \tag{4.243}$$
$$\hat{C}^{(3,4)} = e^{\tilde{u}_3\theta_4} \tag{4.244}$$

4.4 Orientation of the Platform w.r.t Base Frame

$$\text{Let } \hat{C} = \hat{C}^{(0,4)} = \hat{C}^{(0,1)}\hat{C}^{(1,2)}\hat{C}^{(2,3)}\hat{C}^{(3,4)} \tag{4.245}$$
$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\pi} e^{\tilde{u}_3\theta_3} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4} \tag{4.246}$$
$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\pi} e^{\tilde{u}_3\theta_2} \underbrace{e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_1\pi/2}}_{e^{\tilde{u}_2\theta_4}} e^{-\tilde{u}_1\pi/2} \tag{4.247}$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} \underbrace{e^{\tilde{u}_1\pi} e^{\tilde{u}_3\theta_3} e^{-\tilde{u}_1\pi}}_{e^{-\tilde{u}_3\theta_3}} e^{\tilde{u}_1\pi} e^{-\tilde{u}_1\pi} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_1\pi/2} \tag{4.248}$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\pi/2} e^{-\tilde{u}_3\theta_2} \underbrace{e^{\tilde{u}_1\pi} e^{\tilde{u}_2\theta_4} e^{-\tilde{u}_1\pi}}_{e^{-\tilde{u}_2\theta_4}} \underbrace{e^{\tilde{u}_1\pi} e^{-\tilde{u}_1\pi/2}}_{e^{\tilde{u}_1\pi/2}} \tag{4.249}$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} \underbrace{e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{-\tilde{u}_1\pi/2}}_{e^{-\tilde{u}_2\theta_2}} e^{\tilde{u}_1\pi/2} e^{-\tilde{u}_3\theta_3} e^{-\tilde{u}_2\theta_4} e^{\tilde{u}_1\pi/2} \tag{4.250}$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} \underbrace{e^{\tilde{u}_1\pi/2} e^{-\tilde{u}_3\theta_3} e^{-\tilde{u}_1\pi/2}}_{e^{\tilde{u}_2\theta_3}} \underbrace{e^{\tilde{u}_1\pi/2} e^{-\tilde{u}_2\theta_4} e^{-\tilde{u}_1\pi/2}}_{e^{-\tilde{u}_2\theta_4}} \underbrace{e^{\tilde{u}_1\pi/2} e^{\tilde{u}_1\pi/2}}_{e^{\tilde{u}_1\pi}} \tag{4.251}$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_3(\theta_3-\theta_2)} e^{-\tilde{u}_3\theta_4} e^{\tilde{u}_1\pi} \tag{4.252}$$

4.5 Position of the Center of Gravity of the Platform w.r.t Base Frame

Link to link transformations can be expressed as follows:

$$\vec{r}_k = s_k \vec{u}_3^{(k-1)} + a_k \vec{u}_1^{(k)} = \overrightarrow{O_{k-1}O_k} \tag{4.253}$$

The position vectors can be defined as follows:

$$\vec{P}_{01} = \overrightarrow{O_0O_1} = s_1\vec{u}_3^{(0)} + a_1\vec{u}_1^{(1)} = a_1\vec{u}_1^{(1)} \qquad \vec{P}_{01} = a_1\vec{u}_1^{(1)} \tag{4.254}$$

$$\vec{P}_{02} = \overrightarrow{O_1O_2} = s_2\vec{u}_3^{(1)} + a_2\vec{u}_1^{(2)} \qquad \vec{P}_{02} = s_2\vec{u}_3^{(1)} + a_2\vec{u}_1^{(2)} \tag{4.255}$$

$$\vec{P}_{03} = \overrightarrow{O_2O_3} = s_3\vec{u}_3^{(2)} + a_3\vec{u}_1^{(3)} = s_3\vec{u}_3^{(2)} \qquad \vec{P}_{03} = s_3\vec{u}_3^{(2)} \tag{4.256}$$

$$\vec{P}_{04} = \overrightarrow{O_3O_4} = s_4\vec{u}_3^{(3)} + a_4\vec{u}_1^{(4)} = 0 \qquad \vec{P}_{04} = 0 \tag{4.257}$$

$\vec{r}_b$: Position of the center of gravity of the platform w.r.t base frame

$$\vec{r} = \overrightarrow{OR} = \overrightarrow{O_0O_4} = \overrightarrow{O_0O_1} + \overrightarrow{OO_2} + \overrightarrow{O_2O_3} + \overrightarrow{O_3O_4} \tag{4.258}$$

$$\vec{r} = a_1\vec{u}_1^{(1)} + s_2\vec{u}_3^{(1)} + a_2\vec{u}_1^{(2)} + s_3\vec{u}_3^{(2)} \tag{4.259}$$

$$\text{Let } \overline{r} = \overline{r}^{(0)} = \{\vec{r}\}^0 \tag{4.260}$$

$$\overline{r} = a_1\overline{u}_1^{(1/0)} + s_2\overline{u}_3^{(1/0)} + a_2\overline{u}_1^{(2/0)} + s_3\overline{u}_3^{(2/0)} \tag{4.261}$$

$$\overline{r} = a_1\hat{C}^{(0,1)}\overline{u}_1 + s_2\hat{C}^{(0,1)}\overline{u}_3 + a_2\hat{C}^{(0,2)}\overline{u}_1 + s_1\hat{C}^{(0,2)}\overline{u}_3 \tag{4.262}$$

$$\overline{r} = \underbrace{a_1 e^{\tilde{u}_3\theta 1} e^{\tilde{u}_1\pi/2}\overline{u}_1}_{1} + \underbrace{s_2 e^{\tilde{u}_3\theta 1} e^{\tilde{u}_1\pi/2}\overline{u}_3}_{2} + \underbrace{a_2 e^{\tilde{u}_3\theta 1} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\pi}\overline{u}_1}_{3} \tag{4.263}$$

$$+ \underbrace{s_3 e^{\tilde{u}_3\theta 1} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\pi}\overline{u}_3}_{4} \tag{4.264}$$

$$\Rightarrow e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2}\overline{u}_1 = e^{\tilde{u}_3\theta_1}\overline{u}_1 \tag{4.265}$$

$$\begin{aligned}\Rightarrow e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2}\overline{u}_3 &= e^{\tilde{u}_3\theta_1}(\underbrace{\overline{u}_3 c(\pi/2)}_{0} - \underbrace{\overline{u}_2 s(\pi/2)}_{\overline{u}_2}) \\ &= -e^{\tilde{u}_3\theta_1}\overline{u}_2\end{aligned} \tag{4.266}$$

$$\begin{aligned}\Rightarrow e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_2}\overline{u}_1 &= e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2}(\overline{u}_1 c\theta_2 + \overline{u}_2 s\theta_2) \\ &= e^{\tilde{u}_3\theta_1}(\overline{u}_1 c\theta_2 + s\theta_2\,(\underbrace{\overline{u}_2 c(\pi/2)}_{0} + \underbrace{\overline{u}_3 s(\pi/2)}_{\overline{u}_3}) \\ &= e^{\tilde{u}_3\theta_1}(\overline{u}_1 c\theta_2 + \overline{u}_3 s\theta_2)\end{aligned} \tag{4.267}$$

$$\begin{aligned}\Rightarrow e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\pi}\overline{u}_3 &= e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_2}\,(\underbrace{\overline{u}_3 c(\pi)}_{-\overline{u}_3}) - \underbrace{\overline{u}_2 s(\pi))}_{0} \\ &= -e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2}\overline{u}_3 \\ &= -e^{\tilde{u}_3\theta_1}(\overline{u}_3 c(\pi/2) - \underbrace{\overline{u}_2 s(\pi/2)}_{\overline{u}_2}) \\ &= e^{\tilde{u}_3\theta_1}\overline{u}_2\end{aligned} \tag{4.268}$$

Compact form of $\overline{r}$:

$$\overline{r} = e^{\tilde{u}_3\theta_1}(\overline{u}_1(a_1 + a_2 c\theta_2) + \overline{u}_2(s_3 - s_2) + \overline{u}_3(a_2 s\theta_2)) \tag{4.269}$$

Further expansion gives $\overline{r} = \overline{r}^{(0)}$ as follows:

$$\begin{aligned}\overline{r} = {} & (a_1 + a_2 + c\theta_2)(\overline{u}_1 c\theta_1 - \overline{u}_2 s\theta_1) + (s_3 - s_2)(\overline{u}_2 c\theta_1 - \overline{u}_1 s\theta_1) \\ & + \overline{u}_3(a_2 s\theta_2)\end{aligned} \tag{4.270}$$

$$\begin{aligned}\overline{r} = {} & \overline{u}_1(a_1 c\theta_1 + a_2 c\theta_1 c\theta_2) - \overline{u}_2(a_1 s\theta_1 + a_2 s\theta_1 c\theta_2) - \overline{u}_1((s_3 - s_2)s\theta_1) \\ & + \overline{u}_2((s_3 - s_2)c\theta_1) + \overline{u}_3(a_2 s\theta_2)\end{aligned} \tag{4.271}$$

$$\overline{r} = \underbrace{(\overline{u}_1 a_1 c\theta_1 + a_2 c\theta_1 c\theta_2 - (s_3 - s_2) s\theta_1)}_{r_1} + \overline{u_2}(\underbrace{(s_3 - s_2) c\theta_1 - a_1 s\theta_1 + a_2 s\theta_1 c\theta_2}_{r_2})$$
$$+ \overline{u}_3 \underbrace{(a_2 s\theta_2)}_{r_3} \tag{4.272}$$

$$\vec{r} = r_1 \vec{u}_1^{(0)} + r_2 \vec{u}_2^{(0)} + r_3 \vec{u}_3^{(0)} \tag{4.273}$$

4.6 Inverse Kinematics of Disturber and Stabilizer

In this section, the purpose is finding the corresponding joint variables, given $\hat{C}$ & $\overline{r}$
Start with θ_1:

$$\overline{r} = e^{\tilde{u}_3 \theta_1} (\overline{u}_1 (a_1 + a_2 c\theta_2) + \overline{u}_2 (s_3 - s_2) + \overline{u}_3 (a_2 s\theta_2)) \tag{4.274}$$

$$\overline{r}^* = e^{-\tilde{u}_3 \theta_1} \overline{r} = (\overline{u}_1 (a_1 + a_2 c\theta_2) + \overline{u}_2 (s_3 - s_2) + \overline{u}_3 (a_2 s\theta_2)) \tag{4.275}$$

Multiply both sides $\overline{u}_2^t$,

$$\overline{u}_2^t \; e^{-\tilde{u}_3 \theta_1} \; \overline{r} = s_3 - s_2 \tag{4.276}$$

$$(\overline{u}_2^t c\theta_1 - \overline{u}^t s\theta_1) r = s_3 - s_2 \tag{4.277}$$

$$r_2 c\theta_1 - r_1 s\theta_1 = s_3 - s_2 \tag{4.278}$$

$$\text{Let } r_2 = r_{12} sin\gamma_1, \quad r_1 = r_{12} cos\gamma_1, \quad \text{find } r_{12} \;\&\; \gamma_1 \tag{4.279}$$

$$r_{12} = \sqrt{r_1^2 + r_2^2} \rightarrow \text{ known} \Rightarrow \text{ If } r_{12} \neq o \Rightarrow \gamma_1 = atan_2(r_2; r_1) \rightarrow \text{known} \tag{4.280}$$

$$r_{12} s\gamma_1 c\theta_1 - r_{12} c\gamma_1 c\theta_1 = s_3 - s_2 \tag{4.281}$$

$$sin(\gamma_1 - \theta_1) = (s_3 - s_2)/r_{12} = \eta_1 \rightarrow \text{known} \tag{4.282}$$

$$cos(\gamma_1 - \theta_1) = \sigma_1 \sqrt{1 - \eta_1^2} \;\Rightarrow\; \gamma_1 - \theta_1 = atan_2(\eta_1; \sigma_1; \sqrt{1 - \eta_1^2}\,) \tag{4.283}$$

$$\Rightarrow\; \theta_1 = \gamma_1 - \underbrace{atan_2(\Omega_1 \sigma_1; \sqrt{1 - \eta_1^2})}_{\Psi_1} \tag{4.284}$$

Interpretation of r_1:

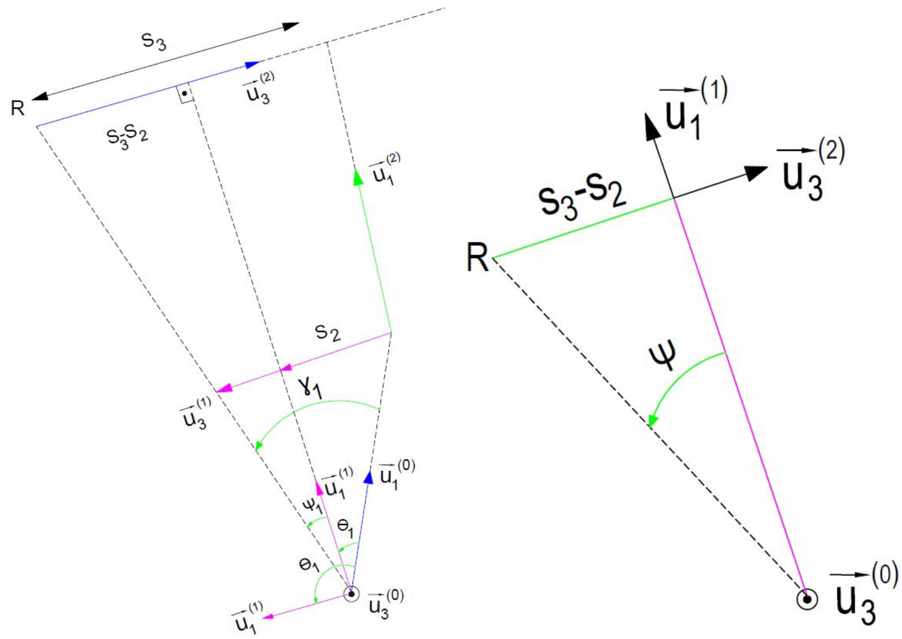


Figure 4.10: Interpretation of r_1

$\sigma_1 = +1 \Rightarrow \Psi_1$: Acute Angle$(cos\psi > 0)$: Left-Shouldered configuration
$\sigma_1 = -1 \Rightarrow \Psi_1$: Obtuse Angle$(cos\psi < 0)$: Right-Shouldered configuration

$$r_1^* = a_1 + a_2 c\theta_2 = \overline{u}_1^t e^{-\tilde{u}_3\theta_1}\overline{r} \tag{4.285}$$

$$(\overline{u}_1^t c\theta_1 + \overline{u}_2^t c\theta_2)\overline{r} = a_1 + a_2\theta_2 \quad \Rightarrow \quad r_1 c\theta_1 + r_2 s\theta_1 = a_1 + a_2 c\theta_2 \tag{4.286}$$

$$r_3^* = a_2 s\theta_2 = \overline{u}_3^t e^{-\tilde{u}_3\theta_1}\overline{r} \quad \Rightarrow \quad r_3 = a_2 s\theta_2 \tag{4.287}$$

$$\Rightarrow s\theta_2 = \frac{r_3}{a_2}, \qquad c\theta_2 = \frac{(r_1 c\theta_1 + r_2 c\theta_1) - a_1}{a_2} \tag{4.288}$$

$$\Rightarrow \theta_2 = atan_2(r_3(r_1 c\theta_1 + r_2 s\theta_1) - a_1) \tag{4.289}$$

We can find θ_3 & θ_4 from $\hat{C}$. Remind that $\hat{C}$ is:

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2(\theta_3-\theta_2)} e^{-\tilde{u}_3\theta_4} e^{\tilde{u}_1\pi} \tag{4.290}$$

$$\underbrace{e^{-\tilde{u}_3\theta_1}\hat{C}e^{-\tilde{u}_1\pi}}_{\hat{C}^*} = e^{\tilde{u}_2(\theta_3-\theta_2)} e^{-\tilde{u}_3\theta_4} \tag{4.291}$$

$$\hat{C}_{13} = \overline{u}_1^t e^{\tilde{u}_2(\theta_3-\theta_2)} e^{\tilde{u}_3\theta_4} \overline{u}_3^t \tag{4.292}$$

$$\hat{C}_{13} = \overline{u}_1^t e^{\tilde{u}_2(\theta_3-\theta_2)} \overline{u}_3^t \tag{4.293}$$

$$\hat{C}_{13} = \overline{u}_1^t(\overline{u}_3 cos(\theta_3 - \theta_2) + \overline{u}_1 sin(\theta_3 - \theta_2)) \tag{4.294}$$

$$\hat{C}_{13} = sin(\theta_3 - \theta_2) \tag{4.295}$$

$$cos(\theta_3 - \theta_2) = \sigma_3\sqrt{1 - \hat{C}_{13}^2} \tag{4.296}$$

$$\theta_3 - \theta_2 = atan_2(C_{13}^*, \sigma_3\sqrt{1 - C_{13}^{*2}}) \tag{4.297}$$

$$\theta_3 = \theta_2 + \underbrace{atan_2(C_{13}^*, \sigma_3\sqrt{1 - C_{13}^{*2}})}_{\alpha} \tag{4.298}$$

Interpretation of σ_3:

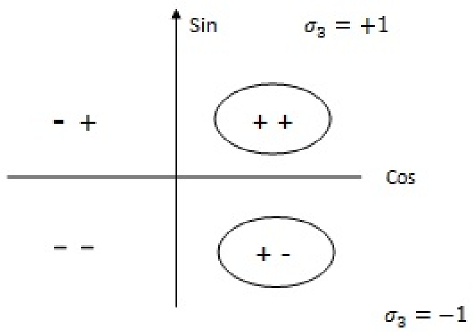


Figure 4.11: Sign convention of σ_3 according to the sign of the angles

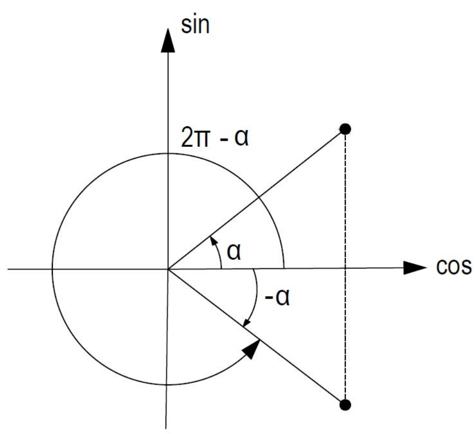


Figure 4.12: Representation of the acute and obtuse angles

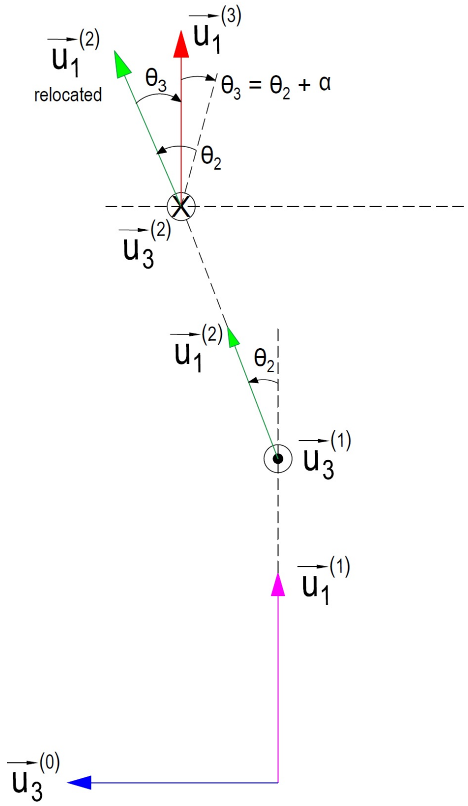


Figure 4.13: Interpretation of σ_3:

$\sigma_3 = +1$ configuration

θ_3 = Acute angle

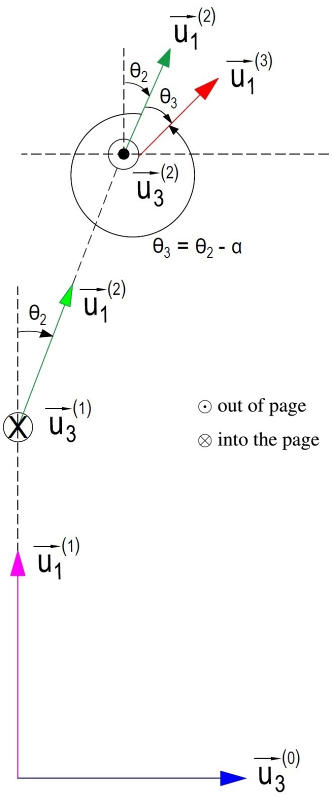


Figure 4.14: Interpretation of σ_3:

$\sigma_3 = -1$ configuration

θ_3 = Obtuse angle

First singularity occurs when $cos(\theta_3 - \theta_2) = 0$

$$\Rightarrow \ \theta_3 - \theta_2 = \pi/2 \tag{4.299}$$

Let's return to $\hat{C}$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2(\theta_3-\theta_2)} e^{-\tilde{u}_3\theta_4} e^{\tilde{u}_1\pi} \tag{4.300}$$

Insert (4.299) into (4.300) to get:

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\pi/2} e^{-\tilde{u}_3\theta_4} e^{\tilde{u}_3\pi} \tag{4.301}$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} \underbrace{e^{\tilde{u}_2\pi/2} e^{-\tilde{u}_3\theta_4} \overbrace{e^{-\tilde{u}_2\pi/2} e^{\tilde{u}_2\pi/2}}^{I}}_{e^{-\tilde{u}_1\theta_4}} e^{\tilde{u}_3\pi} \tag{4.302}$$

$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\theta_4} \Rightarrow$ Note that θ_3 cannot be determined since it has disappeared from the rotation matrix.

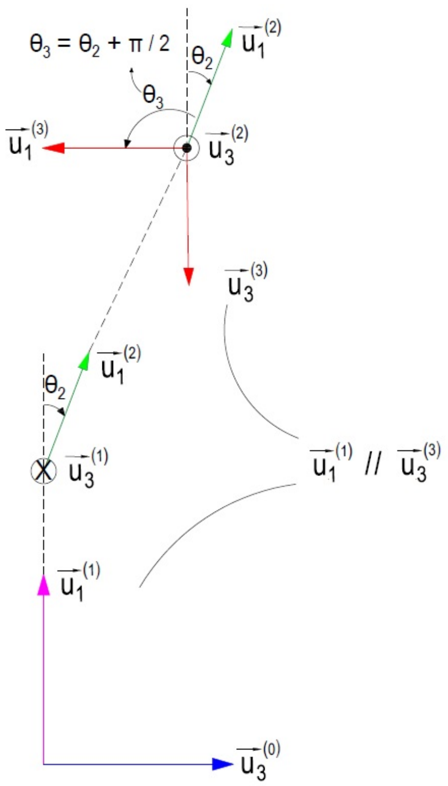


Figure 4.15: Interpretation of the singularity : $cos(\theta_3 - \theta_2) = 0$

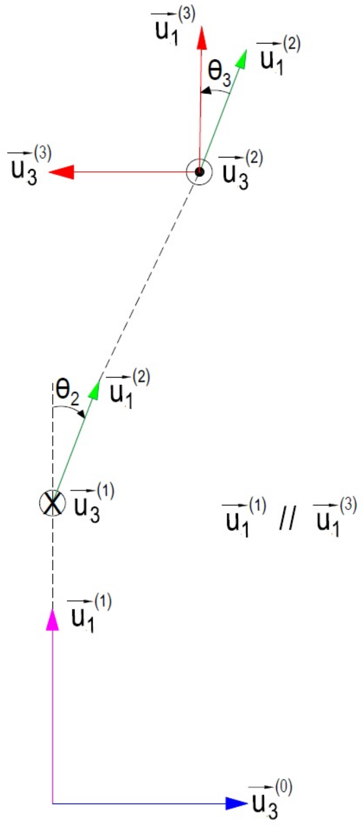


Figure 4.16: Interpretation of the singularity : $sin(\theta_3 - \theta_2) = 0$

Second singularity option is $sin(\theta_3 - \theta_2) = 0$:

$$\Rightarrow \ \theta_3 - \theta_2 = 0 \ \text{ or } \ \theta_3 - \theta_2 = \pi \tag{4.303}$$

The first option for this singularity is as follows:

$$\theta_3 - \theta_2 = 0 \ \Rightarrow \ \theta_3 = \theta_2 \tag{4.304}$$

As it can be seen from the figure $\vec{u}_1^{(1)}$ becomes parallel to $\vec{u}_1^{(3)}$ so θ_3 becomes indistinguishable! Let's return to $\hat{C}$;

$$\hat{C} = e^{\tilde{u}_3\theta_1} \underbrace{e^{\tilde{u}_2(\theta_2-\theta_2)}}_{\hat{I}} e^{-\tilde{u}_3\theta_4} e^{\tilde{u}_1\pi} \tag{4.305}$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_3\theta_4} e^{\tilde{u}_1\pi} \tag{4.306}$$

θ_3 vanished from rotation matrix and cannot be determined. The second option for this singularity is as follows:

$$\theta_3 - \theta_2 = \pi \tag{4.307}$$

From (4.297),

$$\theta_3 - \theta_2 = \underbrace{atan_2\left(\overbrace{sin(\theta_3 - \theta_2)}^{sin(\pi)}, \overbrace{cos(\theta_3 - \theta_2)}^{cos(\pi)}\right)}_{0} \tag{4.308}$$

$$\Rightarrow \theta_3 = \theta_2 \tag{4.309}$$

Since (4.307) & (4.309) contradicts; $\theta_3 = \theta_2 + \pi$ is not a solution.

θ_4 can be extracted from $\hat{C}$ as follows:

$$\hat{C}^*_{21} = \overline{u}^t e^{\overline{u}_2(\theta_3-\theta_2)} e^{-\overline{u}_3\theta_4}\overline{u}_1 \tag{4.310}$$

$$\hat{C}^*_{21} = \overline{u}_2^t e^{-\overline{u}_3\theta_4}\overline{u}_1 = \overline{u}_2^t(\overline{u}_1 c\theta_4) + \overline{u}_2 s\theta_4) \tag{4.311}$$

$$c\theta_4 = \sigma_4\sqrt{1 - \hat{C}^{*2}_{21}} \tag{4.312}$$

$$\theta_4 = atan_2(\hat{C}^*_{21}, \sigma_4\sqrt{\hat{C}^{*2}_{21}}) \tag{4.313}$$

4.7 Forward & Inverse Kinematics of Stabilizer

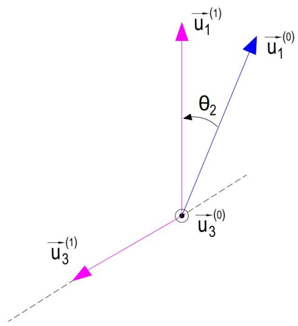


Figure 4.17: Side view

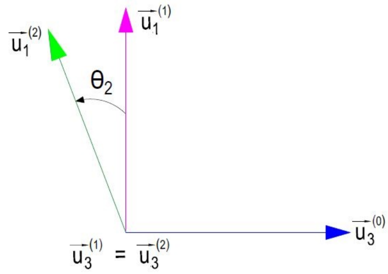


Figure 4.18: Front view

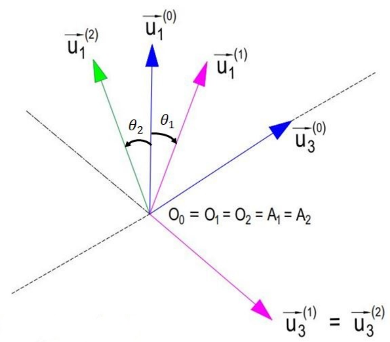


Figure 4.19: Isometric view

A_k : Intersection of $\vec{u}_3^{(k-1)}$ & $\vec{u}_1^{(k)}$

O_k : Intersection of $\vec{u}_1^{(k)}$ & $\vec{u}_3^{(k)}$

Table 4.2: D-H convention for stabilizer

	a_k	α_k	s_k	θ_k
1	0	$-\pi/2$	0	θ_1
2	0	0	0	θ_2

a_k : Distance between $\vec{u}_3^{(k-1)}$ & $\vec{u}_3^{(k)}$ along $\vec{u}_1^{(k)}$
a_1 : $\vec{u}_3^{(0)}$ & $\vec{u}_3^{(1)}$ along $\vec{u}_1^{(1)}$
a_2 : $\vec{u}_3^{(1)}$ & $\vec{u}_3^{(2)}$ along $\vec{u}_1^{(2)}$
α_k : $\measuredangle\,[\vec{u}_3^{(k-1)} \rightarrow \vec{u}_3^{(k)}$ about $\vec{u}_1^{(k)}]$
s_k : Distance between $\vec{u}_1^{(k-1)}$ & $\vec{u}_1^{(k)}$ along $\vec{u}_3^{(k-1)}$
s_1 : $\vec{u}_1^{(0)}$ & $\vec{u}_1^{(1)}$ along $\vec{u}_3^{(0)}$
s_2 : $\vec{u}_1^{(1)}$ & $\vec{u}_1^{(2)}$ along $\vec{u}_3^{(1)}$

In order to obtain the forward kinematics of the system; we'll start with deriving link to link rotation matrices.

$$\hat{C}^{(0,1)} = e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1} = e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_1\pi/2} \tag{4.314}$$

$$\hat{C}^{(1,2)} = e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\alpha_2} = e^{\tilde{u}_3\theta_2} \tag{4.315}$$

Orientation of the platform w.r.t base frame can be expressed as follows:

$$\hat{C}^{(0,2)} = \hat{C}^{(0,1)}\hat{C}^{(1,2)} \tag{4.316}$$

$$\hat{C}^{(0,2)} = e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_2} \tag{4.317}$$

$$\hat{C}^{(2,2)} = e^{\tilde{u}_3\theta_1}\underbrace{e^{u_1\pi/2}e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\pi/2}}_{e^{\tilde{u}_2\theta_2}}e^{\tilde{u}_1\pi/2} \tag{4.318}$$

$$\hat{C}^{(2,2)} = e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}e^{-\tilde{u}_3\pi/2} \tag{4.319}$$

Position of the center of gravity of the platform w.r.t disturber frame can be expressed as follows:

$$\vec{r}_k = s_k\vec{u}_3^{(k-1)} + a_k\vec{u}_1^{(k)} = \overrightarrow{O_{k-1}O_k} \tag{4.320}$$

$$\vec{P}_{01} = \overrightarrow{O_0O_1} = s_1\vec{u}_3^{(0)} + a_1\vec{u}_1^{(1)} = 0 \tag{4.321}$$

$$\vec{P}_{02} = \overrightarrow{O_1O_2} = s_2\vec{u}_3^{(0)} + a_2\vec{u}_1^{(2)} = 0 \tag{4.322}$$

$$\vec{r} = \overrightarrow{OR} = 0 \tag{4.323}$$

In order to obtain inverse kinematics of the system, we can start with transformation matrix.

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_1\pi/2} \tag{4.324}$$

$$\underbrace{\hat{C} e^{-\tilde{u}_1\pi/2}}_{\hat{C}^*} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \tag{4.325}$$

We can obtain θ_1 & θ_2 from the rotation matrix as follows:

$$\hat{C}^*_{12} = \overline{u}_1^t e^{\tilde{u}_2\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_2 \tag{4.326}$$

$$\hat{C}^*_{12} = \overline{u}_1^t \left(e^{\tilde{u}_2\theta_1} \overline{u}_2\right) = \overline{u}_1^t \left(\overline{u}_2 c\theta_1 - \overline{u}_1 s\theta_1\right) \tag{4.327}$$

$$\hat{C}^*_{12} = -s\theta_1, c\theta_1 = \sigma\sqrt{1 - \hat{C}^{*2}_{12}} \tag{4.328}$$

$$\Rightarrow \boxed{\theta_1 = atan_2\left(-\hat{C}^*_{12},\ \sigma_1\sqrt{1 - \hat{C}^{*2}_{12}}\right)} \tag{4.329}$$

Similarly,

$$\hat{C}^*_{31} = \overline{u}_3^t e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 = \overline{u}_3^t \left(e^{\tilde{u}_2\theta_2} \overline{u}_1\right)$$

$$= \overline{u}_3^t \left(\overline{u}_1 c\theta_2 - \overline{u}_3 s\theta_2\right) = -s\theta_2 \tag{4.330}$$

$$s\theta_2 = -\hat{C}^*_{31}, \qquad c\theta_2 = \sigma_2\sqrt{1 - \hat{C}^{*2}_{13}} \tag{4.331}$$

$$\Rightarrow \boxed{\theta_2 = atan_2\left(-\hat{C}^*_{31},\ \sigma_2\sqrt{1 - \hat{C}^{*2}_{13}}\right)} \tag{4.332}$$

Singularity occurs when $c\theta_1 = 0 \ \Rightarrow\ \theta_1 = \pi/2$ or $\theta_1 = 3\pi/2$.

Illustration of this singularity:

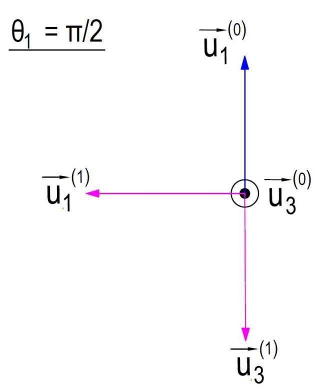


Figure 4.20: Illustration of the singularity: $\vec{u}_1^{(0)}$ & $\vec{u}_3^{(1)}$ becomes co-linear when $\theta_1 = \pi/2$

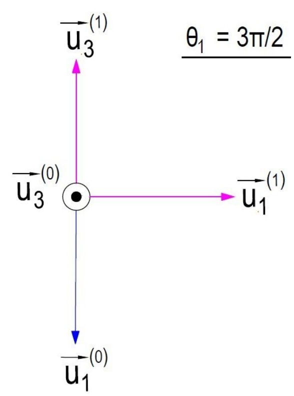


Figure 4.21: Illustration of the singularity: $\vec{u}_1^{(0)}$ & $\vec{u}_3^{(1)}$ becomes co-linear when $\theta_1 = 3\pi/2$

Similarly, another singularity occurs when $c\theta_2 = 0 \Rightarrow \theta_2 = \pi/2$ or $\theta_2 = 3\pi/2$.

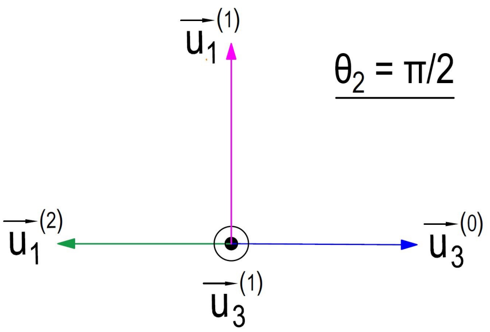


Figure 4.22: Illustration of the singularity: $\vec{u}_1^{(2)}$ & $\vec{u}_3^{(0)}$ becomes co-linear when $\theta_2 = \pi/2$

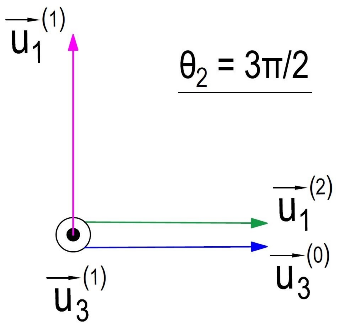


Figure 4.23: Illustration of the singularity: $\vec{u}_1^{(2)}$ & $\vec{u}_3^{(0)}$ becomes co-linear when $\theta_2 = 3\pi/2$

Illustration of σ_1:

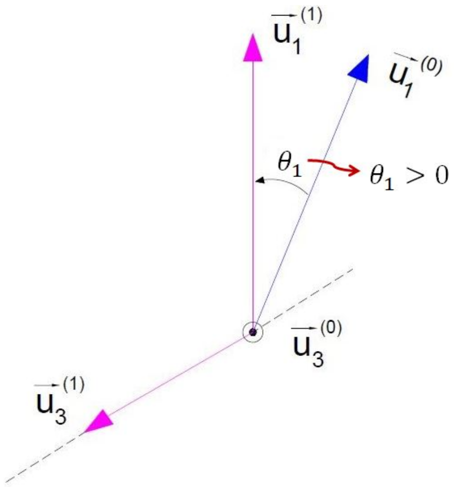


Figure 4.24: Illustration of the σ_1: σ_1 = +1

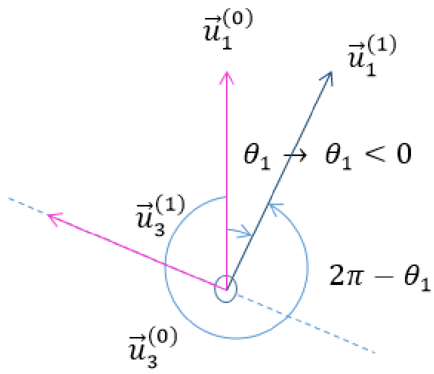


Figure 4.25: Illustration of the σ_1: σ_1 = -1

4.8 Forward & Inverse Dynamic Analysis of Stabilizer without Disturber

The free body diagrams of the bodies of the stabilizer can be found in figures 4.26, 4.27 and 4.28.

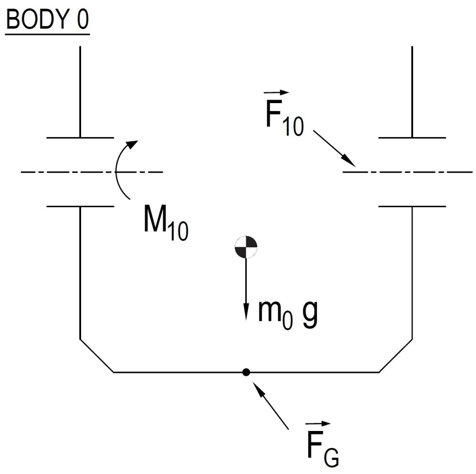


Figure 4.26: Free body diagram of Body 0

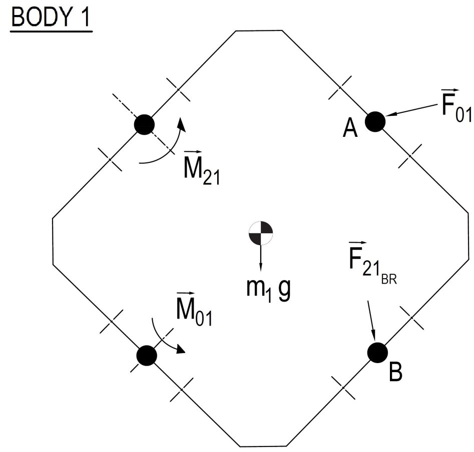


Figure 4.27: Free body diagram of Body 1

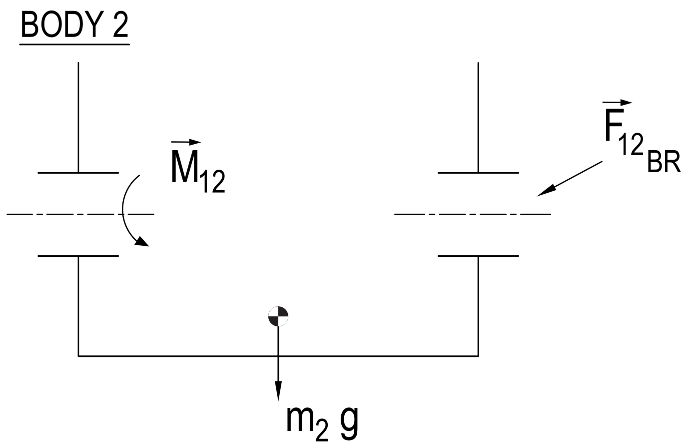


Figure 4.28: Free body diagram of Body 2

The Newton-Euler equations are composed of force and moment equations. The force equations for the bodies can be written as follows:

$$\vec{F}_G + \vec{F}_{10} + m_0\vec{g} = m_0\vec{a}_{c_0}$$

$$m_0\vec{a}_{c_0} = 0 \quad \Rightarrow \quad \vec{F}_G - \vec{F}_{01} + m_0\vec{g} = 0 \tag{4.333}$$

$$\vec{F}_{01} + \vec{F}_{21} + m_1\vec{g} = m_1\vec{a}_{c_1}$$

$$\Rightarrow \quad \vec{F}_{01} - \vec{F}_{12} + m_1\vec{g} = m_1\vec{a}_{c_1} \tag{4.334}$$

$$\Rightarrow \quad \vec{F}_{12} + \vec{F}_2 + m_2\vec{g} = m_2\vec{a}_{c_2} \tag{4.335}$$

Similarly, the moment equations for the bodies can be written as follows:

$$\check{J}_0 \cdot \vec{\alpha}_0 + \vec{w}_0 \times \check{J}_0 \cdot \vec{w}_0 = \vec{M}_G + \vec{M}_{10} + \vec{r}_{0G} \times \vec{F}_G + \vec{r}_{01} \times \vec{F}_{10}$$

$$\Rightarrow \quad \check{J}_0 \cdot \vec{\alpha}_0 + \vec{w}_0 \times \check{J}_0 \cdot \vec{w}_0 = \vec{M}_G - \vec{M}_{10} + \vec{r}_{0G} \times \vec{F}_G - \vec{r}_{01} \times \vec{F}_{01} \tag{4.336}$$

$$\check{J}_1 \cdot \vec{\alpha}_1 + \vec{w}_1 \times \check{J}_1 \cdot \vec{w}_1 = \vec{M}_{01} + \vec{M}_{21} + \vec{r}_{10} \times \vec{F}_{01} + \vec{r}_{12} \times \vec{F}_{21}$$

$$\Rightarrow \quad \check{J}_1 \cdot \vec{\alpha}_1 + \vec{w}_1 \times \check{J}_1 \cdot \vec{w}_1 = \vec{M}_{01} - \vec{M}_{11} + \vec{r}_{10} \times \vec{F}_{01} - \vec{r}_{12} \times \vec{F}_{12} \tag{4.337}$$

$$\Rightarrow \quad \check{J}_2 \cdot \vec{\alpha}_2 + \vec{w}_2 \times \check{J}_2 \cdot \vec{w}_2 = \vec{M}_{12} + \vec{r}_{21} \times \vec{F}_{12} \tag{4.338}$$

Force and moment reactions can be expressed in either of the bodies they act between. Therefore, in order to avoid confusion, the scalar parts of these vectors will be distinguished by means of prime and double prime.

$$\vec{F}_{01} = \sum_{k=1}^{3} F'_{01k}\vec{u}_k^{(0)} = \sum_{k=1}^{3} F''_{01k}\vec{u}_k^{(1)}; \quad \vec{M}_{01} = \sum_{k=1}^{3} M'_{01k}\vec{u}_k^{(0)} = \sum_{k=1}^{3} M''_{01k}\vec{u}_k^{(1)} \tag{4.339}$$

$$\vec{F}_{12} = \sum_{k=1}^{3} F'_{12k}\vec{u}_k^{(1)} = \sum_{k=1}^{3} F''_{12k}\vec{u}_k^{(2)}; \quad \vec{M}_{12} = \sum_{k=1}^{3} M'_{12k}\vec{u}_k^{(1)} = \sum_{k=1}^{3} M''_{12k}\vec{u}_k^{(2)} \tag{4.340}$$

Next step is to derive the kinematic equations of the stabilizer. However, these equations will be valid only if the stabilizer is not attached to the disturber or when the disturber is inactive. More comprehensive kinematic equations which are valid also when the disturber is active will be investigated in the upcoming sections. (Section numarası yazılacak buraya)

$$\vec{w}_{1/0} = \dot{\theta}_1 \vec{u}_3^{(0)}, \quad \vec{\alpha}_{1/0} = \ddot{\theta}_1 \vec{u}_3^{(0)} \tag{4.341}$$

$$\overline{u}_k^{(b/a)} = \hat{C}^{(a,b)} \overline{u}_k^{(b/b)} \tag{4.342}$$

$$\overline{u}_k^{(0/1)} = \hat{C}^{(1,0)} \overline{u}_3^{(0/0)} \tag{4.343}$$

$$\hat{C}^{(0/1)} = e^{\tilde{u}_3 \theta_1} e^{-\tilde{u}_3 \pi/2} \tag{4.344}$$

$$\Rightarrow \hat{C}^{(1/0)} = e^{\tilde{u}_1 \pi/2} e^{-\tilde{u}_3 \theta_1} \tag{4.345}$$

$$\overline{u}_3^{(0/1)} = e^{\tilde{u}_1 \pi/2} e^{-\tilde{u}_3 \theta_1} \overline{u}_3 \tag{4.346}$$

$$\Rightarrow \overline{u}_3^{(0/1)} = e^{\tilde{u}_1 \pi/2} \overline{u}_3 = \overline{u}_3 \underbrace{c(\pi/2)}_{0} - \underbrace{\overline{u}_2 s(\pi/2)}_{\overline{u}_2} \tag{4.347}$$

$$\Rightarrow \overline{u}_3^{(0/1)} = \overline{u}_2^{(1/1)} \Rightarrow \overline{u}_3^{(0)} = -\overline{u}_2^{(1)} \tag{4.348}$$

Inserting (4.348) in to (4.341) will result in:

$$\vec{w}_0 = \vec{w}_{0/G} = \vec{0} \tag{4.349}$$

$$\vec{w}_1 = \vec{w}_{1/0} = -\dot{\theta}\vec{u}_2^{(1)} \Rightarrow \ \vec{\alpha}_1 = -\ddot{\theta}_1\vec{u}_2^{(1)} \tag{4.350}$$

$$\vec{w}_2 = \vec{w}_{2/0} = \vec{w}_{2/1} + \vec{w}_{1/0} = \dot{\theta}_2\vec{u}_3^{(1)} - \dot{\theta}_1\vec{u}_2^{(1)} \tag{4.351}$$

We know that

$$\overline{u}_2^{(1/2)} = \hat{C}^{(2,1)}\overline{u}_2^{(1,1)} \tag{4.352}$$

$$\overline{u}_2^{(1/2)} = e^{-\tilde{u}_3\theta_2}\overline{u}_2 = \overline{u}_2 c\theta_2 + \overline{u}_1 s\theta_2 \tag{4.353}$$

$$\overline{u}_2^{(1/2)} = \overline{u}_1^{(2/2)} s\theta_2 + \overline{u}_2^{(2/2)} c\theta_2 \tag{4.354}$$

$$\vec{u}_2^{(1)} = \vec{u}_1^{(2)} s\theta_2 + \vec{u}_2^{(2)} c\theta_2 \tag{4.355}$$

Also we know that,

$$\vec{u}_3^{(1)} = \vec{u}_3^{(2)} \tag{4.356}$$

Simultaneous solution of (4.351), (4.355), (4.356) yields the following:

$$\vec{w}_2 = \dot{\theta}_2\vec{u}_3^{(2)} - \dot{\theta}_1(\vec{u}_1^{(2)} s\theta_2 + \vec{u}_2^{(2)} c\theta_2) \tag{4.357}$$

$$\vec{w}_2 = -\theta_1 s\theta_2\vec{u}_1^{(2)} - \dot{\theta}_1 c\theta_2\vec{u}_2^{(2)} + \dot{\theta}_2\vec{u}_3^{(2)} \tag{4.358}$$

We can proceed with the acceleration as follows:

$$\vec{\alpha}_2 = -(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2)\vec{u}_1^{(2)} - (\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_2 s\theta_2)\vec{u}_2^{(2)} + \ddot{\theta}_2\vec{u}_3^{(2)} \tag{4.359}$$

Since we have obtained the angular accelerations, we can continue with finding the linear accelerations. The position vector of the center of gravity of the Body 1 with respect to the base frame can be expressed as follows:

$$\vec{r}_1 = \vec{r}_{C1/0} = r_{11}\vec{u}_1^{(1)} + r_{12}\vec{u}_2^{(1)} + r_{13}\vec{u}_3^{(1)} \tag{4.360}$$

In order to obtain the velocity of the mass center of Body-1, we need to use Coriolis(transport) theorem since the velocity expression is defined as the derivative of the position vector with respect to stationary Body-0 reference frame.

$$\begin{aligned}\vec{v}_1 = D_0\vec{r}_1 = D_1\vec{r}_1 + \vec{w}_{1/0} \times \vec{r}_1 \\ = 0 - \dot{\theta}_1\vec{u}_2^{(1)} \times (r_{11}\vec{u}_1^{(1)} + r_{12}\vec{u}_2^{(1)} + r_{13}\vec{u}_3^{(1)}) \\ \Rightarrow \ \vec{v}_1 = -\dot{\theta}_1 r_{13}\vec{u}_1^{(1)} + -\dot{\theta}_1 r_{11}\vec{u}_3^{(1)}\end{aligned} \tag{4.361}$$

Again by using Coriolis theorem,

$$\vec{a}_1 = D_0\vec{v}_1 = D_1\vec{v}_1 + \vec{w}_{1/0} \times \vec{v}_1$$

$$
\begin{aligned}
&= -\ddot{\theta}_1 r_{13}\vec{u}_1^{(1)} + \ddot{\theta}_1 r_{11}\vec{u}_3^{(1)} - \dot{\theta}_1\vec{u}_2^{(1)} \times (-\dot{\theta}_1 r_{13}\vec{u}_1^{(1)} + \dot{\theta}_1 r_{11}\vec{u}_3^{(1)}) \\
&= -\ddot{\theta}_1 r_{13}\vec{u}_1^{(1)} + \ddot{\theta}_1 r_{11}\vec{u}_3^{(1)} - \dot{\theta}_1^2 r_{13}\vec{u}_3^{(1)} - \dot{\theta}_1^2 r_{11}\vec{u}_1^{(1)} \\
\Rightarrow\ \vec{a}_1 &= -(\ddot{\theta}_1 r_{13} + \dot{\theta}_1^2 r_{11})\vec{u}_1^{(1)} + (\ddot{\theta}_1 r_{11} - \dot{\theta}_1^2 r_{13})\vec{u}_3^{(1)} \qquad (4.362)
\end{aligned}
$$

Similarly, the position vector of the mass center of Body-2 can be expressed as:

$$
\vec{r}_2 = \vec{r}_{C_2/0} = r_{21}\vec{u}_1^{(2)} + r_{23}\vec{u}_3^{(2)} \text{ since } r_{22} = 0 \qquad (4.363)
$$

By using Coriolis Theorem,

$$
\vec{v}_2 = D_0\vec{r}_2 = D_2\vec{r}_2 + \vec{w}_{2/0} \times \vec{r}_2 \qquad (4.364)
$$

$$
\vec{v}_2 = 0 + (-\dot{\theta}_1 s\theta_2\vec{u}_1^{(2)} - \dot{\theta}_1 c\theta_2\vec{u}_2^{(2)} + \dot{\theta}_2\vec{u}_3^{(2)}) \times (r_{21}\vec{u}_1^{(2)} + r_{23}\vec{u}_3^{(2)}) \qquad (4.365)
$$

$$
\vec{v}_2 = -\dot{\theta}_1 c\theta_2 r_{23}\vec{u}_2^{(2)} + -\dot{\theta}_1 c\theta_2 r_{21}\vec{u}_3^{(2)} - \dot{\theta}_1 c\theta_2 r_{23}\vec{u}_1^{(2)} + \dot{\theta}_2 r_{21}\vec{u}_2^{(2)} \qquad (4.366)
$$

$$
\vec{v}_2 = (-\dot{\theta}_1 c\theta_2 r_{23})\vec{u}_1^{(2)} + (\dot{\theta}_1 s\theta_2 r_{23} + \dot{\theta}_2 r_{21})\vec{u}_2^{(2)} + (\dot{\theta}_1 c\theta_2 r_{21})\vec{u}_3^{(2)} \qquad (4.367)
$$

In order to obtain the acceleration of the mass center of Body-2, we can use Coriolis Theorem again. Therefore,

$$
\vec{a}_2 = D_0\vec{v}_2 = D_2\vec{v}_2 + \vec{w}_{2/0} \times \vec{v}_2 \qquad (4.368)
$$

$$
\begin{aligned}
\vec{a}_2 = {} & (-\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_1 s\theta_2)r_{23}\vec{u}_1^{(2)} + \big((\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2)r_{23} + \ddot{\theta}_2 r_{21}\big)\vec{u}_2 \\
& + (\ddot{\theta}_1 c\theta_2 - \dot{\theta}_1\dot{\theta}_2 s\theta_2)r_{21}\vec{u}_3^{(2)} + (-\dot{\theta}_1 s\theta_2\vec{u}_1^{(2)} - \dot{\theta}_1 c\theta_2\vec{u}_2^{(2)} + \dot{\theta}_2\vec{u}_3^{(2)}) \\
& \times (-\dot{\theta}_1 c\theta_2 r_{23})\vec{u}_1^{(2)} + (\dot{\theta}_1 s\theta_2 r_{23} + \dot{\theta}_2 r_{21})\vec{u}_3^{(2)} \qquad (4.369)
\end{aligned}
$$

$$
\begin{aligned}
\vec{a}_2 = {} & (-\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_1 s\theta_2)r_{23}\vec{u}_1^{(2)} + \big((\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2\big)r_{23} + \ddot{\theta}_2 r_{21})\vec{u}_2 \\
& + (\ddot{\theta}_1 c\theta_2 - \dot{\theta}_1\dot{\theta}_2 s\theta_2)r_{21}\vec{u}_3^{(2)} - \theta_1^2 s^2\theta_2 r_{23} + (\dot{\theta}\dot{\theta}_2 s\theta_2 r_{21})\vec{u}_3^{(2)} - (\dot{\theta}_1^2 s\theta_2 c\theta_2 r_{21})\vec{u}_2^{(2)} \\
& - (\dot{\theta}_1^2 c^2\theta_2 r_{23})\vec{u}_3^{(2)} - (\dot{\theta}_1^2 c^2\theta_2 r_{21})\vec{u}_1^{(2)} - (\dot{\theta}_1 c\theta_2 r_{23})\vec{u}_2^{(2)} - (\dot{\theta}_1\dot{\theta}_2 s\theta_2 r_{23}) + \dot{\theta}_2^2 r_{21})\vec{u}_1^{(2)} \\
& \qquad (4.370)
\end{aligned}
$$

$$
\begin{aligned}
\vec{a}_2 = {} & \vec{u}_1^{(2)}(\ddot{\theta}_1 c\theta_2 r_{23} + \cancel{\dot{\theta}_1\dot{\theta}_2 s\theta_2 r_{23}} - \dot{\theta}_1^2 c^2\theta_2 r_{21} - \cancel{\dot{\theta}_1\dot{\theta}_2 s\theta_2 r_{23}} - \dot{\theta}_2^2 r_{21}) \\
& + \vec{u}_2^{(2)}(\ddot{\theta}_1 s\theta_2 r_{23} + \cancel{\dot{\theta}_1\dot{\theta}_2 c\theta_2 r_{23}} + \ddot{\theta}_2 r_{21} + \dot{\theta}_1^2 s\theta_2 c\theta_2 r_{21} - \cancel{\dot{\theta}_1\dot{\theta}_2 c\theta_2 r_{23}}) \\
& + \vec{u}_3^{(2)}(\dot{\theta}_1 c\theta_2 r_{21} - \dot{\theta}_1\theta_2 s\theta_2 r_{21} - \dot{\theta}_1^2 s^2\theta_2 r_{23} - \dot{\theta}_1\dot{\theta}_2 s\theta_2 r_{21} - \dot{\theta}_1^2 c^2\theta_2 r_{23}) \qquad (4.371)
\end{aligned}
$$

$$
\begin{aligned}
\vec{a}_2 = {} & \vec{u}_1^{(2)}(\ddot{\theta}_1 c\theta_2 r_{23} - \dot{\theta}_1^2 c^2\dot{\theta}_2 r_{21} - \dot{\theta}_2^2 r_{21}) + \vec{u}_2^{(2)}(\ddot{\theta}_1 s\theta_2 r_{23} + \ddot{\theta}_2 r_{21} + \dot{\theta}_1^2 s\theta_2 c\theta_2 r_{21}) \\
& + \vec{u}_3^{(2)}(\dot{\theta}_1 c\theta_2 r_{21} - 2\dot{\theta}_1\dot{\theta}_2 s\theta_2 r_{21} - \dot{\theta}_1^2 r_{23}) \qquad (4.372)
\end{aligned}
$$

Therefore, from (4.362);

$$
a_{c11} = -(\ddot{\theta}_1 r_{13} + \dot{\theta}_1^2 r_{11}) \qquad (4.373)
$$

$$a_{c12} = 0 \tag{4.374}$$

$$a_{c13} = \ddot{\theta}_1 r_{11} + \dot{\theta}_1^2 r_{13} \tag{4.375}$$

Similarly, from (4.372);

$$a_{c21} = -\ddot{\theta}_1 c\theta_2 r_{23} - \dot{\theta}_1^2 c^2\theta_2 r_{21} - \dot{\theta}_1^2 r_{21} \tag{4.376}$$

$$a_{c22} = -\ddot{\theta}_1 s\theta_2 r_{23} - \ddot{\theta}_2 r_{21} + \dot{\theta}_1^2 s\theta_2 c\theta_2 r_{21} \tag{4.377}$$

$$a_{c23} = -\dot{\theta}_1^2 c\theta_2 r_{21} - 2\dot{\theta}_1 \theta_2 s\theta_2 r_{21} - \dot{\theta}_1^2 r_{23} \tag{4.378}$$

Since we have obtained vector forms of Newton-Euler equations through (4.333)-(4.338), we can proceed with deriving scalar equations for each body. For Body 0, scalar force equations can be expressed as follows:

$$F_{G1} - F_{011}^{'} = 0 \;\Rightarrow\; F_{G1} = F_{011}^{'} \tag{4.379}$$

$$F_{G2} - F_{012}^{'} = 0 \;\Rightarrow\; F_{G2} = F_{012}^{'} \tag{4.380}$$

$$F_{G3} - F_{013}^{'} - m_0 g = 0 \tag{4.381}$$

Similarly, we can proceed with moment equations. For body 0, since both $\vec{w}_o$ & $\vec{\alpha}_0$ are 0, the terms in (4.382) become 0 too.

$$\check{J}_0 \cdot \vec{\alpha}_0 + \vec{w}_0 \times \check{J}_0 \cdot \vec{w}_0 = 0 \tag{4.382}$$

Before moving further, we need to determine $\vec{r}_{01}$, $\vec{r}_{12}$ & $\vec{r}_{21}$. In order to do that we need to define $\vec{r}_{0G}$ & $\vec{r}_{01}$:

$$\vec{r}_{0G} = r_{0G1}\vec{u}_1^{(0)} + r_{0G2}\vec{u}_2^{(0)} + r_{0G3}\vec{u}_3^{(0)} \tag{4.383}$$

Since $r_{0G2} = 0$, $\vec{r}_{0G}$ can be rewritten as:

$$\vec{r}_{0G} = r_{0G1}\vec{u}_1^{(0)} + r_{0G3}\vec{u}_3^{(0)} \tag{4.384}$$

Similarly,

$$\vec{r}_{01} = r_{011}\vec{u}_1^{(0)} + r_{012}\vec{u}_2^{(0)} + r_{013}\vec{u}_3^{(0)} \tag{4.385}$$

Since $\vec{r}_{012} = 0$, $\vec{r}_{01}$ can be rewritten as:

$$\vec{r}_{01} = r_{011}\vec{u}_1^{(0)} + r_{013}\vec{u}_3^{(0)} \tag{4.386}$$

Therefore, the terms in the moment equations can be found as;

$$\vec{r}_{0G} \times \vec{F}_{0G} = (r_{0G1}\vec{u}_1^{(0)} + r_{0G3}\vec{u}_3^{(0)}) \times (F_{G1}\vec{u}_1^{(0)} + F_{G2}\vec{u}_2^{(0)} + F_{G3}\vec{u}_3^{(0)})$$

$$= r_{0G1}F_{G2}\vec{u}_3^{(0)} - r_{0G1}F_{G3}\vec{u}_2^{(0)} + r_{0G3}F_{G1}\vec{u}_2^{(0)} - r_{0G3}F_{G2}\vec{u}_1^{(0)}$$
$$= -r_{0G3}F_{G2}u_1^{(0)} + (\vec{r}_{0G3}F_{G1} - \vec{r}_{0G1}F_{G3})\vec{u}_2^{(0)} + r_{0G1}F_{G2}\vec{u}_3^{(0)} \quad (4.387)$$

Similarly,

$$\vec{r}_{01} \times \vec{F}_{01} = -r_{013}F_{012}\vec{u}_1^{(0)} + (r_{013}F_{011} - r_{011}F_{013})\vec{u}_2^{(0)} + r_{011}F_{012}\vec{u}_3^{(0)} \quad (4.388)$$

Hence, the moment equations are given in (4.389) - (4.391).

$$M_{G1} - M_{011}^{'} + r_{0G3}F_{G2} + r_{013}F_{012} = 0 \quad (4.389)$$
$$M_{G2} - M_{012}^{'} + r_{0G3}F_{G1} - r_{0G1}F_{G3} + r_{013}F_{011} + r_{011}F_{013} = 0 \quad (4.390)$$
$$M_{G3} - M_{013}^{'} + r_{0G1}F_{G2} - r_{011}F_{012} = 0 \quad (4.391)$$

We know that,

$$M_{013}^{'} = T_{01} - c_{01}\dot{\theta}_1 \quad (4.392)$$

Combining (4.391) & (4.392);

$$M_{G3} - T_{01} + c_{01}\dot{\theta}_1 + r_{0G1}F_{G2} - r_{011}F_{012} = 0 \quad (4.393)$$

Since we have obtained scalar equations for Body 0, we will continue with obtaining scalar equations for Body 1. To do that, the very first thing we have to do is to express the gravity term in terms of the body coordinate system.

$$m_1\vec{g} = -m_1 g\vec{u}_1^{(0)} \quad (4.394)$$
$$\overline{u}_1^{(b/a)} = \hat{C}^{(a,b)}\overline{u}_t^{(b/a)} \quad (4.395)$$
$$\overline{u}_1^{(0/1)} = \hat{C}^{(1,0)}\overline{u}_1^{(0/0)} \quad (4.396)$$
$$\overline{u}_1^{(0/1)} = e^{-\tilde{u}_3\theta_1}\overline{u}_1 = \overline{u}_1 c\theta_1 - \overline{u}_2 s\theta_1 \quad (4.397)$$
$$\overline{u}_1^{(0/1)} = \overline{u}_1^{(1/1)}c\theta_1 - \overline{u}_2^{(1/1)}s\theta_1 \quad (4.398)$$
$$\vec{u}_1^{(0)} = \vec{u}_1^{(1)}c\theta_1 - \vec{u}_2^{(1)}s\theta_1 \quad (4.399)$$

Inserting (4.399) into (4.394),

$$m_1\vec{g} = -m_1 g\vec{u}_1^{(0)} = -m_1 gc\theta_1\vec{u}_1^{(1)} + m_1 gs\theta_1\vec{u}_2^{(1)} \quad (4.400)$$

Therefore scalar force equations can be expressed in (4.401) - (4.403):

$$F_{011}^{''} - F_{121}^{'} - m_1 gc\theta_1 = m_1 a_{c11} \quad (4.401)$$

$$F''_{012} - F'_{122} - m_1 g s\theta_1 = m_1 a_{c12} \tag{4.402}$$

$$F''_{013} - F'_{123} = m_1 a_{c13} \tag{4.403}$$

Next step is obtaining the scalar moment equations. The inertia dyadic of Body-1 about its mass center C_1 can be expressed as follows:

$$\check{J}_1(C_1) = \sum_{k=1}^{3}\sum_{l=1}^{3} J_{1kl}\ \vec{u}_k^{(1)}\vec{u}_l^{(1)} \tag{4.404}$$

when we expand (4.404),

$$\begin{aligned}\check{J}_1(C_1) &= J_{111}\ \vec{u}_1^{(1)}\vec{u}_l^{(1)} + J_{112}\ \vec{u}_1^{(1)}\vec{u}_2^{(1)} + J_{113}\ \vec{u}_1^{(1)}\vec{u}_3^{(1)} \\ &+ J_{121}\ \vec{u}_2^{(1)}\vec{u}_l^{(1)} + J_{122}\ \vec{u}_2^{(1)}\vec{u}_2^{(1)} + J_{123}\ \vec{u}_2^{(1)}\vec{u}_3^{(1)} \\ &+ J_{131}\ \vec{u}_3^{(1)}\vec{u}_l^{(1)} + J_{132}\ \vec{u}_3^{(1)}\vec{u}_2^{(1)} + J_{133}\ \vec{u}_3^{(1)}\vec{u}_3^{(1)}\end{aligned} \tag{4.405}$$

Also from (4.350), we know that

$$\vec{w}_1 = -\dot{\theta}_1\vec{u}_2^{(1)} \quad , \quad \vec{\alpha}_1 = -\ddot{\theta}_1\vec{u}_2^{(1)}$$

Then,

$$\check{J}_1 \cdot \vec{\alpha}_1 = -\left(J_{112}\vec{u}_1^{(1)} + J_{122}\vec{u}_2^{(1)} + J_{132}\vec{u}_3^{(1)}\right)\ddot{\theta}_1 \tag{4.406}$$

$$\begin{aligned}\vec{w}_1 \times \check{J}_1 \cdot \vec{w}_1 &= -\dot{\theta}_1\vec{u}_2^{(1)} \times -\left(J_{112}\vec{u}_1^{(1)} + J_{122}\vec{u}_2^{(1)} + J_{132}\vec{u}_3^{(1)}\right)\dot{\theta}_1 \\ &= \dot{\theta}_1^2\left(-J_{112}\vec{u}_3^{(1)} + J_{132}\vec{u}_1^{(1)}\right)\end{aligned} \tag{4.407}$$

Combining (4.406) & (4.407), the left hand of the moment equation can be expressed as:

$$\check{J}_1 \cdot \vec{\alpha} + \vec{w}_1 \times \check{J}_1 \cdot \vec{w} = \vec{u}_1^{(1)}\left(-J_{112}\ddot{\theta}_1 + J_{132}\dot{\theta}_1^2\right) - \vec{u}_2^{(1)}\left(J_{132}\ddot{\theta}_1 + J_{112}\dot{\theta}_1^2\right) \tag{4.408}$$

The moment terms due to reaction forces can be expressed as:

$$\begin{aligned}\vec{r}_{10} \times \vec{F}_{01} &= \left(\vec{r}_{101}\vec{u}_1^{(1)} + \vec{r}_{102}\vec{u}_2^{(1)} + \vec{r}_{103}\vec{u}_3^{(1)}\right) \times \left(\vec{F}''_{011}\vec{u}_1^{(1)} + \vec{F}''_{012}\vec{u}_2^{(1)} + \vec{F}''_{013}\vec{u}_3^{(1)}\right) \\ &= \vec{r}_{101}\vec{F}''_{012}\vec{u}_3^{(1)} - \vec{r}_{101}\vec{F}''_{013}\vec{u}_2^{(1)} - \vec{r}_{102}\vec{F}''_{013}\vec{u}_1^{(1)} + \vec{r}_{103}\vec{F}''_{011}\vec{u}_2^{(1)} - \vec{r}_{103}\vec{F}''_{012}\vec{u}_1^{(1)}\end{aligned}$$

$$\begin{aligned}\Rightarrow \vec{r}_{10} \times \vec{F}_{01} &= \vec{u}_1^{(1)}\left(\vec{r}_{102}\vec{F}''_{013} - \vec{r}_{103}\vec{F}''_{012}\right) + \vec{u}_2^{(1)}\left(\vec{r}_{103}\vec{F}''_{013} - \vec{r}_{101}\vec{F}''_{013}\right) \\ &+ \vec{u}_3^{(1)}\left(\vec{r}_{101}\vec{F}''_{012} - \vec{r}_{102}\vec{F}''_{011}\right)\end{aligned} \tag{4.409}$$

Similarly,

$$\vec{r}_{12} \times \vec{F}_{12} = \vec{u}_1^{(1)}\left(\vec{r}_{122}\vec{F}'_{123} - \vec{r}_{123}\vec{F}'_{122}\right) + \vec{u}_2^{(1)}\left(\vec{r}_{123}\vec{F}''_{121} - \vec{r}_{121}\vec{F}'_{123}\right)$$

$$+\vec{u}_3^{(1)}\left(\vec{r}_{121}\vec{F}'_{122}-\vec{r}_{122}\vec{F}'_{121}\right) \tag{4.410}$$

Therefore, we can proceed by obtaining the following scalar moment equations of Body 1 in (4.411) - (4.413)

$$-J_{112}\ddot{\theta}_1+J_{132}\ddot{\theta}_1^2=M''_{011}-M'_{121}+r_{102}F''_{013}-r_{103}F''_{012}-r_{122}F'_{123}+r_{123}F'_{122} \tag{4.411}$$

$$-J_{122}\ddot{\theta}_1=M''_{012}-M'_{122}+r_{103}F''_{011}-r_{101}F''_{013}-r_{123}F'_{121}+r_{121}F'_{123} \tag{4.412}$$

$$J_{132}\ddot{\theta}_1+J_{112}\ddot{\theta}_1^2=M''_{013}-M'_{123}+r_{101}F''_{012}-r_{102}F''_{011}-r_{121}F'_{122}+r_{122}F'_{121} \tag{4.413}$$

We know that

$$M'_{123}=T_{12}-c_{12}\dot{\theta}_2 \tag{4.414}$$

Combining (4.411) & (4.414)

$$J_{123}\ddot{\theta}_1-J_{112}\dot{\theta}_1^2=M''_{013}-T_{12}+c_{12}\dot{\theta}_2+r_{101}F''_{012}-r_{102}F''_{011}-r_{121}F'_{122}+r_{122}F'_{121} \tag{4.415}$$

Since we have obtained scalar equations of Body 1, we can continue with working on obtaining the scalar equations of Body 2. As we have done previously through (4.394) - (4.399); at first we need to express the gravity term in terms of the components of the Body-2 Reference Frame.

Thus,

$$m_2\vec{g}=m_2g\vec{u}^{(0)} \tag{4.416}$$

$$\overline{u}_k^{(b/a)}=\hat{C}^{(a,b)}\overline{u}_k^{(b/b)} \tag{4.417}$$

$$\overline{u}_1^{(0/2)}=\hat{C}^{(2,0)}\overline{u}_1^{(0/0)} \tag{4.418}$$

We know that

$$\hat{C}^{(0,2)}=e^{\tilde{u}_3\theta_1}\ e^{\tilde{u}_2\theta_2}\ e^{-\tilde{u}_1\pi/2}\ \Rightarrow\ \hat{C}^{(2,0)}=\ e^{\tilde{u}_1\pi/2}\ e^{\tilde{u}_2\theta_2}\ e^{\tilde{u}_3\theta_1} \tag{4.419}$$

From (4.418) & (4.419),

$$\overline{u}_1^{(0/2)}=e^{\tilde{u}_1\pi/2}\,e^{-\tilde{u}_2\theta_2}\,e^{-\tilde{u}_3\theta_1}\,\overline{u}_1 \tag{4.420}$$

$$\overline{u}_1^{(0/2)}=e^{\tilde{u}_1\pi/2}\,e^{-\tilde{u}_2\theta_2}(\overline{u}_1c\theta_1-\overline{u}_2s\theta_1) \tag{4.421}$$

$$\overline{u}_1^{(0/2)} = e^{\tilde{u}_1\pi/2}\left(c\theta_1(\overline{u}_1 c\theta_2 + \overline{u}_3 s\theta_2) - \overline{u}_2 s\theta_1\right) \tag{4.422}$$

$$\overline{u}_1^{(0/2)} = e^{\tilde{u}_1\pi/2}(\overline{u}_1 c\theta_1 c\theta_2 - \overline{u}_2 s\theta_1 + \overline{u}_3 c\theta_1 s\theta_2) \tag{4.423}$$

$$\overline{u}_1^{(0/2)} = \overline{u}_1 c\theta_1 c\theta_2 - s\theta_1(\underbrace{\overline{u}_2 c(\pi/2)}_{0} + \overline{u}_3 s(\pi/2)) + c\theta_1 s\theta_2(\underbrace{\overline{u}_3 c(\pi/2)}_{0} - \overline{u}_2 s(\pi/2)) \tag{4.424}$$

$$\overline{u}_1^{(0/2)} = \overline{u}_1 c\theta_1 c\theta_2 - \overline{u}_3 s\theta_1 - \overline{u}_2 c\theta_1 c\theta_2 \tag{4.425}$$

$$\overline{u}_1^{(0/2)} = \overline{u}_1 c\theta_1 c\theta_2 - \overline{u}_2 c\theta_1 s\theta_2 - \overline{u}_3 s\theta_1 \tag{4.426}$$

$$\overline{u}_1^{(0/2)} = \overline{u}_1^{(2/2)} c\theta_1 c\theta_2 - \overline{u}_2^{(2/2)} c\theta_1 s\theta_2 - \overline{u}_3^{(2/2)} s\theta_1 \tag{4.427}$$

$$\vec{u}_1^{(0)} = \vec{u}_1^{(2)} c\theta_1 c\theta_2 - \vec{u}_2^{(2)} c\theta_1 s\theta_2 - \vec{u}_3^{(2)} s\theta_1 \tag{4.428}$$

Inserting (4.428) into (4.416) gives us the following result:

$$m_2\vec{g} = m_2 g\vec{u}_1^{(0)} = -m_2 g c\theta_1 c\theta_2 \vec{u}_1^{(2)} + m_2 g c\theta_1 s\theta_2 \vec{u}_2^{(2)} + m_2 g s\theta_1 \vec{u}_3^{(2)} \tag{4.429}$$

The scalar force equations of Body-2 can be expressed in

$$m_2 a_{c21} = F''_{121} - m_2 g c\theta_1 c\theta_2 \tag{4.430}$$

$$m_2 a_{c22} = F''_{122} - m_2 g c\theta_1 s\theta_2 \tag{4.431}$$

$$m_2 a_{c23} = F''_{123} - m_2 g s\theta_1 \tag{4.432}$$

Since the scalar force equations are obtained, we can proceed by obtaining the scalar moment equations. Similar to Body-1, the inertia dyadic of Body-2 with respect to its mass center C_2 and aligned with the unit rectors of $F_2(C_2)$ can be expressed as following:

$$\check{J}_2(C_2) = \sum_{k=1}^{3}\sum_{l=1}^{3} J_{2kl}\, \vec{u}_k^{(2)}\vec{u}_l^{(2)} \tag{4.433}$$

When we expand (4.433):

$$\begin{aligned}\check{J}_2(C_2) = {} & J_{211}\vec{u}_1^{(2)}\vec{u}_1^{(2)} + J_{212}\vec{u}_1^{(2)}\vec{u}_2^{(2)} + J_{213}\vec{u}_1^{(2)}\vec{u}_3^{(2)} \\ & + J_{221}\vec{u}_2^{(2)}\vec{u}_1^{(2)} + J_{222}\vec{u}_2^{(2)}\vec{u}_2^{(2)} + J_{223}\vec{u}_2^{(2)}\vec{u}_3^{(2)} \\ & + J_{231}\vec{u}_3^{(2)}\vec{u}_1^{(2)} + J_{232}\vec{u}_3^{(2)}\vec{u}_2^{(2)} + J_{233}\vec{u}_3^{(2)}\vec{u}_3^{(2)}\end{aligned} \tag{4.434}$$

Remind that in (4.358) and (4.359) we have found that:

$$\vec{w}_2 = -\dot{\theta}_1 s\theta_2 \vec{u}_1^{(2)} - \dot{\theta}_1 c\theta_2 \vec{u}_2^{(2)} + \dot{\theta}_2 \vec{u}_3^{(2)}$$

$$\vec{\alpha}_2 = -(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2)\vec{u}_1^{(2)} - (\ddot{\theta}_1 c\theta_2 - \dot{\theta}_1\dot{\theta}_2 s\theta_2)\vec{u}_2^{(2)} + \ddot{\theta}_2\, \vec{u}_3^{(2)}$$

Using (4.434),(4.358) & (4.359), the moment terms can be obtained as following :

$$\begin{aligned}\check{J}_2 \cdot \vec{\alpha}_2 = &- J_{211}(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2)\vec{u}_1^{(2)} - J_{221}(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2)\vec{u}_2^{(2)} \\ &- J_{231}(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2)\vec{u}_3^{(2)} - J_{212}(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_2 s\theta_2)\vec{u}_1^{(2)} \\ &- J_{222}(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_2 s\theta_2)\vec{u}_2^{(2)} - J_{232}(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_2 s\theta_2)\vec{u}_3^{(2)} \\ &+ J_{213}\ddot{\theta}_2\vec{u}_1^{(2)} + J_{223}\ddot{\theta}_2\vec{u}_2^{(2)} + J_{233}\ddot{\theta}_2\vec{u}_3^{(2)} \end{aligned} \tag{4.435}$$

Rearranging the terms in (4.436),

$$\begin{aligned}\check{J}_2 \cdot \vec{\alpha}_2 =& \vec{u}_1^{(2)}\big(- J_{211}(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2) - J_{212}(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_2 s\theta_2) + J_{213}\ddot{\theta}_2\big) \\ &+\vec{u}_2^{(2)}\big(- J_{221}(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2) - J_{222}(\ddot{\theta}_1 c\theta_2 - \dot{\theta}_1\dot{\theta}_2 s\theta_2) + J_{223}\theta_2\big) \\ &+\vec{u}_3^{(2)}\big(- J_{231}(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2) - J_{232}(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_2 s\theta_2) + J_{233}\theta_2\big) \end{aligned} \tag{4.436}$$

The second moment term is $\vec{w}_2 \times \check{J}_2 \cdot \vec{w}_2$; this term is a long one so we will expand it in two parts:

$$\begin{aligned}\check{J}_2 \cdot \vec{w}_2 = &- J_{211}(\dot{\theta}_1 s\theta_2)\vec{u}_1^{(2)} - J_{221}(\dot{\theta}_1 s\theta_2)\vec{u}_2^{(2)} - J_{231}(\dot{\theta}_1 s\theta_2)\vec{u}_3^{(2)} \\ &- J_{212}(\dot{\theta}_1 c\theta_2)\vec{u}_1^{(2)} - J_{222}(\dot{\theta}_1 c\theta_2)\vec{u}_2^{(2)} - J_{232}(\dot{\theta}_1 c\theta_2)\vec{u}_3^{(2)} \\ &+ J_{213}\dot{\theta}_2\vec{u}_1^{(2)} + J_{223}\dot{\theta}_2\vec{u}_2^{(2)} + J_{233}\dot{\theta}_2\vec{u}_3^{(2)} \end{aligned} \tag{4.437}$$

Rearranging the terms in (4.437), we'll obtain the following:

$$\begin{aligned}\dot{J} \cdot \vec{w}_2 = &(-J_{211}(\dot{\theta}_1 s\theta_2) - J_{212}(\dot{\theta}_1 c\theta_2) + J_{213}\dot{\theta}_2)\vec{u}_1^{(2)} \\ &+ (-J_{221}(\dot{\theta}_1 s\theta_2) - J_{222}(\dot{\theta}_1 c\theta_2) + J_{223}\dot{\theta}_2)\vec{u}_2^{(2)} \\ &+ (-J_{231}(\dot{\theta}_1 s\theta_2) - J_{232}(\dot{\theta}_1 c\theta_2) + J_{233}\dot{\theta}_2)\vec{u}_3^{(2)} \end{aligned} \tag{4.438}$$

$$\begin{aligned}\vec{w}_2 \times \check{J} \cdot \vec{w}_2 = &(-\dot{\theta}_1 s\theta_2)\big(- J_{221}(-\dot{\theta}_1 s\theta_2) - J_{222}(-\dot{\theta}_1 c\theta_2) + J_{223}\dot{\theta}_2\big)\vec{u}_3^{(2)} \\ &- (-\dot{\theta}_1 s\theta_2)\big(- J_{231}(\dot{\theta}_1 s\theta_2) - J_{232}(\dot{\theta}_1 c\theta_2) + J_{233}\dot{\theta}_2\big)\vec{u}_2^{(2)} \\ &- (-\dot{\theta}_1 c\theta_2)\big(- J_{211}(\dot{\theta}_1 s\theta_2) - J_{212}(\dot{\theta}_1 c\theta_2) + J_{213}\dot{\theta}_2\big)\vec{u}_3^{(2)} \\ &+ (-\dot{\theta}_1 c\theta_2)\big(- J_{231}(\dot{\theta}_1 s\theta_2) - J_{232}(\dot{\theta}_1 c\theta_2) + J_{233}\dot{\theta}_2\big)\vec{u}_1^{(2)} \\ &+ \dot{\theta}_2\big(- J_{211}(\dot{\theta}_1 s\theta_2) - J_{212}(\dot{\theta}_1 c\theta_2) + J_{213}\dot{\theta}_2\big)\vec{u}_2^{(2)} \\ &- \dot{\theta}_2\big(- J_{221}(\dot{\theta}_1 s\theta_2) - J_{222}(\dot{\theta}_1 c\theta_2) + J_{223}\dot{\theta}_2\big)\vec{u}_1^{(2)} \end{aligned} \tag{4.439}$$

Collecting terms in (4.439),

$$\vec{w}_2 \times \check{J} \cdot \vec{w}_2 = (J_{221}\dot{\theta}_1^2 s^2\theta_2 + J_{222}\dot{\theta}_1^2 s\theta_2 c\theta_2 - J_{223}\dot{\theta}_1\dot{\theta}_2 s_2 c\theta_2$$

$$- J_{211}\dot{\theta}_1^2 s\theta_2 c\theta_2 - J_{212}\dot{\theta}_1^2 c^2\theta_2 + J_{213}\dot{\theta}\dot{\theta}_2 c\theta_2)\vec{u}_3^{(2)}$$
$$+ (-J_{231}\dot{\theta}_1^2 s^2\theta_2 - J_{232}\dot{\theta}_1^2 s\theta_2 c\theta_2 + J_{233}\dot{\theta}_1\dot{\theta}_2 sc\theta_2$$
$$- J_{211}\dot{\theta}_1\theta_2 s\theta_2 - J_{212}\dot{\theta}_1\theta_2 c\theta_2 + J_{213}\dot{\theta}_2^2\dot{\theta}_2 c\theta_2)\vec{u}_2^{(2)}$$
$$+ (-J_{231}\dot{\theta}_1^2 s\theta_2 c\theta_2 + J_{232}\dot{\theta}_1^2 c^2\theta_2 - J_{233}\dot{\theta}_1\dot{\theta} c\theta_2$$
$$+ J_{221}\dot{\theta}_1\theta_2 s\theta_2 - J_{222}\dot{\theta}_1\theta_2 c\theta_2 - J_{223}\dot{\theta}_2^2)\vec{u}_1^{(2)} \tag{4.440}$$

The moment term due to reaction force F_{21} is analogous to the term of "$\vec{r}_{10} \times \vec{F}_{01}$". Therefore, we can express this term directly such as:

$$\vec{r}_{21} \times \vec{F}_{12} = \vec{u}_1^{(2)}(r_{211}\ F''_{123} - r_{213}\ F''_{122}) + \vec{u}_2^{(2)}(r_{213}\ F''_{121} - r_{211}\ F''_{123})$$
$$+ \vec{u}_3^{(2)}(r_{211}\ F''_{122} - r_{212}\ F''_{121}) \tag{4.441}$$

Remind that the Newton-Euler moment equation for Body 2 found in (4.338) was:

$$\check{J}_2.\vec{w}_2 + \vec{w} \times \check{J}_2.\vec{w}_2 = \vec{M}_{12} + \vec{r}_{21} \times \vec{F}_{12}$$

Therefore using (4.436), (4.440) & (4.441), we can write the moment equations as follows:

$$\Rightarrow - J_{211}(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2) - J_{212}(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1\dot{\theta}_2 s\theta_2) + J_{213}(\ddot{\theta}_2) + J_{231}(\dot{\theta}_1^2 s\theta_2 c\theta_2)$$
$$+ J_{232}(\dot{\theta}_1^2 c^2\theta_2 - J_{233}\dot{\theta}_1\dot{\theta}_2 c\theta_2 + J_{221}\dot{\theta}_1\dot{\theta}_2 s\theta_2 + J_{222}\dot{\theta}_1\dot{\theta}_2 c\theta_2 - J_{223}\dot{\theta}_2^2$$
$$= M''_{121} - r_{212}\ F''_{123} - r_{213}F''_{122} \tag{4.442}$$

$$\Rightarrow - J_{221}(\ddot{\theta}_1 s\theta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2) - J_{222}(\ddot{\theta}_1 c\theta_2 - \dot{\theta}_1\dot{\theta}_2 s\theta_2 + J_{223}(\ddot{\theta}_2) - J_{231}(\dot{\theta}_1^2 s\theta_2 c\theta_2)$$
$$- J_{232}(\dot{\theta}_1^2 s\theta_2) + (J_{233} - J211(\dot{\theta}_1\dot{\theta}_2 c\theta_2) - J_{212}\dot{\theta}_1\dot{\theta}_2 c\theta_2 - J_{231}\dot{\theta}_2^2$$
$$= M''_{122} + r_{213}\ F''_{121} - r_{211}F''_{123} \tag{4.443}$$

$$\Rightarrow - J_{231}(\ddot{\theta}_1 s\theta_2 - \dot{\theta}_1\dot{\theta}_2 c\theta_2) - J_{232}(\ddot{\theta}_1 c\theta_2 - \dot{\theta}_1\dot{\theta}_2 s\theta_2) + J_{233}(\ddot{\theta}_2) + J_{221}(\dot{\theta}_1^2 s^2\theta_2)$$
$$+ J_{222}(\dot{\theta}_1^2 s\theta_2 c\theta_2) + J_{223}(\dot{\theta}_1\dot{\theta}_2 s\theta_2) - J_{211}\dot{\theta}_1^2 s\dot{\theta}_2 c\theta_2 - J_{212}\dot{\theta}_1^2 c^2\theta_2 + J_2 13(\dot{\theta}_1\dot{\theta}_2\theta_2$$
$$= M''_{123} + r_{211}\ F''_{122} - r_{212}F''_{121} \tag{4.444}$$

Finally, in order to reduce the number of unknowns, we need to relate the representation of forces and moments in different frames.

Remind that in (4.339), we have found that:

$$\vec{F}_{01} = \sum_{k=1}^{3} F^{'}_{01k}\vec{u}_k^{(0)} = \sum_{k=1}^{3} F^{''}_{01k}\vec{u}_k^{(1)}; \quad \vec{M}_{01} = \sum_{k=1}^{3} M^{'}_{01k}\vec{u}_k^{(0)} = \sum_{k=1}^{3} M^{''}_{01k}\vec{u}_k^{(1)}$$

Expanding (4.339),

$$F^{'}_{011}\vec{u}_1^{(0)} + F^{'}_{012}\vec{u}_2^{(0)} + F^{'}_{013}\vec{u}_3^{(0)} = F^{''}_{011}\vec{u}_1^{(1)} + F^{''}_{012}\vec{u}_2^{(1)} + F^{''}_{013}\vec{u}_3^{(1)} \tag{4.445}$$

If we observe (4.445) in $F_0(0)$;

$$F^{'}_{011}\overline{u}_1^{(0/0)} + F^{'}_{012}\overline{u}_2^{(0/0)} + F^{'}_{013}\overline{u}_3^{(0/0)} = F^{''}_{011}\overline{u}_1^{(1/0)} + F^{''}_{012}\overline{u}_2^{(1/0)} + F^{''}_{013}\overline{u}_3^{(1/0)} \tag{4.446}$$

From (4.342), we know that

$$\begin{aligned}\overline{u}_k^{(b/a)} &= \hat{C}^{(a,b)}\overline{u}_k^{(b/b)} \\ \overline{u}_k^{(1/0)} &= \hat{C}^{(0,1)}\overline{u}_k^{(1/1)} \Rightarrow \overline{u}_k^{(1/0)} = \hat{C}^{(0,1)}\overline{u}_k^{(0/0)}\end{aligned} \tag{4.447}$$

Thus, from (4.447);

$$\overline{u}_1^{(1/0)} = \hat{C}^{(0,1)}\overline{u}_1^{(0/0)}, \quad \overline{u}_2^{(2/0)} = \hat{C}^{(0,1)}\overline{u}_2^{(0/0)}, \quad \overline{u}_3^{(3/0)} = \hat{C}^{(0,3)}\overline{u}_3^{(0/0)} \tag{4.448}$$

Inserting (4.448) into (4.463);

$$\begin{aligned}F^{'}_{011}\overline{u}_1^{(0/0)} + F^{'}_{012}\overline{u}_2^{(0/0)} + F^{'}_{013}\overline{u}_3^{(0/0)} &= F^{''}_{011}\hat{C}^{(0,1)}\overline{u}_1^{(0/1)} + F^{''}_{012}\hat{C}^{(0,1)}\overline{u}_2^{(0/0)} \\ &\quad - F^{''}_{013}\hat{C}^{(0,1)}\overline{u}_3^{(0/0)}\end{aligned} \tag{4.449}$$

$$\Rightarrow F^{'}_{011}\overline{u}_1 + F^{'}_{012}\overline{u}_2 + F^{'}_{013}\overline{u}_3 = e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_1\pi/2}(F^{''}_{011}\overline{u}_1 + F^{''}_{012}\overline{u}_2 + F^{''}_{013}\overline{u}_3) \tag{4.450}$$

Expanding each multiplication, we'll obtain the following expressions:

$$\Rightarrow e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_1\pi/2}\overline{u}_1 = e^{\tilde{u}_3\theta_1}\overline{u}_1 = \overline{u}_1 c\theta_1 + \overline{u}_2 s\theta_1 \tag{4.451}$$

$$\Rightarrow e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_1\pi/2}\overline{u}_1 = e^{\tilde{u}_3\theta_1}(\overline{u}_2 c\pi/2 - \overline{u}_3 s\pi/2) = -\overline{u}_3 \tag{4.452}$$

$$\Rightarrow e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_1\pi/2}\overline{u}_3 = e^{\tilde{u}_3\theta_1}(\overline{u}_3 c\pi/2 + \overline{u}_2 s\pi/2) = e^{\tilde{u}_3\theta_1}\overline{u}_2 = \overline{u}_2 c\theta_1 - \overline{u}_1 s\theta_1 \tag{4.453}$$

Combining (4.450)-(4.453);

$$\begin{aligned}F^{'}_{011}\overline{u}_1 + F^{'}_{012}\overline{u}_2 + F^{'}_{013}\overline{u}_3 &= F^{'}_{011}(\overline{u}_1 c\theta_1 + \overline{u}_2 c\theta_1) - F^{'}_{012}\overline{u}_3 \\ &\quad + F^{'}_{013}(\overline{u}_2 c\theta_1 - \overline{u}_1 s\theta_1)\end{aligned} \tag{4.454}$$

$$F^{'}_{011}\overline{u}_1 + F^{'}_{012}\overline{u}_2 + F^{'}_{013}\overline{u}_3 = \overline{u}_1(F^{''}_{011}c\theta_1 + F^{''}_{013}c\theta_1)$$

$$+ \overline{u}_2(F^{''}_{011}s\theta_1 + F^{''}_{013}c\theta_1) - \overline{u}_3F^{''}_{012} \tag{4.455}$$

$$\Rightarrow \; F^{'}_{011} = F^{''}_{011}c\theta_1 - F^{''}_{013}c\theta_1 \tag{4.456}$$

$$\Rightarrow \; F^{'}_{012} = F^{''}_{011}s\theta_1 - F^{''}_{013}c\theta_1 \tag{4.457}$$

$$\Rightarrow \; F^{'}_{013} = F^{''}_{012} \tag{4.458}$$

Analogously,

$$\Rightarrow \; M^{'}_{011} = M^{''}_{011}c\theta_1 - M^{''}_{013}s\theta_1 \tag{4.459}$$

$$\Rightarrow \; M^{'}_{012} = M^{''}_{011}s\theta_1 + M^{''}_{013}c\theta_1 \tag{4.460}$$

$$\Rightarrow \; M^{'}_{013} = -M^{''}_{012} \tag{4.461}$$

Also remind that in (4.340), we have defined that:

$$\vec{F}_{12} = \sum_{k=1}^{3} F^{'}_{12k}\vec{u}^{(1)}_k = \sum_{k=1}^{3} F^{''}_{12k}\vec{u}^{(2)}_k; \quad \vec{M}_{12} = \sum_{k=1}^{3} M^{'}_{12k}\vec{u}^{(1)}_k = \sum_{k=1}^{3} M^{''}_{12k}\vec{u}^{(2)}_k$$

Expanding (4.340),

$$\vec{F}_{12} = F^{'}_{121}\vec{u}^{(1)}_1 + F^{'}_{122}\vec{u}^{(2)}_1 + F^{'}_{123}\vec{u}^{(3)}_1 = F^{''}_{121}\vec{u}^{(2)}_1 + F^{''}_{122}\vec{u}^{(2)}_2 + F^{''}_{123}\vec{u}^{(2)}_3 \tag{4.462}$$

If we observe (4.462) in $F_1(0)$;

$$F^{'}_{121}\overline{u}^{(1/1)}_1 + F^{'}_{122}\overline{u}^{(1/1)}_2 + F^{'}_{123}\overline{u}^{(1/1)}_3 = F^{''}_{121}\overline{u}^{(2/1)}_1 + F^{''}_{122}\overline{u}^{(2/1)}_2 + F^{''}_{123}\overline{u}^{(2/1)}_3 \tag{4.463}$$

From (4.342), we know that

$$\begin{aligned} \overline{u}^{(b/c)}_k &= \hat{C}^{(a,b)}\overline{u}^{(b/b)}_k \\ \overline{u}^{(2/1)}_k &= \hat{C}^{(a,b)}\overline{u}^{(2/2)}_k \Rightarrow \overline{u}^{(2/1)}_k = \hat{C}^{(1,2)}\overline{u}^{(1/1)}_k \end{aligned} \tag{4.464}$$

Thus, from (4.464);

$$\overline{u}^{(2/1)}_1 = \hat{C}^{(1,2)}\overline{u}^{(1/1)}_1, \quad \overline{u}^{(2/1)}_2 = \hat{C}^{(1,2)}\overline{u}^{(1/1)}_2, \quad \overline{u}^{(2/1)}_3 = \hat{C}^{(1,2)}\overline{u}^{(1/1)}_3 \tag{4.465}$$

Inserting (4.465) into (4.463),

$$\begin{aligned} F^{'}_{121}\overline{u}^{(1/1)}_1 + F^{'}_{122}\overline{u}^{(1/1)}_2 + F^{'}_{123}\overline{u}^{(1/1)}_3 &= F^{''}_{121}\hat{C}^{(1,2)}\overline{u}^{(1/1)}_1 + F^{''}_{122}\hat{C}^{(1,2)}\overline{u}^{(1/1)}_2 \\ &\quad + F^{''}_{123}\hat{C}^{(1,2)}\overline{u}^{(1/1)}_3 \end{aligned} \tag{4.466}$$

$$F^{'}_{121}\overline{u}_1 + F^{'}_{122}\overline{u}_2 + F^{'}_{123}\overline{u}_3 = e^{\tilde{u}^3\theta_2}(F^{''}_{121}\overline{u}_1 + F^{''}_{122}\overline{u}_2 + F^{''}_{123}\overline{u}_3) \tag{4.467}$$

$$\Rightarrow \; e^{\tilde{u}^3\theta_2}\overline{u}_1 = \overline{u}_1c\theta_2 + \overline{u}_2s\theta_2 \tag{4.468}$$

$$\Rightarrow \ e^{\tilde{u}^3\theta_2}\overline{u}_2 = \overline{u}_2 c\theta_2 - \overline{u}_1 s\theta_2 \tag{4.469}$$

$$\Rightarrow \ e^{\tilde{u}^3\theta_2}\overline{u}_2 = \overline{u}_3 \tag{4.470}$$

Combining (4.467)-(4.470),

$$F^{'}_{121}\overline{u}_1 + F^{'}_{122}\overline{u}_2 + F^{'}_{123}\overline{u}_3 = F^{''}_{121}(\overline{u}_1 c\theta_2 + \overline{u}_2 s\theta_2) + F^{''}_{122}(\overline{u}_2 c\theta_2 + \overline{u}_1 s\theta_2) + F^{''}_{123}\overline{u}_3 \tag{4.471}$$

$$\begin{aligned} F^{'}_{121}\overline{u}_1 + F^{'}_{122}\overline{u}_2 + F^{'}_{123}\overline{u}_3 = {} & \overline{u}_1(F^{''}_{121} c\theta_2 - F^{''}_{122} s\theta_2) + \overline{u}_2(F^{''}_{121} s\theta_2 + F^{''}_{122} c\theta_2) \\ & + F^{''}_{123}\overline{u}_3 \end{aligned} \tag{4.472}$$

$$\Rightarrow \ F^{'}_{121} = F^{''}_{121} c\theta_2 - F^{''}_{122} s\theta_2 \tag{4.473}$$

$$\Rightarrow \ F^{'}_{122} = F^{''}_{121} c\theta_2 + F^{''}_{122} c\theta_2 \tag{4.474}$$

$$\Rightarrow \ F^{'}_{123} = F^{''}_{123} \tag{4.475}$$

Analogously,

$$\Rightarrow \ M^{'}_{121} = M^{''}_{121} c\theta_2 - M^{''}_{122} s\theta_2 \tag{4.476}$$

$$\Rightarrow \ M^{'}_{122} = M^{''}_{121} s\theta_2 + M^{''}_{122} c\theta_2 \tag{4.477}$$

$$\Rightarrow \ M^{'}_{121} = M^{''}_{123} \tag{4.478}$$

When we compare the number of unknowns with the number of knowns; we can see that they are equal. Therefore, the set of equations are solvable. We can make a list which shows the unknown parameters and system inputs for both inverse and forward dynamics as follows:

Inverse Dynamics

Knowns: $M_{013}^{'}, M_{123}^{'}$

Unknowns: (18 unknowns)

$$F_{G1}, F_{G2}, F_{G3}; F_{011}^{''}, F_{012}^{''}, F_{013}^{''};$$
$$F_{121}^{''}, F_{122}^{''}, F_{123}^{''};$$
$$M_{G1}, M_{G2}, M_{G3};$$
$$M_{011}^{'}, M_{012}^{'}; M_{121}^{''}, M_{122}^{''};$$
$$\ddot{\theta}_1, \ddot{\theta}_2$$

Main Equation

(4.379-4.381), (4.389-4.381),

(4.401-4.403), (4.411-4.411),

(4.430-4.432), (4.442-4.444)

Forward Dynamics

Knowns: $\ddot{\theta}_1, \ddot{\theta}_2$

Unknowns: (18 unknowns)

$$F_{G1}, F_{G2}, F_{G3}; F_{011}^{''}, F_{012}^{''}, F_{013}^{''};$$
$$F_{121}^{''}, F_{122}^{''}, F_{123}^{''};$$
$$M_{G1}, M_{G2}, M_{G3};$$
$$M_{011}^{'}, M_{012}^{'}; M_{121}^{''}, M_{122}^{''};$$
$$\ddot{\theta}_1, \ddot{\theta}_2$$

Auxiliary Equations

(4.373)-(4.378),

(4.456)-(4.461),

(4.475)-(4.478)

According to the friction assumption we make, the solution method can change slightly. Note that when friction is not ignored, $M_{013}^{'}$ & $M_{123}^{'}$ are dependent on $\dot{\theta}_1$ & $\dot{\theta}_2$. Therefore, 18 knowns and 18 equations matrix solution is not possible; hence recursive solution must be applied. Otherwise if frictionless model is used, linear matrix solution is applicable!

4.9 Lagrange Equations for Stabilizer

General formula for Lagrange equations is:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right) - \frac{\partial K}{\partial q_k} + \frac{\partial U}{\partial q_k} = Q_k \text{ for } k = 1, 2 \tag{4.479}$$

Where K is the total kinetic energy and U is the total potential energy of the system.

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2}\left(\sum_{i=1}^{n} m_i \vec{v_i} \cdot \vec{v_i} + \sum_{i=1}^{n} \vec{\omega_i} \cdot \breve{J}_i \cdot \vec{\omega_i}\right) \text{ and } U = -\sum_{i=1}^{n} m_i \vec{g} \cdot \vec{r_i} \tag{4.480}$$

Where $\vec{r_i}$ is the vector from origin of the inertial frame to the mass center of i^{th} body.

$$q_1 = \theta_5;\ q_2 = \theta_6;\ Q_1 = \tau_5;\ Q_2 = \tau_6 \tag{4.481}$$

There are $n = 3$ bodies:

4.9.1 Body-5

The coordinate transformation matrix between F_0 and F_5 is:

$$\hat{C}^{(0,5)} = e^{\tilde{u}_2\theta_{131}} e^{\tilde{u}_1\theta_{132}} e^{-\tilde{u}_2\pi/2} \tag{4.482}$$

In order to write kinetic end potential energy of the body position of the mass center and linear and rotational velocities are required. The angular velocity of Body-5, $\vec{w}_5$, can be expressed in different reference frames. Thus,

$$\bar{\omega}_5^{(0)} = \begin{bmatrix} \dot{\theta}_{132} \\ \dot{\theta}_{131} \\ 0 \end{bmatrix} = \dot{\theta}_{132}\bar{u}_1 + \dot{\theta}_{131}\bar{u}_2 \tag{4.483}$$

$$\begin{aligned} \bar{\omega}_5 = \bar{\omega}_5^{(5)} &= \hat{C}^{(5,0)}\bar{\omega}_5^{(0)} \\ &= e^{\tilde{u}_2\pi/2} e^{-\tilde{u}_1\theta_{132}} e^{-\tilde{u}_2\theta_{131}} \bar{\omega}_5^{(0)} \\ &= \dot{\theta}_{132}\bar{u}_1^{(0/5)} + \dot{\theta}_{131}\bar{u}_2^{(0/5)} \end{aligned} \tag{4.484}$$

The unit vectors of F_0 can be transformed into unit vectors of F_5 by means of transformation matrix $\hat{C}^{(5,0)}$.

$$\begin{aligned} \therefore\ \bar{u}_1^{(0/5)} &= e^{\tilde{u}_2\pi/2} e^{-\tilde{u}_1\theta_{132}} e^{-\tilde{u}_2\theta_{131}} \bar{u}_1 \\ &= e^{\tilde{u}_2\pi/2} e^{-\tilde{u}_1\theta_{132}} \left(\cos\theta_{131}\bar{u}_1 + \sin\theta_{131}\bar{u}_3\right) \\ &= e^{\tilde{u}_2\pi/2} \left(\cos\theta_{131}\bar{u}_1 + \sin\theta_{131}\left(\cos\theta_{132}\bar{u}_3 + \sin\theta_{132}\bar{u}_2\right)\right) \\ &= e^{\tilde{u}_2\pi/2} \left(\cos\theta_{131}\bar{u}_1 + \sin\theta_{131}\sin\theta_{132}\bar{u}_2 + \sin\theta_{131}\cos\theta_{132}\bar{u}_3\right) \\ &= \sin\theta_{131}\cos\theta_{132}\bar{u}_1 + \sin\theta_{131}\sin\theta_{132}\bar{u}_2 - \cos\theta_{131}\bar{u}_3 \end{aligned} \tag{4.485}$$

$$\begin{aligned} \therefore\ \bar{u}_2^{(0/5)} &= e^{\tilde{u}_2\pi/2} e^{-\tilde{u}_1\theta_{132}} e^{-\tilde{u}_2\theta_{131}} \bar{u}_2 \\ &= e^{\tilde{u}_2\pi/2} e^{-\tilde{u}_1\theta_{132}} \bar{u}_2 \\ &= e^{\tilde{u}_2\pi/2} \left(\cos\theta_{132}\bar{u}_2 - \sin\theta_{132}\bar{u}_3\right) \\ &= -\sin\theta_{132}\bar{u}_1 + \cos\theta_{132}\bar{u}_2 \end{aligned} \tag{4.486}$$

Substituting (4.485) and (4.486) into (4.484); we get,

$$\bar{\omega}_5 = \begin{bmatrix} \sin\theta_{131}\cos\theta_{132}\dot{\theta}_{132} - \sin\theta_{132}\dot{\theta}_{131} \\ \sin\theta_{131}\sin\theta_{132}\dot{\theta}_{132} + \cos\theta_{132}\dot{\theta}_{131} \\ -\cos\theta_{131}\dot{\theta}_{132} \end{bmatrix} \tag{4.487}$$

The position vector $\vec{r}_5$ is the position of center of mass of Body-5 with respect to the base frame. We can resolve it any reference frame we like but for Lagrange equations resolving this term in its own frame looks advantageous. Therefore,

$$\bar{r}_5 = \bar{r}_5^{(5)} = \begin{bmatrix} r_{51} \\ 0 \\ r_{53} \end{bmatrix} \tag{4.488}$$

When taking derivative of position vector in order to obtain the velocity of Body-5 with respect to Body-0, Coriolis theorem should be used. The reason why we cannot take direct derivative is due to the definition of the position vector since it has been defined with respect to the base frame.

$$\vec{v}_5 = D_0\vec{r}_5 = \underbrace{D_5\vec{r}_5}_{=\vec{0}} + \vec{\omega}_5 \times \vec{r}_5 = \vec{\omega}_5 \times \vec{r}_5 \tag{4.489}$$

$$\begin{aligned} \bar{v}_5 &= \bar{v}_5^{(5)} = \widetilde{\omega}_5\bar{r}_5 \\ &= \begin{bmatrix} r_{53}\sin\theta_{131}\sin\theta_{132}\dot{\theta}_{132} + r_{53}\cos\theta_{132}\dot{\theta}_{131} \\ -\left(r_{53}\sin\theta_{131}\cos\theta_{132} + r_{51}\cos\theta_{131}\right)\dot{\theta}_{132} + r_{53}\sin\theta_{132}\dot{\theta}_{131} \\ -r_{51}\sin\theta_{131}\sin\theta_{132}\dot{\theta}_{132} - r_{51}\cos\theta_{132}\dot{\theta}_{131} \end{bmatrix} \end{aligned} \tag{4.490}$$

Then, the kinetic and potential energies of Body-5 can be written as below:

$$K_5 = \frac{1}{2}\left(m_5\bar{v}_5^T\bar{v}_5 + \bar{\omega}_5^T\hat{J}_5^{(5)}\bar{\omega}_5\right) \tag{4.491}$$

$$\begin{aligned} U_5 &= -m_5\bar{g}^{(0)^T}\bar{r}_5^{(0)} = -m_5\bar{g}^{(0)^T}\hat{C}^{(0,5)}\bar{r}_5 \\ &= -m_5\bar{g}^{(0)^T}\left(r_{51}\bar{u}_1^{(5/0)} + r_{53}\bar{u}_3^{(5/0)}\right) \end{aligned} \tag{4.492}$$

As we have done previously, the unit vector transformations can be done as follows:

$$\begin{aligned} \bar{u}_1^{(5/0)} &= e^{\tilde{u}_2\theta_{131}}e^{\tilde{u}_1\theta_{132}}e^{-\tilde{u}_2\pi/2}\bar{u}_1 \\ &= e^{\tilde{u}_2\theta_{131}}e^{\tilde{u}_1\theta_{132}}\bar{u}_3 \\ &= e^{\tilde{u}_2\theta_{131}}\left(\cos\theta_{132}\bar{u}_3 - \sin\theta_{132}\bar{u}_2\right) \\ &= \cos\theta_{132}\left(\cos\theta_{131}\bar{u}_3 + \sin\theta_{131}\bar{u}_1\right) - \sin\theta_{132}\bar{u}_2 \end{aligned}$$

$$= \sin\theta_{131}\cos\theta_{132}\bar{u}_1 - \sin\theta_{132}\bar{u}_2 + \cos\theta_{131}\cos\theta_{132}\bar{u}_3 \tag{4.493}$$

$$\begin{aligned}
\bar{u}_3^{(5/0)} &= e^{\tilde{u}_2\theta_{131}} e^{\tilde{u}_1\theta_{132}} e^{-\tilde{u}_2\pi/2}\bar{u}_3 \\
&= -e^{\tilde{u}_2\theta_{131}} e^{\tilde{u}_1\theta_{132}}\bar{u}_1 \\
&= -e^{\tilde{u}_2\theta_{131}}\bar{u}_1 \\
&= -\cos\theta_{131}\bar{u}_1 + \sin\theta_{131}\bar{u}_3
\end{aligned} \tag{4.494}$$

Substituting (4.493) and (4.494) into (4.492); we get,

$$\begin{aligned}
U_5 &= -m_5\bar{g}^{(0)^T}\left(r_{51}\begin{bmatrix}\sin\theta_{131}\cos\theta_{132} \\ -\sin\theta_{132} \\ \cos\theta_{131}\cos\theta_{132}\end{bmatrix} + r_{53}\begin{bmatrix}-\cos\theta_{131} \\ 0 \\ \sin\theta_{131}\end{bmatrix}\right) \\
&= \begin{bmatrix}0 & 0 & -m_5 g\end{bmatrix}\begin{bmatrix} r_{51}\sin\theta_{131}\cos\theta_{132} - r_{53}\cos\theta_{131} \\ -r_{51}\sin\theta_{132} \\ r_{51}\cos\theta_{131}\cos\theta_{132} + r_{53}\sin\theta_{131}\end{bmatrix} \\
&= m_5 g\left(r_{51}\cos\theta_{131}\cos\theta_{132} + r_{53}\sin\theta_{131}\right)
\end{aligned} \tag{4.495}$$

Since neither K_5 nor U_5 are functions of θ_5 and θ_6, the partial derivatives of these energy expressions in the Lagrange Equations become zero.

4.9.2 Body-6

The coordinate transformation matrix between F_6 and F_5 and between F_6 and F_0 are

$$\hat{C}^{(5,6)} = e^{\tilde{u}_3\theta_5} e^{-\tilde{u}_1\pi/2} \tag{4.496}$$

$$\begin{aligned}
\hat{C}^{(0,6)} &= e^{\tilde{u}_2\theta_{131}} e^{\tilde{u}_1\theta_{132}} e^{-\tilde{u}_2\pi/2} e^{\tilde{u}_3\theta_5} e^{-\tilde{u}_1\pi/2} \\
&= e^{\tilde{u}_2\theta_{131}} e^{\tilde{u}_1\theta_{132}} \underbrace{e^{-\tilde{u}_2\pi/2} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\pi/2}}_{e^{-\tilde{u}_1\theta_5}} e^{-\tilde{u}_2\pi/2} e^{-\tilde{u}_1\pi/2} \\
&= e^{\tilde{u}_2\theta_{131}} e^{-\tilde{u}_1(\theta_5-\theta_{132})} e^{-\tilde{u}_2\pi/2} e^{-\tilde{u}_1\pi/2}
\end{aligned} \tag{4.497}$$

In order to write kinetic end potential energy of the body position of the mass center and linear and rotational velocities are required.

$$\bar{\omega}_6 = \bar{\omega}_6^{(6)} = \hat{C}^{(6,5)}\left(\bar{\omega}_5^{(5)} + \dot{\theta}_5\bar{u}_3\right)$$

$$= e^{\tilde{u}_1\pi/2} e^{-\tilde{u}_3\theta_5} \begin{bmatrix} \sin\theta_{131}\cos\theta_{132}\dot{\theta}_{132} - \sin\theta_{132}\dot{\theta}_{131} \\ \sin\theta_{131}\sin\theta_{132}\dot{\theta}_{132} + \cos\theta_{132}\dot{\theta}_{131} \\ \dot{\theta}_5 - \cos\theta_{131}\dot{\theta}_{132} \end{bmatrix}$$

$$= e^{\tilde{u}_1\pi/2} \begin{bmatrix} \sin\theta_{131}\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132} + \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131} \\ -\sin\theta_{131}\sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132} + \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131} \\ \dot{\theta}_5 - \cos\theta_{131}\dot{\theta}_{132} \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta_{131}\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132} + \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131} \\ -\dot{\theta}_5 + \cos\theta_{131}\dot{\theta}_{132} \\ -\sin\theta_{131}\sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132} + \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131} \end{bmatrix} \tag{4.498}$$

r_{GO1} is the distance between the bottom of U shaped Body-5 to the intersection point of the stabilizer gimbal axes. r_{63} is the $\vec{u}_3^{(6)}$ component of distance vector between center of mass of Body-6 and intersection point of the stabilizer gimbal axes. This intersection is also origin of F_5 & F_6. Please nothe that components in the direction of other unit vector are either too small so that they can be ignored or they are directly zero. Therefore, position vector of Body-6 can be written as follows:

$$\vec{r}_6 = r_{GO1}\vec{u}_1^{(5)} + r_{63}\vec{u}_3^{(6)} \tag{4.499}$$

The unit vectors can be written as:

$$\begin{aligned} \bar{u}_1^{(5/6)} &= \hat{C}^{(6,5)}\bar{u}_1 = e^{\tilde{u}_1\pi/2} e^{-\tilde{u}_3\theta_5}\bar{u}_1 \\ &= e^{\tilde{u}_1\pi/2}\left(\cos\theta_5\bar{u}_1 - \sin\theta_5\bar{u}_2\right) \\ \Rightarrow\ \bar{u}_1^{(5/6)} &= \cos\theta_5\bar{u}_1 - \sin\theta_5\bar{u}_3 \end{aligned} \tag{4.500}$$

Then, 4.499 can be written in matrix form.

$$\bar{r}_6 = \bar{r}_6^{(6)} = r_{GO1}\cos\theta_5\bar{u}_1 + \left(r_{63} - r_{GO1}\sin\theta_5\right)\bar{u}_3 \tag{4.501}$$

Taking derivative of $\vec{r}_6$; we can obtain $\vec{v}_6$. Using Coriolis theorem;

$$\vec{v}_6 = D_0\vec{r}_6 = \underbrace{D_5\left(r_{GO1}\vec{u}_1^{(5)}\right)}_{=\vec{0}} + \vec{\omega}_5 \times r_{GO1}\vec{u}_1^{(5)} + \underbrace{D_6\left(r_{63}\vec{u}_3^{(6)}\right)}_{=\vec{0}} + \vec{\omega}_6 \times r_{63}\vec{u}_3^{(6)} \tag{4.502}$$

When (4.502) resolved in F_6,

$$\bar{v}_6 = \bar{v}_6^{(6)} = r_{GO1}\hat{C}^{(6,5)}\widetilde{\omega}_5\bar{u}_1 + \widetilde{\omega}_6 r_{63}\bar{u}_3$$

$$=r_{GO1}e^{\tilde{u}_1\pi/2}e^{-\tilde{u}_3\theta_5}\begin{bmatrix}0\\-\cos\theta_{131}\dot{\theta}_{132}\\-\sin\theta_{131}\sin\theta_{132}\dot{\theta}_{132}-\cos\theta_{132}\dot{\theta}_{131}\end{bmatrix}$$

$$+r_{63}\begin{bmatrix}-\dot{\theta}_5+\cos\theta_{131}\dot{\theta}_{132}\\-\sin\theta_{131}\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}-\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\\0\end{bmatrix}$$

$$=r_{GO1}\begin{bmatrix}-\sin\theta_5\cos\theta_{131}\dot{\theta}_{132}\\\sin\theta_{131}\sin\theta_{132}\dot{\theta}_{132}+\cos\theta_{132}\dot{\theta}_{131}\\-\cos\theta_5\cos\theta_{131}\dot{\theta}_{132}\end{bmatrix}$$

$$+r_{63}\begin{bmatrix}-\dot{\theta}_5+\cos\theta_{131}\dot{\theta}_{132}\\-\sin\theta_{131}\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}-\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\\0\end{bmatrix} \tag{4.503}$$

Expanding (4.503);

$$\begin{aligned}\bar{v}_6 = &\left(-r_{63}\dot{\theta}_5+(r_{63}-r_{GO1}\sin\theta_5)\cos\theta_{131}\dot{\theta}_{132}\right)\bar{u}_1\\&+\Big(\left(r_{GO1}\cos\theta_{132}-r_{63}\sin\left(\theta_5-\theta_{132}\right)\right)\dot{\theta}_{131}\\&+\left(r_{GO1}\sin\theta_{132}-r_{63}\cos\left(\theta_5-\theta_{132}\right)\right)\sin\theta_{131}\dot{\theta}_{132}\Big)\bar{u}_2\\&+\left(-r_{GO1}\cos\theta_5\cos\theta_{131}\dot{\theta}_{132}\right)\bar{u}_3\end{aligned} \tag{4.504}$$

The kinetic energy of Body-6 can be expressed as below:

$$K_6=\frac{1}{2}(m_6\overline{v}_6^{(6)t}\overline{v}_6+\overline{w}_6^{(6)}\overline{J}_6^{(6)}\overline{w}_6^{(6)}) \tag{4.505}$$

Writing (4.504) in matrix form;

$$\overline{v}_6^{(6)}=\begin{bmatrix}\dot{\theta}_{132}c\theta_{131}(r_{63}-r_{GO1}s\theta_5)-\dot{\theta}_5r_{63}\\\dot{\theta}_{131}\big(r_{GO1}c\theta_{132}-r_{63}s(\theta_5-\theta_{132})\big)\\+\dot{\theta}_{132}s\theta_{131}\big(r_{GO1}s\theta_{132}-r_{63}c(\theta_5-\theta_{132})\big)\\\dot{\theta}_{132}(-r_{GO1}c\theta_{131}c\theta_5)\end{bmatrix}\begin{matrix}\to v_{61}\\\to v_{62}\\\to v_{63}\end{matrix} \tag{4.506}$$

$$\overline{v}_6^{(6)^T}=[v_{61}\quad v_{62}\quad v_{63}] \tag{4.507}$$

$$(\overline{v}_6^{(6)})^T\,(\overline{v}_6^{(6)})=v_{61}^2+v_{62}^2+v_{63}^2 \tag{4.508}$$

$$\begin{aligned}v_{61}^2&=\big(\dot{\theta}_{132}c\theta_{131}(r_{63}-r_{GO1}s\theta_5)-\dot{\theta}_5r_{63}\big)\big(\dot{\theta}_{132}c\theta_{131}(r_{63}-r_{GO1}s\theta_5)-\dot{\theta}_5r_{63}\big)\\&=\dot{\theta}_{132}^2c^2\theta_{131}(r_{63}-r_{GO1}\theta_5)^2-2\dot{\theta}_{132}\dot{\theta}_5r_{63}c\theta_{131}(r_{63}-r_{GO1}s\theta_5)+\dot{\theta}_5^2r_{63}^2\end{aligned} \tag{4.509}$$

$$
\begin{aligned}
v_{62}^2 &= \dot{\theta}_{131}^2\big(r_{GO1}c\theta_{132} - r_{63}s(\theta_5-\theta_{132}) - r_{63}s(\theta_5-\theta_{132})\big)^2 \\
&\quad + \dot{\theta}_{132}^2 s^2\theta_{131}\big(r_{GO1}s\theta_{132} - r_{63}c(\theta_5-\theta_{132})\big)^2 \\
&\quad + 2\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}\big(r_{GO1}c\theta_{132} - r_{63}s(\theta_5-\theta_{132})\big) \\
&\quad \big(r_{GO1}s\theta_{132} - r_{63}c(\theta_5-\theta_{132})\big) \qquad (4.510)
\end{aligned}
$$

$$
v_{63}^2 = \dot{\theta}_{132}^2(-r_{GO1}c\theta_{131}c\theta_5)^2 \qquad (4.511)
$$

The second part of the kinetic energy is due to rotational movement.

$$
\vec{w}_6.\check{J}_6.\vec{w}_6 = (\hat{J}_6^{(6)^T}\overline{w}_6^{(6)})^T\overline{w}_6^{(6)} = \overline{w}_6^{(6)^T}\hat{J}_6^{(6)}\overline{w}_6^{(6)} \qquad (4.512)
$$

$$
\overline{w}_6^{(6)^T}\hat{J}_6^{(6)} = [J_{611}w_{61} + J_{613}w_{63}, \quad J_{622}w_{62}, \quad J_{613}w_{61} + J_{633}w_{63}] \qquad (4.513)
$$

$$
\begin{aligned}
\overline{w}_6^{(6)^T}\hat{J}_6^{(6)}\overline{w}_6^{(6)} &= J_{622}w_{62}^2 + w_{61}(J_{611}w_{61} + J_{613}w_{63}) + w_{63}(J_{613}w_{61} + J_{633}w_{63}) \\
&= J_{611}w_{61}^2 + J_{622}w_{62}^2 + J_{633}w_{63}^2 + 2w_{61}w_{63}J_{613} \qquad (4.514)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ w_{61}^2 &= \dot{\theta}_{131}^2c^2\theta_{131} + \dot{\theta}_{132}^2s^2\theta_{131}c^2(\theta_5-\theta_{132}) \\
&\quad + \underbrace{2\dot{\theta}_{131}\dot{\theta}_{132}s(\theta_5-\theta_{132})c(\theta_5-\theta_{132})}_{\dot{\theta}_{131}\dot{\theta}_{132}s\left(2(\theta_5-\theta_{132})\right)} \qquad (4.515)
\end{aligned}
$$

$$
\therefore\ w_{62}^2 = \dot{\theta}_{132}^2c^2\theta_{131} + \dot{\theta}_5^2 - 2\dot{\theta}_{132}\dot{\theta}_5c\theta_{131} \qquad (4.516)
$$

$$
\begin{aligned}
\therefore\ w_{63}^2 &= \dot{\theta}_{131}^2c^2(\theta_5-\theta_{132}) + \dot{\theta}_{132}^2s^2\big(\theta_5-\theta_{132}\big) \\
&\quad \underbrace{-2\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}s(\theta_5-\theta_{132})s(\theta_5-\theta_{132})}_{-\dot{\theta}_{131}\dot{\theta}_{132}s\left(2(\theta_5-\theta_{132})\right)} \qquad (4.517)
\end{aligned}
$$

$$
\begin{aligned}
\therefore w_{61}w_{63} &= \dot{\theta}_{131}^2s(\theta_5-\theta_{132})c(\theta_5-\theta_{132}) + \dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}c^2(\theta_5-\theta_{132}) \\
&\quad - \dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}s^2(\theta_5-\theta_{132}) + \dot{\theta}_{132}^2s^2\theta_{131}s(\theta_5-\theta_{132}) \qquad (4.518)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ w_{61}w_{63} &= \dot{\theta}_{131}^2s\big(2(\theta_5-\theta_{132})\big) + 2\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}c\big(2(\theta_5-\theta_{132})\big) \\
&\quad - \dot{\theta}_{132}^2\theta_{131}s\big(2(\theta_5-\theta_{132})\big) \qquad (4.519)
\end{aligned}
$$

Since we have obtained the terms of kinetic energy, we can now proceed with finding the potential energy.

$$
U_6 = -m_6\vec{g}\cdot\vec{r} = -m_6\overline{g}^{(0)^T}\overline{r}_6^{(0)} \qquad (4.520)
$$

Writing equation (4.501) in vector form;

$$
\vec{r}_6 = r_{GO1}c\theta_5\vec{u}_1^{(6)} + (r_{63} - r_{GO1}s\theta_5)\vec{u}_3^{(6)} \qquad (4.521)
$$

Insert (4.521) into (4.520) to obtain the following:

$$
U_6 = -m_6\overline{g}^{(0)^T}\left(r_{GO1}c\theta_5\overline{u}_1^{(6/0)} + (r_{63} - r_{GO1}s\theta_5)\overline{u}_3^{(6/0)}\right)
$$

$$= -m_6 g \overline{u}_3^{(0)^T} \left(r_{GO1} c\theta_5 \overline{u}_1^{(6/0)} + (r_{63} - r_{GO1} s\theta_5) \overline{u}_3^{(6/0)} \right) \quad (4.522)$$

The unit vector transitions from F_6 to F_0 can be obtained by means of $\hat{C}^{(0,6)}$.

$$\overline{u}_1^{(6/0)} = \hat{C}^{(0,6)} \overline{u}_1 \quad (4.523)$$

$$\begin{aligned}
\hat{C}^{(0,6)} &= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 \theta_{132}} e^{-\tilde{u}_2 \pi/2} e^{\tilde{u}_3 \theta_5} e^{-\tilde{u}_1 \pi/2} \\
&= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 \theta_{132}} \underbrace{e^{-\tilde{u}_2 \pi/2} e^{\tilde{u}_3 \theta_5} e^{\tilde{u}_2 \pi/2}}_{e^{-\tilde{u}_1 \theta_5}} e^{-\tilde{u}_2 \pi/2} e^{-\tilde{u}_1 \pi/2} \\
&= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_{132} - \theta_5)} e^{-\tilde{u}_2 \pi/2} e^{-\tilde{u}_1 \pi/2} \quad (4.524)
\end{aligned}$$

$$\Rightarrow \hat{C}^{(0,6)} = e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_{132} - \theta_5)} e^{-\tilde{u}_2 \pi/2} e^{-\tilde{u}_1 \pi/2} \quad (4.525)$$

$$\begin{aligned}
\therefore \overline{u}_1^{(6/0)} &= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_{132} - \theta_5)} e^{-\tilde{u}_2 \pi/2} \underbrace{e^{-\tilde{u}_1 \pi/2} \overline{u}_1}_{\overline{u}_1} \quad = e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_{132} - \theta_5)} \overline{u}_3 \\
&= e^{\tilde{u}_2 \theta_{131}} (\overline{u}_3 c(\theta_5 - \theta_{132}) + \overline{u}_2 s(\theta_5 - \theta_{132})) \\
&= c(\theta_5 - \theta_{132})(\overline{u}_3 c\theta_{131} + \overline{u}_1 s\theta_{131}) + \overline{u}_2 s(\theta_5 - \theta_{132})
\end{aligned}$$

$$\Rightarrow \overline{u}_1^{(6/0)} = \overline{u}_1 s\theta_{131} c(\theta_5 - \theta_{132}) + \overline{u}_2 s(\theta_5 - \theta_{132}) + \overline{u}_3 c\theta_{131} c(\theta_5 - \theta_{132}) \quad (4.526)$$

$$\begin{aligned}
\therefore \overline{u}_3^{(6/0)} &= \hat{C}^{(0/6)} \overline{u}_3 \\
&= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_{132} - \theta_5)} e^{-\tilde{u}_1 \pi/2} \underbrace{e^{-\tilde{u}_1 \pi/2} \overline{u}_3}_{\overline{u}_2} = e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_{132} - \theta_5)} \overline{u}_2 \\
&= e^{\tilde{u}_2 \theta_{131}} \left(\overline{u}_2 c(\theta_5 - \theta_{132}) - (\overline{u}_3 s(\theta_5 - \theta_{132}) \right) \\
&= \overline{u}_2 c(\theta_5 - \theta_{132}) - s(\theta_5 - \theta_{132})(\overline{u}_3 c\theta_{131} + \overline{u}_1 s\theta_{131})
\end{aligned}$$

$$\Rightarrow \overline{u}_3^{(6/0)} = -\overline{u}_1 s\theta_{131} s(\theta_5 - \theta_{132}) + \overline{u}_2 c(\theta_5 - \theta_{132}) - \overline{u}_3 s(\theta_5 - \theta_{132}) c\theta_{131} \quad (4.527)$$

Simultaneous solution of (4.522) - (4.527) yields the following result:

$$\begin{aligned}
U_6 &= m_6 g r_{GO1} c\theta_{131} c\theta_5 c(\theta_5 - \theta_{132}) + m_6 g r_{GO1} s\theta_5 s(\theta_5 - \theta_{132}) c\theta_{131} \\
&- m_6 g r_{63} s(\theta_5 - \theta_{132}) c\theta_{131} = m_6 g r_{GO1} c\theta_{131} c\theta_{132} - m_6 g r_{63} s(\theta_5 - \theta_{132}) c\theta_{131}
\end{aligned} \quad (4.528)$$

The derivative terms should be obtained for each independent variable. Since the degree of freedom of the stabilizer is two (θ_5 & θ_6); this process will be repeated for each of them. For $q_1 = \theta_5$;

$$\frac{\partial U_6}{\partial q_k} = \frac{\partial U_6}{\partial \theta_5} = -m_6 g r_{63} c(\theta_5 - \theta_{132}) c\theta_{131} \quad (4.529)$$

$$\frac{\partial K}{\partial q_k} = \frac{\partial K}{\partial \theta_5} = \frac{\partial K_t}{\partial \theta_5} + \frac{\partial K_r}{\partial \theta_5} \quad (4.530)$$

$$\frac{\partial K}{\partial \theta_5} = \frac{\partial v_{61}^2}{\partial \theta_5} + \frac{\partial v_{62}^2}{\partial \theta_5} + \frac{\partial v_{63}^2}{\partial \theta_5} \tag{4.531}$$

Remind that in (4.509), v_{61}^2 has been determined. Taking derivatives of this term:

$$v_{61}^2 = \dot{\theta}_{132}^2 c^2\theta_{131}(r_{63} - r_{GO1}s\theta_5)^2 - 2\dot{\theta}_{132}\dot{\theta}_5 r_{63}c\theta_{131}(r_{63} - r_{GO1}s\theta_5) + \dot{\theta}_5^2 r_{63}$$

$$\therefore \ \frac{\partial v_{61}^2}{\partial \theta_5} = -2\dot{\theta}_{132}^2 c^2\theta_{131}(r_{63} - r_{GO1}s\theta_5)r_{GO1}c\theta_5 + 2\dot{\theta}_{132}\dot{\theta}_5 r_{63}c\theta_{131}r_{GO1}c\theta_5 \tag{4.532}$$

$$\therefore \ \frac{\partial v_{61}^2}{\partial \dot{\theta}_5} = -2\dot{\theta}_{132}r_{63}c\theta_{131}(r_{63} - r_{GO1}s\theta_5) + 2\dot{\theta}_5 r_{63}^2 \tag{4.533}$$

Similarly, from (4.510);

$$\begin{aligned} v_{62}^2 &= \dot{\theta}_{131}^2\big(r_{GO1}c\theta_{132} - r_{63}s(\theta_5 - \theta_{132}) - r_{63}s(\theta_5 - \theta_{132})\big)^2 \\ &+ \dot{\theta}_{132}^2 s^2\theta_{131}\big(r_{GO1}s\theta_{132} - r_{63}c(\theta_5 - \theta_{132})\big)^2 \\ &+ 2\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}\big(r_{GO1}c\theta_{132} - r_{63}s(\theta_5 - \theta_{132})\big)\big(r_{GO1}s\theta_{132} - r_{63}c(\theta_5 - \theta_{132})\big) \\ \therefore \ \frac{\partial v_{62}^2}{\partial \theta_5} &= r_{63}\dot{\theta}_{131}^2 2s(\theta_5 - \theta_{132})c(\theta_5 - \theta_{132}) - r_{GO1}r_{63}\dot{\theta}_{131}^2 c\theta_{132}c(\theta_5 - \theta_{132}) \\ &+ 2\dot{\theta}_{132}^2 s^2\theta_{131}\big(r_{GO1}s\theta_{132} - r_{63}c(\theta_5 - \theta_{132})\big)r_{63}s(\theta_5 - \theta_{132}) \\ &- 2r_{63}\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}c(\theta_5 - \theta_{132})\big(r_{GO1}s\theta_{132} - r_{63}c(\theta_5 - \theta_{132})\big) \\ &+ 2r_{63}s(\theta_5 - \theta_{132})\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}\big(r_{GO1}c\theta_{132} - r_{63}s(\theta_5 - \theta_{132})\big) \end{aligned} \tag{4.534}$$

$$\begin{aligned} \Rightarrow \ \frac{\partial v_{62}^2}{\partial \theta_5} &= r_{63}\dot{\theta}_{131}^2 s\big(2(\theta_5 - \theta_{132})\big) + 2r_{GO1}r_{63}\dot{\theta}_{132}^2 s^2\theta_{131}s\theta_{132}s(\theta_5 - \theta_{132}) \\ &- r_{63}^2\dot{\theta}_{132}^2 s^2\theta_{131}s\big(2(\theta_5 - \theta_{132})\big) + 2r_{GO1}r_{63}\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}s(\theta_5 - \theta_{132}) \\ &+ 2r_{63}^2\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}c\big(2(\theta_5 - \theta_{132})\big) - 2r_{GO1}r_{63}\dot{\theta}_{131}^2 c\theta_{132}c(\theta_5 - \theta_{132}) \end{aligned} \tag{4.535}$$

$$\therefore \ \frac{\partial v_{62}^2}{\partial \dot{\theta}_5} = 0 \tag{4.536}$$

Similarly, from (4.511);

$$\begin{aligned} v_{63}^2 &= \dot{\theta}_{132}^2(-r_{GO1}c\theta_{131}c\theta_5)^2 \\ \frac{\partial v_{63}^2}{\partial \theta_5} &= 2\dot{\theta}_{132}^2(-r_{GO1}c\theta_{131}c\theta_5)r_{GO1}c\theta_{131}s\theta_5 \\ &= -2r_{GO1}^2 c\dot{\theta}_{132}^2 c^2\theta_{131}s\theta_5 s\theta_5 = -r_{GO1}^2\dot{\theta}_{132}^2 c^2\theta_{131}s2\theta_5 \end{aligned} \tag{4.537}$$

$$\frac{\partial v_{63}^2}{\partial \dot{\theta}_5} = 0 \tag{4.538}$$

Summing up (4.536) , (4.535) & (4.538);

$$\frac{\partial K_{6t}}{\partial \theta_5} = \frac{m_6}{2}\left(\frac{\partial v_{61}^2}{\partial \theta_5} + \frac{\partial v_{62}^2}{\partial \theta_5} + \frac{\partial v_{63}^2}{\partial \theta_5}\right)$$

$$
\begin{aligned}
&= \frac{m_6}{2}\Big(-2r_{GO1}r_{63}\dot{\theta}_{132}^2 c^2\theta_{131}c\dot{\theta}_5 + r_{GO1}^2\dot{\theta}_{132}^2 c^2\theta_{131}s2\theta_5 + 2r_{GO1}r_{63}\dot{\theta}_{132}\theta_5 c\theta_{131}c\theta_5 \\
&- r_{63}^2\dot{\theta}_{131}^2 s\big(2(\theta_5-\theta_{132})\big) + 2r_{GO1}r_{63}\dot{\theta}_{132}^2 s^2\theta_{131}s\theta_{132}s(\theta_5-\theta_{132}) \\
&- r_{63}^2\dot{\theta}_{132}^2 s^2\theta_{131}s\big(2(\theta_5-\theta_{132})\big) + 2r_{GO1}^2 r_{63}\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}s(\theta_5-2\theta_{132}) \\
&+ 2r_{63}^2\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131}c\big(2(\theta_5-\theta_{132})\big) - r_{GO1}^2\dot{\theta}_{132}^2 c^2\theta_{131}s2\theta_5\Big)
\end{aligned}
\tag{4.539}
$$

Rewriting (4.541) in terms of $\dot{\theta}_{131},\ \dot{\theta}_{132}$ & $\dot{\theta}_5$;

$$
\begin{aligned}
\frac{\partial K_{6t}}{\partial \theta_5} = \frac{m_6}{2}\Big(&\dot{\theta}_{132}^2\big(-2r_{GO1}r_{63}c^2\theta_{131}c\theta_5 \cancel{+r_{GO1}^2\dot{\theta}_{132}^2 c^2\theta_{131}s2\theta_5} \\
&+ 2r_{GO1}r_{63}s^2\theta_{131}s\theta_{132}s(\theta_5-\theta_{132}) \cancel{-r_{GO1}^2\dot{\theta}_{132}^2 c^2\theta_{131}s2\theta_5} \\
&- r_{63}^2 s^2\theta_{131}s\big(2(\theta_5-\theta_{132})\big) \\
&+ \dot{\theta}_{131}^2\Big(r_{63}^2 s\big(2(\theta_5-\theta_{132})\big) - 2r_{GO1}r_{63}c\theta_{132}c(\theta_5-\theta_{132})\Big) \\
&+ 2\dot{\theta}_{131}\dot{\theta}_{132}r_{63}s\theta_{131}\big(s(\theta_5-2\theta_{132}) + r_{63}c(2(\theta_5-\theta_{132})\big) \\
&+ 2\dot{\theta}_{132}\dot{\theta}_5(r_{GO1}r_{63}c\theta_{131}c\theta_5)\Big)
\end{aligned}
\tag{4.540}
$$

Summing up (4.534), (4.535) & (4.537);

$$
\begin{aligned}
\frac{\partial K_{6t}}{\partial \dot{\theta}_5} &= \frac{m_6}{2}\left(\frac{\partial v_{61}^2}{\partial \dot{\theta}_5} + \frac{\partial v_{62}^2}{\partial \dot{\theta}_5} + \frac{\partial v_{63}^2}{\partial \dot{\theta}_5}\right) \\
&= \frac{m_6}{2}\big(-2\dot{\theta}_{132}r_{63}c\theta_{131}(r_{63} - r_{GO1}s\theta_5) + 2\dot{\theta}_5 r_{63}^2\big) \\
&= m_6(-r_{63}^2\dot{\theta}_{132}c\theta_{131} + r_{GO1}r_{63}\dot{\theta}_{132}s\theta_5 c\theta_{131} + \dot{\theta}_5 r_{63}^2) \qquad (4.541) \\
\frac{d}{dt}\left(\frac{\partial K_{6t}}{\partial \dot{\theta}_5}\right) &= m_6(-r_{63}^2\ddot{\theta}_{132}c\theta_{131} + r_{63}^2\dot{\theta}_{131}\dot{\theta}_{132}s\theta_{131} + r_{GO1}r_{63}\dot{\theta}_{132}s\theta_5 c\theta_{131} \\
&+ r_{GO1}r_{63}\ddot{\theta}_{132}\dot{\theta}_5 c\theta_5 c\theta_{131} - r_{GO1}r_{63}\dot{\theta}_{131}\dot{\theta}_{132}s\theta_5 s\theta_{131} + \ddot{\theta}_5 r_{63}^2) \qquad (4.542)
\end{aligned}
$$

Remind that in (4.528) potential energy, U_6 has been found. It's derivative with respect to θ_5 can be found as:

$$
\begin{aligned}
U_6 &= m_6 g r_{GO1}c\theta_{131}c\theta_{132} - m_6 g r_{63}s\left(\theta_5-\theta_{132}\right)c\theta_{131} \\
\frac{\partial U_6}{\partial \theta_5} &= -m_6 g r_{63}c\left(\theta_5-\theta_{132}\right)c\theta_{131}
\end{aligned}
\tag{4.543}
$$

Next, we can go through (4.515) - (4.519) to find the derivatives of the terms involved in rotational kinetic energy. Therefore,

$$
\begin{aligned}
w_{61}^2 = &\dot{\theta}_{131}^2 s^2\left(\theta_5-\theta_{132}\right) + \dot{\theta}_{132}^2 s^2\theta_{131}^2 c^2\left(\theta_5-\theta_{132}\right) \\
&+ \dot{\theta}_{131}\theta_{132}s\left(2\left(\theta_5-\theta_{132}\right)\right)
\end{aligned}
$$

$$\therefore \frac{w_{61}^2}{\partial\theta_5} = -2\dot{\theta}_{132}^2 s^2\theta_{131} c\left(\theta_5 - \theta_{132}\right) s\left(\theta_5 - \theta_{132}\right) + 2\dot{\theta}_{131}\dot{\theta}_{132} c\left(2\left(\theta_5 - \theta_{132}\right)\right)$$
$$+ 2\dot{\theta}_{131}^2 s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right)$$
$$= -\dot{\theta}_{132}^2 s^2\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right) + 2\dot{\theta}_{131} s^2\theta_{132} c\left(2\left(\theta_5 - \theta_{132}\right)\right)$$
$$+ \dot{\theta}_{131}^2 s\left(2\left(\theta_5 - \theta_{132}\right)\right) \tag{4.544}$$
$$\therefore \frac{\partial w_{61}^2}{\partial\dot{\theta}_5} = 0 \tag{4.545}$$

Nextly,

$$w_{62}^2 = \dot{\theta}_{132}^2 c^2\theta_{131} + \dot{\theta}_5^2 - 2\dot{\theta}_{132}\dot{\theta}_5 c\theta_{131}$$
$$\therefore \frac{\partial w_{62}^2}{\partial\theta_5} = 0 \tag{4.546}$$
$$\therefore \frac{\partial w_{62}^2}{\partial\theta_5} = 2\dot{\theta}_5 - 2\dot{\theta}_{132} c\theta_{131} \tag{4.547}$$

w_{63}^2 will follow next:

$$w_{63}^2 = \dot{\theta}_{131}^2 c^2\left(\theta_5 - \theta_{132}\right) + \dot{\theta}_{132}^2 s^2\theta_{131}^2 s^2\left(\theta_5 - \theta_{132}\right)$$
$$- \dot{\theta}_{131}^2\theta_{132}^2 s\left(2\left(\theta_5 - \theta_{132}\right)\right)$$
$$\therefore \frac{\partial w_{63}^2}{\partial\theta_5} = -2\dot{\theta}_{131}^2 c^2 c\left(\theta_5 - \theta_{132}\right) s\left(\theta_5 - \theta_{132}\right)$$
$$+ 2\dot{\theta}_{132}^2 s^2\dot{\theta}_{131} s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right)$$
$$- 2\dot{\theta}_{131}\dot{\theta}_{132} c\left(2\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right)\right) \tag{4.548}$$
$$\therefore \frac{\partial w_{63}^2}{\partial\dot{\theta}_5} = 0 \tag{4.549}$$

The last terms is $2w_{61}w_{63}$:

$$2w_{61}w_{63} = \dot{\theta}_{161}^2 s\left(2\left(\theta_5 - \theta_{132}\right)\right) + 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} c\left(2\left(\theta_5 - \theta_{132}\right)\right)$$
$$- \dot{\theta}_{132}^2 s^2\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right)$$
$$\therefore \frac{\partial\left(2w_{61}w_{63}\right)}{\partial\theta_5} = 2\dot{\theta}_{131}^2\left(2\left(\theta_5 - \theta_{132}\right)\right) - 4\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right)$$
$$- 2\dot{\theta}_{132}^2 s^2\theta_{131} c\left(2\left(\theta_5 - \theta_{132}\right)\right) \tag{4.550}$$
$$\therefore \frac{\partial\left(2w_{61}w_{63}\right)}{\partial\dot{\theta}_5} = 0 \tag{4.551}$$

Remind that in (4.514), it has been found that:

$$\overline{w}_6^{(6)^T}\hat{J}_6^{(6)}\overline{w}_6^{(6)} = J_{611}w_{61}^2 + J_{622}w_{62}^2 + J_{633}w_{63}^2 + 2w_{61}w_{63}J_{613}$$

$$\frac{\partial K_{6r}}{\partial \theta_5} = \frac{1}{2}\left(J_{611}\frac{\partial(w_{61}^2)}{\partial \theta_5} + J_{622}\frac{\partial(w_{62}^2)}{\partial \theta_5} + J_{613}\frac{\partial(2w_{61}w_{63})}{\partial \theta_5}\right)$$
$$= \frac{1}{2}\left(\dot{\theta}_{132}^2\left(-J_{611}s^2\theta_{131}s\left(2\left(\theta_5-\theta_{132}\right)\right) + J_{633}s^2\theta_{131}s\left(2\left(\theta_5-\theta_{132}\right)\right)\right.\right.$$
$$- 2J_{613}s^2\theta_{131}c\left(2\left(\theta_5-\theta_{132}\right)\right) + \theta_{131}^2\left(-J_{633}s\left(2\left(\theta_5-\theta_{132}\right)\right)\right.$$
$$+ 2J_{613}c\left(2\left(\theta_5-\theta_{132}\right)\right) + J_{611}s\left(2\left(\theta_5-\theta_{132}\right)\right)$$
$$+ 2\dot{\theta}_{131}\dot{\theta}_{132}\left(J_{611}c\left(2\left(\theta_5-\theta_{132}\right)\right) - J_{633}c\left(2\left(\theta_5-\theta_{132}\right)\right)\right.$$
$$\left.\left.\left.-4J_{613}s\theta_{131}s\left(2\left(\theta_5-\theta_{132}\right)\right)\right)\right)\right) \tag{4.552}$$

Similarly,

$$\frac{\partial K_{6r}}{\partial \dot{\theta}_5} = \frac{\left(2J_{622}\dot{\theta}_5 - 2J_{622}\dot{\theta}_{132}c\theta_{131}\right)}{2}$$
$$= J_{622}\dot{\theta}_5 - J_{622}\dot{\theta}_{132}c\theta_{131} \tag{4.553}$$
$$\frac{d}{dt}\left(\frac{\partial K_{6r}}{\partial \dot{\theta}_5}\right) = J_{622}\left(\ddot{\theta}_5 - \ddot{\theta}_{132}c\theta_{131} + \dot{\theta}_{131}\dot{\theta}_{132}s\dot{\theta}_{131}\right) \tag{4.554}$$

Alternative to the brute force method we have followed up to now, we can simplify derivative terms with some manipulations. Let $\overline{w}_6^T = \left(\overline{w}_6^{(6)}\right)^T$ & $\overline{v}_6 = \overline{v}_6^{(6)}$,

$$\frac{\partial}{\partial \theta_5}\left(\overline{v}_6^T\overline{v}_6\right) = \overbrace{\frac{\partial \overline{v}_6}{\partial \theta_5}\overline{v}_6^T + \left(\frac{\partial \overline{v}_6}{\partial \theta_5}\right)^T\overline{v}_6}^{Inner\ product\ sequence\ \ does\ not\ change\ result!} = 2\overline{v}_6^T\frac{\partial v_6}{\partial \theta_5} \tag{4.555}$$
$$\frac{\partial}{\partial \theta_5}\left(\overline{w}_6^T\hat{J}_6\overline{w}_6\right) = \left(\frac{\partial \overline{w}_6}{\partial \theta_5}\right)\hat{J}_6\overline{w}_6 + \overline{w}_6^T\hat{J}_6\left(\frac{\partial \overline{w}_6}{\partial \theta_5}\right)$$
$$= 2\left(\frac{\partial \overline{w}_6}{\partial \theta_5}\right)^T\hat{J}\overline{w}_6 = 2\overline{w}_6^T\hat{J}_6\left(\frac{\partial \overline{w}_6}{\partial \theta_5}\right) \tag{4.556}$$

Remind that in (4.506), $\overline{v}_6^{(6)}$ has been found as:

$$\overline{v}_6^{(6)} = \begin{bmatrix} \dot{\theta}_{132}c\theta_{131}(r_{63} - r_{GO1}s\theta_5) - \dot{\theta}_5 r_{63} \\ \dot{\theta}_{131}\big(r_{GO1}c\theta_{132} - r_{63}s(\theta_5-\theta_{132})\big) \\ +\dot{\theta}_{132}s\theta_{131}\big(r_{GO1}s\theta_{132} - r_{63}c(\theta_5-\theta_{132})\big) \\ \dot{\theta}_{132}(-r_{GO1}c\theta_{131}c\theta_5) \end{bmatrix} \tag{4.557}$$

Expanding (4.556);

$$\frac{\partial \overline{v}_6}{\partial \theta_5} = \begin{bmatrix} -r_{GO1}\dot{\theta}_{132}c\theta_{131}c\theta_5 \\ -r_{63}\dot{\theta}_{131}c(\theta_5-\theta_{132}) + r_{63}\dot{\theta}_{132}s\theta_{131}c(\theta_5-\theta_{132}) \\ \dot{\theta}_{132}(-r_{GO1}c\theta_{131}c\theta_5) \end{bmatrix} \tag{4.558}$$

$$\frac{\partial}{\partial \theta_5}\left(v_6^T v_6\right) = 2 \times \Big(-r_{GO1} r_{63} \dot{\theta}_{132}^2 c^2\theta_{131} c\theta_5 + r_{GO1}^2 \dot{\theta}_{132}^2 c^2\theta_{131} s\theta_5 c\theta_5$$
$$+ r_{GO1} r_{63} \dot{\theta}_5 \dot{\theta}_{132} c\theta_{131} c\theta_5 - r_{GO1} r_{63} \dot{\theta}_{131}^2 c\theta_{132} c\left(\theta_5 - \theta_{132}\right)$$
$$+ r_{63}^2 \dot{\theta}_{131}^2 s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right) - r_{GO1} r_{63} \dot{\theta}_{131} \dot{\theta}_{132} s\theta_{131} \theta_{132} c\left(\theta_5 - \theta_{132}\right)$$
$$+ r_{63}^2 \dot{\theta}_{131} \dot{\theta}_{132} s\theta_{131} c^2\left(\theta_5 - \theta_{132}\right) + r_{GO1} r_{63} \dot{\theta}_{131} \dot{\theta}_{132} s\theta_{131} s\left(\theta_5 - \theta_{132}\right) c\theta_{132}$$
$$- r_{63}^2 \dot{\theta}_{131} \dot{\theta}_{132} s\theta_{131} s^2\left(\theta_5 - \theta_{132}\right) + r_{GO1} r_{63} \dot{\theta}_{132}^2 s^2\theta_{131} s\theta_{132} s\left(\theta_5 - \theta_{132}\right)$$
$$- r_{63}^2 \dot{\theta}_{132}^2 s^2\theta_{131} s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right) - r_{GO1}^2 \dot{\theta}_{132}^2 c^2\theta_{131} s\theta_5 c\theta_5\Big) \tag{4.559}$$

$$\Rightarrow \frac{\partial}{\partial \theta_5}(v_6^T v_6) = \dot{\theta}_{132}^2\Big(- 2r_{GO1} r_{63} c^2\theta_{131} c\theta_5 + 2r_{GO1} r_{63} s^2\theta_{131} s\theta_{132} s(\theta_5 - \theta_{132})$$
$$- r_{63} s^2\theta_{131} s\big(2(\theta_5 - \theta_{132})\big)\Big)$$
$$+ \dot{\theta}_{131}^2 r_{63}\Big(- 2r_{GO1} c\theta_{132} c(\theta_5 - \theta_{132}) + r_{63} s\big(2(\theta_5 - \theta_{132})\big)\Big)$$
$$+ 2r_{63} \dot{\theta}_{131} \dot{\theta}_{132} s\theta_{131}\Big(r_{GO1} s(\theta_5 - 2\theta_{132}) + r_{63} c\big(2(\theta_5 - \theta_{132})\big)\Big)$$
$$+ 2r_{GO1} r_{63} \dot{\theta}_5 \dot{\theta}_{132} c\theta_{131} c\theta_5 \tag{4.560}$$

From (4.555), we know that:

$$\frac{\partial K_{6t}}{\partial \theta_5} = \frac{m_6}{2} \frac{\partial}{\partial \theta_5}\left(v_6^T v_6\right)$$

Substitution of (4.560) into (4.555) yields the following result:

$$\frac{\partial K_{6t}}{\partial \theta_5} = \frac{m_6}{2}\Big(\dot{\theta}_{132}^2\big(- 2r_{GO1} r_{63} c^2\theta_{131} c\theta_5 + 2r_{GO1} r_{63} s^2\theta_{131} s\theta_{132} s(\theta_5 - \theta_{132})$$
$$- r_{63}^2 s^2\theta_{131} s\big(2(\theta_5 - \theta_{132})\big)$$
$$+ \dot{\theta}_{131}^2\Big(r_{63}^2 s\big(2(\theta_5 - \theta_{132})\big) - 2r_{GO1} r_{63} c\theta_{132} c(\theta_5 - \theta_{132})\Big)$$
$$+ 2\dot{\theta}_{131} \dot{\theta}_{132} r_{63} s\theta_{131}\big(s(\theta_5 - 2\theta_{132}) + r_{63} c(2(\theta_5 - \theta_{132})\big)$$
$$+ 2\dot{\theta}_{132} \dot{\theta}_5 (r_{GO1} r_{63} c\theta_{131} c\theta_5)\Big) \tag{4.561}$$

As we expect, the results of (4.540) & (4.561) are identical! As found in (4.556),

$$\frac{\partial K_{6r}}{\partial \theta_5} = \frac{1}{2}.2\overline{w}_6^T \hat{J}_6 \left(\frac{\partial \overline{w}_6}{\partial \theta_5}\right)$$

We have determined $\overline{w}_6$ as:

$$\overline{w}_6 = \begin{bmatrix} \dot{\theta}_{131} s\left(\theta_5 - \theta_{132}\right) + \dot{\theta}_{132} s\theta_{131} c\left(\theta_5 - \theta_{132}\right) \\ \\ \dot{\theta}_{132} c\theta_{131} - \dot{\theta}_5 \\ \\ \dot{\theta}_{131} c\left(\theta_5 - \theta_{132}\right) + \dot{\theta}_{132} s\theta_{131} s\left(\theta_5 - \theta_{132}\right) \end{bmatrix} \begin{matrix} \rightarrow \ w_{61} \\ \\ \rightarrow \ w_{62} \\ \\ \rightarrow \ w_{63} \end{matrix}$$

$$\frac{\partial \overline{w}_6}{\partial \theta_5} = \begin{bmatrix} \dot{\theta}_{131} c\left(\theta_5 - \theta_{132}\right) - \dot{\theta}_{132} s\theta_{131} s\left(\theta_5 - \theta_{132}\right) \\ \\ 0 \\ \\ -\dot{\theta}_{131} c\left(\theta_5 - \theta_{132}\right) - \dot{\theta}_{132} s\theta_{131} c\left(\theta_5 - \theta_{132}\right) \end{bmatrix} \begin{matrix} \rightarrow \ w_{d61} \\ \\ \rightarrow \ w_{d62} \\ \\ \rightarrow \ w_{d63} \end{matrix} \quad (4.562)$$

$$\hat{J}_6 \left(\frac{\partial \overline{w}_6}{\partial \theta_5} \right) = \begin{bmatrix} J_{611} w_{611} + J_{d61} w_{d63} \\ \\ J_{622} w_{d62} \\ \\ J_{613} w_{d61} + J_{633} w_{d63} \end{bmatrix} \quad (4.563)$$

$$\overline{w}_6^T \hat{J}_6 \left(\frac{\partial \overline{w}_6}{\partial \theta_5} \right) = w_{61} \left(J_{611} w_{d61} + J_{613} w_{d63} \right) + w_{63} \left(J_{613} w_{d61} + J_{633} w_{d33} \right) + J_{622} w_{d62} w_{d62} \quad (4.564)$$

Rewriting (4.564), we'll obtain that

$$\overline{w}_6^T \hat{J}_6 \left(\frac{\partial \overline{w}_6}{\partial \theta_5} \right) = J_{611} w_{61} w_{d61} + J_{613} w_{61} w_{db3} + J_{613} w_{63} w_{db1} + J_{633} w_{63} w_{db3} + J_{622} w_{62} w_{db2} \quad (4.565)$$

Rewriting the multiplication terms:

$$* \ 2 w_{61} w_{d61} = \dot{\theta}_{131}^2 s\left(2\left(\theta_5 - \theta_{132}\right)\right) - 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s^2\left(\theta_5 - \theta_{132}\right) + 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} c^2\left(\theta_5 - \theta_{132}\right) - \dot{\theta}_{132}^2 s^2\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right)$$

$$\Rightarrow \ 2 w_{61} w_{d61} = \dot{\theta}_{131}^2 s\left(2\left(\theta_5 - \theta_{132}\right)\right) - \dot{\theta}_{132}^2 s^2\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right) + 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} c\left(2\left(\theta_5 - \theta_{132}\right)\right) \quad (4.566)$$

$$* \ 2 w_{61} w_{db3} = -2\dot{\theta}_{131}^2 s^2\left(\theta_5 - \theta_{132}\right) - \dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right) - \dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right) - 2\dot{\theta}_{132}^2 s^2\theta_{131} c^2\left(\theta_5 - \theta_{132}\right)$$

$$\Rightarrow \ 2 w_{61} w_{db3} = -2\dot{\theta}_{131}^2 s^2\left(\theta_5 - \theta_{132}\right) - 2\dot{\theta}_{132}^2 s^2\theta_{131} c^2\left(\theta_5 - \theta_{132}\right)$$

$$
\begin{aligned}
& - 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s\left(2\left(\theta_5-\theta_{132}\right)\right) && (4.567)\\
* \ 2w_{63}w_{db1} &= 2\dot{\theta}_{131}^2 c^2\left(\theta_5-\theta_{132}\right) - \dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s\left(2\left(\theta_5-\theta_{132}\right)\right)\\
& - \dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s\left(2\left(\theta_5-\theta_{132}\right)\right) + 2\dot{\theta}_{132}^2 s^2\theta_{131} s^2(\theta_5-\theta_{132})\\
\Rightarrow \ 2w_{63}w_{db1} &= 2\dot{\theta}_{131}^2 c^2\left(\theta_5-\theta_{132}\right) + 2\dot{\theta}_{132}^2 s^2\theta_{131} s^2\left(\theta_5-\theta_{132}\right)\\
& - 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s\left(2\left(\theta_5-\theta_{132}\right)\right) && (4.568)\\
* \ 2w_{63}w_{db3} &= -\theta_{131}^2 s\left(2\left(\theta_5-\theta_{132}\right)\right) - 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} c^2\left(\theta_5-\theta_{132}\right)\\
& + 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} s^2\left(\theta_5-\theta_{132}\right) + \dot{\theta}_{132}^2 s^2\theta_{131} s\left(2\left(\theta_5-\theta_{132}\right)\right)\\
\Rightarrow \ 2w_{63}w_{db3} &= -\dot{\theta}_{131}^2 s\left(2\left(\theta_5-\theta_{132}\right)\right) + \dot{\theta}_{132}^2 s^2\theta_{131} s\left(2\left(\theta_5-\theta_{132}\right)\right)\\
& - 2\dot{\theta}_{131}\dot{\theta}_{132} s\theta_{131} c\left(2\left(\theta_5-\theta_{132}\right)\right) && (4.569)\\
* \ 2w_{62}w_{db2} &= 0 && (4.570)
\end{aligned}
$$

Therefore, substitution of (4.565) - (4.570) into (4.556) yields that:

$$
\begin{aligned}
\frac{\partial K_{6r}}{\partial \theta_5} &= \frac{1}{2}.2\overline{w}_6^T \hat{J}\left(\frac{\partial \overline{w}_6}{\partial \theta_5}\right)\\
&= \frac{1}{2}\Bigg(\dot{\theta}_{131}^2\Big(2J_{613}c\big(2(\theta_5-\theta_{132})\big) + (J_{611}-J_{633})s\big(2(\theta_5-\theta_{132})\big)\Big)\\
&+ \dot{\theta}_{132}^2\Big((J_{633}-J_{611})s^2\theta_{131}s\big(2(\theta_5-\theta_{132})\big) - 2J_{613}s^2\theta_{131}c\big(2(\theta_5-\theta_{132})\big)\Big)\\
&+ \dot{\theta}_{131}\dot{\theta}_{132}\Big(2(J_{611}-J_{633})s\theta_{131}c\big(2(\theta_5-\theta_{132})\big) - 4J_{613}s\theta_{131}s\big(2(\theta_5-\theta_{132})\big)\Big)\Bigg)
\end{aligned} \tag{4.571}
$$

As we have expected (4.552) & (4.571) are identical.

4.9.3 Body-7

The coordinate transformation matrix between F_7 and F_6, between F_7 and F_5 and between F_7 and F_0 are:

$$
\hat{C}^{(6,7)} = e^{\tilde{u}_3\theta_6} \tag{4.572}
$$

$$
\begin{aligned}
\hat{C}^{(5,7)} &= e^{\tilde{u}_3\theta_5}e^{-\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_6}\\
&= e^{\tilde{u}_3\theta_5}e^{\tilde{u}_2\theta_6}e^{-\tilde{u}_1\pi/2}
\end{aligned} \tag{4.573}
$$

$$
\begin{aligned}
\hat{C}^{(0,7)} &= e^{\tilde{u}_2\theta_{131}}e^{-\tilde{u}_1(\theta_5-\theta_{132})}e^{-\tilde{u}_2\pi/2}e^{-\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_6}\\
&= e^{\tilde{u}_2\theta_{131}}e^{-\tilde{u}_1(\theta_5-\theta_{132})}e^{\tilde{u}_2\theta_6}e^{-\tilde{u}_2\pi/2}e^{-\tilde{u}_1\pi/2}
\end{aligned} \tag{4.574}
$$

In order to write kinetic end potential energy of the body position of the mass center and linear and rotational velocities are required.

$$
\begin{aligned}
\bar{\omega}_7 = \bar{\omega}_7^{(7)} &= \hat{C}^{(7,6)} \left(\bar{\omega}_6^{(6)} + \dot{\theta}_6 \bar{u}_3\right) \\
&= e^{-\tilde{u}_3\theta_6} \Big[\Big(\sin\theta_{131}\cos(\theta_5 - \theta_{132})\dot{\theta}_{132} + \sin(\theta_5 - \theta_{132})\dot{\theta}_{131}\Big)\bar{u}_1 \\
&+ \Big(-\dot{\theta}_5 + \cos\theta_{131}\dot{\theta}_{132}\Big)\bar{u}_2 \\
&+ \Big(\dot{\theta}_6 - \sin\theta_{131}\sin(\theta_5 - \theta_{132})\dot{\theta}_{132} + \cos(\theta_5 - \theta_{132})\dot{\theta}_{131}\Big)\bar{u}_3\Big] \\
&= \Big(\cos\theta_6\Big(\sin\theta_{131}\cos(\theta_5 - \theta_{132})\dot{\theta}_{132} + \sin(\theta_5 - \theta_{132})\dot{\theta}_{131}\Big) \\
&+ \sin\theta_6\Big(-\dot{\theta}_5 + \cos\theta_{131}\dot{\theta}_{132}\Big)\Big)\bar{u}_1 \\
&+ \Big(-\sin\theta_6\Big(\sin\theta_{131}\cos(\theta_5 - \theta_{132})\dot{\theta}_{132} + \sin(\theta_5 - \theta_{132})\dot{\theta}_{131}\Big) \\
&+ \cos\theta_6\Big(-\dot{\theta}_5 + \cos\theta_{131}\dot{\theta}_{132}\Big)\Big)\bar{u}_2 \\
&+ \Big(\dot{\theta}_6 - \sin\theta_{131}\sin(\theta_5 - \theta_{132})\dot{\theta}_{132} + \cos(\theta_5 - \theta_{132})\dot{\theta}_{131}\Big)\bar{u}_3 \\
&= \Big(-\sin\theta_6\dot{\theta}_5 + \cos\theta_6\sin(\theta_5 - \theta_{132})\dot{\theta}_{131} \\
&+ (\cos\theta_6\sin\theta_{131}\cos(\theta_5 - \theta_{132}) + \sin\theta_6\cos\theta_{131})\dot{\theta}_{132}\Big)\bar{u}_1 \\
&+ \Big(-\cos\theta_6\dot{\theta}_5 - \sin\theta_6\sin(\theta_5 - \theta_{132})\dot{\theta}_{131} \\
&+ (\cos\theta_6\cos\theta_{131} - \sin\theta_6\sin\theta_{131}\cos(\theta_5 - \theta_{132}))\dot{\theta}_{132}\Big)\bar{u}_2 \\
&+ \Big(\dot{\theta}_6 + \cos(\theta_5 - \theta_{132})\dot{\theta}_{131} - \sin\theta_{131}\sin(\theta_5 - \theta_{132})\dot{\theta}_{132}\Big)\bar{u}_3
\end{aligned}
\tag{4.575}
$$

Position vector of Body-7 can be written as follows:

$$
\vec{r}_7 = r_{GO1}\vec{u}_1^{(5)} + r_{71}\vec{u}_1^{(7)} + r_{73}\vec{u}_3^{(7)}
\tag{4.576}
$$

Necessary unit vector transformations can be done as follows:

$$
\begin{aligned}
\bar{u}_1^{(5/7)} =& \hat{C}^{(7,5)}\bar{u}_1 = e^{\tilde{u}_1\pi/2} e^{-\tilde{u}_2\theta_6} e^{-\tilde{u}_3\theta_5}\bar{u}_1 \\
=& e^{\tilde{u}_1\pi/2} e^{-\tilde{u}_2\theta_6}\left(\cos\theta_5\bar{u}_1 - \sin\theta_5\bar{u}_2\right) \\
=& e^{\tilde{u}_1\pi/2}\left(\cos\theta_5\cos\theta_6\bar{u}_1 + \cos\theta_5\sin\theta_6\bar{u}_3 - \sin\theta_5\bar{u}_2\right) \\
\bar{u}_1^{(5/7)} =& \cos\theta_5\cos\theta_6\bar{u}_1 - \cos\theta_5\sin\theta_6\bar{u}_2 - \sin\theta_5\bar{u}_3
\end{aligned}
\tag{4.577}
$$

Then, 4.576 can be written in matrix form.

$$
\bar{r}_7 = \bar{r}_7^{(7)} = \left(r_{71} + r_{GO1}\cos\theta_5\cos\theta_6\right)\bar{u}_1 - r_{GO1}\cos\theta_5\sin\theta_6\bar{u}_2
$$

$$+ \left(r_{73} - r_{GO1}\sin\theta_5\right)\bar{u}_3 \tag{4.578}$$

By using Coriolis theorem:

$$\vec{v}_7 = D_0\vec{r}_7 = D_7\vec{r}_7 + \vec{\omega}_7 \times \vec{r}_7 \tag{4.579}$$

$$\bar{v}_7 = \bar{v}_7^{(7)} = \dot{\bar{r}}_7 + \widetilde{\omega}_7\bar{r}_7 \tag{4.580}$$

$$\begin{aligned}\dot{\bar{r}}_7 =& \frac{d}{dt}\left(\left(r_{71} + r_{GO1}\cos\theta_5\cos\theta_6\right)\bar{u}_1\right.\\ &\left.- r_{GO1}\cos\theta_5\sin\theta_6\bar{u}_2 + \left(r_{73} - r_{GO1}\sin\theta_5\right)\bar{u}_3\right)\\ &= \left(-r_{GO1}\sin\theta_5\cos\theta_6\dot{\theta}_5 - r_{GO1}\cos\theta_5\sin\theta_6\dot{\theta}_6\right)\bar{u}_1\\ &+ \left(r_{GO1}\sin\theta_5\sin\theta_6\dot{\theta}_5 - r_{GO1}\cos\theta_5\cos\theta_6\dot{\theta}_6\right)\bar{u}_2\\ &-r_{GO1}\cos\theta_5\dot{\theta}_5\bar{u}_3\end{aligned} \tag{4.581}$$

$$\begin{aligned}\widetilde{\omega}_7\bar{r}_7 = &\Big[\Big(-\cos\theta_6\dot{\theta}_5 - \sin\theta_6\sin\left(\theta_5 - \theta_{132}\right)\dot{\theta}_{131}\\ &+ \left(\cos\theta_6\cos\theta_{131} - \sin\theta_6\sin\theta_{131}\cos\left(\theta_5 - \theta_{132}\right)\right)\dot{\theta}_{132}\Big)\left(r_{73} - r_{GO1}\sin\theta_5\right)\\ &- \Big(\dot{\theta}_6 + \cos\left(\theta_5 - \theta_{132}\right)\dot{\theta}_{131} - \sin\theta_{131}\sin\left(\theta_5 - \theta_{132}\right)\dot{\theta}_{132}\Big)\\ &\left(-r_{GO1}\cos\theta_5\sin\theta_6\right)\Big]\bar{u}_1\\ &+ \Big[\Big(\dot{\theta}_6 + \cos\left(\theta_5 - \theta_{132}\right)\dot{\theta}_{131} - \sin\theta_{131}\sin\left(\theta_5 - \theta_{132}\right)\dot{\theta}_{132}\Big)\\ &\left(r_{71} + r_{GO1}\cos\theta_5\cos\theta_6\right) - \Big(-\sin\theta_6\dot{\theta}_5 + \cos\theta_6\sin\left(\theta_5 - \theta_{132}\right)\dot{\theta}_{131}\\ &+ \left(\cos\theta_6\sin\theta_{131}\cos\left(\theta_5 - \theta_{132}\right) + \sin\theta_6\cos\theta_{131}\right)\dot{\theta}_{132}\Big)\left(r_{73} - r_{GO1}\sin\theta_5\right)\Big]\bar{u}_2\\ &+ \Big[\Big(-\sin\theta_6\dot{\theta}_5 + \cos\theta_6\sin\left(\theta_5 - \theta_{132}\right)\dot{\theta}_{131}\\ &+ \left(\cos\theta_6\sin\theta_{131}\cos\left(\theta_5 - \theta_{132}\right) + \sin\theta_6\cos\theta_{131}\right)\dot{\theta}_{132}\Big)\left(-r_{GO1}\cos\theta_5\sin\theta_6\right)\\ &- \Big(-\cos\theta_6\dot{\theta}_5 - \sin\theta_6\sin\left(\theta_5 - \theta_{132}\right)\dot{\theta}_{131}\\ &+ \left(\cos\theta_6\cos\theta_{131} - \sin\theta_6\sin\theta_{131}\cos\left(\theta_5 - \theta_{132}\right)\right)\dot{\theta}_{132}\Big)\\ &\left(r_{71} + r_{GO1}\cos\theta_5\cos\theta_6\right)\Big]\bar{u}_3\end{aligned} \tag{4.582}$$

Let's define $\bar{v}_7$ as,

$$\bar{v}_7 = v_{71}\bar{u}_1 + v_{72}\bar{u}_2 + v_{73}\bar{u}_3 \tag{4.583}$$

$$
\begin{aligned}
v_{71} &= -r_{GO1}\sin\theta_5\cos\theta_6\dot{\theta}_5 - r_{GO1}\cos\theta_5\sin\theta_6\dot{\theta}_6 \\
&+ \Big(-\cos\theta_6\dot{\theta}_5 - \sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131} \\
&+ (\cos\theta_6\cos\theta_{131} - \sin\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132}))\,\dot{\theta}_{132}\Big)\,(r_{73} - r_{GO1}\sin\theta_5) \\
&- \Big(\dot{\theta}_6 + \cos(\theta_5-\theta_{132})\,\dot{\theta}_{131} - \sin\theta_{131}\sin(\theta_5-\theta_{132})\,\dot{\theta}_{132}\Big)\,(-r_{GO1}\cos\theta_5\sin\theta_6) \\
&= (-\sin(\theta_5-\theta_{132})\,r_{73} + \sin(\theta_5-\theta_{132})\,r_{GO1}\sin\theta_5 \\
&+ \cos(\theta_5-\theta_{132})\,r_{GO1}\cos\theta_5)\sin\theta_6\dot{\theta}_{131} \\
&+ ((\cos\theta_6\cos\theta_{131} - \sin\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132}))\,(r_{73} - r_{GO1}\sin\theta_5) \\
&- \sin\theta_{131}\sin(\theta_5-\theta_{132})\,(r_{GO1}\cos\theta_5\sin\theta_6))\,\dot{\theta}_{132} - r_{73}\cos\theta_6\dot{\theta}_5 \\
&+ (r_{GO1}\cos\theta_5\sin\theta_6 - r_{GO1}\cos\theta_5\sin\theta_6)\,\dot{\theta}_6
\end{aligned}
$$

$$
\begin{aligned}
v_{71} &= \big(r_{GO1}\cos\theta_{132}\sin\theta_6 - r_{73}\sin\theta_6\sin(\theta_5-\theta_{132})\,\big)\dot{\theta}_{131} \\
&+ \big(r_{73}\cos\theta_{131}\cos\theta_6 - r_{73}\sin\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132}) \\
&- r_{GO1}\cos\theta_{131}\sin\theta_5\cos\theta_6 + r_{GO1}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\big)\dot{\theta}_{132} - r_{73}\cos\theta_6\dot{\theta}_5
\end{aligned}
\tag{4.584}
$$

$$
\begin{aligned}
v_{72} &= r_{GO1}\sin\theta_5\sin\theta_6\dot{\theta}_5 - r_{GO1}\cos\theta_5\cos\theta_6\dot{\theta}_6 \\
&+ \Big(\dot{\theta}_6 + \cos(\theta_5-\theta_{132})\,\dot{\theta}_{131} - \sin\theta_{131}\sin(\theta_5-\theta_{132})\,\dot{\theta}_{132}\Big) \\
&(r_{71} + r_{GO1}\cos\theta_5\cos\theta_6) - \Big(-\sin\theta_6\dot{\theta}_5 + \cos\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131} \\
&+ (\cos\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132}) + \sin\theta_6\cos\theta_{131})\,\dot{\theta}_{132}\Big)\,(r_{73} - r_{GO1}\sin\theta_5) \\
&= (r_{71}\cos(\theta_5-\theta_{132}) + r_{GO1}\cos\theta_5\cos\theta_6\cos(\theta_5-\theta_{132}) \\
&- r_{73}\cos\theta_6\sin(\theta_5-\theta_{132}) + r_{GO1}\sin\theta_5\cos\theta_6\sin(\theta_5-\theta_{132}))\,\dot{\theta}_{131} \\
&+ (-r_{71}\sin\theta_{131}\sin(\theta_5-\theta_{132}) - r_{GO1}\sin\theta_{131}\cos\theta_5\cos\theta_6\sin(\theta_5-\theta_{132}) \\
&+ r_{GO1}\sin\theta_{131}\sin\theta_5\cos\theta_6\cos(\theta_5-\theta_{132}) + r_{GO1}\cos\theta_{131}\sin\theta_5\sin\theta_6 \\
&- r_{73}\sin\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132}) - r_{73}\cos\theta_{131}\sin\theta_6)\,\dot{\theta}_{132} \\
&+ (r_{GO1}\sin\theta_5\sin\theta_6 + r_{73}\sin\theta_6 - r_{GO1}\sin\theta_5\sin\theta_6)\,\dot{\theta}_5 \\
&+ (-r_{GO1}\cos\theta_5\cos\theta_6 + r_{71} + r_{GO1}\cos\theta_5\cos\theta_6)\,\dot{\theta}_5
\end{aligned}
$$

$$
\begin{aligned}
v_{72} =& \left(r_{71}\cos\left(\theta_5-\theta_{132}\right)+r_{GO1}\cos\theta_{132}\cos\theta_6-r_{73}\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\right)\dot{\theta}_{131} \\
&+\left(r_{GO1}\sin\theta_{131}\sin\theta_{132}\cos\theta_6 - r_{71}\sin\theta_{131}\sin\left(\theta_5-\theta_{132}\right)\right. \\
&+r_{GO1}\cos\theta_{131}\sin\theta_5\sin\theta_6-r_{73}\sin\theta_{131}\cos\theta_6\cos\left(\theta_5-\theta_{132}\right) \\
&\left.- r_{73}\cos\theta_{131}\sin\theta_6\right)\dot{\theta}_{132}+r_{73}\sin\theta_6\dot{\theta}_5+r_{71}\dot{\theta}_6
\end{aligned}
\tag{4.585}
$$

$$
\begin{aligned}
v_{73} =& -r_{GO1}\cos\theta_5\dot{\theta}_5+\Big(-\sin\theta_6\dot{\theta}_5+\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131} \\
&+ \left(\cos\theta_6\sin\theta_{131}\cos\left(\theta_5-\theta_{132}\right)+\sin\theta_6\cos\theta_{131}\right)\dot{\theta}_{132}\Big)\left(-r_{GO1}\cos\theta_5\sin\theta_6\right) \\
&- \Big(-\cos\theta_6\dot{\theta}_5-\sin\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131} \\
&+ \left(\cos\theta_6\cos\theta_{131}-\sin\theta_6\sin\theta_{131}\cos\left(\theta_5-\theta_{132}\right)\right)\dot{\theta}_{132}\Big) \\
&\left(r_{71}+r_{GO1}\cos\theta_5\cos\theta_6\right) \\
=& \left(-r_{GO1}\cos\theta_5\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)+r_{71}\sin\theta_6\sin\left(\theta_5-\theta_{132}\right)\right. \\
&\left.+ r_{GO1}\cos\theta_5\cos\theta_6\sin\theta_6\sin\left(\theta_5-\theta_{132}\right)\right)\dot{\theta}_{131} \\
&+\left(-r_{GO1}\sin\theta_{131}\cos\theta_5\sin\theta_6\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\right. \\
&- r_{GO1}\cos\theta_{131}\cos\theta_5\sin\theta_6\sin\theta_6-r_{71}\cos\theta_{131}\cos\theta_6 \\
&+r_{71}\sin\theta_{131}\sin\theta_6\cos\left(\theta_5-\theta_{132}\right)-r_{GO1}\cos\theta_{131}\cos\theta_5\cos\theta_6\cos\theta_6 \\
&\left.+r_{GO1}\sin\theta_{131}\cos\theta_5\cos\theta_6\sin\theta_6\cos\left(\theta_5-\theta_{132}\right)\ \right)\dot{\theta}_{132} \\
&+\left(r_{GO1}\cos\theta_5\sin\theta_6\sin\theta_6-r_{GO1}\cos\theta_5+r_{71}\cos\theta_6\right. \\
&\left.+ r_{GO1}\cos\theta_5\cos\theta_6\cos\theta_6\right)\dot{\theta}_5
\end{aligned}
$$

$$
\begin{aligned}
v_{73} =& r_{71}\sin\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}+\left(\ r_{71}\sin\theta_{131}\sin\theta_6\cos\left(\theta_5-\theta_{132}\right)\right. \\
&\left.- r_{GO1}\cos\theta_{131}\cos\theta_5-r_{71}\cos\theta_{131}\cos\theta_6\ \right)\dot{\theta}_{132}+r_{71}\cos\theta_6\dot{\theta}_5
\end{aligned}
\tag{4.586}
$$

The potential energy can be expressed as follows:

$$U_7 = -m_7 g^{-(0)T}\overline{r}_7^{(7/0)} \tag{4.587}$$

The transition of position vector of Body-7 from F_7 to F_0 can be done by the transformation matrix $\hat{C}^(0,7)$.

$$\overline{r}_7^{(7/0)} = \hat{C}^{(0,7)}\overline{r}_7^{(7/7)} \tag{4.588}$$

Combine (4.587) & (4.588) to obtain:

$$U_7 = -m_7 \overline{g}^{(0)T} \hat{C}^{(0,7)} \overline{r}_7 \tag{4.589}$$

Writing (4.578) in vector form:

$$\vec{r}_7 = \vec{u}_1^{(7)} \left(r_{GO1} c\theta_5 c\theta_6 + r_{71} \right) + \vec{u}_2^{(7)} \left(-r_{GO1} c\theta_5 s\theta_6 \right) - \vec{u}_3^{(7)} \left(r_{GO1} s\theta_5 - r_{73} \right)$$

Then, let's resolve $\vec{r}_7$ in F_0:

$$\overline{r}_7^{(0)} = \overline{u}_1^{(7/0)} \left(r_{GO1} c\theta_5 c\theta_6 + r_{71} \right) + \overline{u}_2^{(7/0)} \left(-r_{GO1} c\theta_5 c\theta_6 \right) - \overline{u}_3^{(7/0)} \left(r_{GO1} s\theta_5 - r_{73} \right) \tag{4.590}$$

We know that

$$\hat{C}^{(0,5)} = e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 \theta_{132}} e^{-\tilde{u}_2 \pi/2} \quad ; \quad \hat{C}^{(5,7)} = e^{\tilde{u}_3 \theta_5} e^{-\tilde{u}_1 \pi/2} e^{-\tilde{u}_3 \theta_6}$$

The transformation matrix from Body-0 to Body-7 can be expressed as follows:

$$\begin{aligned}
\hat{C}^{(0,7)} &= \hat{C}^{(0,5)} \hat{C}^{(5,7)} \\
&= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 \theta_{132}} e^{-\tilde{u}_2 \pi/2} e^{\tilde{u}_3 \theta_5} \underbrace{e^{-\tilde{u}_1 \pi/2} e^{\tilde{u}_3 \theta_6} e^{\tilde{u}_1 \pi/2}}_{e^{\tilde{u}_2 \theta_6}} e^{\tilde{u}_1 \pi/2} \\
&= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 \theta_{132}} \underbrace{e^{-\tilde{u}_2 \pi/2} e^{\tilde{u}_3 \theta_5} e^{\tilde{u}_3 \pi/2}}_{e^{-\tilde{u}_1 \theta_5}} e^{-\tilde{u}_2 \pi/2} e^{\tilde{u}_2 \theta_6} e^{-\tilde{u}_1 \pi/2} \\
&= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_5 - \theta_{132})} e^{\tilde{u}_2 (\theta_6 - \pi/2)} e^{-\tilde{u}_1 \pi/2} \\
&= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_5 - \theta_{132})} e^{\tilde{u}_2 \theta_6} e^{-\tilde{u}_2 \pi/2} e^{-\tilde{u}_1 \pi/2}
\end{aligned} \tag{4.591}$$

The unit vectors can be transformed as follows:

$$\begin{aligned}
\therefore \ \overline{u}_1^{(7/0)} &= \hat{C}^{(0,7)} \overline{u}_1 \\
&= e^{\tilde{u}_2 \theta_{131}} e^{\tilde{u}_1 (\theta_5 - \theta_{132})} e^{\tilde{u}_2 \theta_6} e^{-\tilde{u}_2 \pi/2} \overbrace{e^{-\tilde{u}_1 \pi/2} \overline{u}_1}^{\overline{u}_1} \\
&= e^{\tilde{u}_2 \theta_{131}} e^{-\tilde{u}_1 (\theta_5 - \theta_{132})} \left(\overline{u}_3 c\theta_6 + \overline{u}_1 s\theta_6 \right) \\
&= e^{-\tilde{u}_2 \theta_{131}} \left(\overline{u}_1 s\theta_6 + c\theta_6 \left(\overline{u}_3 c \left(\theta_5 - \theta_{132} \right) + \overline{u}_2 s \left(\theta_5 - \theta_{132} \right) \right) \right) \\
&= e^{\tilde{u}_2 \theta_{131}} \left(\overline{u}_1 s\theta_6 + \overline{u}_2 s \left(\theta_5 - \theta_{132} \right) c\theta_6 + \overline{u}_3 c \left(\theta_5 - \theta_{132} \right) c\theta_6 \right) \\
&= s\theta_6 \left(\overline{u}_1 c\theta_{131} - \overline{u}_3 s\theta_{131} \right) + \overline{u}_2 s \left(\theta_5 - \theta_{132} \right) c\theta_6 \\
&\quad + c \left(\theta_5 - \theta_{132} \right) c\theta_6 \left(\overline{u}_3 c\theta_{131} + \overline{u}_1 s\theta_{131} \right) \\
\Rightarrow \ \overline{u}_1^{(7/0)} &= \overline{u}_1 \left(s\theta_6 c\theta_{131} + s\theta_{131} c \left(\theta_5 - \theta_{132} \right) c\theta_6 \right) + \overline{u}_2 s \left(\theta_5 - \theta_{132} \right) c\theta_6
\end{aligned}$$

$$
\begin{aligned}
&+ \overline{u}_3 \left(c\theta_{131} c\left(\theta_5 - \theta_{132}\right) c\theta_6 - s\theta_{131} s\theta_6\right) && (4.592)\\
\therefore\ \overline{u}_2^{(7/0)} &= \hat{C}^{(0/7)} \overline{u}_2 \\
&= e^{\tilde{u}_2\theta_{131}} e^{-\tilde{u}_1(\theta_5-\theta_{132})} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_2\pi/2} e^{-\tilde{u}_1\pi/2} \overline{u}_2 \\
&= e^{\tilde{u}_2\theta_{131}} e^{-\tilde{u}_1(\theta_5-\theta_{132})} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_2\pi/2} \left(-\overline{u}_3\right) \\
&= e^{\tilde{u}_2\theta_{131}} e^{-\tilde{u}_1(\theta_5-\theta_{132})} e^{\tilde{u}_2\theta_6} \overline{u}_1 = e^{\tilde{u}_2\theta_{131}} e^{-\tilde{u}_1(\theta_5-\theta_{132})} \left(\overline{u}_1 c\theta_6 - \overline{u}_3 s\theta_6\right) \\
&= e^{\tilde{u}_2\theta_{131}} \left(\overline{u}_1 c\theta_6 - s\theta_6 \left(\overline{u}_3 c\left(\theta_5 - \theta_{132}\right) - \overline{u}_2 s\left(\theta_5 - \theta_{132}\right)\right)\right) \\
&= e^{\tilde{u}_2\theta_{131}} \left(\overline{u}_1 c\theta_6 - \overline{u}_2 s\left(\theta_5 - \theta_{132}\right) s\theta_6 - \overline{u}_3 s\theta_6 c\left(\theta_5 - \theta_{132}\right)\right) \\
&= c\theta_6 \left(\overline{u}_1 c\theta_{131} - \overline{u}_3 s\theta_{131}\right) - \overline{u}_2 s\left(\theta_5 - \theta_{132}\right) s\theta_6 \\
&- s\theta_6 c\left(\theta_5 - \theta_{132}\right) \left(\overline{u}_3 c\theta_{131} + \overline{u}_1 s\theta_{131}\right) \\
\Rightarrow\ \overline{u}_2^{(7/0)} &= \overline{u}_1 \left(c\theta_{131} c\theta_6 - s\theta_{131} s\theta_6 c\left(\theta_5 - \theta_{132}\right)\right) - \overline{u}_2 s\left(\theta_5 - \theta_{132}\right) s\theta_6 \\
&- \overline{u}_3 \left(s\theta_{131} c\theta_6 + s\theta_6 c\theta_{131} c\left(\theta_5 - \theta_{132}\right)\right) && (4.593)\\
\therefore\ \overline{u}_3^{(7/0)} &= \hat{C}^{(0,7)} \overline{u}_3 \\
&= e^{\tilde{u}_2\theta_{131}} e^{\tilde{u}_1(\theta_5-\theta_{132})} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_2\pi/2} \overbrace{e^{-\tilde{u}_1\pi/2} \overline{u}_3}^{\overline{u}_2} \\
&= e^{\tilde{u}_2\theta_{131}} e^{-\tilde{u}_1(\theta_5-\theta_{132})} \overline{u}_2 \\
&= e^{\tilde{u}_2\theta_{131}} \left(\overline{u}_2 c\left(\theta_5 - \theta_{132}\right) - \overline{u}_3 s\left(\theta_5 - \theta_{132}\right)\right) \\
&= \overline{u}_2 c\left(\theta_5 - \theta_{132}\right) - s\left(\theta_5 - \theta_{132}\right) \left(\overline{u}_3 c\theta_{131} + \overline{u}_1 s\theta_{131}\right) \\
\Rightarrow\ \overline{u}_3^{(0/7)} &= -\tilde{u}_1 s\theta_{131} s\left(\theta_5 - \theta_{132}\right) + \overline{u}_2 c\left(\theta_5 - \theta_{132}\right) - \overline{u}_3 s\left(\theta_5 - \theta_{132}\right) c\theta_{131} \\
& && (4.594)
\end{aligned}
$$

Substitution of (4.592), (4.593) & (4.594) into (4.590) yields the following results;

$$
\begin{aligned}
\overline{r}_7 = &\left(\overline{u}_1 \left(s\theta_6 c\theta_{131} + s\theta_{131} c\left(\theta_5 - \theta_{132}\right) c\theta_6\right) + \overline{u}_2 s\left(\theta_5 - \theta_{132}\right) c\theta_6\right. && (4.595)\\
&\left. + \overline{u}_3 \left(c\theta_{131} c\left(\theta_5 - \theta_{132}\right) c\theta_6 - s\theta_{131} s\theta_6\right)\right) \left(r_{GO1} c\theta_5 c\theta_6 + r_{71}\right) \\
&+ \left(\overline{u}_1 \left(c\theta_{131} c\theta_6 - s\theta_{131} s\theta_6 c\left(\theta_5 - \theta_{132}\right) - \overline{u}_2 s\left(\theta_5 - \theta_{132}\right) s\theta_6\right.\right. \\
&\left.\left. - \overline{u}_3 \left(s\theta_{131} c\theta_6 + s\theta_6 c\theta_{131} c\left(\theta_5 - \theta_{132}\right)\right)\right)\right) \left(-r_{GO1} c\theta_5 s\theta_6\right) \\
&\left(-\overline{u}_1 s\theta_{131} s\left(\theta_5 - \theta_{132}\right) + \overline{u}_2 c\left(\theta_5 - \theta_{132}\right) - \overline{u}_3 s\left(\theta_5 - \theta_{132}\right) c\theta_{131}\right) \left(-r_{GO1} s\theta_5 + r_{73}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ \overline{r}_7 = \overline{u}_1 &\left(r_{GO1} s\theta_6 c\theta_{131} c\theta_5 c\theta_6 + r_{GO1} s\theta_{131} c\theta_5 c\left(\theta_5 - \theta_{132}\right) c^2\theta_6\right. \\
&+ r_{71} s\theta_6 c\theta_{131} + r_{71} s\theta_{131} c\left(\theta_5 - \theta_{132}\right) c\theta_6 - r_{GO1} c\theta_{131} c\theta_5 s\theta_6 c\theta_6
\end{aligned}
$$

$$
\begin{aligned}
&+ r_{GO1} s\theta_{131} s^2\theta_6 c\theta_5 c\left(\theta_5-\theta_{132}\right)+r_{GO1} s\theta_{131} s\theta_5 s\left(\theta_5-\theta_{132}\right)\\
&-r_{73} s\theta_{131} s\left(\theta_5-\theta_{132}\right)\Big)\\
&+\bar{u}_2\left(r_{GO1} s\left(\theta_5-\theta_{132}\right) c\theta_5 c^2\theta_6+r_{71} s\left(\theta_5-\theta_{132}\right) c\theta_6\right.\\
&+r_{GO1} s\left(\theta_5-\theta_{132}\right) s^2\theta_6 c\theta_5\\
&\left.-r_{GO1} s\theta_5 c\left(\theta_5-\theta_{132}\right)+r_{73} c\left(\theta_5-\theta_{132}\right)\right)\\
&+\bar{u}_3\left(r_{GO1} c\theta_{131} c\theta_5 c\left(\theta_5-\theta_{132}\right) c^2\theta_6+r_{71} c\theta_{131} c\left(\theta_5-\theta_{132}\right) c\theta_6\right.\\
&-r_{GO1} s\theta_{131} s\theta_6 c\theta_5 c\theta_6-r_{71} s\theta_{131} s\theta_{131} s\theta_6+r_{GO1} s\theta_{131} s\theta_6 c\theta_5 c\theta_6\\
&+r_{GO1} s^2\theta_6 c\theta_{131} c\theta_5 c\left(\theta_5-\theta_{132}\right)+r_{GO1} s\theta_5 s\left(\theta_5-\theta_{132}\right) c\theta_{131}\\
&\left.-r_{73} s\left(\theta_5-\theta_{132}\right) c\theta_{131}\right)
\end{aligned}
\tag{4.596}
$$

$$
\begin{aligned}
\Rightarrow\ \bar{r}_{73} &= r_{GO1} c\theta_{131} c\theta_5 c\left(\theta_5-\theta_{132}\right)+r_{71} c\theta_{131} c\left(\theta_5-\theta_{132}\right) c\theta_6\\
&-r_{71} s\theta_{131} s\theta_6+r_{GO1} s\theta_5 s\left(\theta_5-\theta_{132}\right) c\theta_{131}-r_{73} s\left(\theta_5-\theta_{132}\right) c\theta_{131}
\end{aligned}
$$

$$
\Rightarrow\ \bar{r}_{73} = r_{GO1} c\theta_{131} \underbrace{\left(c\theta_5 c\left(\theta_5-\theta_{132}\right)+s\theta_5 s\left(\theta_5-\theta_{132}\right)\right)}_{c(\theta_5-\theta_5+\theta_{132})} \tag{4.597}
$$

$$
\begin{aligned}
\Rightarrow\ \bar{r}_{73} &= r_{GO1} c\theta_{131} c\theta_{132}+r_{71} c\theta_{131} c\left(\theta_5-\theta_{132}\right) c\theta_6-r_{71} s\theta_{131} s\theta_6\\
&-r_{73} s\left(\theta_5-\theta_{132}\right) c\theta_{131}
\end{aligned}
\tag{4.598}
$$

$$
\begin{aligned}
U_7 &= -m_7 \bar{g}^{(0)^T} \bar{r}_7^{(0)} = -m_7 \bar{g}^{(0)^T} \hat{C}^{(0,7)} \bar{r}_7\\
&= -m_7 \bar{g}^{(0)^T}\Big(\left(r_{71}+r_{GO1}\cos\theta_5\cos\theta_6\right)\bar{u}_1^{(7/0)}-r_{GO1}\cos\theta_5\sin\theta_6\bar{u}_2^{(7/0)}\\
&+\left(r_{73}-r_{GO1}\sin\theta_5\right)\bar{u}_3^{(7/0)}\Big)
\end{aligned}
\tag{4.599}
$$

Substituting (4.592), (4.593) and (4.594) into (4.599), we get

$$
\begin{aligned}
U_7 &= m_7 g\left[r_{71}\cos\theta_{131}\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)-r_{71}\sin\theta_{131}\sin\theta_6\right.\\
&+r_{GO1}\cos\theta_{131}\cos\theta_5\cos\theta_6\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\\
&-r_{GO1}\sin\theta_{131}\cos\theta_5\cos\theta_6\sin\theta_6+r_{GO1}\sin\theta_{131}\cos\theta_5\sin\theta_6\cos\theta_6\\
&-r_{73}\cos\theta_{131}\sin\left(\theta_5-\theta_{132}\right)+r_{GO1}\cos\theta_{131}\sin\theta_5\sin\left(\theta_5-\theta_{132}\right)\\
&\left.+r_{GO1}\cos\theta_{131}\cos\theta_5\sin\theta_6\sin\theta_6\cos\left(\theta_5-\theta_{132}\right)\right]\\
&= m_7 g\left[r_{71}\cos\theta_{131}\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)-r_{71}\sin\theta_{131}\sin\theta_6\right.\\
&+r_{GO1}\cos\theta_{131}\cos\theta_5\cos\left(\theta_5-\theta_{132}\right)-r_{73}\cos\theta_{131}\sin\left(\theta_5-\theta_{132}\right)
\end{aligned}
$$

$$+ r_{GO1}\cos\theta_{131}\sin\theta_5\sin(\theta_5-\theta_{132})]$$

$$= m_7 g\,[r_{71}\cos\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132}) - r_{71}\sin\theta_{131}\sin\theta_6$$

$$+ r_{GO1}\cos\theta_{131}\cos\theta_{132} - r_{73}\cos\theta_{131}\sin(\theta_5-\theta_{132})] \tag{4.600}$$

For $q_1 = \theta_5$;

$$\Rightarrow \frac{\partial U_7}{\partial\theta_5} = -m_7 g\,(r_{71}s(\theta_5-\theta_{132})\,c\theta_{131}c\theta_6 + r_{73}c\theta_{131}c(\theta_5-\theta_{132}) \tag{4.601}$$

For $q_2 = \theta_6$;

$$\Rightarrow \frac{\partial U_7}{\partial\theta_6} = -m_7 g\,(r_{71}s\theta_6 c\theta_{131}c(\theta_5-\theta_{132}) + r_{71}s\theta_{131}c\theta_6 \tag{4.602}$$

The whole potential energy can be expressed as follows:

$$* \frac{\partial U}{\partial\theta_5} = \frac{\partial U_6}{\partial\theta_5} + \frac{\partial U_7}{\partial\theta_5} \tag{4.603}$$

$$= -m_6 g r_{63} c\theta_{131} c(\theta_5-\theta_{132}) \tag{4.604}$$

$$= -m_7 g\,(r_{71}s(\theta_5-\theta_{132})\,c\theta_{131}c\theta_6 + r_{73}c\theta_{131}c(\theta_5-\theta_{132})) \tag{4.605}$$

$$* \frac{\partial U}{\partial\theta_6} = \frac{\partial U_6}{\partial\theta_6} + \frac{\partial U_7}{\partial\theta_6} = \frac{\partial U_7}{\partial\theta_6} \tag{4.606}$$

$$= -m_7 g\,(r_{71}s\theta_6 c\theta_{131}c(\theta_5-\theta_{132}) + r_{71}s\theta_{131}c\theta_6) \tag{4.607}$$

Before we proceed with Body-7 velocities; we need to find the derivatives of velocity terms of Body-6 w.r.t θ_6.

$$\frac{\partial}{\partial\theta_6}\left(v_6^T v_6\right) = 2v_6^T \underbrace{\frac{\partial v_6}{\partial\theta_6}}_{0} = 0 \quad\Rightarrow\quad \frac{\partial K_{6t}}{\partial\theta_6} = 0 \tag{4.608}$$

$$\frac{\partial}{\partial\theta_6}\left(w_6^T \hat{J}_6 \overline{w}_6\right) = 2\overline{w}_6^T \hat{J}_6 \underbrace{\left(\frac{\partial\overline{w}_6}{\partial\theta_6}\right)}_{0} = 0 \quad\Rightarrow\quad \frac{\partial K_{6r}}{\partial\theta_6} = 0 \tag{4.609}$$

$$\frac{\partial K_6}{\partial\theta_6} = \frac{\partial K_{6t}}{\partial\theta_6} + \frac{\partial K_6 r}{\partial\theta_6} = 0 \tag{4.610}$$

Similarly,

$$\frac{\partial K_{6r}}{\partial\dot\theta_6} = 0, \quad \frac{\partial K_{6t}}{\partial\dot\theta_6} = 0, \quad\Rightarrow\quad \frac{\partial K_6}{\partial\dot\theta_6} = \frac{\partial K_{6r}}{\partial\dot\theta_6} + \frac{\partial K_{6t}}{\partial\dot\theta_6} = 0 \tag{4.611}$$

The kinetic energy of Body-7 can be expressed as follows;

$$K_7 = \frac{1}{2}\left(m_7 v_7^T v_7 + \overline{w}_7^T \hat{J}_7^{(7)} \overline{w}_7\right)$$

$$\frac{\partial K_{7t}}{\partial \theta_5} = 2\overline{v}_7^T \frac{\partial \overline{v}_7}{\partial \theta_5} \quad ; \quad K_{7t} = \frac{1}{2}.2v_7^T \frac{\partial \overline{v}_7}{\partial \theta_6} \quad \Rightarrow \quad K_{7t} = v_7^T \frac{\partial \overline{v}_7}{\partial \theta_5}$$

For $q_1 = \theta_5$;

$$\frac{\partial V_{71}}{\partial \theta_5} = -r_{73}\dot{\theta}_{131}s\theta_6 c\left(\theta_5 - \theta_{132}\right) + \dot{\theta}_{132}\left(r_{73}s\theta_{131}s\left(\theta_5 - \theta_{132}\right)s\theta_6 - r_{GO1}c\theta_{131}c\theta_5 c\theta_6\right) \tag{4.612}$$

$$\begin{aligned}\frac{\partial V_{72}}{\partial \theta_5} &= \dot{\theta}_{131}\left(-r_{71}s\left(\theta_5 - \theta_{132}\right) - r_{73}c\left(\theta_5 - \theta_{132}\right)c\theta_6\right) \\ &+ \dot{\theta}_{132}\left(-r_{71}s\theta_{131}c\left(\theta_5 - \theta_{132}\right) + r_{GO1}s\theta_6 c\theta_{131}c\theta_5 + r_{73}s\theta_{131}s\left(\theta_5 - \theta_{132}\right)c\theta_6\right)\end{aligned} \tag{4.613}$$

$$\frac{\partial V_{73}}{\partial \theta_5} = \dot{\theta}_{131}\left(r_{71}s\theta_6 c\left(\theta_5 - \theta_{132}\right)\right) + \dot{\theta}_{132}\left(r_{GO1}s\theta_5 c\theta_{131} - r_{71}s\theta_{131}s\left(\theta_5 - \theta_{132}\right)s\theta_6\right) \tag{4.614}$$

Derivative w.r.t θ_5 are:

$$\frac{\partial V_{71}}{\partial \dot{\theta}_5} = -r_{73}c\theta_6, \quad \frac{\partial V_{72}}{\partial \dot{\theta}_5} = r_{73}s\theta_6, \quad \frac{\partial V_{73}}{\partial \dot{\theta}_5} = r_{71}c\theta_6 \tag{4.615}$$

For $q_2 = \theta_6$;

$$\begin{aligned}\frac{\partial V_{71}}{\partial \dot{\theta}_6} &= \dot{\theta}_{131}c\theta_6\left(r_{GO1}c\theta_{132} - r_{73}s\left(\theta_5 - \theta_{132}\right)\right) + \dot{\theta}_5 r_{73}s\theta_6 \\ &+ \dot{\theta}_{132}\left(r_{GO1}s\theta_{131}s\theta_{132}c\theta_6 + r_{GO1}s\theta_5 s\theta_6 c\theta_{131} - r_{73}s\theta_{131}c\left(\theta_5 - \theta_{132}\right)c\theta_6\right.\end{aligned} \tag{4.616}$$

$$\left. - \; r_{73}s\theta_6 c\theta_{131}\right)$$

$$\begin{aligned}\frac{\partial V_{72}}{\partial \theta_6} &= \dot{\theta}_{131}\left(-r_{GO1}s\theta_6 c\theta_{132} + r_{73}s\left(\theta_5 - \theta_{132}\right)s\theta_6\right) + \dot{\theta}_5 r_{73}c\theta_6 \\ &+ \dot{\theta}_{132}\left(-r_{GO1}s\theta_{131}s\theta_{132}s\theta_6 + r_{GO1}s\theta_5 c\theta_{131}c\theta_6 + r_{73}s\theta_{131}s\theta_6 c\left(\theta_5 - \theta_{132}\right)\right. \\ &\left. -r_{73}c\theta_{131}c\theta_6\right)\end{aligned} \tag{4.617}$$

$$\begin{aligned}\frac{\partial V_{73}}{\partial \theta_6} &= \dot{\theta}_{131}r_{71}s\left(\theta_5 - \theta_{132}\right)c\theta_6 + \dot{\theta}_{132}\left(r_{71}s\theta_{131}c\left(\theta_5 - \theta_{132}\right)c\theta_6 + r_{71}s\theta_6 c\theta_{131}\right) \\ &- \dot{\theta}_5 r_{71}s\theta_6\end{aligned} \tag{4.618}$$

Derivatives w.r.t $\dot{\theta}_6$ are;

$$\frac{\partial V_{71}}{\partial \dot{\theta}_6} = 0, \quad \frac{\partial V_{72}}{\partial \dot{\theta}_6} = r_{71}, \quad \frac{\partial V_{73}}{\partial \dot{\theta}_6} = 0 \tag{4.619}$$

Similarly, next step will be taking derivative of rotational kinetic energy terms w.r.t independent variable. For $q_1 = \theta_5$;

$$\frac{\partial \overline{w}_{71}}{\partial \theta_5} = \dot{\theta}_{131}c\left(\theta_5 - \theta_{132}\right)c\theta_6 - \dot{\theta}_{132}s\theta_{131}s\left(\theta_5 - \theta_{132}\right)c\theta_6 \tag{4.620}$$

$$\frac{\partial \overline{w}_{72}}{\partial \theta_5} = -\dot{\theta}_{131} c\left(\theta_5 - \theta_{132}\right) s\theta_6 + \dot{\theta}_{132} s\theta_{131} s\left(\theta_5 - \theta_{132}\right) s\theta_6 \quad (4.621)$$

$$\frac{\partial \overline{w}_{73}}{\partial \theta_5} = -\dot{\theta}_{131} s\left(\theta_5 - \theta_{132}\right) + \dot{\theta}_{132} s\theta_{131} s\left(\theta_5 - \theta_{132}\right) \quad (4.622)$$

$$(4.623)$$

For $\dot{q}_1 = \dot{\theta}_5$;

$$\frac{\partial \overline{w}_{71}}{\partial \dot{\theta}_5} = -s\theta_6 \quad (4.624)$$

$$\frac{\partial \overline{w}_{72}}{\partial \dot{\theta}_5} = c\theta_6 \quad (4.625)$$

$$\frac{\partial \overline{w}_{73}}{\partial \dot{\theta}_5} = 0 \quad (4.626)$$

For $q_2 = \theta_6$;

$$\frac{\partial \overline{w}_{71}}{\partial \theta_6} = -\dot{\theta}_{131} s\left(\theta_5 - \theta_{132}\right) s\theta_6 + \dot{\theta}_{132}\left(c\theta_{131} c\theta_6 - s\theta_{131} s\theta_6 c\left(\theta_5 - \theta_{132}\right)\right) - \dot{\theta}_5 c\theta_6 \quad (4.627)$$

$$\frac{\partial \overline{w}_{72}}{\partial \theta_6} = -\dot{\theta}_{131} s\left(\theta_5 - \theta_{132}\right) c\theta_6 - \dot{\theta}_{132}\left(s\theta_{131} c\left(\theta_5 - \theta_{132}\right) c\theta_6 + s\theta_6 c\theta_{131}\right) + \dot{\theta}_5 s\theta_6 \quad (4.628)$$

$$\frac{\partial \overline{w}_{73}}{\partial \theta_6} = 0 \quad (4.629)$$

Now, we can start evaluating the dot product terms;

$$\frac{\partial K_{7t}}{\partial \theta_5} = m_7 v_7^T \frac{\partial v_7}{\partial \theta_5} = m_7\left(v_{71}\frac{\partial v_{71}}{\partial \theta_5} + v_{72}\frac{\partial v_{72}}{\partial \theta_5} + v_{73}\frac{\partial v_{73}}{\partial \theta_5}\right) \quad (4.630)$$

$$\begin{aligned}
v_{71}\frac{\partial v_{71}}{\partial \theta_5} &= \dot{\theta}_{131}^2 r_{73} s^2\theta_6 c\left(\theta_5 - \theta_{132}\right)\left(-r_{GO1} c\theta_{132} + r_{73} s\left(\theta_5 - \theta_{132}\right)\right) \\
&+ \dot{\theta}_{131}\dot{\theta}_{132}\big(r_{GO1} r_{73} s\theta_{131} s^2\theta_6 s\left(\theta_5 - 2\theta_{132}\right) + r_{73}^2 s\theta_{131} s^2\theta_6 c\left(2\left(\theta_5 - \theta_{132}\right)\right) \\
&+ r_{GO1} r_{73} s\left(2\theta_5 - \theta_{132}\right) s\theta_6 c\theta_{131} c\theta_6 - r_{73}^2 s\theta_6 c\theta_{131} c\left(\theta_5 - \theta_{132}\right) c\theta_6 \\
&- r_{GO1}^2 s\theta_6 c\theta_{131} c\theta_{132} c\theta_5 c\theta_6 \\
&+ \dot{\theta}_{132}^2\big(r_{GO1} r_{73}\big(s^2\theta_{131} s\theta_{132} s\left(\theta_5 - \theta_{132}\right) s^2\theta_6 \\
&+ s\theta_{131} s\theta_6 c\theta_{131} c\left(2\theta_5 - \theta_{132}\right) c\theta_6 - c^2\theta_{131} c\theta_5 c^2\theta_6\big) \\
&+ r_{73}^2 s\theta_{131} s\left(\theta_5 - \theta_{132}\right) s\theta_6\left(-s\theta_{131} s\theta_6 c\left(\theta_5 - \theta_{132}\right) + c\theta_{131} c\theta_6\right) \\
&+ r_{GO1}^2 c\theta_{131} c\theta_5 c\theta_6\left(s\theta_5 c\theta_{131} c\theta_6 - s\theta_{131} s\theta_{132} s\theta_6\right)\big)
\end{aligned}$$

$$+ \dot{\theta}_{131}\dot{\theta}_5 \left(r_{73}^2 s\theta_6 c\left(\theta_5 - \theta_{132}\right) c\theta_6\right)$$
$$+ \dot{\theta}_{132}\dot{\theta}_5 r_{73} c\theta_6 \left(-r_{73} s\theta_{131} s\left(\theta_5 - \theta_{132}\right) s\theta_6 + r_{GO1} c\theta_{131} c\theta_5 c\theta_6\right) \quad (4.631)$$

$$v_{72}\frac{\partial v_{72}}{\partial \theta_5} = \dot{\theta}_{131}^2 \left(-r_{GO1} r_{71} s\left(\theta_5 - \theta_{132}\right) c\theta_{132} c\theta_6 - r_{71}^2 s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right)\right.$$
$$- r_{71} r_{73} c\left(2\left(\theta_5 - \theta_{132}\right)\right) c\theta_6 - r_{GO1} r_{73} c\theta_{132} c\left(\theta_5 - \theta_{132}\right) c^2\theta_6$$
$$\left. + r_{73}^2 s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right) c^2\theta_6\right)$$
$$+ \dot{\theta}_{131}\dot{\theta}_{132} \left(-r_{GO1} r_{71} s\theta_{131} c\left(2\theta_{132} - \theta_5\right) c\theta_6\right.$$
$$+ r_{GO1} r_{71} s\theta_6 c\theta_{131} c\left(2\theta_5 - \theta_{132}\right) - r_{GO1} r_{73} s\left(2\theta_5 - \theta_{132}\right) s\theta_6 c\theta_{131} c\theta_6$$
$$+ r_{GO1} r_{73} s\theta_{131} s\left(\theta_5 - 2\theta_{132}\right) c^2\theta_6 + r_{71} r_{73} s\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right) c\theta_6$$
$$+ r_{73}^2 s\theta_{131} c\left(2\left(\theta_5 - \theta_{132}\right)\right) c^2\theta_6 - r_{71} s\theta_{131} c\left(2\left(\theta_5 - \theta_{132}\right)\right)$$
$$+ r_{71} r_{73} s\theta_{131} s\left(2\left(\theta_5 - \theta_{132}\right)\right) c\theta_6 + r_{71} r_{73} s\left(\theta_5 - \theta_{132}\right) s\theta_6 c\theta_{131}$$
$$\left. + r_{73}^2 s\theta_6 c\theta_{131} c\left(\theta_5 - \theta_{132}\right) c\theta_6 + r_{GO1}^2 s\theta_6 c\theta_{131} c\theta_{132} c\theta_5 c\theta_6\right)$$
$$+ \dot{\theta}_{132}^2 \left(-r_{GO1} r_{71} s^2\theta_{131} s\theta_{132} c\left(\theta_5 - \theta_{132}\right) c\theta_6\right.$$
$$+ r_{71}^2 s^2\theta_{131} s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right) + r_{71} r_{73} s\theta_{131} s\theta_6 c\theta_{131} c\left(\theta_5 - \theta_{132}\right)$$
$$- r_{GO1} r_{71} s\theta_{131} s\left(2\theta_5 - \theta_{132}\right) s\theta_6 c\theta_{131} + r_{71} r_{73} s^2\theta_{131} c\left(2\left(\theta_5 - \theta_{132}\right)\right) c\theta_6$$
$$- r_{GO1} r_{73} s\theta_{131} s\theta_6 c\theta_{131} c\left(2\theta_5 - \theta_{132}\right) c\theta_6 + r_{GO1}^2 s\theta_{131} s\theta_{132} s\theta_6 c\theta_{131} c\theta_5 c\theta_6$$
$$+ r_{GO1}^2 s\theta_5 s^2\theta_6 c^2\theta_{131} c\theta_5 - r_{GO1} r_{73} s^2\theta_6 c^2\theta_{131} c\theta_5$$
$$+ r_{GO1} r_{73} s^2\theta_{131} s\theta_{132} s\left(\theta_5 - \theta_{132}\right) c^2\theta_6$$
$$- r_{73}^2 s^2\theta_{131} s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right) c^2\theta_6$$
$$\left. - r_{73}^2 s\theta_{131} s\left(\theta_5 - \theta_{132}\right) s\theta_6 c\theta_{131} c\theta_6\right)$$
$$+ \dot{\theta}_{131}\dot{\theta}_5 \left(-r_{71} r_{73} s\left(\theta_5 - \theta_{132}\right) s\theta_6 - r_{73}^2 s\theta_6 c\left(\theta_5 - \theta_{132}\right) c\theta_6\right)$$
$$+ \dot{\theta}_{132}\dot{\theta}_5 \left(-r_{71} r_{73} s\theta_{131} s\theta_6 c\left(\theta_5 - \theta_{132}\right) + r_{GO1} r_{73} s^2\theta_6 c\theta_{131} c\theta_5\right.$$
$$\left. + r_{73}^2 s\theta_{131} s\left(\theta_5 - \theta_{132}\right) s\theta_6 c\theta_6\right)$$
$$+ \dot{\theta}_{131}\dot{\theta}_6 \left(-r_{71}^2 s\left(\theta_5 - \theta_{132}\right) - r_{71} r_{73} c\left(\theta_5 - \theta_{132}\right) c\theta_6\right)$$
$$+ \dot{\theta}_{132}\dot{\theta}_6 \left(-r_{71}^2 s\theta_{131} c\left(\theta_5 - \theta_{132}\right) + r_{GO1} r_{71} s\theta_6 c\theta_{131} c\theta_5\right.$$
$$\left. + r_{71} r_{73} s\theta_{131} s\left(\theta_5 - \theta_{132}\right) c\theta_6\right) \quad (4.632)$$

$$v_{73}\frac{\partial v_{73}}{\partial \theta_5} = \dot{\theta}_{131}^2 \left(r_{71}^2 s^2\theta_6 s\left(\theta_5 - \theta_{132}\right) c\left(\theta_5 - \theta_{132}\right)\right)$$

$$
\begin{aligned}
&+ \dot{\theta}_{131}\dot{\theta}_{132} \left(-r_{GO1}r_{71}s\theta_6 c\theta_{131}c\left(\theta_5-\theta_{132}\right)c\theta_5\right.\\
&+ r_{71}^2 s\theta_{131}s^2\theta_6 c\left(2\left(\theta_5-\theta_{132}\right)\right) - r_{71}^2 s\theta_6 c\theta_{131}c\left(\theta_5-\theta_{132}\right)c\theta_6\\
&\left.+\ r_{GO1}r_{71}s\left(\theta_5-\theta_{132}\right)s\theta_5 s\theta_6 c\theta_{131}\right) + \dot{\theta}_{132}^2 \left(-r_{GO1}^2 s\theta_5 c^2\theta_{131}c\theta_5\right.\\
&+ r_{GO1}r_{71}s\theta_{131}s\theta_5 s\theta_6 c\theta_{131}c\left(\theta_5-\theta_{132}\right) - r_{GO1}r_{71}s\theta_5 c^2\theta_{131}c\theta_6\\
&+ r_{GO1}r_{71}s\theta_{131}s\left(\theta_5-\theta_{132}\right)s\theta_6 c\theta_{131}c\theta_5\\
&- r_{71}^2 s^2\theta_{131}s\left(\theta_5-\theta_{132}\right)s^2\theta_6 c\left(\theta_5-\theta_{132}\right)\\
&\left.+ r_{71}^2 s\theta_{131}s\left(\theta_5-\theta_{132}\right)s\theta_6 c\theta_{131}c\theta_6\right) + \dot{\theta}_{131}\dot{\theta}_5\left(r_{71}^2 s\theta_6 c\left(\theta_5-\theta_{132}\right)c\theta_6\right)\\
&+ \dot{\theta}_{132}\dot{\theta}_5\left(r_{GO1}r_{71}s\theta_5 c\theta_{131}c\theta_6 - r_{71}^2 s\theta_{131}s\left(\theta_5-\theta_{132}\right)s\theta_6 c\theta_6\right) \qquad (4.633)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial K_{7t}}{\partial \theta_5} &= m_7\left(v_{71}\frac{\partial v_{71}}{\partial \theta_5} + v_{72}\frac{\partial v_{72}}{\partial \theta_5} + v_{73}\frac{\partial v_{73}}{\partial \theta_5}\right)\\
&= \dot{\theta}_{131}^2\left(-r_{GO1}r_{73}s^2\theta_6 c\theta_{132}c\left(\theta_5-\theta_{132}\right) + r_{73}^2 s\left(\theta_5-\theta_{132}\right)c\left(\theta_5-\theta_{132}\right)\right.\\
&- r_{GO1}r_{71}s\left(\theta_5-\theta_{132}\right)c\theta_{132}c\theta_6 - r_{71}^2 s\left(\theta_5-\theta_{132}\right)c\left(\theta_5-\theta_{132}\right)c^2\theta_6\\
&\left.- r_{71}r_{73}c\left(2\left(\theta_5-\theta_{132}\right)\right)c\theta_6\right)\\
&+ \dot{\theta}_{131}\dot{\theta}_{132}\left(-r_{GO1}r_{71}s\theta_{131}c\left(2\theta_{132}-\theta_5\right)c\theta_6 + r_{71}r_{73}s\left(\theta_5-\theta_{132}\right)s\theta_6 c\theta_{131}\right.\\
&- r_{71}^2 s\theta_6 c\theta_{131}c\left(\theta_5-\theta_{132}\right)c\theta_6 - r_{71}^2 s\theta_{131}c\left(2\left(\theta_5-\theta_{132}\right)\right)c^2\theta_6\\
&+ 2r_{71}r_{73}s\theta_{131}s\left(2\left(\theta_5-\theta_{132}\right)\right)c\theta_6 + r_{GO1}r_{73}s\theta_{131}s\left(\theta_5-2\theta_{132}\right)\\
&\left.+ r_{73}^2 s\theta_{131}c\left(2\left(\theta_5-\theta_{132}\right)\right)\right)\\
&+ \dot{\theta}_{132}^2\left(-r_{GO1}r_{71}s^2\theta_{131}s\theta_{132}c\left(\theta_5-\theta_{132}\right)c\theta_6 + r_{71}r_{73}s\theta_{131}s\theta_6 c\theta_{131}c\left(\theta_5-\theta_{132}\right)\right.\\
&- r_{GO1}r_{71}s\theta_{131}s\left(2\theta_5-\theta_{132}\right)s\theta_6 c\theta_{131} + r_{71}r_{73}s^2\theta_{131}c\left(2\left(\theta_5-\theta_{132}\right)\right)c\theta_6\\
&- r_{GO1}s\theta_5 c^2\theta_{131}c\theta_5 - r_{GO1}r_{71}s\theta_5 c^2\theta_{131}c\theta_6 + r_{71}^2 s\theta_{131}s\left(\theta_5-\theta_{132}\right)s\theta_6 c\theta_{131}c\theta_6\\
&+ r_{GO1}r_{71}s\theta_{131}s\theta_6 s\left(2\theta_5-\theta_{132}\right)c\theta_{131} - r_{71}^2 s^2\theta_{131}s\left(\theta_5-\theta_{132}\right)c\left(\theta_5-\theta_{132}\right)c^2\theta_6\\
&- r_{GO1}r_{73}c^2\theta_{131}c\theta_5 - r_{73}^2 s^2\theta_{131}s\left(\theta_5-\theta_{132}\right)c\left(\theta_5-\theta_{132}\right) + r_{GO1}^2 s\theta_5 c^2\theta_{131}c\theta_5\\
&\left.+ r_{GO1}r_{73}s^2\theta_{131}s\theta_{132}s\left(\theta_5-\theta_{132}\right)\right)\\
&+ \dot{\theta}_{131}\dot{\theta}_5\left(r_{71}^2 s\theta_6 c\left(\theta_5-\theta_{132}\right)c\theta_6 - r_{71}r_{73}s\left(\theta_5-\theta_{132}\right)s\theta_6\right.\\
&+ \dot{\theta}_{132}\dot{\theta}_5\left(r_{GO1}r_{73}c\theta_{131}c\theta_5 - r_{71}r_{73}s\theta_{131}s\theta_6 c\left(\theta_5-\theta_{132}\right) + r_{GO1}r_{71}s\theta_5 c\theta_{131}c\theta_6\right.\\
&- r_{71}^2 s\theta_{131}s\left(\theta_5-\theta_{132}\right)s\theta_6 c\theta_6\\
&+ \dot{\theta}_{131}\dot{\theta}_6\left(-r_{71}^2 s\left(\theta_5-\theta_{132}\right) - r_{71}r_{73}c\left(\theta_5-\theta_{132}\right)c\theta_6\right)\\
&+ \dot{\theta}_{132}\dot{\theta}_6\left(-r_{71}^2 s\theta_{131}c\left(\theta_5-\theta_{132}\right) + r_{GO1}r_{71}s\theta_6 c\theta_{131}c\theta_5\right.
\end{aligned}
$$

$$+ \, r_{71} r_{73} s\theta_{131} s\left(\theta_5 - \theta_{132}\right) c\theta_6) \tag{4.634}$$

As an alternative but a more practical way; partial derivatives of K_6 and U_6 will be calculated.

$$\frac{\partial K_7}{\partial \theta_5} = m_7 \bar{v}_7^T \frac{\partial \bar{v}_7}{\partial \theta_5} + \bar{\omega}_7^T \hat{J}_7^{(7)} \frac{\partial \bar{\omega}_7}{\partial \theta_5} \tag{4.635}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial K_7}{\partial \dot{\theta}_5}\right) &= \frac{d}{dt}\left(m_7 \bar{v}_7^T \frac{\partial \bar{v}_7}{\partial \dot{\theta}_5} + \bar{\omega}_7^T \hat{J}_7^{(7)} \frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_5}\right) = m_7 \left(\dot{\bar{v}}_7^T \frac{\partial \bar{v}_7}{\partial \dot{\theta}_5} + \bar{v}_7^T \frac{d}{dt}\left(\frac{\partial \bar{v}_7}{\partial \dot{\theta}_5}\right)\right) \\ &+ \dot{\bar{\omega}}_7^T \hat{J}_7^{(7)} \frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_5} + \bar{\omega}_7^T \hat{J}_7^{(7)} \frac{d}{dt}\left(\frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_5}\right) \end{aligned} \tag{4.636}$$

$$\frac{\partial U_7}{\partial \theta_5} = m_7 g \left(-r_{71} \cos\theta_{131} \cos\theta_6 \sin\left(\theta_5 - \theta_{132}\right) - r_{73} \cos\theta_{131} \cos\left(\theta_5 - \theta_{132}\right)\right) \tag{4.637}$$

$$\frac{\partial K_7}{\partial \theta_6} = m_7 \bar{v}_7^T \frac{\partial \bar{v}_7}{\partial \theta_6} + \bar{\omega}_7^T \hat{J}_7^{(7)} \frac{\partial \bar{\omega}_7}{\partial \theta_6} \tag{4.638}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial K_7}{\partial \dot{\theta}_6}\right) &= \frac{d}{dt}\left(m_7 \bar{v}_7^T \frac{\partial \bar{v}_7}{\partial \dot{\theta}_6} + \bar{\omega}_7^T \hat{J}_7^{(7)} \frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_6}\right) = m_7 \left(\dot{\bar{v}}_7^T \frac{\partial \bar{v}_7}{\partial \dot{\theta}_6} + \bar{v}_7^T \frac{d}{dt}\left(\frac{\partial \bar{v}_7}{\partial \dot{\theta}_6}\right)\right) \\ &+ \dot{\bar{\omega}}_7^T \hat{J}_7^{(7)} \frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_6} + \bar{\omega}_7^T \hat{J}_7^{(7)} \frac{d}{dt}\left(\frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_6}\right) \end{aligned} \tag{4.639}$$

$$\frac{\partial U_7}{\partial \theta_6} = m_7 g \left(-r_{71} \cos\theta_{131} \sin\theta_6 \cos\left(\theta_5 - \theta_{132}\right) - r_{71} \sin\theta_{131} \cos\theta_6\right) \tag{4.640}$$

where

$$\begin{aligned} \frac{\partial \bar{v}_7}{\partial \theta_5} = \Big(&- r_{73} \sin\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} + \left(r_{73} \sin\theta_{131} \sin\theta_6 \sin\left(\theta_5 - \theta_{132}\right)\right. \\ &- \left. r_{GO1} \cos\theta_{131} \cos\theta_5 \cos\theta_6\right) \dot{\theta}_{132}\Big) \bar{u}_1 \end{aligned}$$

$$+ \Big(- (r_{71} \sin(\theta_5 - \theta_{132}) - r_{73} \cos\theta_6 \cos(\theta_5 - \theta_{132}))\, \dot\theta_{131}$$
$$+ (-r_{71} \sin\theta_{131} \cos(\theta_5 - \theta_{132}) + r_{GO1} \cos\theta_{131} \cos\theta_5 \sin\theta_6$$
$$+ r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}))\, \dot\theta_{132}\Big)\, \bar u_2$$
$$+ \Big(r_{71} \sin\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131} + (- r_{71} \sin\theta_{131} \sin\theta_6 \sin(\theta_5 - \theta_{132})$$
$$+ r_{GO1} \cos\theta_{131} \sin\theta_5)\, \dot\theta_{132}\Big)\, \bar u_3 \quad (4.641)$$

$$\frac{\partial \bar\omega_7}{\partial \theta_5} = \Big(\cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131} - \cos\theta_6 \sin\theta_{131} \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}\Big)\, \bar u_1$$
$$+ \Big(- \sin\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131} + \sin\theta_6 \sin\theta_{131} \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}\Big)\, \bar u_2$$
$$+ \Big(- \sin(\theta_5 - \theta_{132})\, \dot\theta_{131} - \sin\theta_{131} \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}\Big)\, \bar u_3 \quad (4.642)$$

$$\frac{\partial \bar v_7}{\partial \theta_6} = \Big((r_{GO1} \cos\theta_{132} \cos\theta_6 - r_{73} \cos\theta_6 \sin(\theta_5 - \theta_{132}))\, \dot\theta_{131}$$
$$+ (-r_{73} \cos\theta_{131} \sin\theta_6 - r_{73} \sin\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132})$$
$$+ r_{GO1} \cos\theta_{131} \sin\theta_5 \sin\theta_6 + r_{GO1} \sin\theta_{131} \sin\theta_{132} \cos\theta_6)\, \dot\theta_{132}$$
$$+ r_{73} \sin\theta_6 \dot\theta_5\Big) \bar u_1$$
$$+ \Big((-r_{GO1} \cos\theta_{132} \sin\theta_6 + r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132}))\dot\theta_{131}$$
$$+ (-r_{GO1} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 + r_{GO1} \cos\theta_{131} \sin\theta_5 \cos\theta_6$$
$$+ r_{73} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) - r_{73} \cos\theta_{131} \cos\theta_6)\dot\theta_{132} + r_{73} \cos\theta_6 \dot\theta_5\Big) \bar u_2$$
$$+ \Big(r_{71} \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{131} + (r_{71} \sin\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132})$$
$$+ r_{71} \cos\theta_{131} \sin\theta_6)\dot\theta_{132} - r_{71} \sin\theta_6 \dot\theta_5\Big) \bar u_3 \quad (4.643)$$

$$\frac{\partial \bar\omega_7}{\partial \theta_6} = \Big(- \cos\theta_6 \dot\theta_5 - \sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{131}$$
$$+ (- \sin\theta_6 \sin\theta_{131} \cos(\theta_5 - \theta_{132}) + \cos\theta_6 \cos\theta_{131})\, \dot\theta_{132}\Big)\, \bar u_1$$
$$+ \Big(\sin\theta_6 \dot\theta_5 - \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{131}$$
$$+ (- \sin\theta_6 \cos\theta_{131} - \cos\theta_6 \sin\theta_{131} \cos(\theta_5 - \theta_{132}))\, \dot\theta_{132}\Big)\, \bar u_2 \quad (4.644)$$

$$\dot{\bar{v}}_7 = \dot{v}_{71}\bar{u}_1 + \dot{v}_{72}\bar{u}_2 + \dot{v}_{73}\bar{u}_3 \tag{4.645}$$

$$\begin{aligned}
\dot{v}_{71} = &\big(r_{GO1}\cos\theta_{132}\sin\theta_6 - r_{73}\sin\theta_6\sin(\theta_5-\theta_{132})\big)\ddot{\theta}_{131} + \big(r_{73}\cos\theta_{131}\cos\theta_6 \\
&- r_{73}\sin\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132}) - r_{GO1}\cos\theta_{131}\sin\theta_5\cos\theta_6 \\
&+ r_{GO1}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\big)\ddot{\theta}_{132} - r_{73}\cos\theta_6\ddot{\theta}_5 \\
&+ \Big(- r_{GO1}\sin\theta_{132}\sin\theta_6\dot{\theta}_{132} + r_{GO1}\cos\theta_{132}\cos\theta_6\dot{\theta}_6 \\
&- r_{73}\cos\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_6 - r_{73}\sin\theta_6\cos(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right)\Big)\dot{\theta}_{131} \\
&+ \Big(- r_{73}\sin\theta_{131}\cos\theta_6\dot{\theta}_{131} - r_{73}\cos\theta_{131}\sin\theta_6\dot{\theta}_6 \\
&- r_{73}\cos\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131} - r_{73}\sin\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_6 \\
&+ r_{73}\sin\theta_{131}\sin\theta_6\sin(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right) + r_{GO1}\sin\theta_{131}\sin\theta_5\cos\theta_6\dot{\theta}_{131} \\
&- r_{GO1}\cos\theta_{131}\cos\theta_5\cos\theta_6\dot{\theta}_5 + r_{GO1}\cos\theta_{131}\sin\theta_5\sin\theta_6\dot{\theta}_6 \\
&+ r_{GO1}\cos\theta_{131}\sin\theta_{132}\sin\theta_6\dot{\theta}_{131} + r_{GO1}\sin\theta_{131}\cos\theta_{132}\sin\theta_6\dot{\theta}_{132} \\
&+ r_{GO1}\sin\theta_{131}\sin\theta_{132}\cos\theta_6\dot{\theta}_6\Big)\dot{\theta}_{132} + r_{73}\sin\theta_6\dot{\theta}_6\dot{\theta}_5
\end{aligned}$$

$$\begin{aligned}
\dot{v}_{71} = &\big(r_{GO1}\cos\theta_{132}\sin\theta_6 - r_{73}\sin\theta_6\sin(\theta_5-\theta_{132})\big)\ddot{\theta}_{131} + \big(r_{73}\cos\theta_{131}\cos\theta_6 \\
&- r_{73}\sin\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132}) - r_{GO1}\cos\theta_{131}\sin\theta_5\cos\theta_6 \\
&+ r_{GO1}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\big)\ddot{\theta}_{132} - r_{73}\cos\theta_6\ddot{\theta}_5 - r_{GO1}\sin\theta_{132}\sin\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{GO1}\sin\theta_{131}\sin\theta_5\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} + r_{GO1}\cos\theta_{131}\sin\theta_{132}\sin\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{GO1}\cos\theta_{132}\cos\theta_6\dot{\theta}_{131}\dot{\theta}_6 + r_{GO1}\sin\theta_{131}\cos\theta_{132}\sin\theta_6\dot{\theta}_{132}^2 \\
&- r_{GO1}\cos\theta_{131}\cos\theta_5\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5 + r_{GO1}\cos\theta_{131}\sin\theta_5\sin\theta_6\dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{GO1}\sin\theta_{131}\sin\theta_{132}\cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 + r_{73}\sin\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{73}\cos\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} - r_{73}\sin\theta_{131}\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{73}\sin\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 - r_{73}\cos\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6 \\
&- r_{73}\sin\theta_{131}\sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 + r_{73}\sin\theta_{131}\sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{73}\cos\theta_{131}\sin\theta_6\dot{\theta}_{132}\dot{\theta}_6 - r_{73}\sin\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{73}\sin\theta_6\dot{\theta}_5\dot{\theta}_6
\end{aligned} \tag{4.646}$$

$$
\begin{aligned}
\dot{v}_{72} = {} & (r_{71}\cos(\theta_5-\theta_{132}) + r_{GO1}\cos\theta_{132}\cos\theta_6 - r_{73}\cos\theta_6\sin(\theta_5-\theta_{132}))\ddot{\theta}_{131} \\
& + (r_{GO1}\sin\theta_{131}\sin\theta_{132}\cos\theta_6 - r_{71}\sin\theta_{131}\sin(\theta_5-\theta_{132}) \\
& + r_{GO1}\cos\theta_{131}\sin\theta_5\sin\theta_6 - r_{73}\sin\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132}) \\
& - r_{73}\cos\theta_{131}\sin\theta_6)\ddot{\theta}_{132} + r_{73}\sin\theta_6\ddot{\theta}_5 + r_{71}\ddot{\theta}_6 \\
& + \Big(- r_{71}\sin(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right) - r_{GO1}\sin\theta_{132}\cos\theta_6\dot{\theta}_{132} \\
& - r_{GO1}\cos\theta_{132}\sin\theta_6\dot{\theta}_6 + r_{73}\sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_6 \\
& - r_{73}\cos\theta_6\cos(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right)\Big)\dot{\theta}_{131} + \Big(r_{GO1}\cos\theta_{131}\sin\theta_{132}\cos\theta_6\dot{\theta}_{131} \\
& + r_{GO1}\sin\theta_{131}\cos\theta_{132}\cos\theta_6\dot{\theta}_{132} - r_{GO1}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\dot{\theta}_6 \\
& - r_{71}\cos\theta_{131}\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131} - r_{71}\sin\theta_{131}\cos(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right) \\
& - r_{GO1}\sin\theta_{131}\sin\theta_5\sin\theta_6\dot{\theta}_{131} + r_{GO1}\cos\theta_{131}\cos\theta_5\sin\theta_6\dot{\theta}_5 \\
& + r_{GO1}\cos\theta_{131}\sin\theta_5\cos\theta_6\dot{\theta}_6 - r_{73}\cos\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131} \\
& + r_{73}\sin\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_6 \\
& - r_{73}\sin\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right) + r_{73}\sin\theta_{131}\sin\theta_6\dot{\theta}_{131} \\
& - r_{73}\cos\theta_{131}\cos\theta_6\dot{\theta}_6\Big)\dot{\theta}_{132} + r_{73}\cos\theta_6\dot{\theta}_5\dot{\theta}_6
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}_{72} = {} & (r_{71}\cos(\theta_5-\theta_{132}) + r_{GO1}\cos\theta_{132}\cos\theta_6 - r_{73}\cos\theta_6\sin(\theta_5-\theta_{132}))\ddot{\theta}_{131} \\
& + (r_{GO1}\sin\theta_{131}\sin\theta_{132}\cos\theta_6 - r_{71}\sin\theta_{131}\sin(\theta_5-\theta_{132}) \\
& + r_{GO1}\cos\theta_{131}\sin\theta_5\sin\theta_6 - r_{73}\sin\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132}) \\
& - r_{73}\cos\theta_{131}\sin\theta_6)\ddot{\theta}_{132} + r_{73}\sin\theta_6\ddot{\theta}_5 + r_{71}\ddot{\theta}_6 - r_{GO1}\sin\theta_{132}\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
& + r_{GO1}\cos\theta_{131}\sin\theta_{132}\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} - r_{GO1}\sin\theta_{131}\sin\theta_5\sin\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
& - r_{GO1}\cos\theta_{132}\sin\theta_6\dot{\theta}_{131}\dot{\theta}_6 + r_{GO1}\sin\theta_{131}\cos\theta_{132}\cos\theta_6\dot{\theta}_{132}^2 \\
& + r_{GO1}\cos\theta_{131}\cos\theta_5\sin\theta_6\dot{\theta}_{132}\dot{\theta}_5 - r_{GO1}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\dot{\theta}_{132}\dot{\theta}_6 \\
& + r_{GO1}\cos\theta_{131}\sin\theta_5\cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 + r_{71}\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
& - r_{71}\cos\theta_{131}\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} - r_{71}\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
& - r_{71}\sin\theta_{131}\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 - r_{71}\sin\theta_{131}\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
& - r_{73}\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} - r_{73}\cos\theta_{131}\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}
\end{aligned}
$$

$$
\begin{aligned}
&+ r_{73} \sin\theta_{131} \sin\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} - r_{73}\cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_5 \\
&+ r_{73}\sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 - r_{73}\sin\theta_{131}\cos\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- r_{73}\sin\theta_{131}\cos\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{73}\sin\theta_{131}\sin\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 - r_{73}\cos\theta_{131}\cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{73}\cos\theta_6\dot{\theta}_5\dot{\theta}_6
\end{aligned} \tag{4.647}
$$

$$
\begin{aligned}
\dot{v}_{73} = {} & r_{71}\sin\theta_6\sin(\theta_5 - \theta_{132})\, \ddot{\theta}_{131} + (r_{71}\sin\theta_{131}\sin\theta_6\cos(\theta_5 - \theta_{132}) \\
&- r_{GO1}\cos\theta_{131}\cos\theta_5 - r_{71}\cos\theta_{131}\cos\theta_6)\ddot{\theta}_{132} + r_{71}\cos\theta_6\ddot{\theta}_5 \\
&+ \Big(r_{71}\cos\theta_6\sin(\theta_5 - \theta_{132})\, \dot{\theta}_6 + r_{71}\sin\theta_6\cos(\theta_5 - \theta_{132})\left(\dot{\theta}_5 - \dot{\theta}_{132}\right)\Big)\dot{\theta}_{131} \\
&+ \Big(r_{71}\cos\theta_{131}\sin\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131} + r_{71}\sin\theta_{131}\cos\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_6 \\
&- r_{71}\sin\theta_{131}\sin\theta_6\sin(\theta_5 - \theta_{132})\left(\dot{\theta}_5 - \dot{\theta}_{132}\right) + r_{GO1}\sin\theta_{131}\cos\theta_5\dot{\theta}_{131} \\
&+ r_{GO1}\cos\theta_{131}\sin\theta_5\dot{\theta}_5 + r_{71}\sin\theta_{131}\cos\theta_6\dot{\theta}_{131} + r_{71}\cos\theta_{131}\sin\theta_6\dot{\theta}_6\Big)\dot{\theta}_{132} \\
&- r_{71}\sin\theta_6\dot{\theta}_5\dot{\theta}_6
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}_{73} = {} & r_{71}\sin\theta_6\sin(\theta_5 - \theta_{132})\, \ddot{\theta}_{131} + (r_{71}\sin\theta_{131}\sin\theta_6\cos(\theta_5 - \theta_{132}) \\
&- r_{GO1}\cos\theta_{131}\cos\theta_5 - r_{71}\cos\theta_{131}\cos\theta_6)\ddot{\theta}_{132} + r_{71}\cos\theta_6\ddot{\theta}_5 \\
&+ r_{GO1}\sin\theta_{131}\cos\theta_5\dot{\theta}_{131}\dot{\theta}_{132} + r_{GO1}\cos\theta_{131}\sin\theta_5\dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{71}\sin\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} + r_{71}\cos\theta_{131}\sin\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{71}\sin\theta_{131}\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} + r_{71}\sin\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_5 \\
&+ r_{71}\cos\theta_6\sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 + r_{71}\sin\theta_{131}\sin\theta_6\sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- r_{71}\sin\theta_{131}\sin\theta_6\sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{71}\sin\theta_{131}\cos\theta_6\cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 + r_{71}\cos\theta_{131}\sin\theta_6\dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{71}\sin\theta_6\dot{\theta}_5\dot{\theta}_6
\end{aligned} \tag{4.648}
$$

$$
\frac{\partial \bar{v}_7}{\partial \dot{\theta}_5} = -r_{73}\cos\theta_6\bar{u}_1 + r_{73}\sin\theta_6\bar{u}_2 + r_{71}\cos\theta_6\bar{u}_3 \tag{4.649}
$$

$$\frac{d}{dt}\left(\frac{\partial \bar{v}_7}{\partial \dot{\theta}_5}\right) = r_{73}\sin\theta_6\dot{\theta}_6\bar{u}_1 + r_{73}\cos\theta_6\dot{\theta}_6\bar{u}_2 - r_{71}\sin\theta_6\dot{\theta}_6\bar{u}_3 \tag{4.650}$$

$$\frac{\partial \bar{v}_7}{\partial \dot{\theta}_6} = r_{71}\bar{u}_2 \tag{4.651}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{v}_7}{\partial \dot{\theta}_6}\right) = \bar{0} \tag{4.652}$$

$$\begin{aligned}
\dot{\bar{\omega}}_7 = \Big(&- \sin\theta_6\ddot{\theta}_5 + \cos\theta_6\sin(\theta_5-\theta_{132})\,\ddot{\theta}_{131} + (\cos\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132}) \\
&+ \sin\theta_6\cos\theta_{131})\ddot{\theta}_{132} - \cos\theta_6\dot{\theta}_5\dot{\theta}_6 + \Big(- \sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_6 \\
&+ \cos\theta_6\cos(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right)\Big)\dot{\theta}_{131} \\
&+ \Big(- \sin\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132})\,\dot{\theta}_6 + \cos\theta_6\cos\theta_{131}\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131} \\
&- \cos\theta_6\sin\theta_{131}\sin(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right) + \cos\theta_6\cos\theta_{131}\dot{\theta}_6 \\
&- \sin\theta_6\sin\theta_{131}\dot{\theta}_{131}\Big)\dot{\theta}_{132}\Big)\bar{u}_1 \\
&+ \Big(- \cos\theta_6\ddot{\theta}_5 - \sin\theta_6\sin(\theta_5-\theta_{132})\,\ddot{\theta}_{131} + (\cos\theta_6\cos\theta_{131} \\
&- \sin\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132}))\ddot{\theta}_{132} + \sin\theta_6\dot{\theta}_5\dot{\theta}_6 \\
&+ \Big(- \cos\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_6 - \sin\theta_6\cos(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right)\Big)\dot{\theta}_{131} \\
&+ \Big(- \sin\theta_6\cos\theta_{131}\dot{\theta}_6 - \cos\theta_6\sin\theta_{131}\dot{\theta}_{131} - \cos\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132})\,\dot{\theta}_6 \\
&- \sin\theta_6\cos\theta_{131}\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131} \\
&+ \sin\theta_6\sin\theta_{131}\sin(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right)\Big)\dot{\theta}_{132}\Big)\bar{u}_2 \\
&+ \Big(\ddot{\theta}_6 + \cos(\theta_5-\theta_{132})\,\ddot{\theta}_{131} - \sin\theta_{131}\sin(\theta_5-\theta_{132})\,\ddot{\theta}_{132} \\
&- \sin(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right)\dot{\theta}_{131} + \Big(- \cos\theta_{131}\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131} \\
&- \sin\theta_{131}\cos(\theta_5-\theta_{132})\left(\dot{\theta}_5-\dot{\theta}_{132}\right)\Big)\dot{\theta}_{132}\Big)\bar{u}_3
\end{aligned} \tag{4.653}$$

$$\frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_5} = -\sin\theta_6\bar{u}_1 - \cos\theta_6\bar{u}_2 \tag{4.654}$$

$$\frac{d}{dt}\left(\frac{\partial\bar{\omega}_7}{\partial\dot{\theta}_5}\right) = -\cos\theta_6\dot{\theta}_6\bar{u}_1 + \sin\theta_6\dot{\theta}_6\bar{u}_2 \tag{4.655}$$

$$\frac{\partial\bar{\omega}_7}{\partial\dot{\theta}_6} = \bar{u}_3 \tag{4.656}$$

$$\frac{d}{dt}\left(\frac{\partial\bar{\omega}_7}{\partial\dot{\theta}_6}\right) = \bar{0} \tag{4.657}$$

$$\hat{J}_7^{(7)} = \begin{bmatrix} J_{711} & J_{712} & 0 \\ J_{712} & J_{722} & 0 \\ 0 & 0 & J_{733} \end{bmatrix} \tag{4.658}$$

$$\bar{v}_7^T\frac{\partial\bar{v}_7}{\partial\theta_5} = v_{71}\frac{\partial v_{71}}{\partial\theta_5} + v_{72}\frac{\partial v_{72}}{\partial\theta_5} + v_{73}\frac{\partial v_{73}}{\partial\theta_5} \tag{4.659}$$

$$\begin{aligned}
v_{71}\frac{\partial v_{71}}{\partial\theta_5} = &\Big((r_{GO1}\cos\theta_{132}\cos\theta_5 - r_{73}\cos(\theta_5-\theta_{132}))\,\dot{\theta}_{131} \\
&+ (r_{73}\sin\theta_{131}\sin\theta_6\sin(\theta_5-\theta_{132}) - r_{GO1}\cos\theta_{131}\cos\theta_5\cos\theta_6 \\
&- \; r_{GO1}\sin\theta_{131}\sin\theta_{132}\sin\theta_5\sin\theta_6)\,\dot{\theta}_{132}\Big) \\
&\Big(\,(r_{GO1}\cos\theta_{132}\sin\theta_5 - r_{73}\sin(\theta_5-\theta_{132}))\,\dot{\theta}_{131} \\
&+ (r_{73}\cos\theta_{131}\cos\theta_6 - r_{73}\sin\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132}) \\
&- \; r_{GO1}\cos\theta_{131}\sin\theta_5\cos\theta_6 + r_{GO1}\sin\theta_{131}\sin\theta_{132}\cos\theta_5\sin\theta_6)\,\dot{\theta}_{132} \\
&- r_{73}\cos\theta_6\dot{\theta}_5\Big) \\
&= r_{GO1}^2\cos^2\theta_{132}\sin\theta_5\cos\theta_5\dot{\theta}_{131}^2 \\
&- r_{GO1}^2\cos\theta_{131}\cos\theta_{132}\sin\theta_5\cos\theta_5\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{GO1}^2\sin\theta_{131}\sin\theta_{132}\cos\theta_{132}\cos^2\theta_5\sin\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1}^2\cos\theta_{131}\cos\theta_{132}\sin\theta_5\cos\theta_5\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1}^2\sin\theta_{131}\sin\theta_{132}\cos\theta_{132}\sin^2\theta_5\sin\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{GO1}^2\cos^2\theta_{131}\sin\theta_5\cos\theta_5\cos^2\theta_6\dot{\theta}_{132}^2
\end{aligned}$$

$$
\begin{aligned}
&- r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \cos^2\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2 \\
&+ r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \sin^2\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2 \\
&- r_{GO1}^2 \sin^2\theta_{131} \sin^2\theta_{132} \sin\theta_5 \cos\theta_5 \sin^2\theta_6 \dot{\theta}_{132}^2 \\
&- r_{GO1} r_{73} \cos\theta_{132} \cos\theta_5 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2 \\
&- r_{GO1} r_{73} \cos\theta_{132} \sin\theta_5 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_{132} \cos\theta_5 \cos\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \cos\theta_5 \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \cos\theta_5 \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \sin\theta_5 \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_5 \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{GO1} r_{73} \cos\theta_{132} \cos\theta_5 \cos\theta_6 \dot{\theta}_{131} \dot{\theta}_5 \\
&- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2 \\
&+ r_{GO1} r_{73} \sin^2\theta_{131} \sin\theta_{132} \cos\theta_5 \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2 \\
&- r_{GO1} r_{73} \cos^2\theta_{131} \cos\theta_5 \cos^2\theta_6 \dot{\theta}_{132}^2 \\
&+ r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2 \\
&- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2 \\
&+ r_{GO1} r_{73} \sin^2\theta_{131} \sin\theta_{132} \sin\theta_5 \sin^2\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \cos^2\theta_6 \dot{\theta}_{132} \dot{\theta}_5 \\
&+ r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132} \dot{\theta}_5 \\
&+ r_{73}^2 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2 \\
&- r_{73}^2 \cos\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos^2(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \sin^2(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{73}^2 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_5 \\
&+ r_{73}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2 \\
&- r_{73}^2 \sin^2\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2
\end{aligned}
$$

$$
\begin{aligned}
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}\dot\theta_5 \\
&= r_{GO1}^2 \cos^2\theta_{132} \sin\theta_5 \cos\theta_5 \dot\theta_{131}^2 \\
&- 2r_{GO1}^2 \cos\theta_{131} \cos\theta_{132} \sin\theta_5 \cos\theta_5 \cos\theta_6 \dot\theta_{131}\dot\theta_{132} \\
&+ r_{GO1}^2 \sin\theta_{131} \sin\theta_{132} \cos\theta_{132} \cos 2\theta_5 \sin\theta_6 \dot\theta_{131}\dot\theta_{132} \\
&+ r_{GO1}^2 \cos^2\theta_{131} \sin\theta_5 \cos\theta_5 \cos^2\theta_6 \dot\theta_{132}^2 \\
&- r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \cos 2\theta_5 \sin\theta_6 \cos\theta_6 \dot\theta_{132}^2 \\
&- r_{GO1}^2 \sin^2\theta_{131} \sin^2\theta_{132} \sin\theta_5 \cos\theta_5 \sin^2\theta_6 \dot\theta_{132}^2 \\
&- r_{GO1} r_{73} \cos\theta_{132} \sin(2\theta_5 - \theta_{132})\, \dot\theta_{131}^2 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_{132} \cos\theta_5 \cos\theta_6 \dot\theta_{131}\dot\theta_{132} \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_6 \sin(2\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \sin\theta_6 \cos(2\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos(2\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1} r_{73} \cos\theta_{132} \cos\theta_5 \cos\theta_6 \dot\theta_{131}\dot\theta_5 - r_{GO1} r_{73} \cos^2\theta_{131} \cos\theta_5 \cos^2\theta_6 \dot\theta_{132}^2 \\
&- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot\theta_{132}^2 \\
&+ r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(2\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&+ r_{GO1} r_{73} \sin^2\theta_{131} \sin\theta_{132} \sin^2\theta_6 \sin(2\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \cos^2\theta_6 \dot\theta_{132}\dot\theta_5 \\
&+ r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot\theta_{132}\dot\theta_5 \\
&+ r_{73}^2 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}^2 \\
&- r_{73}^2 \cos\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&+ r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos(2\theta_5 - 2\theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&+ r_{73}^2 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_5 \\
&+ r_{73}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- r_{73}^2 \sin^2\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}\dot\theta_5 \qquad (4.660)
\end{aligned}
$$

$$
\begin{aligned}
v_{72}\frac{\partial v_{72}}{\partial \theta_5} &= \Big(-\left(r_{71}\sin(\theta_5 - \theta_{132}) - r_{73}\cos\theta_6 \cos(\theta_5 - \theta_{132})\right)\dot\theta_{131} \\
&+ \left(-r_{71}\sin\theta_{131}\cos(\theta_5 - \theta_{132}) + r_{GO1}\cos\theta_{131}\cos\theta_5 \sin\theta_6\right.
\end{aligned}
$$

$$+ \; r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132})) \, \dot\theta_{132} \Big)$$

$$\Big((r_{71} \cos(\theta_5 - \theta_{132}) + r_{GO1} \cos\theta_{132} \cos\theta_6 - r_{73} \cos\theta_6 \sin(\theta_5 - \theta_{132})) \, \dot\theta_{131}$$

$$+ (r_{GO1} \sin\theta_{131} \sin\theta_{132} \cos\theta_6 \; - r_{71} \sin\theta_{131} \sin(\theta_5 - \theta_{132})$$

$$+ r_{GO1} \cos\theta_{131} \sin\theta_5 \sin\theta_6 - r_{73} \sin\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132})$$

$$- \; r_{73} \cos\theta_{131} \sin\theta_6) \, \dot\theta_{132} + r_{73} \sin\theta_6 \dot\theta_5 + r_{71} \dot\theta_6 \Big)$$

$$= r_{GO1}^2 \cos\theta_{131} \cos\theta_{132} \cos\theta_5 \sin\theta_6 \cos\theta_6 \dot\theta_{131} \dot\theta_{132}$$

$$+ r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \cos\theta_5 \sin\theta_6 \cos\theta_6 \dot\theta_{132}^2$$

$$+ r_{GO1}^2 \cos^2\theta_{131} \sin\theta_5 \cos\theta_5 \sin^2\theta_6 \dot\theta_{132}^2$$

$$- r_{GO1} r_{71} \cos\theta_{132} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \, \dot\theta_{131}^2$$

$$- r_{GO1} r_{71} \sin\theta_{131} \sin\theta_{132} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \, \dot\theta_{131} \dot\theta_{132}$$

$$- r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \sin(\theta_5 - \theta_{132}) \, \dot\theta_{131} \dot\theta_{132}$$

$$- r_{GO1} r_{71} \sin\theta_{131} \cos\theta_{132} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \, \dot\theta_{131} \dot\theta_{132}$$

$$+ r_{GO1} r_{71} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \cos(\theta_5 - \theta_{132}) \, \dot\theta_{131} \dot\theta_{132}$$

$$- r_{GO1} r_{71} \sin^2\theta_{131} \sin\theta_{132} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \, \dot\theta_{132}^2$$

$$- r_{GO1} r_{71} \sin\theta_{131} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos(\theta_5 - \theta_{132}) \, \dot\theta_{132}^2$$

$$- r_{GO1} r_{71} \sin\theta_{131} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \sin(\theta_5 - \theta_{132}) \, \dot\theta_{132}^2$$

$$+ r_{GO1} r_{71} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \dot\theta_{132} \dot\theta_6$$

$$- r_{GO1} r_{73} \cos\theta_{132} \cos^2\theta_6 \cos(\theta_5 - \theta_{132}) \, \dot\theta_{131}^2$$

$$- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \cos^2\theta_6 \cos(\theta_5 - \theta_{132}) \, \dot\theta_{131} \dot\theta_{132}$$

$$- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \, \dot\theta_{131} \dot\theta_{132}$$

$$- r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \, \dot\theta_{131} \dot\theta_{132}$$

$$+ r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \, \dot\theta_{131} \dot\theta_{132}$$

$$- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \, \dot\theta_{132}^2$$

$$- r_{GO1} r_{73} \cos^2\theta_{131} \cos\theta_5 \sin^2\theta_6 \dot\theta_{132}^2$$

$$+ r_{GO1} r_{73} \sin^2\theta_{131} \sin\theta_{132} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \, \dot\theta_{132}^2$$

$$+ r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \, \dot\theta_{132}^2$$

$$+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \sin^2\theta_6 \dot\theta_{132} \dot\theta_5$$

$$- r_{71}^2 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \, \dot\theta_{131}^2$$

$$+ r_{71}^2 \sin\theta_{131} \sin^2(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} - r_{71}^2 \sin\theta_{131} \cos^2(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{71}^2 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6 + r_{71}^2 \sin^2\theta_{131} \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$- r_{71}^2 \sin\theta_{131} \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6 + r_{71} r_{73} \cos\theta_6 \sin^2(\theta_5 - \theta_{132})\,\dot{\theta}_{131}^2$$
$$- r_{71} r_{73} \cos\theta_6 \cos^2(\theta_5 - \theta_{132})\,\dot{\theta}_{131}^2$$
$$+ r_{71} r_{73} \cos\theta_{131} \sin\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{71} r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 - r_{71} r_{73} \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6$$
$$+ r_{71} r_{73} \sin^2\theta_{131} \cos\theta_6 \cos^2(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$- r_{71} r_{73} \sin^2\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$- r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6$$
$$+ r_{73}^2 \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}^2$$
$$+ r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \cos^2(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{73}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5$$
$$- r_{73}^2 \sin^2\theta_{131} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$- r_{73}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$+ r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5$$
$$= r_{GO1}^2 \cos\theta_{131} \cos\theta_{132} \cos\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \cos\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2$$
$$+ r_{GO1}^2 \cos^2\theta_{131} \sin\theta_5 \cos\theta_5 \sin^2\theta_6 \dot{\theta}_{132}^2$$
$$- r_{GO1} r_{71} \cos\theta_{132} \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}^2$$
$$- r_{GO1} r_{71} \sin\theta_{131} \cos\theta_6 \cos(\theta_5 - 2\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$

$$+ r_{GO1} r_{71} \cos\theta_{131} \sin\theta_6 \cos(2\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$- r_{GO1} r_{71} \sin^2\theta_{131} \sin\theta_{132} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$- r_{GO1} r_{71} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \sin(2\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$+ r_{GO1} r_{71} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \dot{\theta}_{132} \dot{\theta}_6$$
$$- r_{GO1} r_{73} \cos\theta_{132} \cos^2\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$
$$+ r_{GO1} r_{73} \sin\theta_{131} \cos^2\theta_6 \sin(\theta_5 - 2\theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(2\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$- r_{GO1} r_{73} \cos^2\theta_{131} \cos\theta_5 \sin^2\theta_6 \dot{\theta}_{132}^2$$
$$+ r_{GO1} r_{73} \sin^2\theta_{131} \sin\theta_{132} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(2\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \sin^2\theta_6 \dot{\theta}_{132} \dot{\theta}_5$$
$$- r_{71}^2 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$
$$- r_{71}^2 \sin\theta_{131} \cos(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132} - r_{71}^2 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_6$$
$$+ r_{71}^2 \sin^2\theta_{131} \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$- r_{71}^2 \sin\theta_{131} \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132} \dot{\theta}_6 - r_{71} r_{73} \cos\theta_6 \cos(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131}^2$$
$$+ r_{71} r_{73} \cos\theta_{131} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$+ 2 r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$- r_{71} r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_5 - r_{71} r_{73} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_6$$
$$+ r_{71} r_{73} \sin^2\theta_{131} \cos\theta_6 \cos^2(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$- r_{71} r_{73} \sin^2\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$- r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132} \dot{\theta}_5$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132} \dot{\theta}_6$$
$$+ r_{73}^2 \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$
$$+ r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \cos^2(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$+ r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$- r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$- r_{73}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_5$$

$$
\begin{aligned}
&- r_{73}^2 \sin^2\theta_{131} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- r_{73}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&+ r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132} \dot\theta_5
\end{aligned}
\tag{4.661}
$$

$$
\begin{aligned}
v_{73} \frac{\partial v_{73}}{\partial \theta_5} = &\Big(r_{71} \sin\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131} + (- r_{71} \sin\theta_{131} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \\
&+ r_{GO1} \cos\theta_{131} \sin\theta_5)\, \dot\theta_{132} \Big) \Big(r_{71} \sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{131} \\
&+ (r_{71} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) - r_{GO1} \cos\theta_{131} \cos\theta_5 \\
&- r_{71} \cos\theta_{131} \cos\theta_6)\, \dot\theta_{132} + r_{71} \cos\theta_6 \dot\theta_5 \Big) \\
&= -r_{GO1}^2 \cos^2\theta_{131} \sin\theta_5 \cos\theta_5 \dot\theta_{132}^2 \\
&- r_{GO1} r_{71} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131} \dot\theta_{132} \\
&+ r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{131} \dot\theta_{132} \\
&+ r_{GO1} r_{71} \sin\theta_{131} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&+ r_{GO1} r_{71} \sin\theta_{131} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- r_{GO1} r_{71} \cos^2\theta_{131} \sin\theta_5 \cos\theta_6 \dot\theta_{132}^2 + r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \dot\theta_{132} \dot\theta_5 \\
&+ r_{71}^2 \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}^2 \\
&+ r_{71}^2 \sin\theta_{131} \sin^2\theta_6 \cos^2(\theta_5 - \theta_{132})\, \dot\theta_{131} \dot\theta_{132} \\
&- r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131} \dot\theta_{132} \\
&- r_{71}^2 \sin\theta_{131} \sin^2\theta_6 \sin^2(\theta_5 - \theta_{132})\, \dot\theta_{131} \dot\theta_{132} \\
&+ r_{71}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131} \dot\theta_5 \\
&- r_{71}^2 \sin\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \sin\theta_{131} \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&+ r_{71}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132} \dot\theta_5 \\
&= -r_{GO1}^2 \cos^2\theta_{131} \sin\theta_5 \cos\theta_5 \dot\theta_{132}^2 \\
&- r_{GO1} r_{71} \cos\theta_{131} \sin\theta_6 \cos(2\theta_5 - \theta_{132})\, \dot\theta_{131} \dot\theta_{132} \\
&+ r_{GO1} r_{71} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \sin(2\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- r_{GO1} r_{71} \cos^2\theta_{131} \sin\theta_5 \cos\theta_6 \dot\theta_{132}^2 + r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \dot\theta_{132} \dot\theta_5 \\
&+ r_{71}^2 \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}^2
\end{aligned}
$$

$$+ r_{71}^2 \sin\theta_{131} \sin^2\theta_6 \cos(2\theta_5 - 2\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{71}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5$$
$$- r_{71}^2 \sin\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \sin\theta_{131} \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$+ r_{71}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$- r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \tag{4.662}$$

$$\bar{v}_7^T \frac{\partial \bar{v}_7}{\partial \theta_5} = r_{GO1}^2 \cos^2\theta_{132} \sin\theta_5 \cos\theta_5 \dot{\theta}_{131}^2$$
$$- r_{GO1}^2 \cos\theta_{131} \cos\theta_{132} \sin 2\theta_5 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{GO1}^2 \sin\theta_{131} \sin\theta_{132} \cos\theta_{132} \cos 2\theta_5 \sin\theta_6 \dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{GO1}^2 \cos\theta_{131} \cos\theta_{132} \cos\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \cos 2\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2$$
$$+ r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \cos\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2$$
$$- r_{GO1}^2 \sin^2\theta_{131} \sin^2\theta_{132} \sin\theta_5 \cos\theta_5 \sin^2\theta_6 \dot{\theta}_{132}^2$$
$$- r_{GO1} r_{71} \cos\theta_{132} \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}^2$$
$$- r_{GO1} r_{71} \sin\theta_{131} \cos\theta_6 \cos(\theta_5 - 2\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{GO1} r_{71} \sin^2\theta_{131} \sin\theta_{132} \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2$$
$$- r_{GO1} r_{71} \cos^2\theta_{131} \sin\theta_5 \cos\theta_6 \dot{\theta}_{132}^2 + r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_5$$
$$+ r_{GO1} r_{71} \cos\theta_{131} \cos\theta_5 \sin\theta_6 \dot{\theta}_{132}\dot{\theta}_6 - r_{GO1} r_{73} \cos\theta_{132} \sin(2\theta_5 - \theta_{132})\,\dot{\theta}_{131}^2$$
$$- r_{GO1} r_{73} \cos\theta_{132} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}^2$$
$$+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_{132} \cos\theta_5 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_6 \sin(2\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(2\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \sin\theta_6 \cos(2\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos(2\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{GO1} r_{73} \sin\theta_{131} \cos^2\theta_6 \sin(\theta_5 - 2\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{GO1} r_{73} \cos\theta_{132} \cos\theta_5 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_5 - r_{GO1} r_{73} \cos^2\theta_{131} \cos\theta_5 \dot{\theta}_{132}^2$$
$$- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2$$

$$+ r_{GO1} r_{73} \sin^2\theta_{131} \sin\theta_{132} \sin^2\theta_6 \sin(2\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$+ r_{GO1} r_{73} \sin^2\theta_{131} \sin\theta_{132} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \dot{\theta}_{132} \dot{\theta}_5$$
$$+ r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132} \dot{\theta}_5$$
$$- r_{71}^2 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$
$$+ r_{71}^2 \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$
$$- r_{71}^2 \sin\theta_{131} \cos(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$+ r_{71}^2 \sin\theta_{131} \sin^2\theta_6 \cos(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$- r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$+ r_{71}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_5 - r_{71}^2 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_6$$
$$+ r_{71}^2 \sin^2\theta_{131} \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$- r_{71}^2 \sin\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \sin\theta_{131} \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$+ r_{71}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$- r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132} \dot{\theta}_5$$
$$- r_{71}^2 \sin\theta_{131} \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132} \dot{\theta}_6 - r_{71} r_{73} \cos\theta_6 \cos(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131}^2$$
$$+ r_{71} r_{73} \cos\theta_{131} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$+ 2 r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$- r_{71} r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_5 - r_{71} r_{73} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_6$$
$$+ r_{71} r_{73} \sin^2\theta_{131} \cos\theta_6 \cos(2\theta_5 - 2\theta_{132}) \dot{\theta}_{132}^2$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$
$$- r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132} \dot{\theta}_5$$
$$+ r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132} \dot{\theta}_6$$
$$+ r_{73}^2 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$
$$+ r_{73}^2 \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$
$$- r_{73}^2 \cos\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$+ r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$+ r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \cos(2\theta_5 - 2\theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$
$$+ r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$+ r_{73}^2 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot\theta_{131}\dot\theta_5 - r_{73}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot\theta_{131}\dot\theta_5$$
$$- r_{73}^2 \sin^2\theta_{131} \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot\theta_{132}^2 \quad (4.663)$$

$$\begin{aligned}
\bar\omega_7^T \hat J_7^{(7)} \frac{\partial\bar\omega_7}{\partial\theta_5} = &\Big(J_{711}\Big(\cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot\theta_{131} \\
&- \cos\theta_6 \sin\theta_{131} \sin(\theta_5 - \theta_{132}) \dot\theta_{132}\Big) \\
&+ J_{712}\Big(- \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot\theta_{131} \\
&+ \sin\theta_6 \sin\theta_{131} \sin(\theta_5 - \theta_{132}) \dot\theta_{132}\Big)\Big) \\
&\Big(- \sin\theta_6 \dot\theta_5 + \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot\theta_{131} \\
&+ (\cos\theta_6 \sin\theta_{131} \cos(\theta_5 - \theta_{132}) + \sin\theta_6 \cos\theta_{131}) \dot\theta_{132}\Big) \\
&+ \Big(J_{712}\Big(\cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot\theta_{131} \\
&- \cos\theta_6 \sin\theta_{131} \sin(\theta_5 - \theta_{132}) \dot\theta_{132}\Big) \\
&+ J_{722}\Big(- \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot\theta_{131} \\
&+ \sin\theta_6 \sin\theta_{131} \sin(\theta_5 - \theta_{132}) \dot\theta_{132}\Big)\Big) \\
&\Big(- \cos\theta_6 \dot\theta_5 - \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot\theta_{131} \\
&+ (\cos\theta_6 \cos\theta_{131} - \sin\theta_6 \sin\theta_{131} \cos(\theta_5 - \theta_{132})) \dot\theta_{132}\Big) \\
&+ J_{733}\Big(- \sin(\theta_5 - \theta_{132}) \dot\theta_{131} - \sin\theta_{131} \cos(\theta_5 - \theta_{132}) \dot\theta_{132}\Big) \\
&\Big(\dot\theta_6 + \cos(\theta_5 - \theta_{132}) \dot\theta_{131} - \sin\theta_{131} \sin(\theta_5 - \theta_{132}) \dot\theta_{132}\Big) \quad (4.664)
\end{aligned}$$

$$\begin{aligned}
\bar\omega_7^T \hat J_7^{(7)} \frac{\partial\bar\omega_7}{\partial\theta_5} = &- J_{711} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot\theta_{131}\dot\theta_5 \\
&+ J_{711} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot\theta_{132}\dot\theta_5 \\
&+ J_{711} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot\theta_{131}^2 \\
&- J_{711} \sin\theta_{131} \cos^2\theta_6 \sin^2(\theta_5 - \theta_{132}) \dot\theta_{131}\dot\theta_{132} \\
&+ J_{711} \sin\theta_{131} \cos^2\theta_6 \cos^2(\theta_5 - \theta_{132}) \dot\theta_{131}\dot\theta_{132} \\
&+ J_{711} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot\theta_{131}\dot\theta_{132} \\
&- J_{711} \sin^2\theta_{131} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot\theta_{132}^2
\end{aligned}$$

$$
\begin{aligned}
&- J_{711} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&+ J_{712} \sin\theta_6 \sin\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_5 \\
&- J_{712} \sin\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}\dot\theta_5 \\
&- J_{712} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_5 \\
&+ J_{712} \sin\theta_{131} \cos^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}\dot\theta_5 \\
&- J_{712} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}^2 \\
&+ J_{712} \sin^2\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&+ J_{712} \sin\theta_{131} \cos\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- J_{712} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}^2 \\
&- J_{712} \sin\theta_{131} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos\theta_{131}\dot\theta_{132}^2 \\
&+ J_{712} \sin^2\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&+ J_{712} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin^2(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&- J_{712} \sin\theta_{131} \sin\theta_6 \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&- J_{712} \cos\theta_{131} \sin^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&+ J_{712} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin^2(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&+ J_{712} \cos\theta_{131} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&- J_{712} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos^2(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&+ J_{722} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_5 \\
&- J_{722} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}\dot\theta_5 \\
&+ J_{722} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}^2 \\
&- J_{722} \sin\theta_{131} \sin^2\theta_6 \sin^2(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&- J_{722} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&+ J_{722} \sin\theta_{131\,\sin^2\theta_6} \cos^2(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132} \\
&+ J_{722} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- J_{722} \sin^2\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}^2 \\
&- J_{733} \sin(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_6 - J_{733} \sin\theta_{131} \cos(\theta_5 - \theta_{132})\, \dot\theta_{132}\dot\theta_6 \\
&- J_{733} \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot\theta_{131}^2 \\
&- J_{733} \sin\theta_{131} \cos^2(\theta_5 - \theta_{132})\, \dot\theta_{131}\dot\theta_{132}
\end{aligned}
$$

$$
\begin{aligned}
&+ J_{733} \sin\theta_{131} \sin^2(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{733} \sin^2\theta_{131} \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \qquad (4.665)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\omega}_7^T \hat{J}_7^{(7)} \frac{\partial \bar{\omega}_7}{\partial \theta_5} = &-J_{711} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_5 \\
&+ J_{711} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&+ J_{711} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}^2 \\
&+ J_{711} \sin\theta_{131} \cos^2\theta_6 \cos(2\theta_5 - 2\theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{711} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&- J_{711} \sin^2\theta_{131} \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- J_{711} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- J_{712} \cos 2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_5 \\
&+ J_{712} \sin\theta_{131} \cos 2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&- 2J_{712} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}^2 \\
&+ 2J_{712} \sin^2\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- J_{712} \sin\theta_{131} \cos\theta_{131} \cos 2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&+ J_{712} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin^2(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&- J_{712} \sin\theta_{131} \sin\theta_6 \cos 2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{712} \cos\theta_{131} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&- J_{712} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(2\theta_5 - 2\theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{722} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_5 \\
&- J_{722} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&+ J_{722} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}^2 \\
&- J_{722} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{722} \sin\theta_{131} \sin^2\theta_6 \cos(2\theta_5 - 2\theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{722} \sin\theta_{131} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- J_{722} \sin^2\theta_{131} \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- J_{733} \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 - J_{733} \sin\theta_{131} \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \\
&- J_{733} \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}^2
\end{aligned}
$$

$$
\begin{aligned}
&- J_{733} \sin\theta_{131} \cos(2\theta_5 - 2\theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{733} \sin^2\theta_{131} \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2
\end{aligned}
\tag{4.666}
$$

$$
\frac{d}{dt}\left(\frac{\partial K_7}{\partial \dot{\theta}_5}\right) = m_7 \left(\dot{\bar{v}}_7^T \frac{\partial \bar{v}_7}{\partial \dot{\theta}_5} + \bar{v}_7^T \frac{d}{dt}\left(\frac{\partial \bar{v}_7}{\partial \dot{\theta}_5}\right)\right) + \dot{\bar{\omega}}_7^T \hat{J}_7^{(7)} \frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_5} + \bar{\omega}_7^T \hat{J}_7^{(7)} \frac{d}{dt}\left(\frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_5}\right)
\tag{4.667}
$$

$$
\dot{\bar{v}}_7^T \frac{\partial \bar{v}_7}{\partial \dot{\theta}_5} = \dot{v}_{71} \frac{\partial v_{71}}{\partial \dot{\theta}_5} + \dot{v}_{72} \frac{\partial v_{72}}{\partial \dot{\theta}_5} + \dot{v}_{73} \frac{\partial v_{73}}{\partial \dot{\theta}_5}
\tag{4.668}
$$

$$
\begin{aligned}
\dot{v}_{71} \frac{\partial v_{71}}{\partial \dot{\theta}_5} = &\big(- r_{GO1} r_{73} \cos\theta_{132} \sin\theta_6 \cos\theta_6 + r_{73}^2 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \big) \ddot{\theta}_{131} \\
&+ \big(- r_{73}^2 \cos\theta_{131} \cos^2\theta_6 + r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \\
&+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \cos^2\theta_6 - r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \big) \ddot{\theta}_{132} \\
&+ r_{73}^2 \cos^2\theta_6 \ddot{\theta}_5 + r_{GO1} r_{73} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_5 \cos^2\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1} r_{73} \cos\theta_{132} \cos^2\theta_6 \dot{\theta}_{131}\dot{\theta}_6 - r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \cos^2\theta_6 \dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \cos^2\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{73}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} + r_{73}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_5 \\
&+ r_{73}^2 \cos^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 \\
&+ r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 + r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{73}^2 \sin\theta_6 \cos\theta_6 \dot{\theta}_5\dot{\theta}_6
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}_{72}\frac{\partial v_{72}}{\partial \dot{\theta}_5} = &\big(r_{71}r_{73}\sin\theta_6\cos(\theta_5-\theta_{132}) + r_{GO1}r_{73}\cos\theta_{132}\sin\theta_6\cos\theta_6 \\
&- r_{73}^2\sin\theta_6\cos\theta_6\sin(\theta_5-\theta_{132})\big)\ddot{\theta}_{131} \\
&+ \big(r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\cos\theta_6 \\
&- r_{71}r_{73}\sin\theta_{131}\sin\theta_6\sin(\theta_5-\theta_{132}) + r_{GO1}r_{73}\cos\theta_{131}\sin\theta_5\sin^2\theta_6 \\
&- r_{73}^2\sin\theta_{131}\sin\theta_6\cos\theta_6\cos(\theta_5-\theta_{132}) - r_{73}^2\sin\theta_6\cos\theta_{131}\sin\theta_6\big)\ddot{\theta}_{132} \\
&+ r_{73}^2\sin^2\theta_6\ddot{\theta}_5 + r_{71}r_{73}\sin\theta_6\ddot{\theta}_6 - r_{GO1}r_{73}\sin\theta_{132}\sin\theta_6\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{GO1}r_{73}\cos\theta_{131}\sin\theta_{132}\sin\theta_6\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1}r_{73}\sin\theta_{131}\sin\theta_5\sin^2\theta_6\dot{\theta}_{131}\dot{\theta}_{132} - r_{GO1}r_{73}\cos\theta_{132}\sin^2\theta_6\dot{\theta}_{131}\dot{\theta}_6 \\
&+ r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{132}\sin\theta_6\cos\theta_6\dot{\theta}_{132}^2 \\
&+ r_{GO1}r_{73}\cos\theta_{131}\cos\theta_5\sin^2\theta_6\dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\sin^2\theta_6\dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{GO1}r_{73}\cos\theta_{131}\sin\theta_5\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{71}r_{73}\sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{71}r_{73}\cos\theta_{131}\sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{71}r_{73}\sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
&- r_{71}r_{73}\sin\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 \\
&- r_{71}r_{73}\sin\theta_{131}\sin\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{73}^2\sin\theta_6\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{73}^2\cos\theta_{131}\sin\theta_6\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{73}^2\sin\theta_{131}\sin^2\theta_6\dot{\theta}_{131}\dot{\theta}_{132} - r_{73}^2\sin\theta_6\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
&+ r_{73}^2\sin^2\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6 \\
&- r_{73}^2\sin\theta_{131}\sin\theta_6\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 \\
&- r_{73}^2\sin\theta_{131}\sin\theta_6\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{73}^2\sin\theta_{131}\sin^2\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6 - r_{73}^2\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{73}^2\sin\theta_6\cos\theta_6\dot{\theta}_5\dot{\theta}_6
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}_{73}\frac{\partial v_{73}}{\partial \dot{\theta}_5} &= r_{71}^2 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\ddot{\theta}_{131} \\
&+ \big(r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) - r_{GO1} r_{71} \cos\theta_{131} \cos\theta_5 \cos\theta_6 \\
&- r_{71}^2 \cos\theta_{131} \cos^2\theta_6\big)\ddot{\theta}_{132} + r_{71}^2 \cos^2\theta_6 \ddot{\theta}_5 \\
&+ r_{GO1} r_{71} \sin\theta_{131} \cos\theta_5 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} + r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{71}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{71}^2 \sin\theta_{131} \cos^2\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} + r_{71}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
&+ r_{71}^2 \cos^2\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6 \\
&+ r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2 \\
&- r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{71}^2 \sin\theta_{131} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6 + r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{71}^2 \sin\theta_6 \cos\theta_6 \dot{\theta}_5\dot{\theta}_6
\end{aligned}
$$

$$
\begin{aligned}
\dot{\bar{v}}_7^T \frac{\partial \bar{v}_7}{\partial \dot{\theta}_5} &= \big(- r_{GO1} r_{73} \cos\theta_{132} \sin\theta_6 \cos\theta_6 + r_{GO1} r_{73} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \\
&+ r_{71}^2 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\big)\ddot{\theta}_{131} + r_{71} r_{73} \sin\theta_6 \cos(\theta_5 - \theta_{132}) \\
&+ r_{73}^2 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) - r_{73}^2 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \\
&+ \big(- r_{GO1} r_{71} \cos\theta_{131} \cos\theta_5 \cos\theta_6 + r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \cos^2\theta_6 \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 + r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin^2\theta_6 + r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \\
&- r_{71}^2 \cos\theta_{131} \cos^2\theta_6\big)\ddot{\theta}_{132} - r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \\
&- r_{73}^2 \cos\theta_{131} \cos^2\theta_6 + r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) - r_{73}^2 \sin\theta_6 \cos\theta_{131} \sin\theta_6 \\
&+ \big(r_{73}^2 \cos^2\theta_6 + r_{73}^2 \sin^2\theta_6 + r_{71}^2 \cos^2\theta_6\big)\ddot{\theta}_5 + r_{71} r_{73} \sin\theta_6 \ddot{\theta}_6 \\
&+ r_{GO1} r_{71} \sin\theta_{131} \cos\theta_5 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} + r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{GO1} r_{73} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_5 \cos^2\theta_6 \dot{\theta}_{131}\dot{\theta}_{132}
\end{aligned}
$$

$$
\begin{aligned}
&- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{GO1} r_{73} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_5 \sin^2\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} - r_{GO1} r_{73} \cos\theta_{132} \cos^2\theta_6 \dot{\theta}_{131} \dot{\theta}_6 \\
&- r_{GO1} r_{73} \cos\theta_{132} \sin^2\theta_6 \dot{\theta}_{131} \dot{\theta}_6 - r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2 \\
&+ r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \cos^2\theta_6 \dot{\theta}_{132} \dot{\theta}_5 + r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5 \sin^2\theta_6 \dot{\theta}_{132} \dot{\theta}_5 \\
&- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132} \dot{\theta}_6 \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \cos^2\theta_6 \dot{\theta}_{132} \dot{\theta}_6 \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin^2\theta_6 \dot{\theta}_{132} \dot{\theta}_6 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132} \dot{\theta}_6 \\
&- r_{71}^2 \sin\theta_6 \cos\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{71}^2 \sin\theta_{131} \cos^2\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} + r_{71}^2 \sin\theta_6 \cos\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_5 \\
&+ r_{71}^2 \cos^2\theta_6 \sin\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_6 \\
&+ r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{132}^2 \\
&- r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{132} \dot{\theta}_5 \\
&+ r_{71}^2 \sin\theta_{131} \cos^2\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{132} \dot{\theta}_6 + r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132} \dot{\theta}_6 \\
&- r_{71}^2 \sin\theta_6 \cos\theta_6 \dot{\theta}_5 \dot{\theta}_6 + r_{71} r_{73} \sin\theta_6 \sin\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{71} r_{73} \cos\theta_{131} \sin\theta_6 \sin\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{71} r_{73} \sin\theta_6 \sin\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_5 \\
&- r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{132}^2 \\
&- r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{132} \dot{\theta}_5 \\
&- r_{73}^2 \sin\theta_6 \cos\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} - r_{73}^2 \sin\theta_6 \cos\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_{132} \\
&- r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_{132} \\
&+ r_{73}^2 \sin\theta_{131} \sin^2\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} + r_{73}^2 \sin\theta_6 \cos\theta_6 \cos\left(\theta_5 - \theta_{132}\right) \dot{\theta}_{131} \dot{\theta}_5
\end{aligned}
$$

$$
\begin{aligned}
&- r_{73}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 + r_{73}^2 \sin^2\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6 \\
&+ r_{73}^2 \cos^2\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6 \\
&+ r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2 \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2 \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 + r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{73}^2 \sin\theta_{131} \sin^2\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6 - r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{73}^2 \sin\theta_6 \cos\theta_6\dot{\theta}_5\dot{\theta}_6 + r_{73}^2 \sin\theta_6 \cos\theta_6\dot{\theta}_5\dot{\theta}_6 \qquad (4.669)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\bar{v}}_7^T \frac{\partial \bar{v}_7}{\partial \dot{\theta}_5} &= \big(- r_{GO1} r_{73} \cos\theta_{132} \sin\theta_6 \cos\theta_6 + r_{GO1} r_{73} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \\
&+ r_{71}^2 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\,\big)\ddot{\theta}_{131} + r_{71} r_{73} \sin\theta_6 \cos(\theta_5 - \theta_{132}) \\
&+ r_{73}^2 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) - r_{73}^2 \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \\
&+ \big(- r_{GO1} r_{71} \cos\theta_{131} \cos\theta_5 \cos\theta_6 + r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \cos^2\theta_6 \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 + r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin^2\theta_6 + r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \\
&- r_{71}^2 \cos\theta_{131} \cos^2\theta_6\big)\ddot{\theta}_{132} - r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \\
&- r_{73}^2 \cos\theta_{131} \cos^2\theta_6 + r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) - r_{73}^2 \sin\theta_6 \cos\theta_{131} \sin\theta_6 \\
&+ \big(r_{73}^2 \cos^2\theta_6 + r_{73}^2 \sin^2\theta_6 + r_{71}^2 \cos^2\theta_6\big)\ddot{\theta}_5 + r_{71} r_{73} \sin\theta_6\ddot{\theta}_6 \\
&+ r_{GO1} r_{71} \sin\theta_{131} \cos\theta_5 \cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} + r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \cos\theta_6\dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_5\dot{\theta}_{131}\dot{\theta}_{132} - r_{GO1} r_{73} \cos\theta_{132}\dot{\theta}_{131}\dot{\theta}_6 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_5\dot{\theta}_{132}\dot{\theta}_5 - r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132}\dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{71}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{71}^2 \sin\theta_{131} \cos^2\theta_6\dot{\theta}_{131}\dot{\theta}_{132} + r_{71}^2 \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
&+ r_{71}^2 \cos^2\theta_6 \sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6
\end{aligned}
$$

$$
\begin{aligned}
&+ r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- r_{71}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{71}^2 \sin\theta_{131} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 + r_{71}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{71}^2 \sin\theta_6 \cos\theta_6 \dot{\theta}_5\dot{\theta}_6 + r_{71} r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{71} r_{73} \cos\theta_{131} \sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{71} r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_5 \\
&- r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{73}^2 \sin 2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} + r_{73}^2 \sin\theta_{131} \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{73}^2 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 + r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{73}^2 \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{73}^2 \sin\theta_{131} \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \qquad (4.670)
\end{aligned}
$$

$$
\begin{aligned}
\bar{v}_7^T \frac{d}{dt}\left(\frac{\partial \bar{v}_7}{\partial \dot{\theta}_5}\right) &= r_{GO1} r_{73} \cos\theta_{132} \sin^2\theta_6 \dot{\theta}_{131}\dot{\theta}_6 - r_{73}^2 \sin^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 \\
&+ r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{73}^2 \sin\theta_{131} \sin^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin^2\theta_6 \dot{\theta}_{132}\dot{\theta}_6 - r_{73}^2 \sin\theta_6 \cos\theta_6 \dot{\theta}_5\dot{\theta}_6 \\
&+ r_{71} r_{73} \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 + r_{GO1} r_{73} \cos\theta_{132} \cos^2\theta_6 \dot{\theta}_{131}\dot{\theta}_6 \\
&- r_{73}^2 \cos^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 \\
&+ r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \cos^2\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{71} r_{73} \sin\theta_{131} \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \\
&+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{73}^2 \sin\theta_{131} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \\
&- r_{73}^2 \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 + r_{73}^2 \sin\theta_6 \cos\theta_6 \dot{\theta}_5\dot{\theta}_6
\end{aligned}
$$

$$+ r_{71}r_{73}\cos\theta_6\dot{\theta}_6\dot{\theta}_6 - r_{71}^2\sin^2\theta_6\sin(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_6$$
$$- r_{71}^2\sin\theta_{131}\sin^2\theta_6\cos(\theta_5-\theta_{132})\dot{\theta}_{132}\dot{\theta}_6$$
$$+ r_{GO1}r_{71}\cos\theta_{131}\cos\theta_5\sin\theta_6\dot{\theta}_{132}\dot{\theta}_6 + r_{71}^2\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_6\dot{\theta}_{132}$$
$$- r_{71}^2\sin\theta_6\cos\theta_6\dot{\theta}_5\dot{\theta}_6$$

$$\begin{aligned}
\bar{v}_7^T\frac{d}{dt}\left(\frac{\partial\bar{v}_7}{\partial\dot{\theta}_5}\right) &= r_{GO1}r_{71}\cos\theta_{131}\cos\theta_5\sin\theta_6\dot{\theta}_{132}\dot{\theta}_6 + r_{GO1}r_{73}\cos\theta_{132}\dot{\theta}_{131}\dot{\theta}_6\\
&+ r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\dot{\theta}_{132}\dot{\theta}_6 - r_{71}^2\sin^2\theta_6\sin(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_6\\
&- r_{71}^2\sin\theta_{131}\sin^2\theta_6\cos(\theta_5-\theta_{132})\dot{\theta}_{132}\dot{\theta}_6\\
&+ r_{71}^2\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 - r_{71}^2\sin\theta_6\cos\theta_6\dot{\theta}_5\dot{\theta}_6\\
&+ r_{71}r_{73}\cos\theta_6\cos(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_6\\
&- r_{71}r_{73}\sin\theta_{131}\cos\theta_6\sin(\theta_5-\theta_{132})\dot{\theta}_{132}\dot{\theta}_6 + r_{71}r_{73}\cos\theta_6\dot{\theta}_6^2\\
&- r_{73}^2\sin(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_6 - r_{73}^2\sin\theta_{131}\cos(\theta_5-\theta_{132})\dot{\theta}_{132}\dot{\theta}_6
\end{aligned} \tag{4.671}$$

$$\begin{aligned}
\dot{\bar{\omega}}_7^T\hat{J}_7^{(7)}\frac{\partial\bar{\omega}_7}{\partial\dot{\theta}_5} = \Big(& -\sin\theta_6\ddot{\theta}_5 + \cos\theta_6\sin(\theta_5-\theta_{132})\ddot{\theta}_{131}\\
&+ \cos\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132})\ddot{\theta}_{132} + \sin\theta_6\cos\theta_{131}\ddot{\theta}_{132} - \cos\theta_6\dot{\theta}_5\dot{\theta}_6\\
&- \sin\theta_6\sin(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_6 + \cos\theta_6\cos(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_5\\
&- \cos\theta_6\cos(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_{132} - \sin\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132})\dot{\theta}_{132}\dot{\theta}_6\\
&+ \cos\theta_6\cos\theta_{131}\cos(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_{132}\\
&- \cos\theta_6\sin\theta_{131}\sin(\theta_5-\theta_{132})\dot{\theta}_{132}\dot{\theta}_5\\
&+ \cos\theta_6\sin\theta_{131}\sin(\theta_5-\theta_{132})\dot{\theta}_{132}^2 + \cos\theta_6\cos\theta_{131}\dot{\theta}_{132}\dot{\theta}_6\\
&- \sin\theta_6\sin\theta_{131}\dot{\theta}_{131}\dot{\theta}_{132}\Big)(-J_{711}\sin\theta_6 - J_{712}\cos\theta_6) + \Big(-\cos\theta_6\ddot{\theta}_5\\
&- \sin\theta_6\sin(\theta_5-\theta_{132})\ddot{\theta}_{131} + (\cos\theta_6\cos\theta_{131}\\
&- \sin\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132}))\ddot{\theta}_{132} + \sin\theta_6\dot{\theta}_5\dot{\theta}_6\\
&- \cos\theta_6\sin(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_6 - \sin\theta_6\cos(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_5\\
&+ \sin\theta_6\cos(\theta_5-\theta_{132})\dot{\theta}_{131}\dot{\theta}_{132} - \sin\theta_6\cos\theta_{131}\dot{\theta}_{132}\dot{\theta}_6\\
&- \cos\theta_6\sin\theta_{131}\dot{\theta}_{131}\dot{\theta}_{132} - \cos\theta_6\sin\theta_{131}\cos(\theta_5-\theta_{132})\dot{\theta}_{132}\dot{\theta}_6
\end{aligned}$$

$$
\begin{aligned}
&- \sin\theta_6 \cos\theta_{131} \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ \sin\theta_6 \sin\theta_{131} \sin(\theta_5 - \theta_{132})\, \dot{\theta}_5\dot{\theta}_{132} \\
&+ \sin\theta_6 \sin\theta_{131} \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2\Big) \left(-J_{712}\sin\theta_6 - J_{722}\cos\theta_6\right)
\end{aligned}
\tag{4.672}
$$

$$
\begin{aligned}
\dot{\bar{\omega}}_7^T \hat{J}_7^{(7)} \frac{\partial\bar{\omega}_7}{\partial\dot{\theta}_5} = {} & J_{711}\sin^2\theta_6\ddot{\theta}_5 - J_{711}\sin\theta_6\cos\theta_6\sin(\theta_5 - \theta_{132})\,\ddot{\theta}_{131} \\
& - J_{711}\sin\theta_{131}\sin\theta_6\cos\theta_6\cos(\theta_5 - \theta_{132})\,\ddot{\theta}_{132} \\
& - J_{711}\cos\theta_{131}\sin\theta_6\sin\theta_6\ddot{\theta}_{132} \\
& + J_{711}\sin\theta_6\cos\theta_6\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
& - J_{711}\cos\theta_{131}\sin\theta_6\cos\theta_6\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
& + J_{711}\sin\theta_{131}\sin^2\theta_6\dot{\theta}_{131}\dot{\theta}_{132} - J_{711}\sin\theta_6\cos\theta_6\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
& + J_{711}\sin^2\theta_6\sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6 \\
& - J_{711}\sin\theta_{131}\sin\theta_6\cos\theta_6\sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2 \\
& + J_{711}\sin\theta_{131}\sin\theta_6\cos\theta_6\sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
& + J_{711}\sin\theta_{131}\sin^2\theta_6\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6 \\
& - J_{711}\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 + J_{711}\sin\theta_6\cos\theta_6\dot{\theta}_5\dot{\theta}_6 \\
& + J_{712}\sin\theta_6\cos\theta_6\ddot{\theta}_5 - J_{712}\cos^2\theta_6\sin(\theta_5 - \theta_{132})\,\ddot{\theta}_{131} \\
& - J_{712}\cos^2\theta_6\sin\theta_{131}\cos(\theta_5 - \theta_{132})\,\ddot{\theta}_{132} - J_{712}\cos\theta_{131}\sin\theta_6\cos\theta_6\ddot{\theta}_{132} \\
& + J_{712}\cos 2\theta_6\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
& - J_{712}\cos\theta_{131}\cos 2\theta_6\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
& + 2J_{712}\sin\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} - J_{712}\cos 2\theta_6\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
& + 2J_{712}\sin\theta_6\cos\theta_6\sin(\theta_5 - \theta_{132})\,\dot{\theta}_{131}\dot{\theta}_6 \\
& - J_{712}\sin\theta_{131}\sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}^2 \\
& + J_{712}\sin\theta_{131}\cos 2\theta_6\sin(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
& + 2J_{712}\sin\theta_{131}\sin\theta_6\cos\theta_6\cos(\theta_5 - \theta_{132})\,\dot{\theta}_{132}\dot{\theta}_6 \\
& - J_{712}\cos\theta_{131}\cos 2\theta_6\dot{\theta}_{132}\dot{\theta}_6 + J_{712}\cos 2\theta_6\dot{\theta}_5\dot{\theta}_6 + J_{722}\cos^2\theta_6\ddot{\theta}_5 \\
& + J_{722}\sin\theta_6\cos\theta_6\sin(\theta_5 - \theta_{132})\,\ddot{\theta}_{131} - J_{722}(\cos\theta_{131}\cos^2\theta_6 \\
& - \sin\theta_{131}\sin\theta_6\cos\theta_6\cos(\theta_5 - \theta_{132}))\ddot{\theta}_{132}
\end{aligned}
$$

$$
\begin{aligned}
&- J_{722} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} + J_{722} \sin\theta_{131} \cos^2\theta_6 \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{722} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_{132} \\
&+ J_{722} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_5 \\
&+ J_{722} \cos^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 \\
&- J_{722} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}^2 \\
&- J_{722} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_5 \\
&+ J_{722} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&+ J_{722} \sin\theta_{131} \cos^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 - J_{722} \sin\theta_6 \cos\theta_6 \dot{\theta}_5\dot{\theta}_6
\end{aligned} \tag{4.673}
$$

$$
\begin{aligned}
\bar{\omega}_7^T \hat{J}_7^{(7)} \frac{d}{dt}\left(\frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_5}\right) &= \Big(- \sin\theta_6 \dot{\theta}_5 + \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131} \\
&+ (\cos\theta_6 \sin\theta_{131} \cos(\theta_5 - \theta_{132}) + \sin\theta_6 \cos\theta_{131})\dot{\theta}_{132}\Big) \\
&\Big(- J_{711} \cos\theta_6 \dot{\theta}_6 + J_{712} \sin\theta_6 \dot{\theta}_6\Big) \\
&+ \Big(- \cos\theta_6 \dot{\theta}_5 - \sin\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131} \\
&+ (\cos\theta_6 \cos\theta_{131} - \sin\theta_6 \sin\theta_{131} \cos(\theta_5 - \theta_{132}))\dot{\theta}_{132}\Big) \\
&\Big(- J_{712} \cos\theta_6 \dot{\theta}_6 + J_{722} \sin\theta_6 \dot{\theta}_6\Big)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\omega}_7^T \hat{J}_7^{(7)} \frac{d}{dt}\left(\frac{\partial \bar{\omega}_7}{\partial \dot{\theta}_5}\right) &= -J_{711} \cos^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 \\
&- J_{711} \cos^2\theta_6 \sin\theta_{131} \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \\
&- J_{711} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 + J_{711} \sin\theta_6 \cos\theta_6 \dot{\theta}_5\dot{\theta}_6 \\
&+ 2J_{712} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 \\
&+ 2J_{712} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \\
&- J_{712} \cos\theta_{131} \cos 2\theta_6 \dot{\theta}_{132}\dot{\theta}_6 + J_{712} \cos 2\theta_6 \dot{\theta}_5\dot{\theta}_6 \\
&- J_{722} \sin^2\theta_6 \sin(\theta_5 - \theta_{132})\, \dot{\theta}_{131}\dot{\theta}_6 \\
&+ J_{722} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}\dot{\theta}_6 \\
&- J_{722} \sin\theta_{131} \sin^2\theta_6 \cos(\theta_5 - \theta_{132})\, \dot{\theta}_{132}\dot{\theta}_6 \\
&- J_{722} \sin\theta_6 \cos\theta_6 \dot{\theta}_5\dot{\theta}_6
\end{aligned}
$$

$$
\begin{aligned}
&= \big(- J_{711} \cos^2 \theta_6 \sin(\theta_5 - \theta_{132}) \\
&+ 2J_{712} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \\
&- J_{722} \sin^2 \theta_6 \sin(\theta_5 - \theta_{132}) \big) \dot{\theta}_{131} \dot{\theta}_6 \\
&- \big(J_{711} \cos^2 \theta_6 \sin\theta_{131} \cos(\theta_5 - \theta_{132}) - J_{711} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \\
&+ 2J_{712} \sin\theta_{131} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) - J_{712} \cos\theta_{131} \cos 2\theta_6 \\
&+ J_{722} \cos\theta_{131} \sin\theta_6 \cos\theta_6 \\
&- J_{722} \sin\theta_{131} \sin^2 \theta_6 \cos(\theta_5 - \theta_{132}) \big) \dot{\theta}_{132} \dot{\theta}_6 \\
&+ \big(J_{711} \sin\theta_6 \cos\theta_6 + J_{712} \cos 2\theta_6 - J_{722} \sin\theta_6 \cos\theta_6 \big) \dot{\theta}_5 \dot{\theta}_6
\end{aligned} \tag{4.674}
$$

$$
\begin{aligned}
\bar{v}_7^T \frac{\partial \bar{v}_7}{\partial \theta_6} &= \Big(\left(r_{GO1} \cos\theta_{132} \cos\theta_6 - r_{73} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \right) \dot{\theta}_{131} \\
&+ (-r_{73} \cos\theta_{131} \sin\theta_6 - r_{73} \sin\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \\
&+ r_{GO1} \cos\theta_{131} \sin\theta_5 \sin\theta_6 + r_{GO1} \sin\theta_{131} \sin\theta_{132} \cos\theta_6) \dot{\theta}_{132} \\
&+ r_{73} \sin\theta_6 \dot{\theta}_5 \Big) \Big(\big(r_{GO1} \cos\theta_{132} \sin\theta_6 - r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \big) \dot{\theta}_{131} \\
&+ \big(r_{73} \cos\theta_{131} \cos\theta_6 - r_{73} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) \\
&- r_{GO1} \cos\theta_{131} \sin\theta_5 \cos\theta_6 + r_{GO1} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \big) \dot{\theta}_{132} \\
&- r_{73} \cos\theta_6 \dot{\theta}_5 \Big) \\
&+ \Big((-r_{GO1} \cos\theta_{132} \sin\theta_6 + r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132})) \dot{\theta}_{131} \\
&+ (-r_{GO1} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 + r_{GO1} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \\
&+ r_{73} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) - r_{73} \cos\theta_{131} \cos\theta_6) \dot{\theta}_{132} + r_{73} \cos\theta_6 \dot{\theta}_5 \Big) \\
&\Big((r_{71} \cos(\theta_5 - \theta_{132}) + r_{GO1} \cos\theta_{132} \cos\theta_6 - r_{73} \cos\theta_6 \sin(\theta_5 - \theta_{132})) \dot{\theta}_{131} \\
&+ (r_{GO1} \sin\theta_{131} \sin\theta_{132} \cos\theta_6 - r_{71} \sin\theta_{131} \sin(\theta_5 - \theta_{132}) \\
&+ r_{GO1} \cos\theta_{131} \sin\theta_5 \sin\theta_6 - r_{73} \sin\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \\
&- r_{73} \cos\theta_{131} \sin\theta_6) \dot{\theta}_{132} + r_{73} \sin\theta_6 \dot{\theta}_5 + r_{71} \dot{\theta}_6 \Big) \\
&+ \Big(r_{71} \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} + (r_{71} \sin\theta_{131} \cos\theta_6 \cos(\theta_5 - \theta_{132}) \\
&+ r_{71} \cos\theta_{131} \sin\theta_6) \dot{\theta}_{132} - r_{71} \sin\theta_6 \dot{\theta}_5 \Big) \Big(r_{71} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \\
&+ (r_{71} \sin\theta_{131} \sin\theta_6 \cos(\theta_5 - \theta_{132}) - r_{GO1} \cos\theta_{131} \cos\theta_5
\end{aligned}
$$

$$- r_{71}\cos\theta_{131}\cos\theta_6\ )\,\dot\theta_{132} + r_{71}\cos\theta_6\dot\theta_5\Big) \tag{4.675}$$

$$\begin{aligned}
v_{71}\frac{\partial v_{71}}{\partial\theta_6} &= r_{GO1}^2\cos^2\theta_{132}\sin\theta_6\cos\theta_6\dot\theta_{131}^2 \\
&- r_{GO1}^2\cos\theta_{131}\cos\theta_{132}\sin\theta_5\cos^2\theta_6\dot\theta_{131}\dot\theta_{132} \\
&+ r_{GO1}^2\sin\theta_{131}\sin\theta_{132}\cos\theta_{132}\sin\theta_6\cos\theta_6\dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1}r_{73}\cos\theta_{132}\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot\theta_{131}^2 \\
&+ r_{GO1}r_{73}\cos\theta_{131}\cos\theta_{132}\cos^2\theta_6\dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{132}\sin\theta_6\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1}r_{73}\cos\theta_{132}\cos^2\theta_6\dot\theta_{131}\dot\theta_5 \\
&+ r_{GO1}^2\cos\theta_{131}\cos\theta_{132}\sin\theta_5\sin^2\theta_6\dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1}^2\cos^2\theta_{131}\sin^2\theta_5\sin\theta_6\cos\theta_6\dot\theta_{132}^2 \\
&+ r_{GO1}^2\sin\theta_{131}\cos\theta_{131}\sin\theta_{132}\sin\theta_5\sin^2\theta_6\dot\theta_{132}^2 \\
&- r_{GO1}r_{73}\cos\theta_{131}\sin\theta_5\sin^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot\theta_{131}\dot\theta_{132} \\
&+ r_{GO1}r_{73}\cos^2\theta_{131}\sin\theta_5\sin\theta_6\cos\theta_6\dot\theta_{132}^2 \\
&- r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{131}\sin\theta_5\sin^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot\theta_{132}^2 \\
&- r_{GO1}r_{73}\cos\theta_{131}\sin\theta_5\sin\theta_6\cos\theta_6\dot\theta_{132}\dot\theta_5 \\
&+ r_{GO1}^2\sin\theta_{131}\sin\theta_{132}\cos\theta_{132}\sin\theta_6\cos\theta_6\dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1}^2\sin\theta_{131}\cos\theta_{131}\sin\theta_{132}\sin\theta_5\cos^2\theta_6\dot\theta_{132}^2 \\
&+ r_{GO1}^2\sin^2\theta_{131}\sin^2\theta_{132}\sin\theta_6\cos\theta_6\dot\theta_{132}^2 \\
&- r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot\theta_{131}\dot\theta_{132} \\
&+ r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{131}\sin\theta_{132}\cos^2\theta_6\dot\theta_{132}^2 \\
&- r_{GO1}r_{73}\sin^2\theta_{131}\sin\theta_{132}\sin\theta_6\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot\theta_{132}^2 \\
&- r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\cos^2\theta_6\dot\theta_{132}\dot\theta_5 \\
&- r_{GO1}r_{73}\cos\theta_{132}\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot\theta_{131}^2 \\
&+ r_{GO1}r_{73}\cos\theta_{131}\sin\theta_5\cos^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot\theta_{131}\dot\theta_{132} \\
&- r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot\theta_{131}\dot\theta_{132} \\
&+ r_{73}r_{73}\sin\theta_6\cos\theta_6\sin^2\left(\theta_5-\theta_{132}\right)\dot\theta_{131}^2 \\
&- r_{73}r_{73}\cos\theta_{131}\cos^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot\theta_{131}\dot\theta_{132}
\end{aligned}$$

$$
\begin{aligned}
&+ r_{73}r_{73}\sin\theta_{131}\sin\theta_6\cos\theta_6\sin(\theta_5-\theta_{132})\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{73}r_{73}\cos^2\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
&- r_{GO1}r_{73}\cos\theta_{131}\cos\theta_{132}\sin^2\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{GO1}r_{73}\cos^2\theta_{131}\sin\theta_5\sin\theta_6\cos\theta_6\dot{\theta}_{132}^2 \\
&- r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{131}\sin\theta_{132}\sin^2\theta_6\dot{\theta}_{132}^2 \\
&+ r_{73}r_{73}\cos\theta_{131}\sin^2\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{73}r_{73}\cos^2\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}^2 \\
&+ r_{73}r_{73}\sin\theta_{131}\cos\theta_{131}\sin^2\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 \\
&+ r_{73}r_{73}\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{132}\sin\theta_6\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&+ r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{131}\sin\theta_5\cos^2\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 \\
&- r_{GO1}r_{73}\sin^2\theta_{131}\sin\theta_{132}\sin\theta_6\cos\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 \\
&+ r_{73}r_{73}\sin\theta_{131}\sin\theta_6\cos\theta_6\sin(\theta_5-\theta_{132})\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{73}r_{73}\sin\theta_{131}\cos\theta_{131}\cos^2\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 \\
&+ r_{73}r_{73}\sin^2\theta_{131}\sin\theta_6\cos\theta_6\cos^2(\theta_5-\theta_{132})\,\dot{\theta}_{132}^2 \\
&+ r_{73}r_{73}\sin\theta_{131}\cos^2\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{GO1}r_{73}\cos\theta_{132}\sin^2\theta_6\dot{\theta}_{131}\dot{\theta}_5 - r_{GO1}r_{73}\cos\theta_{131}\sin\theta_5\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5 \\
&+ r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\sin^2\theta_6\dot{\theta}_{132}\dot{\theta}_5 - r_{73}r_{73}\sin^2\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_5 \\
&+ r_{73}r_{73}\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5 \\
&- r_{73}r_{73}\sin\theta_{131}\sin^2\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{132}\dot{\theta}_5 - r_{73}r_{73}\sin\theta_6\cos\theta_6\dot{\theta}_5^2
\end{aligned}
\tag{4.676}
$$

$$
\begin{aligned}
v_{72}\frac{\partial v_{72}}{\partial\theta_6} &= -r_{GO1}^2\cos^2\theta_{132}\sin\theta_6\cos\theta_6\dot{\theta}_{131}^2 \\
&- r_{GO1}^2\sin\theta_{131}\sin\theta_{132}\cos\theta_{132}\sin\theta_6\cos\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1}^2\cos\theta_{131}\cos\theta_{132}\sin\theta_5\sin^2\theta_6\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1}r_{71}\cos\theta_{132}\sin\theta_6\cos(\theta_5-\theta_{132})\,\dot{\theta}_{131}^2 \\
&+ r_{GO1}r_{71}\sin\theta_{131}\cos\theta_{132}\sin\theta_6\sin(\theta_5-\theta_{132})\,\dot{\theta}_{131}\dot{\theta}_{132} \\
&- r_{GO1}r_{71}\cos\theta_{132}\sin\theta_6\dot{\theta}_{131}\dot{\theta}_6
\end{aligned}
$$

$$+ r_{GO1} r_{73} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$

$$+ r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$+ r_{GO1} r_{73} \cos\theta_{131} \cos\theta_{132} \sin^2\theta_6 \dot{\theta}_{131} \dot{\theta}_{132} - r_{GO1} r_{73} \cos\theta_{132} \sin^2\theta_6 \dot{\theta}_{131} \dot{\theta}_5$$

$$- r_{GO1}^2 \sin\theta_{131} \sin\theta_{132} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{131} \dot{\theta}_{132}$$

$$- r_{GO1}^2 \sin^2\theta_{131} \sin^2\theta_{132} \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2$$

$$- r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \sin\theta_5 \sin^2\theta_6 \dot{\theta}_{132}^2$$

$$- r_{GO1} r_{71} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$+ r_{GO1} r_{71} \sin^2\theta_{131} \sin\theta_{132} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$

$$- r_{GO1} r_{71} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \dot{\theta}_{132} \dot{\theta}_6$$

$$+ r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$+ r_{GO1} r_{73} \sin^2\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$

$$+ r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \sin^2\theta_6 \dot{\theta}_{132}^2$$

$$- r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin^2\theta_6 \dot{\theta}_{132} \dot{\theta}_5$$

$$+ r_{GO1}^2 \cos\theta_{131} \cos\theta_{132} \sin\theta_5 \cos^2\theta_6 \dot{\theta}_{131} \dot{\theta}_{132}$$

$$+ r_{GO1}^2 \sin\theta_{131} \cos\theta_{131} \sin\theta_{132} \sin\theta_5 \cos^2\theta_6 \dot{\theta}_{132}^2$$

$$+ r_{GO1}^2 \cos^2\theta_{131} \sin^2\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2$$

$$+ r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$- r_{GO1} r_{71} \sin\theta_{131} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$

$$+ r_{GO1} r_{71} \cos\theta_{131} \sin\theta_5 \cos\theta_6 \dot{\theta}_{132} \dot{\theta}_6$$

$$- r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \cos^2\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$- r_{GO1} r_{73} \sin\theta_{131} \cos\theta_{131} \sin\theta_5 \cos^2\theta_6 \cos(\theta_5 - \theta_{132}) \dot{\theta}_{132}^2$$

$$- r_{GO1} r_{73} \cos^2\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132}^2$$

$$+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin\theta_6 \cos\theta_6 \dot{\theta}_{132} \dot{\theta}_5$$

$$+ r_{GO1} r_{73} \cos\theta_{132} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$

$$+ r_{GO1} r_{73} \sin\theta_{131} \sin\theta_{132} \sin\theta_6 \cos\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$+ r_{GO1} r_{73} \cos\theta_{131} \sin\theta_5 \sin^2\theta_6 \sin(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$+ r_{71} r_{73} \sin\theta_6 \sin(\theta_5 - \theta_{132}) \cos(\theta_5 - \theta_{132}) \dot{\theta}_{131}^2$$

$$- r_{71} r_{73} \sin\theta_{131} \sin\theta_6 \sin^2(\theta_5 - \theta_{132}) \dot{\theta}_{131} \dot{\theta}_{132}$$

$$+ r_{71}r_{73}\sin\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_6$$
$$- r_{73}r_{73}\sin\theta_6\cos\theta_6\sin^2\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}^2$$
$$- r_{73}r_{73}\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\cos\left(\theta_5-\theta_{132}\right)\sin\theta_{131}\dot{\theta}_{132}$$
$$- r_{73}r_{73}\cos\theta_{131}\sin^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{73}r_{73}\sin^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_5$$
$$+ r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{132}\sin\theta_6\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\sin\theta_6\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}\sin\theta_{131}\dot{\theta}_{132}$$
$$+ r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{131}\sin\theta_5\sin^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2$$
$$+ r_{71}r_{73}\sin\theta_{131}\dot{\theta}_{132}\sin\theta_6\cos^2\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}$$
$$- r_{71}r_{73}\sin^2\theta_{131}\sin\theta_6\sin\left(\theta_5-\theta_{132}\right)\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2$$
$$+ r_{71}r_{73}\sin\theta_{131}\sin\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}\dot{\theta}_6$$
$$- r_{73}r_{73}\sin\theta_{131}\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{73}r_{73}\sin^2\theta_{131}\sin\theta_6\cos\theta_6\cos^2\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2$$
$$- r_{73}r_{73}\sin\theta_{131}\cos\theta_{131}\sin^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2$$
$$+ r_{73}r_{73}\sin\theta_{131}\sin^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}\dot{\theta}_5$$
$$- r_{GO1}r_{73}\cos\theta_{131}\cos\theta_{132}\cos^2\theta_6\dot{\theta}_{131}\dot{\theta}_{132}$$
$$- r_{GO1}r_{73}\sin\theta_{131}\cos\theta_{131}\sin\theta_{132}\cos^2\theta_6\dot{\theta}_{132}^2$$
$$- r_{GO1}r_{73}\cos^2\theta_{131}\sin\theta_5\sin\theta_6\cos\theta_6\dot{\theta}_{132}^2$$
$$- r_{71}r_{73}\cos\theta_{131}\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{71}r_{73}\sin\theta_{131}\cos\theta_{131}\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2$$
$$- r_{71}r_{73}\cos\theta_{131}\cos\theta_6\dot{\theta}_{132}\dot{\theta}_6 + r_{73}r_{73}\cos\theta_{131}\cos^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}$$
$$+ r_{73}r_{73}\sin\theta_{131}\cos\theta_{131}\cos^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2$$
$$+ r_{73}r_{73}\cos^2\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}^2 - r_{73}r_{73}\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5$$
$$+ r_{GO1}r_{73}\cos\theta_{132}\cos^2\theta_6\dot{\theta}_{131}\dot{\theta}_5 + r_{GO1}r_{73}\sin\theta_{131}\sin\theta_{132}\cos^2\theta_6\dot{\theta}_{132}\dot{\theta}_5$$
$$+ r_{GO1}r_{73}\cos\theta_{131}\sin\theta_5\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5$$
$$+ r_{71}r_{73}\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_5$$
$$- r_{71}r_{73}\sin\theta_{131}\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}\dot{\theta}_5 + r_{71}r_{73}\cos\theta_6\dot{\theta}_5\dot{\theta}_6$$
$$- r_{73}r_{73}\cos^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_5$$

$$- r_{73}r_{73}\sin\theta_{131}\cos^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}\dot{\theta}_5$$
$$- r_{73}r_{73}\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5 + r_{73}r_{73}\sin\theta_6\cos\theta_6\dot{\theta}_5^2 \tag{4.677}$$

$$\begin{aligned}
v_{73}\frac{\partial v_{73}}{\partial\theta_6} = &-r_{GO1}r_{71}\cos\theta_{131}\cos\theta_5\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}\\
&+ r_{71}r_{71}\sin\theta_6\cos\theta_6\sin^2\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}^2\\
&+ r_{71}r_{71}\sin\theta_{131}\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}\\
&- r_{71}r_{71}\cos\theta_{131}\cos^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}\\
&+ r_{71}r_{71}\cos^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_5\\
&- r_{GO1}r_{71}\sin\theta_{131}\cos\theta_{131}\cos\theta_5\cos\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2\\
&+ r_{71}r_{71}\sin\theta_{131}\sin\theta_6\cos\theta_6\sin\left(\theta_5-\theta_{132}\right)\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}\\
&+ r_{71}r_{71}\sin^2\theta_{131}\sin\theta_6\cos\theta_6\cos^2\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2\\
&- r_{71}r_{71}\sin\theta_{131}\cos\theta_{131}\cos^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2\\
&+ r_{71}r_{71}\sin\theta_{131}\cos^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}\dot{\theta}_5\\
&- r_{GO1}r_{71}\cos^2\theta_{131}\cos\theta_5\sin\theta_6\dot{\theta}_{132}^2\\
&+ r_{71}r_{71}\cos\theta_{131}\sin^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_{132}\\
&+ r_{71}r_{71}\sin\theta_{131}\cos\theta_{131}\sin^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}^2\\
&- r_{71}r_{71}\cos^2\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}^2 + r_{71}r_{71}\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5\\
&+ r_{GO1}r_{71}\cos\theta_{131}\cos\theta_5\sin\theta_6\dot{\theta}_{132}\dot{\theta}_5 - r_{71}r_{71}\sin^2\theta_6\sin\left(\theta_5-\theta_{132}\right)\dot{\theta}_{131}\dot{\theta}_5\\
&- r_{71}r_{71}\sin\theta_{131}\sin^2\theta_6\cos\left(\theta_5-\theta_{132}\right)\dot{\theta}_{132}\dot{\theta}_5\\
&+ r_{71}r_{71}\cos\theta_{131}\sin\theta_6\cos\theta_6\dot{\theta}_{132}\dot{\theta}_5 - r_{71}r_{71}\sin\theta_6\cos\theta_6\dot{\theta}_5^2
\end{aligned} \tag{4.678}$$

As it can be seen from the equations above; the equations are getting bigger and bigger and we have to end derivation of these equations here in order to prevent overloading. In Appendix D; the relevant Matlab codes are included and the results can be obtained by running these scripts. However, explanation of equations up to this point are especially important in order to understand the code progression and manual interventions to the parametric scripts. Note that both full parametric and semi parametric versions of the codes exist. Semi parametric version code simplifies the end result al lot by further implications and using mathematical theorems that are not included in Matlab's simplify and combine functions.

CHAPTER 5

CONTROLLER DESIGN AND RESULTS

In order to design various controllers, the very first thing we need is a mathematical model. The mathematical models for ball and beam model, disturber and stabilizer are derived in Chapter 4.

For ball and beam model, state feedback controller, output feedback controller, LQR, sliding mode controller are designed in addition to the observer designs. The analysis results were promising; however during real time tests, we have faced various problems mainly due to the touchpad sensor which is responsible for sending the ball's position to the controller. Therefore, no real life test results could be obtained to compare with the controller design analysis results. This model has no direct relation with the scope of the thesis; instead it was designed as an intermediate stage to gain know how for stabilizer controller design. In that sense, it has done a good job; with the experience gained from the failure during tests; the engagement process of stabilizer controller has been much smoother. In 5.1, the controller designs, its analysis results and the problems faced in real life tests will be explained.

For the disturber module, a PID controller has been designed. Due to simple but effective geometry based mathematical model; this controller has worked pretty well from the beginning. Nevertheless, according to the results, the optimizations in parameters have been made in order to achieve the best results. In 5.2, the controller designs, its analysis results and the problems faced in real life tests will be explained.

Finally, for the stabilizer module, LQR and sliding mode PID controllers have been designed. Both controllers have worked pretty well in spite of numerous sensor malfunctions, noise problems and backlash. In 5.3.2, the controller designs, its analysis results and the problems faced in real life tests will be explained.

The common thing for all these controller designs is their initial step, which is linearization. All of the controller types we have used require linear models. The mathematical models obtained in Section 4.9.3 on the other hand are highly non-linear. Therefore, at first Taylor series expansions of the governing equations are obtained. The higher order terms are omitted and the remaining expression is evaluated around the equilibrium points. For ball and beam system, this equilibrium is pretty simple which is the geometric center of the touchpad. Similarly, for the disturber the equilibrium point is the corresponding actuator strokes which makes the platform angles zero.

It is obvious that the controllers are expected to work much better around the equilibrium points and as you go further from the equilibrium point, the performance is expected to decrease. This is no different in our case too; but for ball and beam and disturber implementations the results were satisfactory even in the most disadvantageous positions. However, since the requirements of the stabilizer module are more demanding; extra efforts have been made to improve the performance of the controllers. One of them is to use a library of controller parameters tuned specifically for every possible payload angle combination of θ_{131} & θ_{132}. In other words, by this way the controllers become adaptive to every equilibrium point. According to the sensed base angle, the stabilizer controllers choose appropriate gains from the previously generated controller gain libraries. The reason why this method has been applied is the due to the dependence of the linearized model on the payload angles θ_{131} & θ_{132} and their derivative terms.

5.1 Ball & Beam Controller Design

This part of the study has been designed as a preliminary stage for the stabilization process. Many different controllers and observers have been designed by means of Matlab and Simulink. All these designs were made according to assumption that the feedback sensor of this model, which is the touchpad, is working properly and good feedback measurements will be taken. This was the case at first; but after some experiments, the ball on the touchpad has jumped a few times while we were optimizing the controllers and the touchpad was broken. As a resut of this accident, the test procedure of these models could not be completed. Since it is not directly related to the scope of the thesis; this part of the study has been left as it is and we have carried on the study with developing the stabilizer controllers. Simulation models and simulation results are shared in Appendix B. On the other hand, the Matlab codes are given in Appendix D.

5.2 Disturber Controller Design

5.2.1 Linearization of the Mathematical Model

Linearization of the model can be done in three ways. The first method is taking Taylor series expansion of the model and evaluating it around an equilibrium point. This method can be applied purely by hand calculations or by the help of Matlab. The second method is the linearize the model option of Matlab. When the block is right clicked a linearization option is available for that block. However, this method only works in relatively simple models. It takes too much time to calculate for the complex models and sometimes even then the solution cannot be found. The figure for demonstrating this feature is given in 5. The third option is using PID tuner block of Simulink. Once the tune button is pressed, the plant is automatically linearized again by Simulink. The figure for demonstrating this feature is given in Figure 5.1.

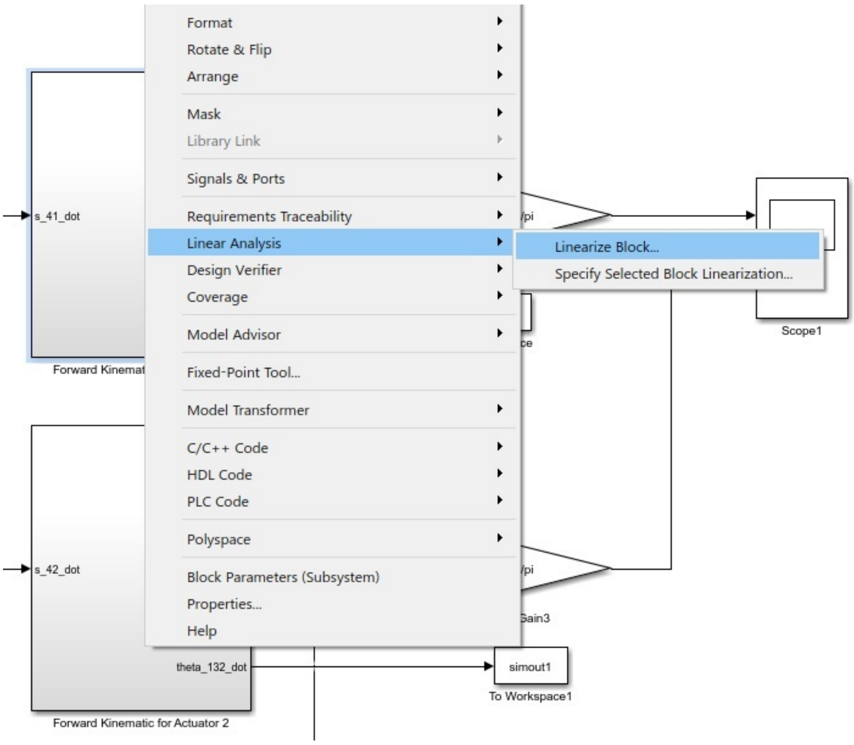


Figure 5.1: Simulation model of disturber - linearize block option

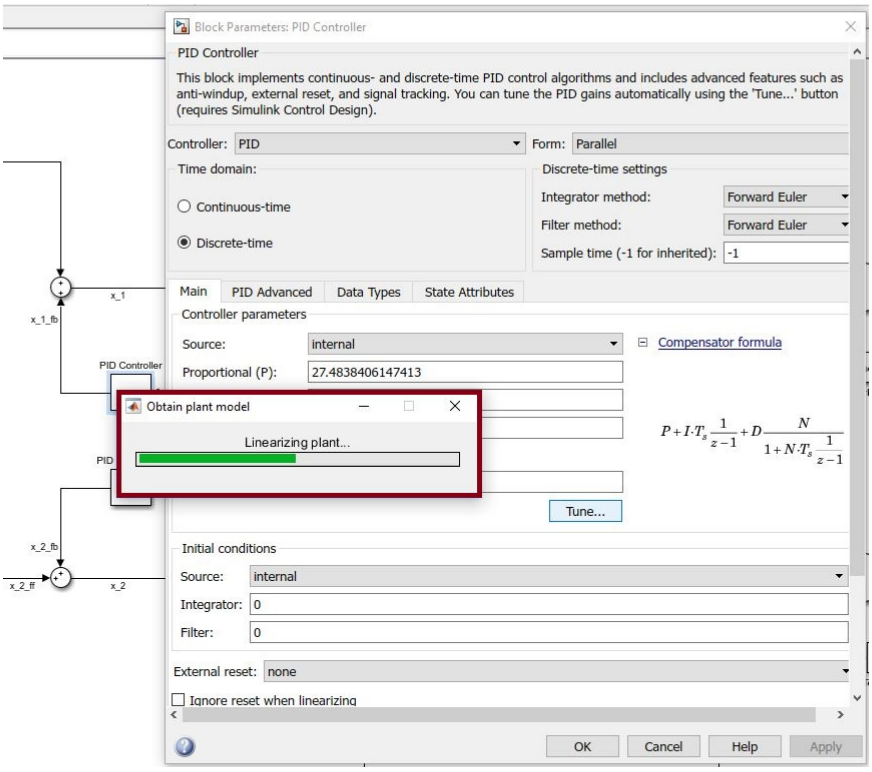


Figure 5.2: Simulation model of disturber - PID tuner linearization

5.2.2 PID Controller Design

PID controller is a relatively simple, efficient and effective in a wide array of applications. In fact, it is the majority of the controller types and industrial applications. When it comes to design a PID controller, first thing you'll need is a well behaved system. What we mean by the well behaved system is a system that is not highly nonlinear, not open loop unstable, does not have lots of delay and is not a non-minimum phase. Our system cannot be called a well-behaved system entirely since it is highly nonlinear and open loop unstable. On the other hand, it does not have delay problems and not a non-minimum phase. Therefore, it is suggested to proceed with advanced methods but nevertheless we took our chances with this controller and the results came up to be satisfactory.

A variety of methods can be used to tune the parameters. These methods can be classified as model based tuning which we have used in this study and designing with a physical system. We also have a prototype so we could also have used this approach. Firstly, designing with a physical system involves tweaking the gains manually while the system is running. Then by observing the system in real time; the parameters are changed until a satisfactory result is achieved. For this method to work, the designer has to have a good command on the properties of each parameter and has to know how each parameter affects requirements; therefore it is highly intuitive. Otherwise it becomes just a trial and error case. A more systematic approach is to run some predefined input sequence on the hardware and then observe the open loop response. Often this is a step input but it can be any arbitrary command. From the response heuristic techniques like Ziegler-Nichols or Cohen-Coon method can be used. The advantages of these methods is that you don't need a model which can be very tough to obtain in some cases. Only certain aspects of the response like time constant or oscillation period should be measured. Using these values, initial gain set can be calculated. The heuristic methods will provide the initial guess and then again manual tweaking should be done to get the required performance.

Secondly, in order to use model based techniques to tune the controller a mathematical model is required. This model is derived by means of first principles. This requires a good understanding of kinematics and dynamics of the system so that one can obtain the equations of motion. That model should be able closely simulate the physical hardware as close as possible.

An alternative method to these two is to use system identification techniques. This technique uses a measured response from the hardware, often a step response. Then it finds an optimal set of model coefficients to match as close as possible these two responses. For simple system ID tools, the model structure needs to be defined ahead of time; which means that the designer needs to specify how to fit the system. For instance, it can be specified to fit the system to a first order transfer function. More advanced tools will find the optimal structure as well as the coefficients itself. The utilization of this method also does not require any knowledge of the system dynamics and yet you could benefit from the advantages of the model based techniques.

For both system identification and model based model where the mathematical model is derived by the designer, next step is tuning the parameters. There are two methods to accomplish this. First one is manual tuning. Knowledge of control theory is used to choose the proper gains. Through all of the possible methods, which there are far too many of them, pole placement, loop shaping and heuristic methods and are the most popular ones. In the pole placement, the aim is to determine where to place the closed loop poles so that the dominant poles produce the system stability and required response. Solving the equation to find the proper gains sounds pretty straightforward but in order to do that one needs to know where to put the poles and how the zeros in the system affect the response. In sliding mode PID controller design pole placement technique has been studied; however, its success has been overshadowed by Simulink's tuner block and therefore the studies regarding this method is not included in this thesis content. Another option is called loop shaping. For this method, open loop transfer function is used in conjunction with Bode and Nyquist plots to shape the loop function so that the closed loop has the desired frequency or time domain response. This method requires the knowledge of how open loop system compares to closed loop behaviour. Moreover, adjusting two zeros of a PID controller affects this behaviour. The third option is heuristic methods like Ziegler-Nichols or Cohen-Coon methods and this one does not require intimate knowledge of control theory. These methods can be used with a simulation of the system to predict an initial gain set. This method is similar to the one described in physical hardware tuning methods but it is much safer to force the model to go unstable than to force the physical hardware instead which is a part of the Ziegler-Nichols procedure.

Lastly, again if mathematical model is derived one can rely on software tools for automatic tuning. This method is used to design final version of the controllers of both disturber and stabilizer sliding mode PID controller. Therefore, all PID results obtained in this study are obtained by means of Simulink's PID tuner block. In case mathematical model is not derived, system identification technique again be combined with this software tool by using input and output response to find optimal set of controller parameters. It is known that auto tuning can even run real time on the hardware; constantly adjusting and optimizing the gains. However, this method requires compatible tools of Matlab and Simulink real time which is not included in our experimental set-up.

Regardless of the method used to obtain the initial gains, all of these procedures end up with manually tweaking the parameters in order to achieve the best performance and exact desired response. This can be done on simulation model or physical hardware. Both these methods are used to determine the manual adjustments required for the sliding mode PID controller of stabilizer which is explained in section 5.3.3. Being able to tweak the gains is a strong reason to use PID controller rather than another controller form because it can be an intuitive process for well behaved systems. The hardware can be sent with an initial gain set and the users can later on tweak the gains for their particular situation and it can be done without much control theory knowledge.

5.2.2.1 Determining PID Parameters

The methodology for obtaining the parameters is using Simulink PID parameter tuner which has two adjustment slides. One of these bars change the system's transient behaviour; it can be adjusted to be more aggressive or robust on the limiting cases. The second slider is used to change the response time of the system, as it can be guessed it determines how fast the controller response is going to be. While changing these sliders, some additional assistant plots can also be drawn to decide on the optimal controller. These additional plots include plant, open loop, reference tracking, controller effort, input disturbance rejection and output disturbance rejection. As these sliders are adjusted, the plots are upgraded and you can observe the effects of your adjustments on the system characteristics. By this way, it becomes much easier to optimize the controller parameters. If desired the pole locations and their changes according to the adjustments can also be tracked. All in all, these features make this tool a very useful and comprehensive one. In the figure 5.3, you can see the GUI of this block. More detailed explanation about GUI and additional assistant system characteristics blocks is done in section 5.3.3.1.

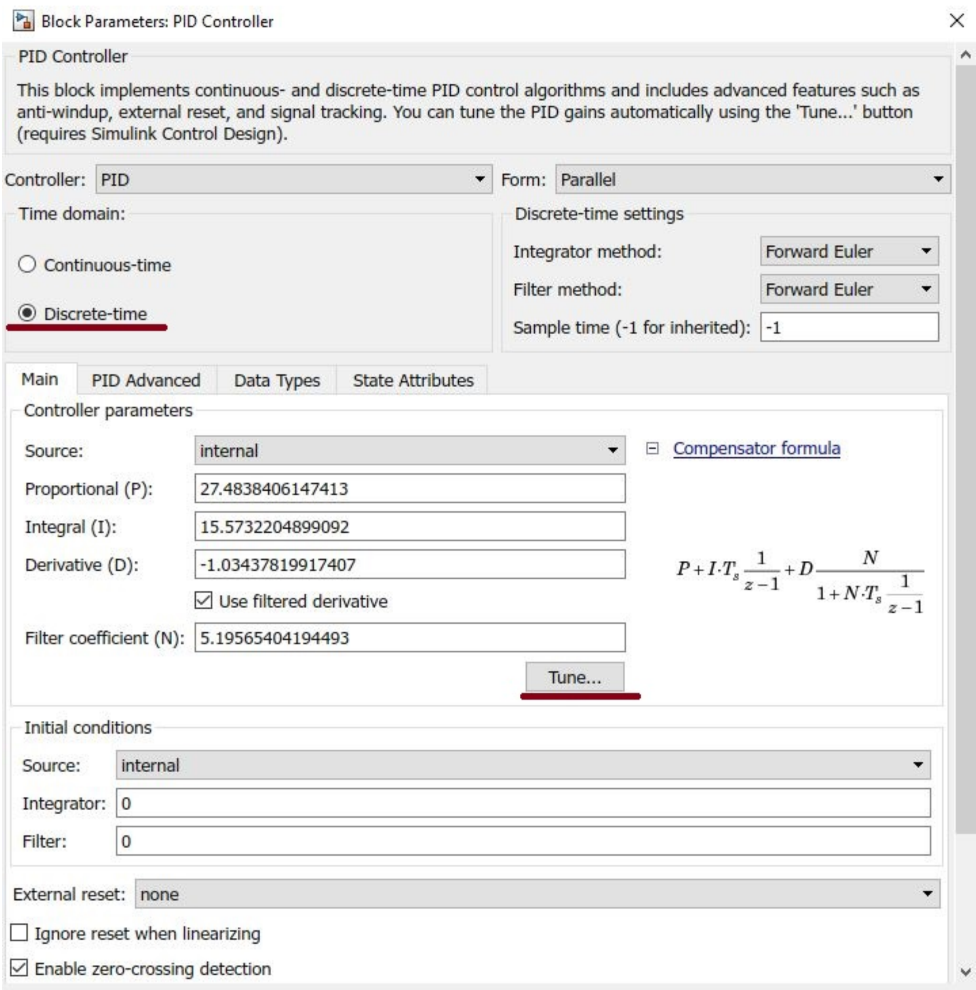


Figure 5.3: Simulation model of disturber - PID Tuner main tab

5.2.2.2 Description of the Simulation Model

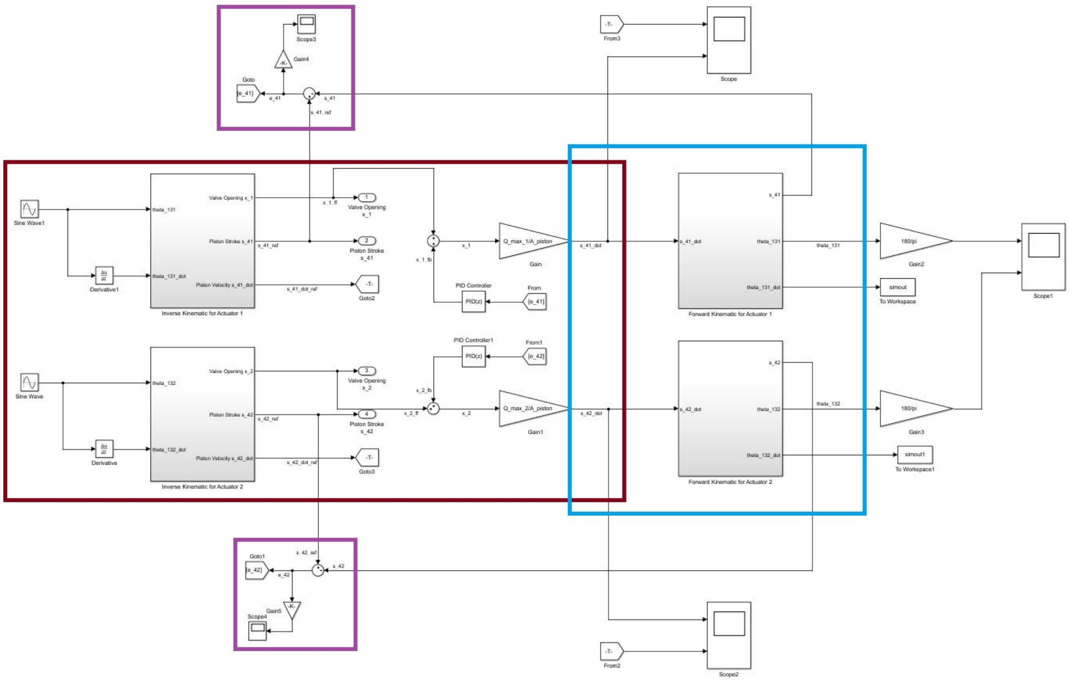


Figure 5.4: Simulation model of disturber

The Simulink model of the disturber is comprised of mainly controller(red region) and plant model(cyan region) and essential feedback branches(purple regions). Detailed schematics of these regions are presented in figures 5.5, 5.7, 5.9 & 5.10.

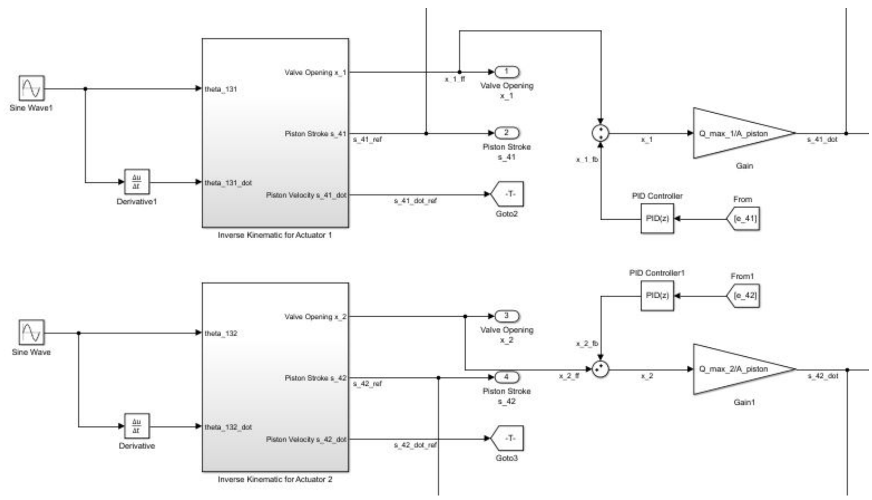


Figure 5.5: Simulation model of disturber - highlighted controller module

On the very left, motion generating Simulink blocks are included. The inverse kinematics blocks take in the payload angles and its first derivatives to calculate corresponding valve openings. These reference valve openings are combined with the compensation term of the PID controller to generate the valve opening command which directly controls the flowrate. By controlling flowrate of the oil entering to the pistons, linear actuator speed is controlled. By controlling the actuator speed, the position of the actuators can be controlled which designates the orientation of the payload. If we look under the mask of the inverse kinematics block, the model given in Figure 5.6 can be seen.

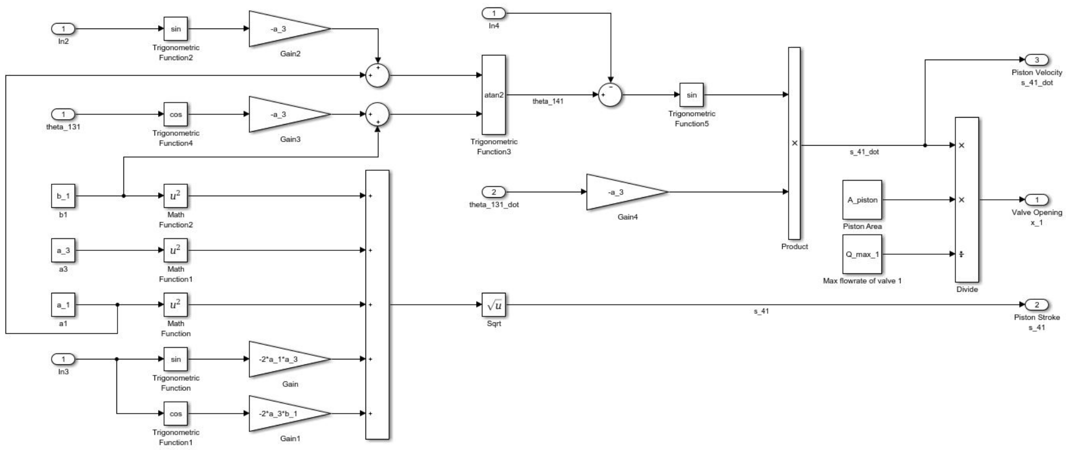


Figure 5.6: Simulation model of disturber - inverse kinematics block

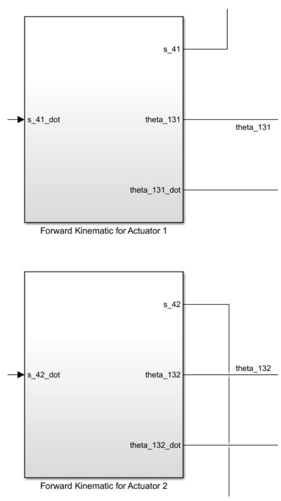


Figure 5.7: Simulation model of disturber - highlighted plant module

On the right of the Simulink disturber model, there lies the plant model. The plant model is the forward kinematics of the disturber mechanism. It takes in the actuator velocities as an input and calculates the orientation angles of the payload. If we look under the mask of the inverse kinematics block, the model given in Figure 5.8 can be seen.

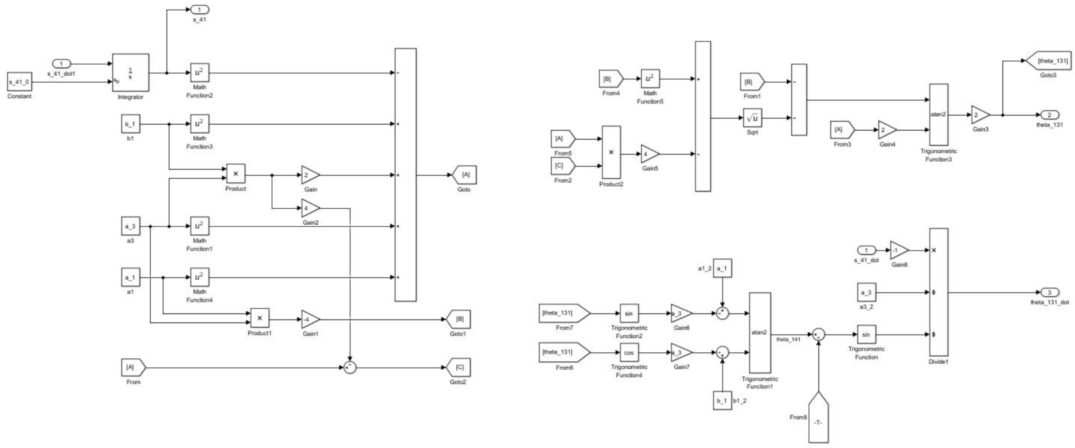


Figure 5.8: Simulation model of disturber - forward kinematics block

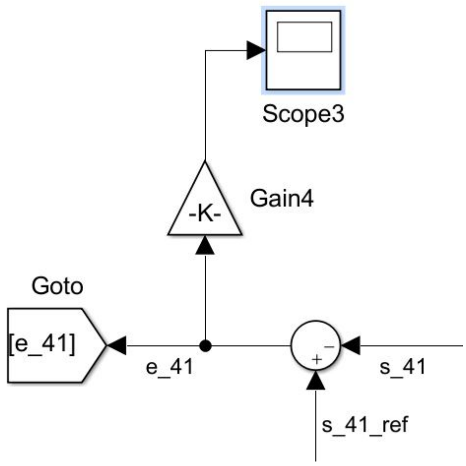


Figure 5.9: Simulation model of disturber - highlighted s_{41} error module

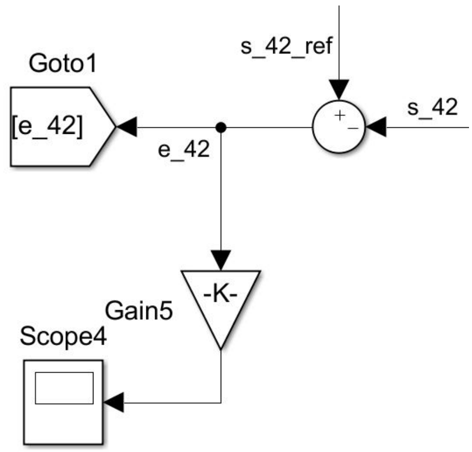


Figure 5.10: Simulation model of disturber - highlighted s_{42} error module

Lastly, we will talk about the how errors are calculated. As it can be seen in Figure 5.5, PID controller block takes in the error term to generate the controller signals.

Reference strokes of the actuators are obtained by means of the inverse kinematics blocks and actual strokes of real life is simulated and calculated by means of the forward kinematics blocks. The difference of them gives us the error terms.

5.2.2.3 Analysis and Experiment Results

4 different motion profiles are tracked by the disturber mechanism to test its adaptability and robustness. These profiles are sine, trapezoidal, random and Roketsan vehicle motion profile. The test results with the reference values are given as follows:

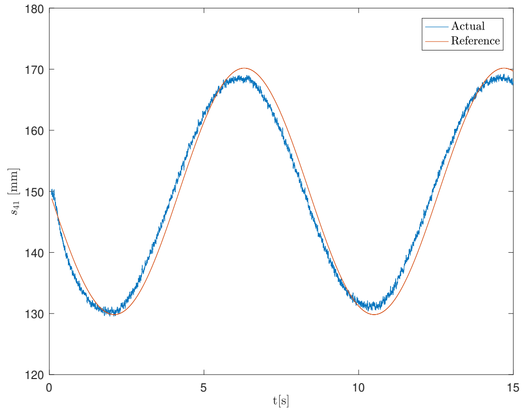


Figure 5.11: Reference tracking of the first disturber actuator - sine profile

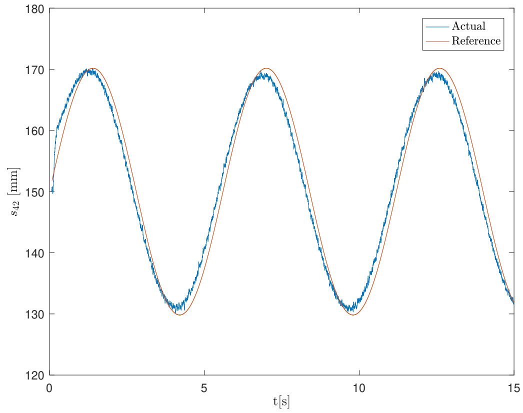


Figure 5.12: Reference tracking of the second disturber actuator - sine profile

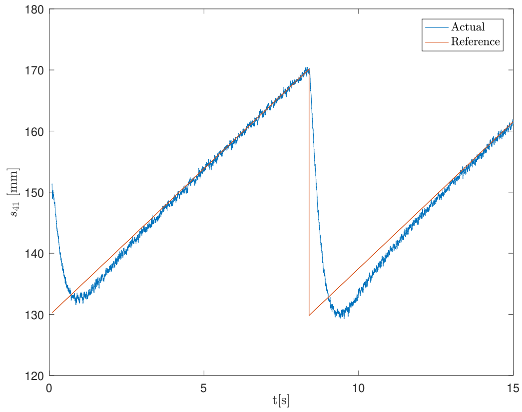


Figure 5.13: Reference tracking of the first disturber actuator - trapezoidal profile

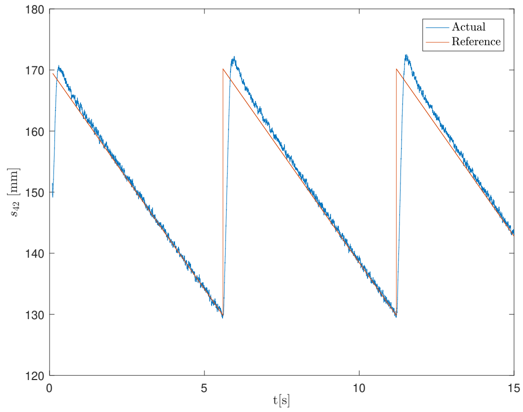


Figure 5.14: Reference tracking of the second disturber actuator - trapezoidal profile

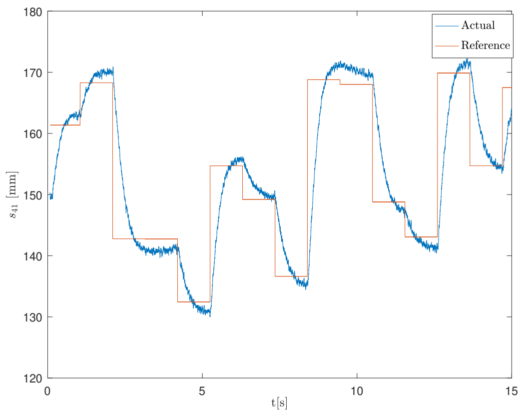


Figure 5.15: Reference tracking of the first disturber actuator - random profile

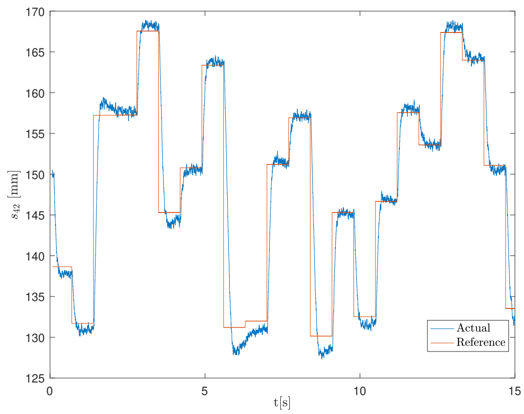


Figure 5.16: Reference tracking of the second disturber actuator - random profile

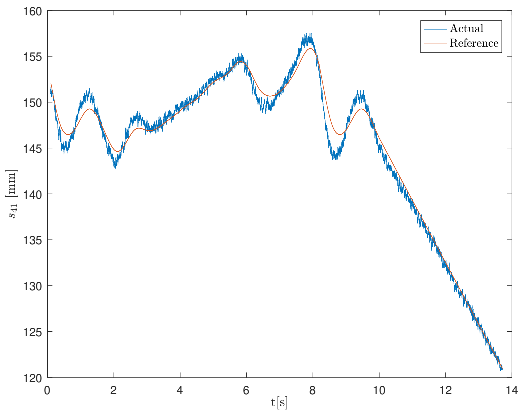


Figure 5.17: Reference tracking of the first disturber actuator - Roketsan vehicle profile

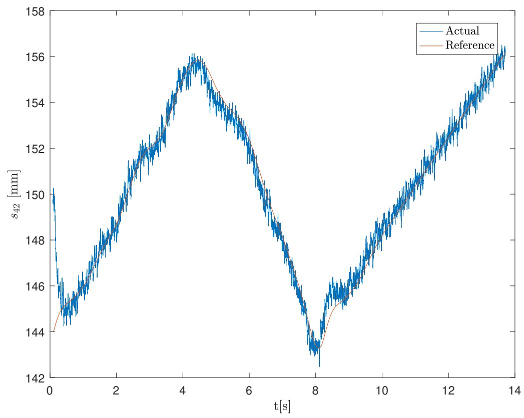


Figure 5.18: Reference tracking of the second disturber actuator - Roketsan vehicle profile

5.3 Stabilizer Controller Design

5.3.1 Linearization of the Mathematical Model

Two attempts have been made in order to obtain the governing equations, which will relate the inputs and outputs of the system. The first attempt is through Newton-Euler equations. For that purpose, after all of the FBD's are drawn; 3 force and 3 moment equations are written for each body. In order to obtain the differential equations for the disturber this process is repeated for 3 bodies. All these equations are represented in matrix form and all 18 unknowns are isolated by algebraic manipulations.

Both mathematical models from N-E equations and Lagrange equations are obtained and the end results are almost the same. At the end of N-E equations, 18 equations are obtained. However, we only need 2 differential equations which need to relate the angular accelerations to the motor torques. Obtaining these equations from 18 equations is a very cumbersome process since these variables appear in the majority of the 18 equations and all of these equations are very long. Therefore, after some time in order to save from computational time, we had to switch to Lagrange equations which are direct unlike N-E equations. However, the efforts devoted to obtaining the N-E equations did not go in vain. Firstly, the results of Lagrange equations have been justified with the results of N-E equations. Secondly, the structural forces obtained from the solution of these forces are used to size the machine elements used in the design.

The Lagrange equations are obtained in Section 4.9.3. The results are double checked with the results of Matlab code. However, the built in mathematical simplification functions such as "collect" and "simplify" are sometimes inadequate to cancel out every available expression, collect different times in a practical way and thus fully shorten the equations. After each mathematical operation the situation only worsens; the equations get bigger and bigger and after some time it is almost impossible to keep track of the equations. Therefore, at each step manual interventions have been made to the intermediate expressions if necessary; of course while making sure the end results are same with both hand written equations and Matlab code's own results. By this way, at the end we were able to obtain manageable equations. The linearization procedure have been shown in 4.1.1 in detail, therefore this time the main equations will be given. The state space matrix for the stabilizer can be expressed as follows:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \theta_5^*(t) \\ \dot{\theta}_5^*(t) \\ \theta_6^*(t) \\ \dot{\theta}_6^*(t) \end{bmatrix} \tag{5.1}$$

where θ_5^* & θ_6^* are defined as follows:

$$\theta_5^* = \theta_5 - \theta_{132} \tag{5.2}$$

$$\theta_6^* = \theta_6 + \theta_{131} \tag{5.3}$$

Note that these angles determine the platform orientation with respect to the ground and therefore they are chosen as state variables. The reason why we are subtracting θ_{132} while adding θ_{131} to the stabilizer angles θ_5 & θ_6, respectively is due to the their definition of positive direction. Remind that classical convention which defines each angle in CCW direction from the ground has been applied while defining θ_{131} & θ_{132} angles whereas Denavit-Hartenberg convention has been applied in Stabilizer module where θ_5 & θ_6 are defined. Therefore, end result of angle combinations come up to be as shown in (5.2) and (5.3). Moving further the state equations can be defined as follows:

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t) \tag{5.4}$$

where,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.5}$$

Expanding (5.4);

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ A(2,1) & A(2,2) & A(2,3) & A(2,4) \\ 0 & 0 & 0 & 1 \\ A(4,1) & A(4,2) & A(4,3) & A(4,4) \end{bmatrix}}_{Jacobian\ matrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \partial\dot{x}_1/\partial T_5 & \partial\dot{x}_1/\partial T_6 \\ \partial\dot{x}_2/\partial T_5 & \partial\dot{x}_2/\partial T_6 \\ \partial\dot{x}_3/\partial T_5 & \partial\dot{x}_3/\partial T_6 \\ \partial\dot{x}_4/\partial T_5 & \partial\dot{x}_4/\partial T_6 \end{bmatrix} \begin{bmatrix} T_5 \\ T_6 \end{bmatrix} \tag{5.6}$$

The elements of the Jacobian matrix are rather long, derivation and final results of them will not be given here. Derivation of these elements are done both by hand and by Matlab. Relevant Matlab code is included in Appendix D.

5.3.2 LQR Controller Design

LQR is a type of optimal control and is based on state space representation. Like pole placement method used in PID controllers; this controller is also a full state feedback controller having an identical block diagram with pole placement method. The implementation of the gain matrix $\underline{K}$ is the same but the difference here is how we choose $\underline{K}$. In LQR design, the optimal $\underline{K}$ is found by choosing closed loop characteristics. The block diagram of LQR controller is given in Figure 5.19. As stated before and also seen in the figure, LQR controller is a full state feedback controller.

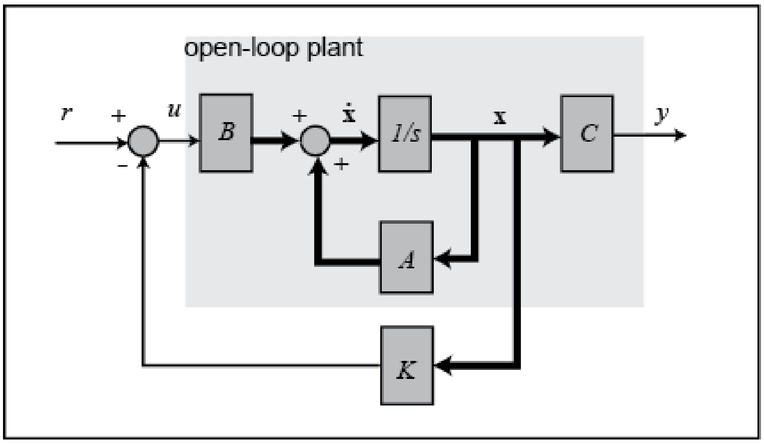


Figure 5.19: Block diagram of LQR controller [4]

In LQR design we choose the weights for every state variable and from their combination we obtain the cost function. The gain matrix $\underline{K}$ is obtained from the solution of this cost function. The weights we choose indicate how we penalize each state and as a result we specify the system characteristics. By this way, actually we specify the maximum error we expect from the controller. There are two main sources we have which are the controller effort, in this case the servo motor torques and time. We can also say that these two sources are related and we can simply call it a single parameter, energy. We have to match these sources with what we want from the system. Therefore, meanwhile the controller effort is observed while forcing the system to satisfy desired performance. If these two match, then we are good to go but if the demand from the actuators, in this case the servo motors, are more than the supplied torque and velocity capacity; then the performance criteria of the system has to be readjusted. Hence, we aim to minimize the following cost function:

$$J = \int_0^\infty (\underline{x}^T \underline{Q} \underline{x} + \underline{u}^T \underline{R}\, \underline{u}) dt \tag{5.7}$$

Here, the matrix $\underline{Q}$ penalize bad performance where $\underline{R}$ penalize actuator effort. The integral of $\underline{x}^T \underline{Q} \underline{x}$ inspects on the area under state vs time graph which is a good indicator of performance. By this way, it tests how quickly the states reach to the desired point. However, the states can also be negative. Therefore, $\underline{x}^T \underline{Q} \underline{x}$ ensures positive outcome by taking square of the states. The downside of this methodology is that it punishes larger errors proportionally more than smaller ones; but in overall it is good a compromise since it turns the function into a quadratic function. This is a greatly beneficial feature since quadratic functions are convexed so they have a definite minimum. Moreover, quadratic functions that are subject to linear dynamics remain quadratic; therefore it is also ensured the cost function and the thus the system also have a definite minimum value.

$\underline{Q}$ is chosen to be square matrix with non-zero diagonal elements and zero non-diagonal elements and same number of rows with states so that it allows targeting every individual state. We can penalize the corresponding state by increasing corresponding $\underline{Q}_n$ component. This means that we want really low error on that state; this is the logic while we are selecting the $\underline{Q}$ matrix. $\underline{R}$ is similar but it acts on input matrix. If we want to make a deeper analysis, we can also relate state variables and inputs by simply replacing zero terms in matrix notation of cost function by $\underline{N}$ and $\underline{N}^T$. Thus, now we have the ability to also punish the cross product of $\underline{x}$ & $\underline{u}$ by altering $\underline{N}$. Then, the corresponding cost function becomes as follows:

$$J = \int_0^\infty \underbrace{(\underline{x}^T \underline{Q} \underline{x} + \underline{u}^T \underline{R}\, \underline{u})}_{\begin{bmatrix} \underline{x}^T & \underline{u}^T \end{bmatrix} \begin{bmatrix} \underline{Q} & \overbrace{0}^{\underline{N}} \\ \underbrace{0}_{\underline{N}^T} & \underline{R} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{u} \end{bmatrix}} \, dt \tag{5.8}$$

$$J = \int_0^\infty (\underline{x}^T \underline{Q} \underline{x} + \underline{u}^T \underline{R}\, \underline{u} + 2\, \underline{x}^T \underline{N}\, \underline{u}) dt \tag{5.9}$$

However, as you will see in selecting the weights paragraph; there will be no need for this move since only careful adjustment of $\underline{Q}$ will create satisfactory results.

Since we have completed the first step which is developing a mathematical model and linearization of it; the next step is to adjust $\underline{Q}$ & $\underline{R}$. Then, by means of ***LQRD*** function of Matlab we can solve Riccati equation to obtain the optimal gains. By means of Simulink, we can simulate the results and if we are not satisfied we can return back to adjusting $\underline{Q}$ & $\underline{R}$.

5.3.2.1 Choosing LQR Weights

The general approach for choosing LQR weights is intuitive and heuristic. Actually, that was our first attempt too. We were trying to observe how the performance of the system changes and whether the controller effort saturates or not by adjusting the weights. Changing all the weights all at once and trying to decide on the next step was rather confusing so we isolated weights and started changing them one by one. After some time, with the help of the knowledge of dynamics of the system; we have developed an intuition and was able to come up with a satisfactory controller design. However, at this point we were not sure whether there is better solution or not since there were many possible combinations and we could be hanging around a local maximum instead of a global maximum.

In [8], a procedure for choosing LQR weights has been presented. The second method of the procedure they presented suggests to choose each q_i for same badness. For instance, the unit of $q_1 = \theta_5^*$ is rad and $q_2 = \dot{\theta}_5^*$ is rad/s. Then,

$$\underbrace{0.001745 \text{ rad}}_{0.1 \text{ deg}} \text{error OK} \Rightarrow q_1 = \left(\frac{1}{0.001745}\right)^2, \quad q_1 x_1^2 = 1 \text{ when } x_1 = 0.001745 \text{ rad} \tag{5.10}$$

$$\underbrace{2.2361 \text{ rad/s}}_{128.1 \text{ deg/s}} \text{error OK} \Rightarrow q_2 = \left(\frac{1}{2.2361}\right)^2, \qquad q_2 x_2^2 = 1 \text{ when } x_2 = 2.2361 \text{ rad/s} \tag{5.11}$$

Note that the design limits of the disturber is ± 16. Therefore, the limit of the angular velocity error, which in this case it corresponds to the maximum allowed angular velocity itself, is limited in order to prevent chattering effect and overshoot. In this case, the maximum steady state error can be compensated in $16/128 = 0.12$ seconds. This angular velocity corresponds to 21.4 rpm. The maximum angular velocity of the servo motor on the hand is 3000 rpm and the gearbox ratio is 25. Therefore maximum angular velocity of the system is 120 rpm. Therefore, for smooth motion characteristic the velocity potential of the system is restricted which leads to higher response time. Nevertheless, even if we enlarge this limit; the torque required turns out to be more than the system can provide. Therefore, the lost potential is not that great and a good compromise has been achieved in that sense. From this result we can say that the gearbox ratio of the outer system can be selected to be larger. This could also be more advantageous since there is a well known rule of thumb which says that the system becomes more controllable when the inertia ratio of the drive part to the drive part is between 10-20. Also it is known that the more this ratio decreases, the more system becomes controllable. Although we have foreseen this detail in the design phase, much longer lead time of this higher ratio gearbox forced our hand to continue with this configuration.

5.3.2.2 Description of the Simulation Model

In order to simulate the results before real life tests on the experimental set-up; the following Simulink model has been created:

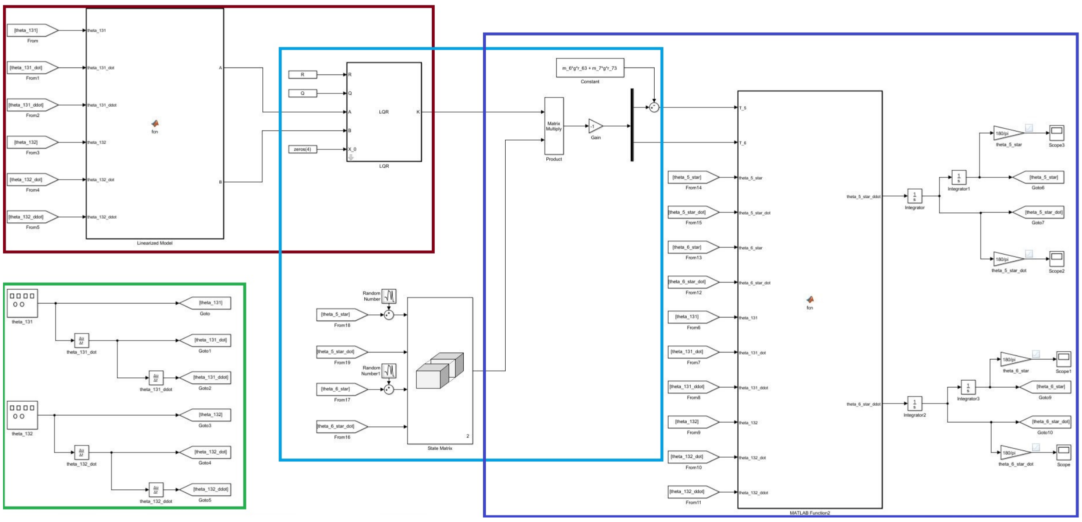


Figure 5.20: Simulation model of stabilizer with LQR controller

This model is nothing but an expansion of the block diagram shown in 5.19. From left to right we can see the linearized model providing $\underline{A}$ & $\underline{B}$ matrices to LQR block. The LQR block solves the Riccati equation and provides the gain matrix $\underline{K}$ as an outcome. This $\underline{K}$ matrix is multiplied with the states and subtracted from the reference matrix multiplication. Please note that since reference matrix is composed of zeros due to stabilization criteria; torques become equal to $-\underline{K}\ \underline{x}$ directly. Then these torques are used in plant model, which is again derived from Lagrange equations, to calculate the acceleration terms ($\ddot{\theta}_5^*$ & $\ddot{\theta}_6^*$). Note that the plant block is obtained from forward dynamics whereas the linearized model is obtained from the inverse dynamics. We can observe the platform angles with respect to the ground (θ_5^* & θ_6^*) by integrating twice the acceleration terms ($\ddot{\theta}_5^*$ & $\ddot{\theta}_6^*$) that are outputs of the plant model. Finally, on the left bottom disturber profile blocks are included to simulate the disturber motion. In Figure 5.20 you can see different coloured rectangles. All these rectangles will be explained separately for sake of clarity.

Nextly, we will explain in more detail the highlighted boxes. We will start with the green box which is enlarged in Figure 5.21.

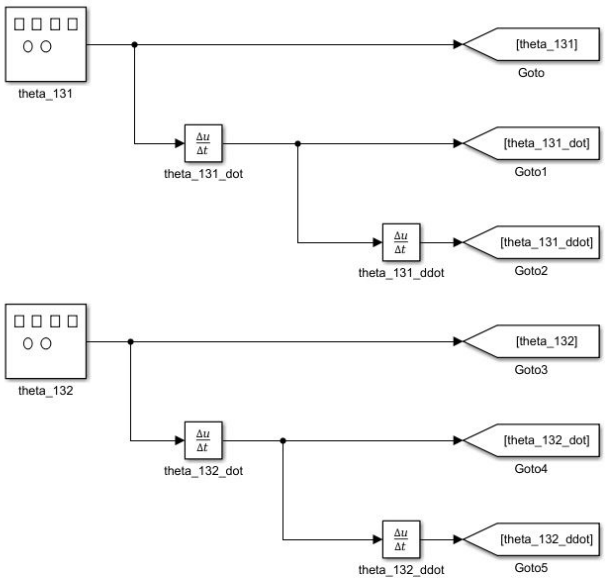


Figure 5.21: Simulation model of stabilizer with LQR controller - highlighted disturber module

As it has been stated in 5.2.2.3, the performance of the disturber has been tested under four motion profiles. These same profiles are used also while disturbing the LQR model. On the left, motion generating blocks can be seen. The rest is pretty straightforward; first and second derivatives of the base(payload) angles θ_{131} & θ_{132} to be used in both linearized model and the plant model.

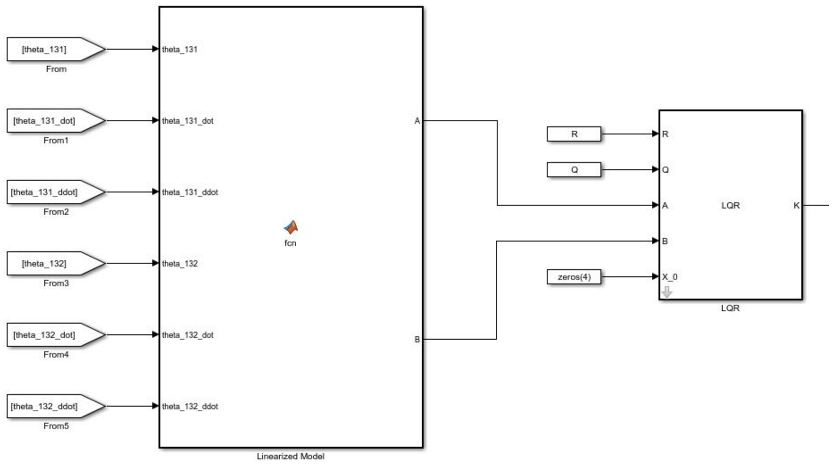


Figure 5.22: Simulation model of stabilizer with LQR controller - highlighted controller module

On the left, linearized model can be seen. The derivation of the matrix elements can be found in 5.3.1 and in Appendix D, respectively. The linearized model generates state matrices $\underline{A}$ & $\underline{B}$ matrices which is as an input of the LQR block. When we look under the mask of the LQR block, we can observe the following combination of simpler blocks. These blocks together are responsible for solving the Riccati equation to obtain the optimal gains. The script of the Riccati Matlab functon block is given in Appendix D.

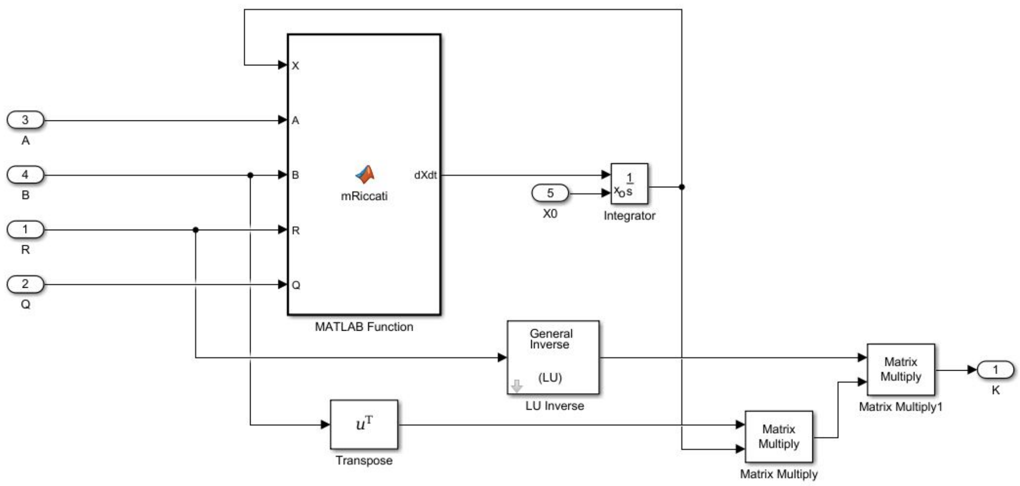


Figure 5.23: Simulation model of stabilizer with LQR controller - looking under the mask of LQR block

At this point there is a detail worth mentioning. The experimental setup utilizes Simulink Real Time environment. Also, before proceeding to real life test; we are designing the controller in Matlab / Simulink environment to predict its performance beforehand. The Matlab has a built in function capable of solving Riccati equation for the optimization of the cost function called *LQRD*; however the Simulink environment does not have a built in *LQRD* block. Furthermore, it also does not support any Matlab function block that contains this function in Simulink environment. Therefore, in script analysis of Matlab we can use *LQRD* but for Simulink we had to create our own mask for simulation purposes. This has done a great job in simulation environment but it turned out to be computationally expensive. In order to prevent that, by using *LQRD* command of Matlab all the gain matrices are calculated in a Matlab script for all θ_{131} & θ_{132} payload angle combinations. This gain library is loaded to real time environment before experiments and the controller chooses the appropriate gains according to the payload angles. Hence, the system becomes adaptive to not only a single equilibrium point but to all points. The reason for this adaptation is due to the fact that the linearized model is a function of both θ_{131} & θ_{132} and their derivatives. The Matlab script for calculating the LQR gains is included in Appendix D.

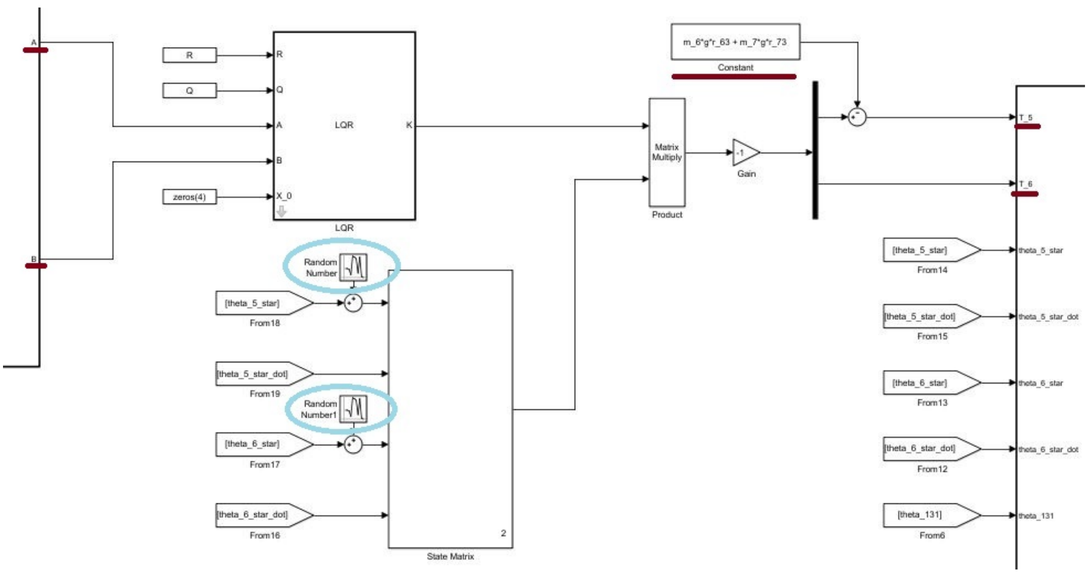


Figure 5.24: Simulation model of stabilizer with LQR controller - highlighted feedback module

As it has been stated before when the gain matrix is multiplied with the state matrix the torques are obtained. The static torque necessary to keep the outer assembly in balance is added to this value for feedforward purposes. This block is underlined in Figure 5.24 with claret red. The system works well either this imbalance of the center of gravity is compensated by the feedforward term or not but nevertheless it has been found useful and kept its place in the final model. Moreover, in order observe the effects of sensor noise disturbance block has been added with random number block. These blocks are circled in cyan.

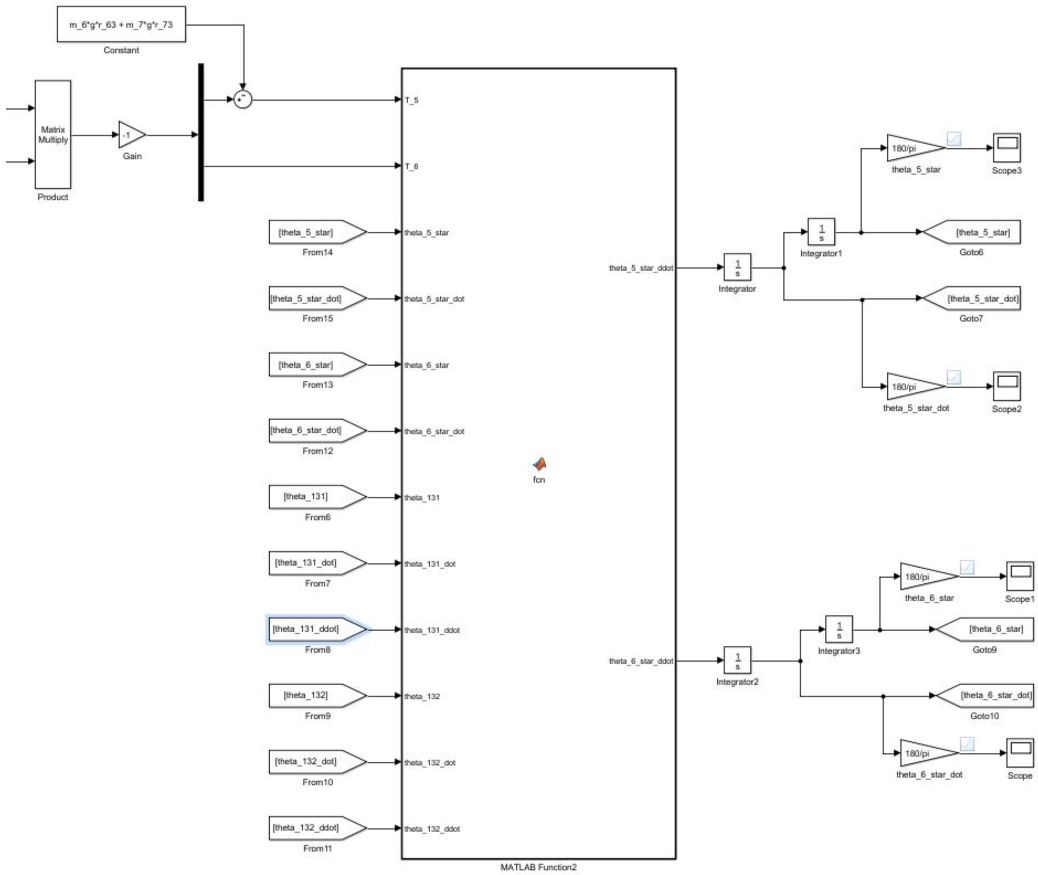


Figure 5.25: Simulation model of stabilizer with LQR controller - highlighted plant module

Lastly, the plant module represents the system dynamics. It takes the generated torque values and calculates the corresponding acceleration values. The platform angles (θ_5 & θ_6) are obtained by integration the angular acceleration terms ($\ddot{\theta}_5^*$ & $\ddot{\theta}_6^*$) twice.

5.3.2.3 Analysis and Experiment Results

As it has been stated in 5.2.2.3, the performance of the disturber has been tested under four motion profiles. These same profiles are used also while disturbing the LQR model. This section is about demonstration and comparison of these results.

The first profile is sine profile. In order to test the limits of both disturber and stabilizer controllers; different sine profiles have been applied to the system from each disturber actuator. These sine profiles vary both in amplitude and frequency. Applied sine profiles are shown in Figure 5.26.

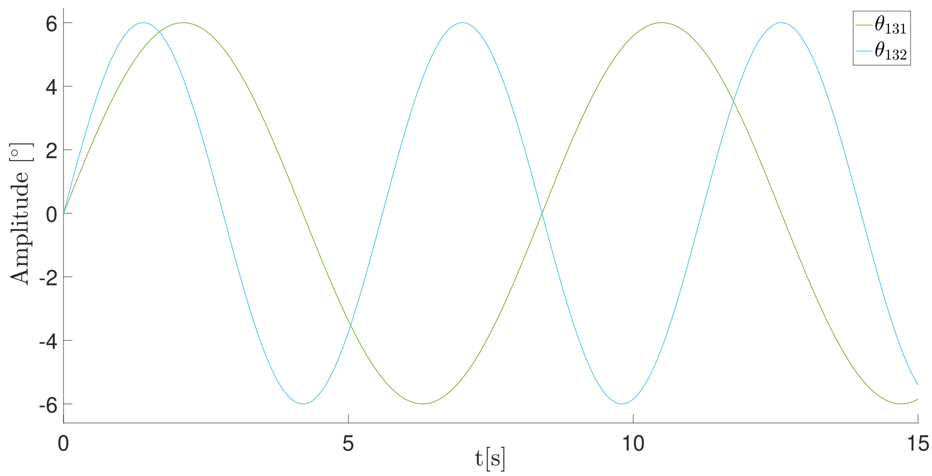


Figure 5.26: Stabilizer with LQR controller - disturber sine profiles

The simulation model is based on frictionless plant model, linearized mathematical model and noiseless sensor data. In other words, simulation environment is idealized from every aspect. However, the real world experimental set-up environment is far from linear, has friction effect and is exposed to noise from every feedback sensor. Therefore, in order to demonstrate the variance at first pure simulation results will be given in figures 5.29 & 5.30.

After that, the results of the noisy simulation data will be given in figures 5.31, 5.33, 5.35 & 5.37. It is no surprise that the latter case represents the actual dynamics of the system better. However, before all these let's start with the real time step response of the system. The step response of the gimbal axes will be as follows:

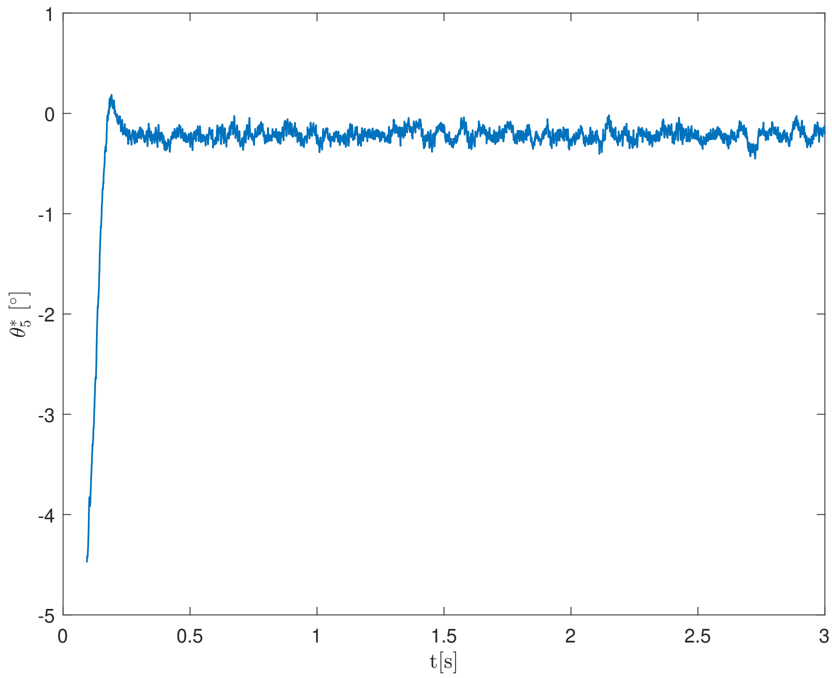


Figure 5.27: Step response of θ_5^* with LQR controller

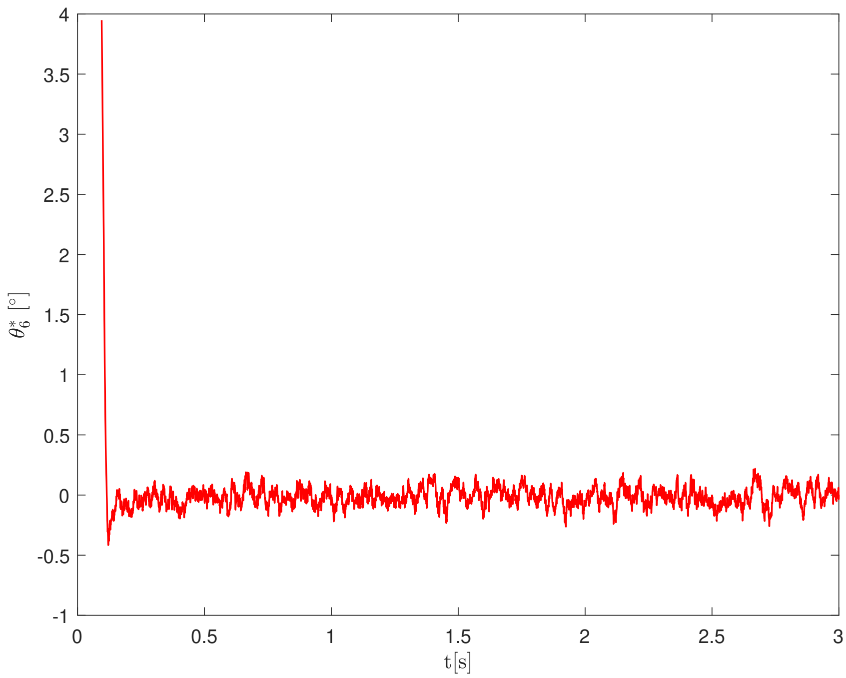


Figure 5.28: Step response of θ_6^* with LQR controller

Then, according to the simulation model under sine profiles, platform angles will be as follows:

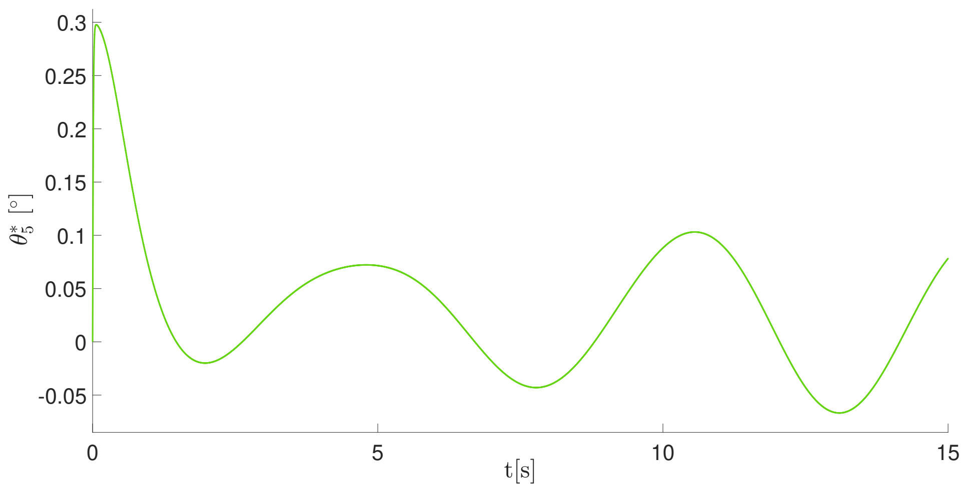


Figure 5.29: Simulation results of stabilizer with LQR controller - θ_5^* vs $t(s)$

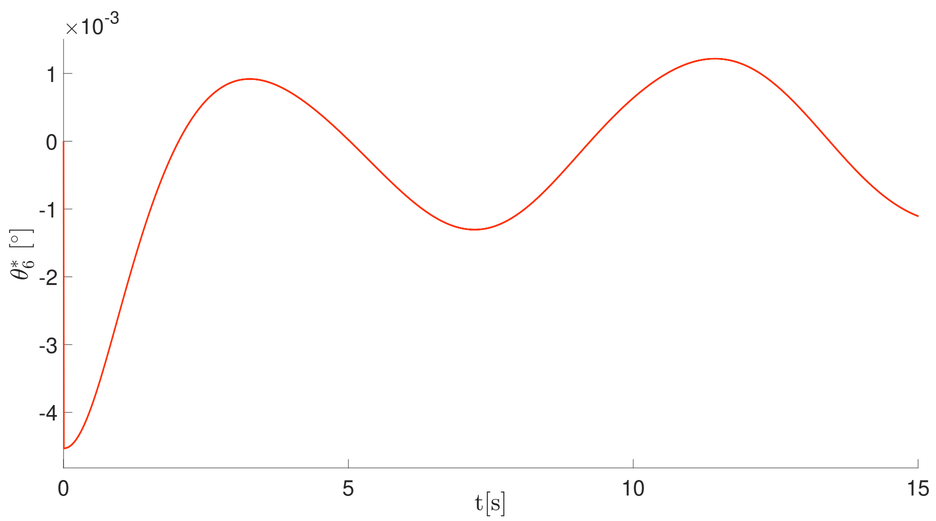


Figure 5.30: Simulation results of stabilizer with LQR controller under sine profile - θ_6^* vs $t(s)$

Similarly, when noise is added to the system; the platform angle about the outer platform axis (θ_5^*) behave as shown in figures 5.31 & 5.32.

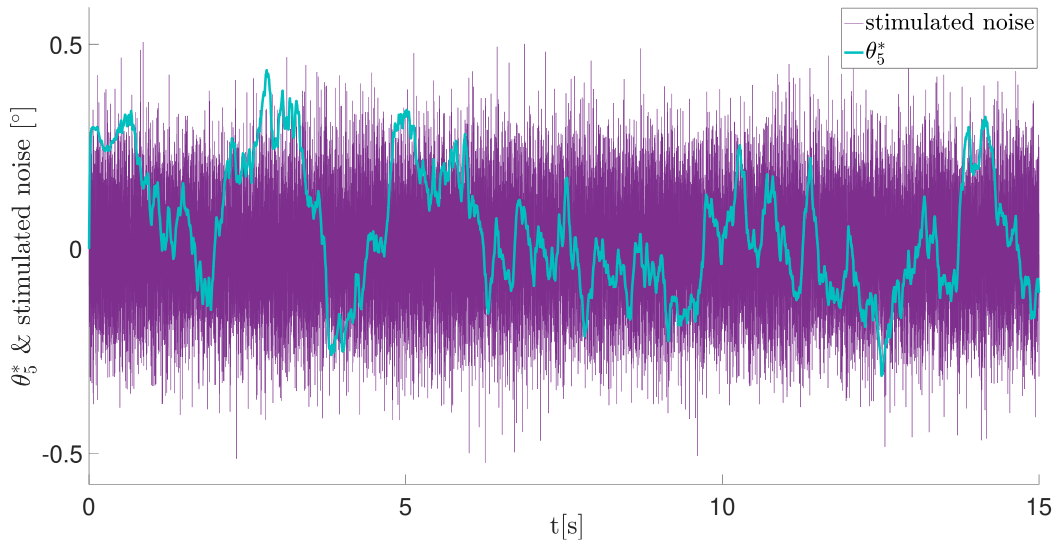


Figure 5.31: Simulation results of stabilizer with LQR controller and sensor noise under sine profile - θ_5^* vs $t(s)$

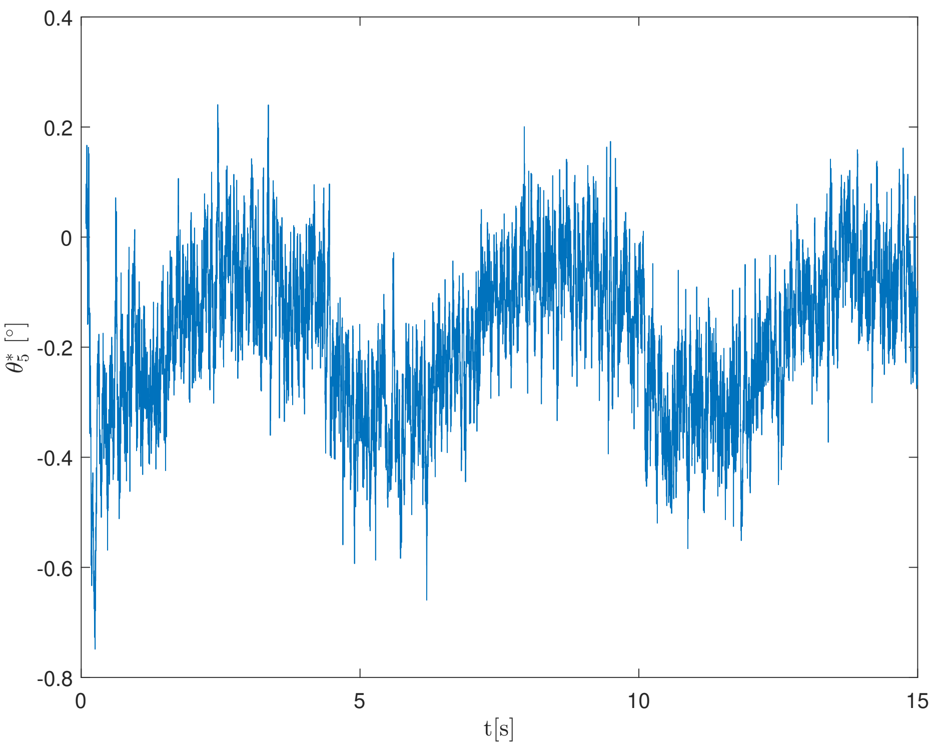


Figure 5.32: Test results of stabilizer with LQR controller and sensor noise under sine profile - θ_5^* vs $t(s)$

The behaviour of the other platform angle (θ_6^*), which is about the inner axis, is given in figures 5.33 & 5.34.

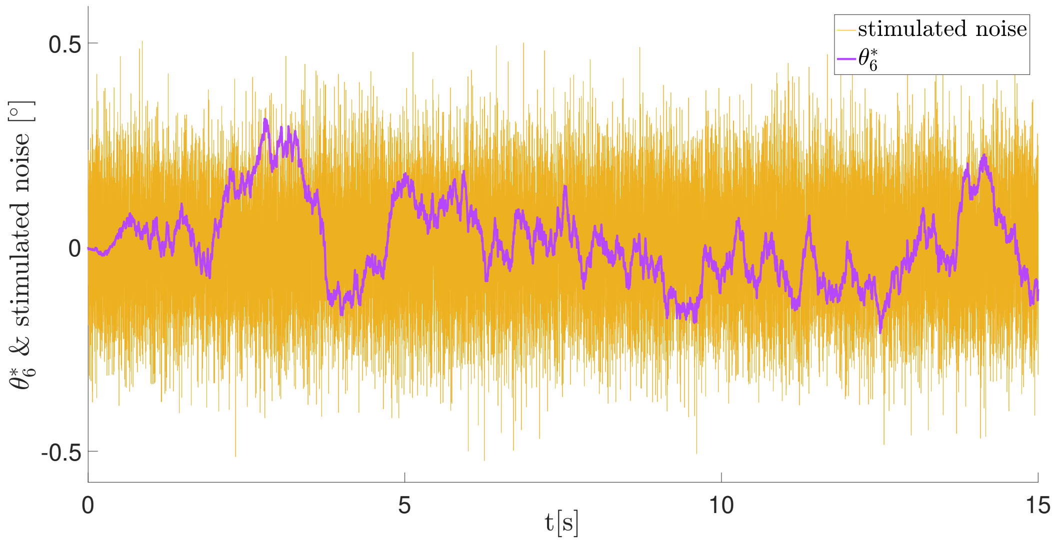


Figure 5.33: Simulation results of stabilizer with LQR controller and sensor noise under sine profile - θ_6^* vs $t(s)$

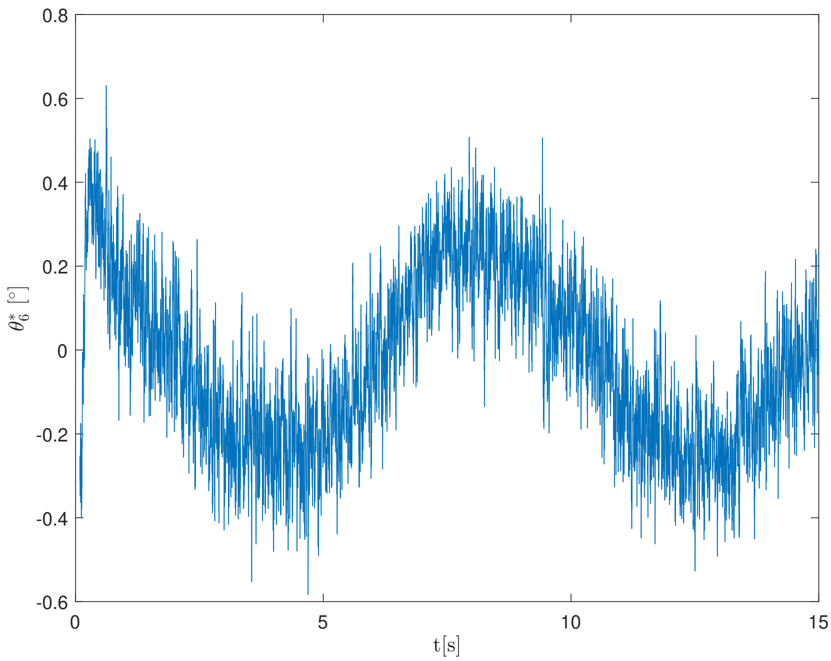


Figure 5.34: Test results of stabilizer with LQR controller and sensor noise under sine profile - θ_6^* vs $t(s)$

The torques generated by the outer gimbal axis (T_5) in simulation environment and tests are presented in figures 5.35 & 5.36, respectively.

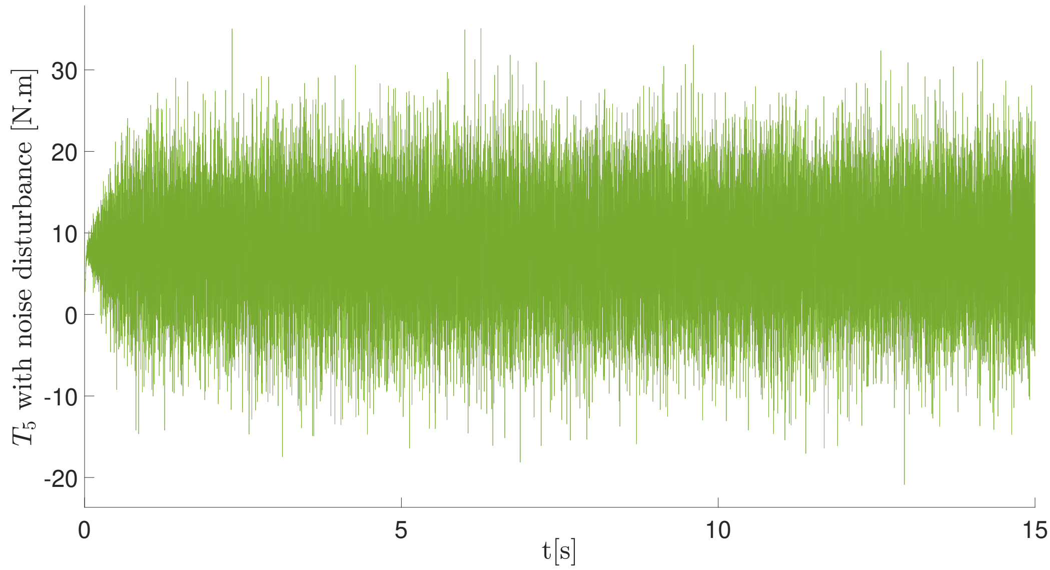


Figure 5.35: Simulation results of stabilizer with LQR controller and sensor noise under sine profile - T_5 vs $t(s)$

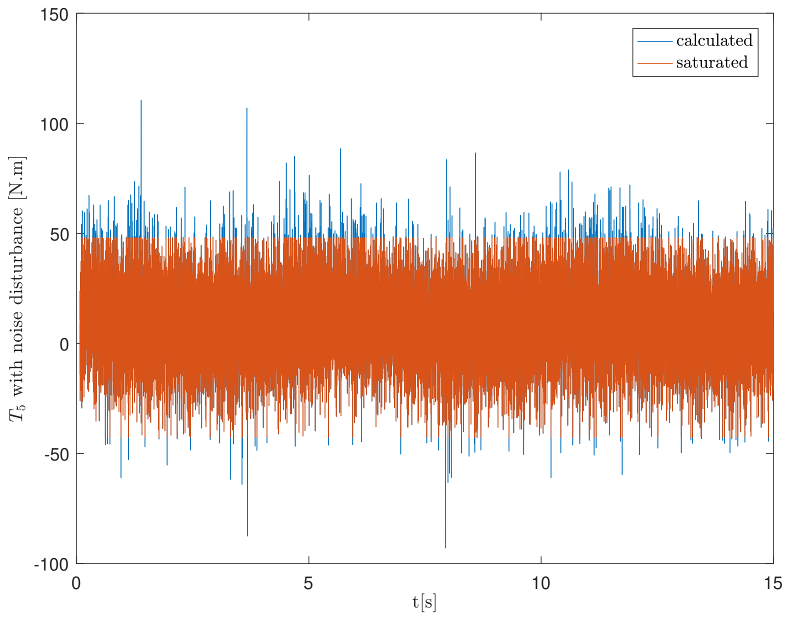


Figure 5.36: Test results of stabilizer with LQR controller and sensor noise under sine profile - T_5 vs $t(s)$

The torques generated by the inner gimbal axis (T_6) in simulation environment and tests are presented in figures 5.37 & 5.38, respectively.

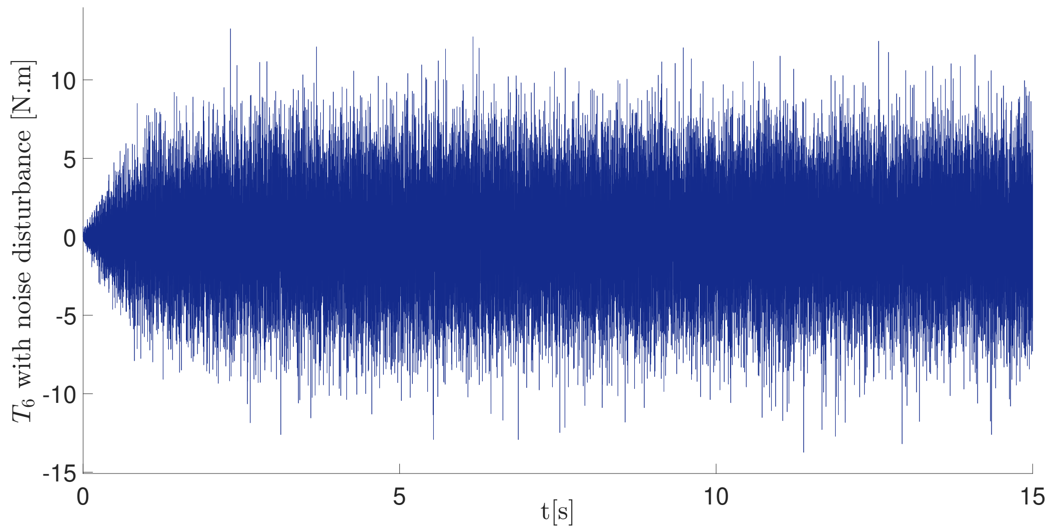


Figure 5.37: Simulation results of stabilizer with LQR controller and sensor noise under sine profile - T_6 vs $t(s)$

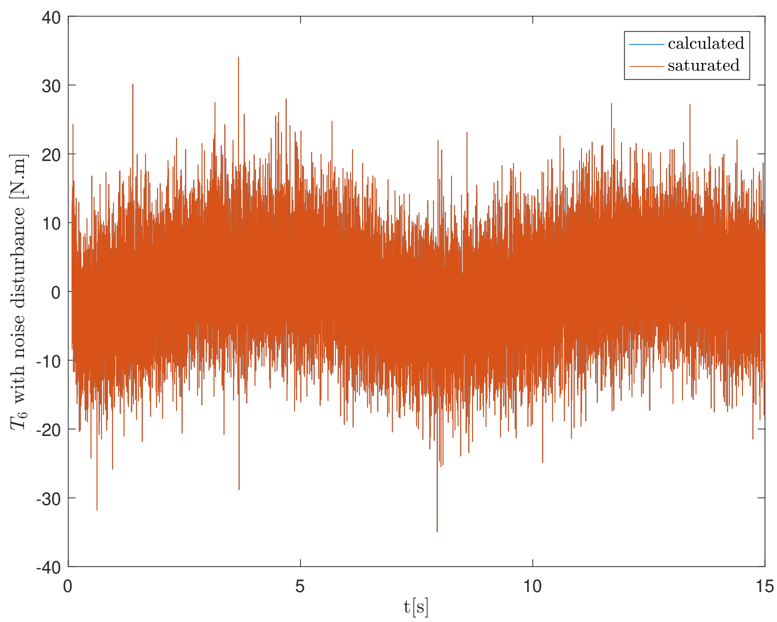


Figure 5.38: Test results of stabilizer with LQR controller and sensor noise under sine profile - T_6 vs $t(s)$

The second profile is trapezoidal profile. In order to test the limits of both disturber and stabilizer controllers; again different trapezoidal profiles have been applied to the system from each disturber actuator. These trapezoidal profiles vary both in amplitude and frequency. Applied trapezoidal profiles are shown in Figure 5.39. Since nothing new can be examined here from the comparison of test results with the simulation results, in order not to overload the section; only test results will be given for the remaining profiles. The results are given in figures 5.40 - 5.43.

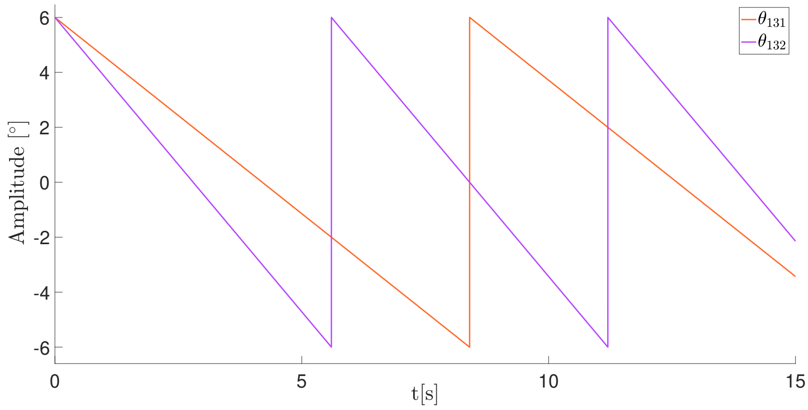


Figure 5.39: Stabilizer with LQR controller - disturber trapezoidal profiles

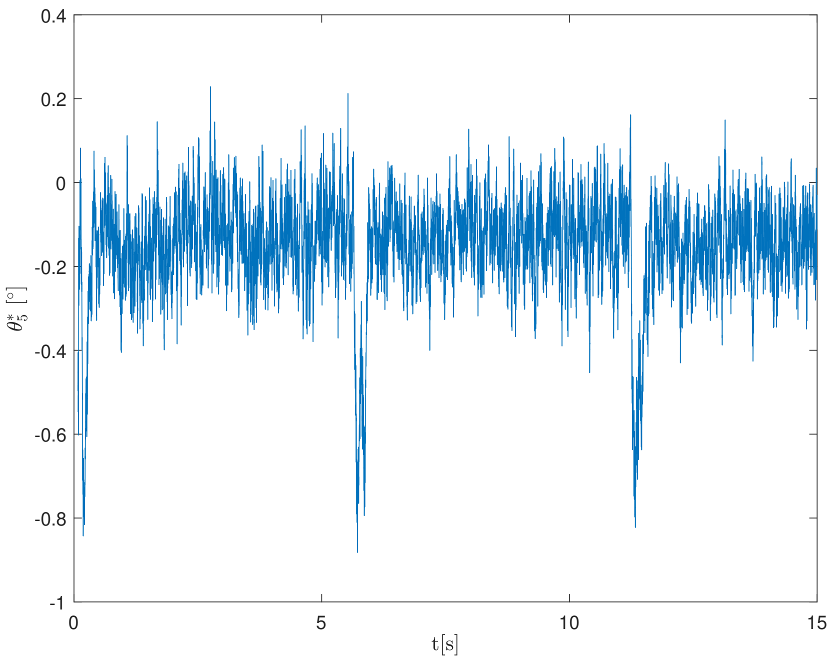


Figure 5.40: Test results of stabilizer with LQR controller under trapezoidal disturber profile - θ_5^* vs $t(s)$

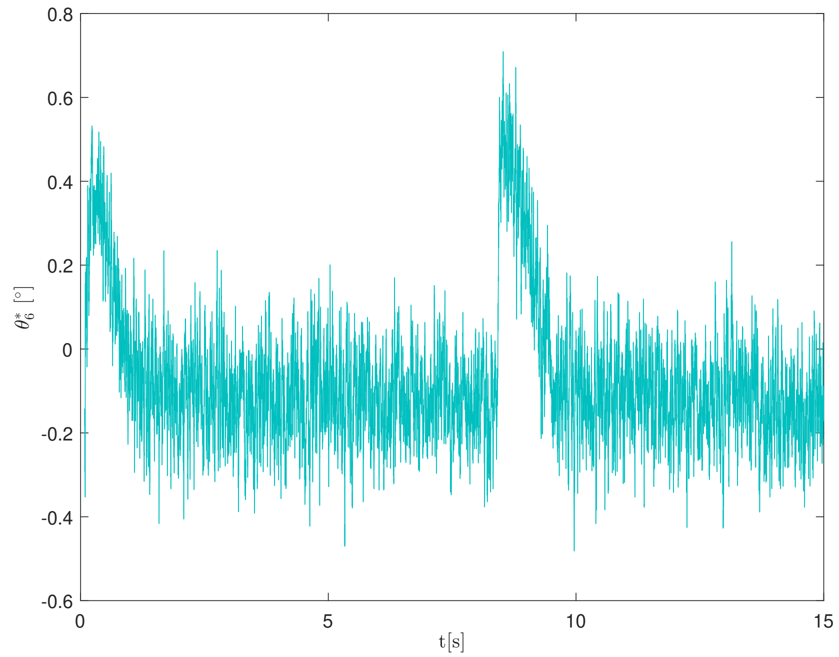


Figure 5.41: Test results of stabilizer with LQR controller under trapezoidal disturber profile - θ_6^* vs $t(s)$

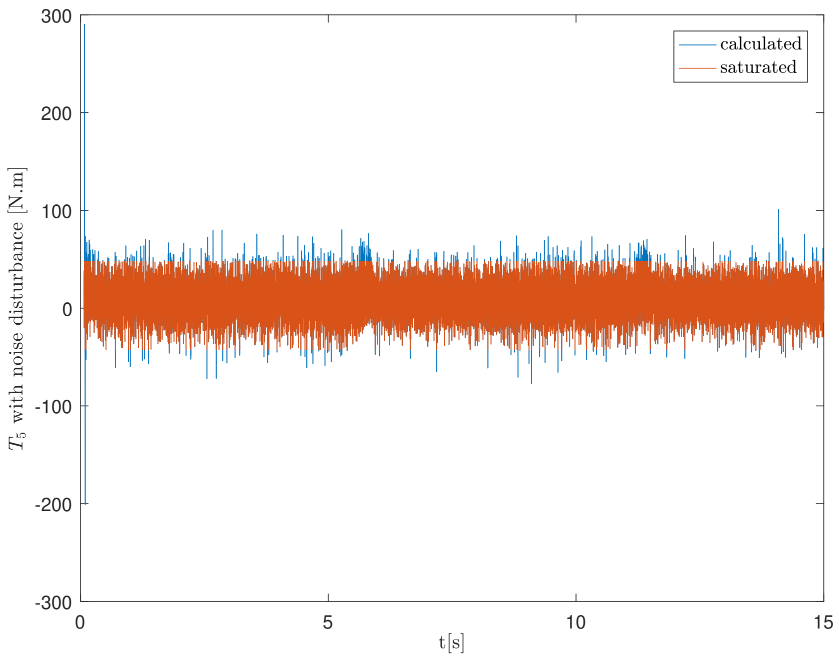


Figure 5.42: Test results of stabilizer with LQR controller under trapezoidal disturber profile - T_5 vs $t(s)$

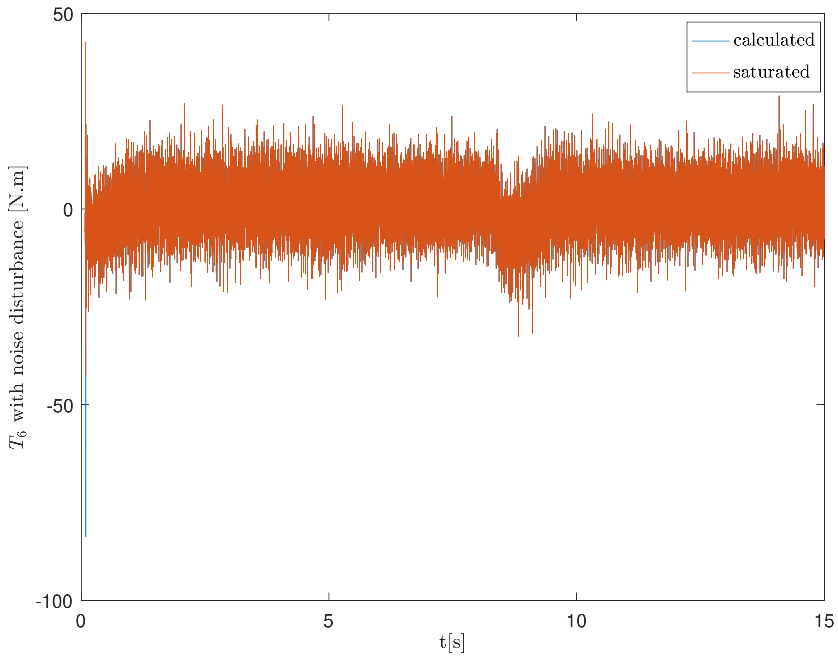


Figure 5.43: Test results of stabilizer with LQR controller under trapezoidal disturber profile - T_6 vs $t(s)$

The third profile is random profile. In order to test the limits of both disturber and stabilizer controllers; again different random profiles have been applied to the system from each disturber actuator. These random profiles vary both in amplitude and frequency. Applied random profiles are shown in Figure 5.44.

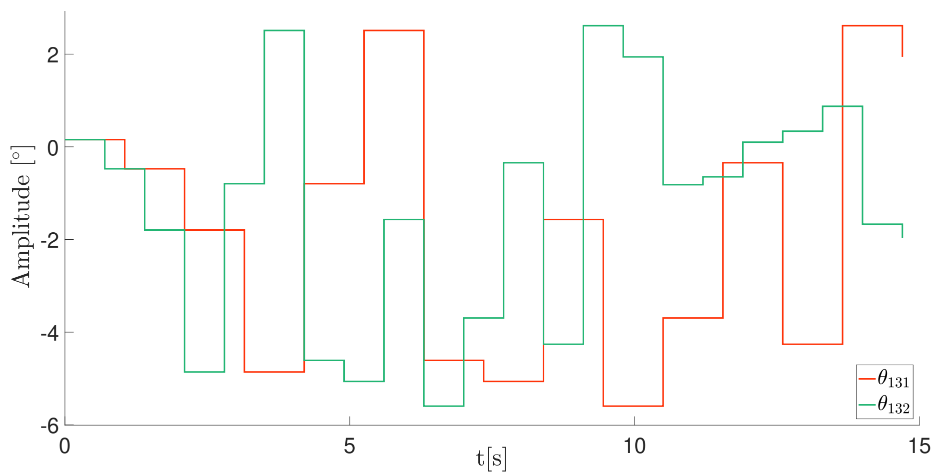


Figure 5.44: Stabilizer with LQR controller - disturber random profiles

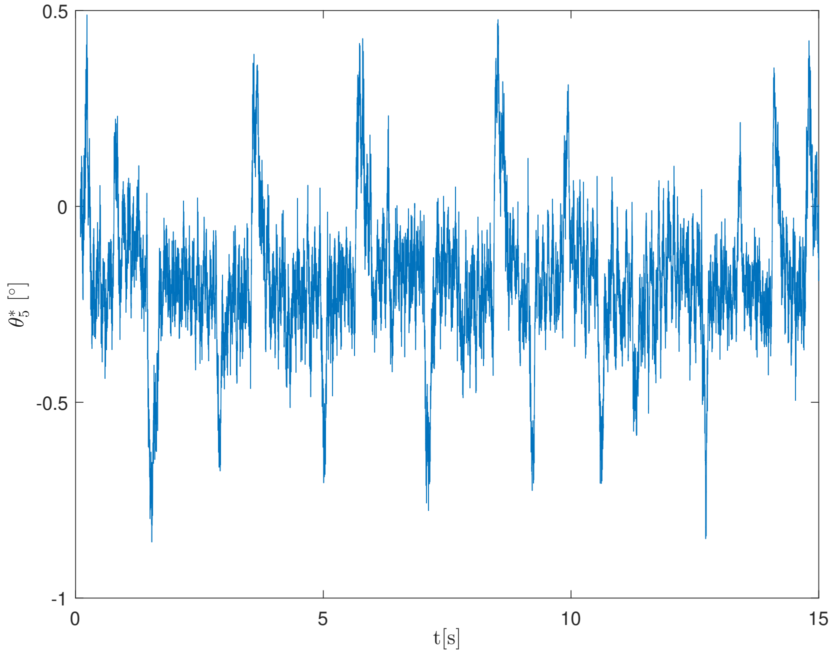


Figure 5.45: Test results of stabilizer with LQR controller under random disturber profile - θ_5^* vs $t(s)$

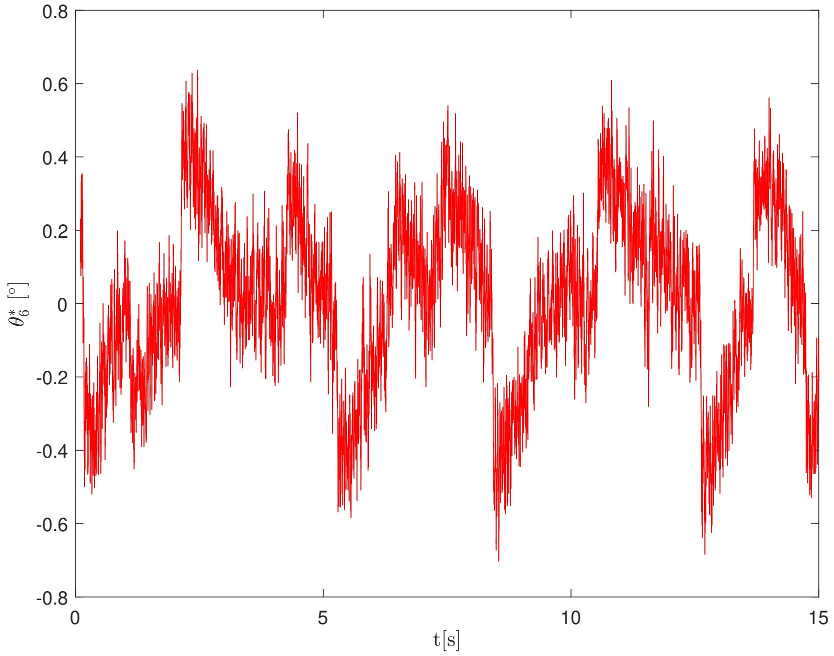


Figure 5.46: Test results of stabilizer with LQR controller under random disturber profile - θ_6^* vs $t(s)$

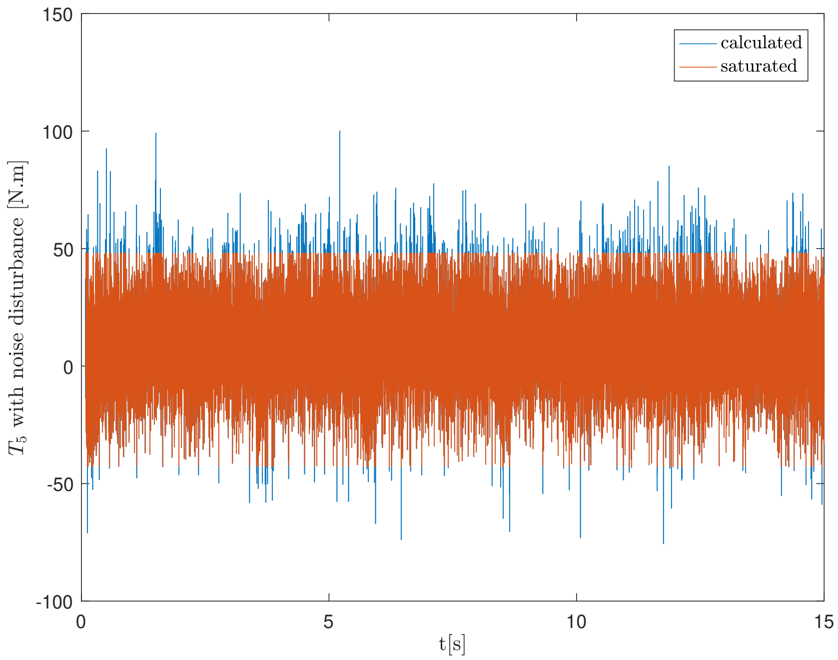


Figure 5.47: Test results of stabilizer with LQR controller under random disturber profile - T_5 vs $t(s)$

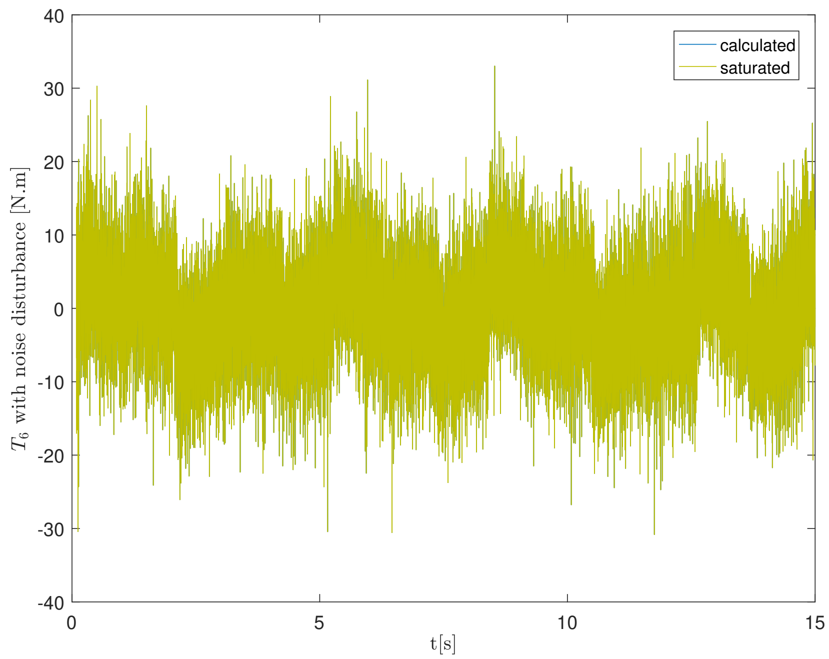


Figure 5.48: Test results of stabilizer with LQR controller under random disturber profile - T_6 vs $t(s)$

The final results are derived from the Roketsan vehicle profile.

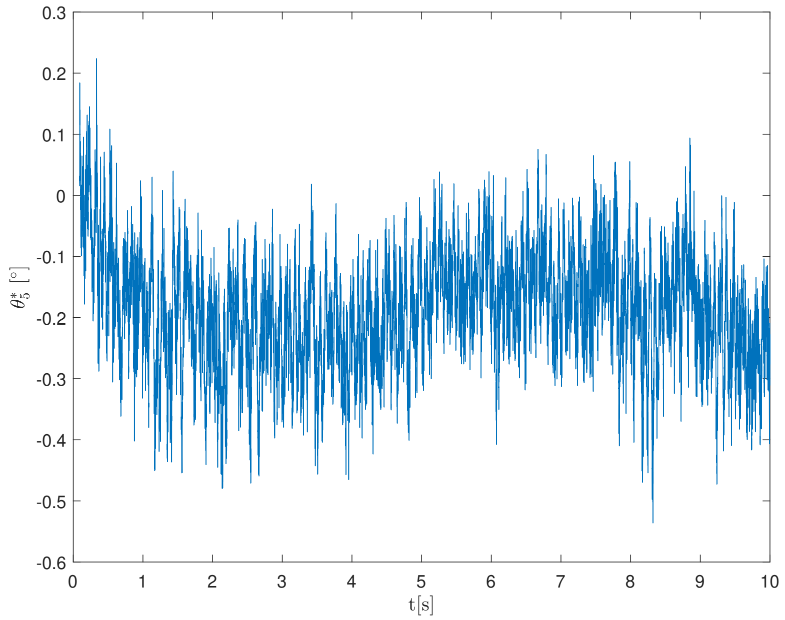


Figure 5.49: Test results of stabilizer with LQR controller under Roketsan vehicle disturber profile - θ_5^* vs $t(s)$

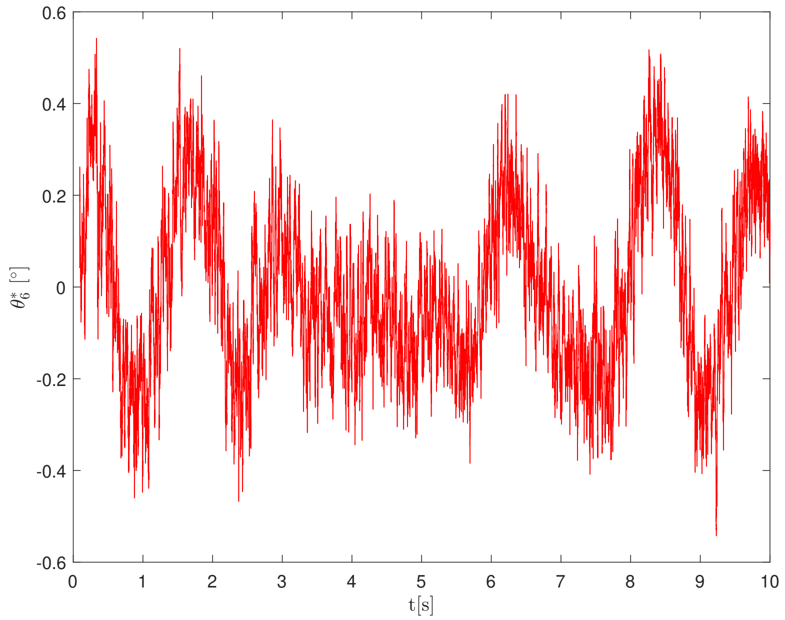


Figure 5.50: Test results of stabilizer with LQR controller under Roketsan vehicle disturber profile - θ_6^* vs $t(s)$

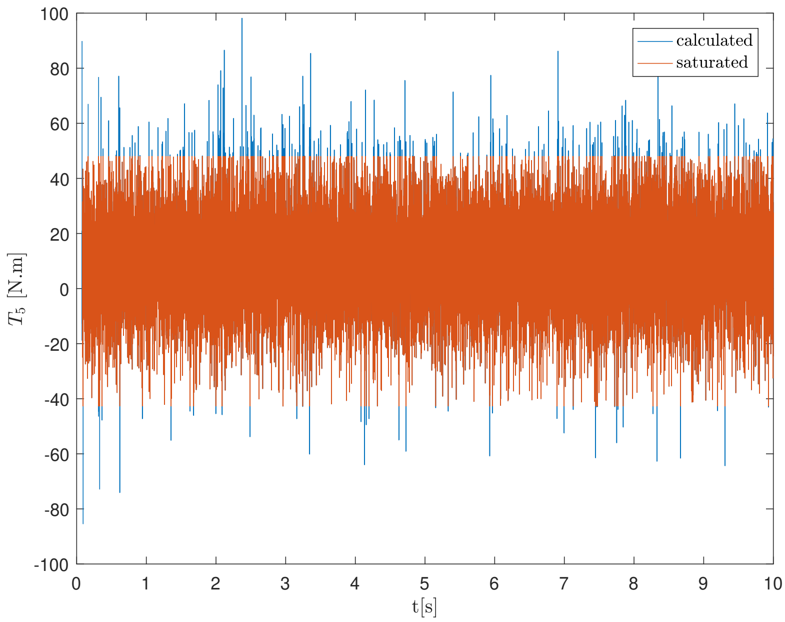


Figure 5.51: Test results of stabilizer with LQR controller and under Roketsan vehicle disturber profile - T_5 vs $t(s)$

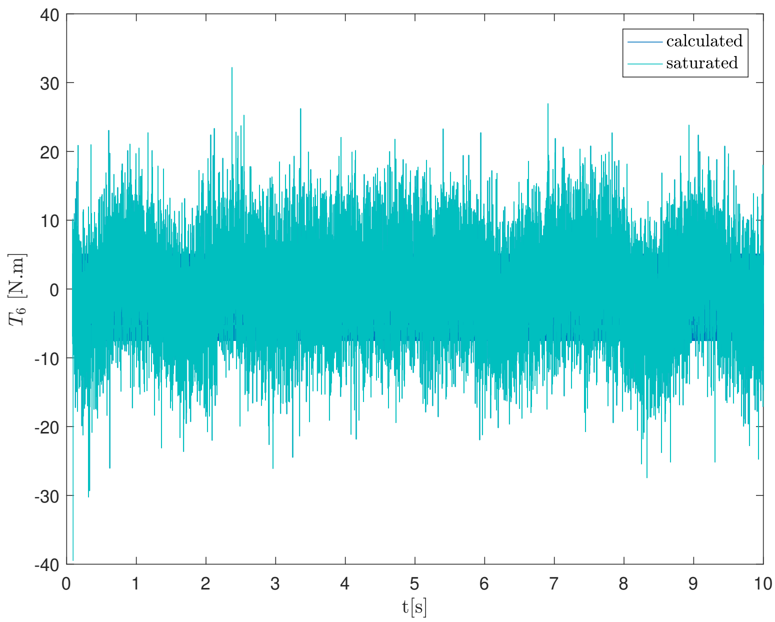


Figure 5.52: Test results of stabilizer with LQR controller under Roketsan vehicle disturber profile - T_6 vs $t(s)$

5.3.3 Sliding Mode PID Controller Design

5.3.3.1 Determination of Sliding Mode PID Controller Parameters

A general definition of the PID controllers and methods for tuning the controllers has been explained in Section 5.2.2. Same methodology for tuning the PID parameters in disturber module have been applied while designing stabilizer sliding mode controller. One key difference is that the PID controller of the disturber has been optimized for a single equilibrium point which is the zero angle position of the payload; whereas for stabilizer PID controller, multiple PID controllers have been designed for an array of θ_{131} & θ_{132} angles. This method has also been applied in LQR controller. Therefore, these two controllers have become more adaptive. That's why, unlike the disturber PID controller; this stabilizer controller has been called as sliding mode PID controller. In total 289 different control parameters for a each controller have been calculated for both LQR and sliding mode PID controllers. As mentioned before, these parameters have been placed in a library to be later used in tests. Obtaining all these parameters is a long process; therefore for automation of this task a Matlab script has been developed. This script can be found in Appendix D.

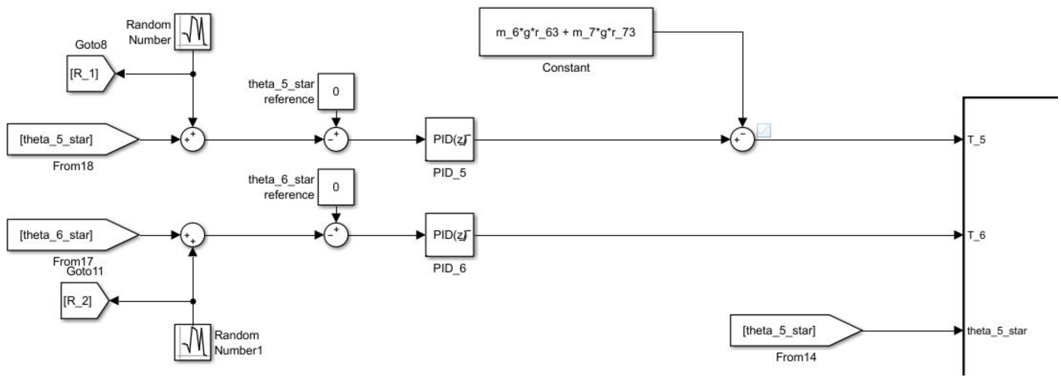


Figure 5.53: Stabilizer PID controller tuning model

In 5.53, it can be seen that Simulink discrete PID blocks are between the output signal of torques T_5 & T_6 and the error signal $\theta_5^* - \theta_{5_{ref}}^*$ & $\theta_6^* - \theta_{6_{ref}}^*$. Note that the reference signals are set to zero due to stabilization purposes. Once the PID parameter tuner of these blocks are opened, the block initially linearizes the plant model automatically. This feature only works for the PID tuner block; therefore for LQR controller, this linearization has been done manually. After linearization the following GUI's appears on the screen.

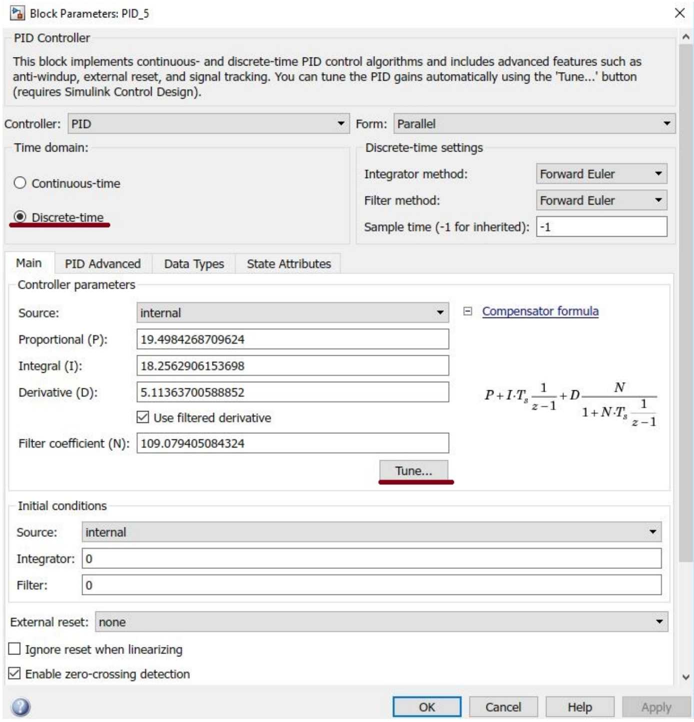


Figure 5.54: Stabilizer PID controller GUI main tab

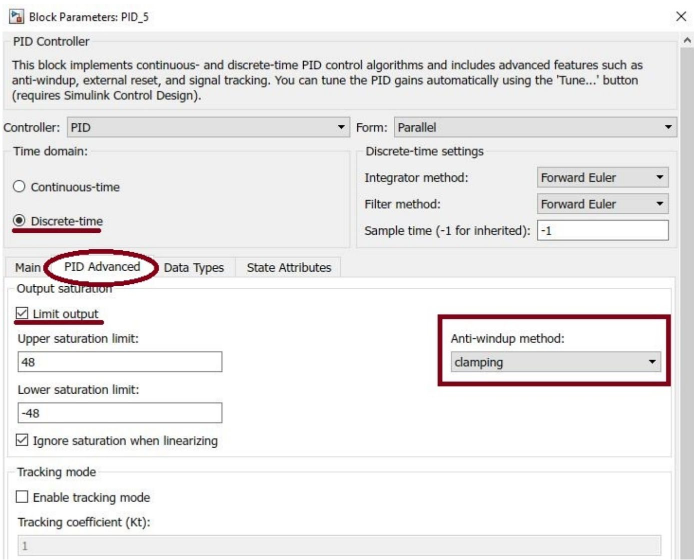


Figure 5.55: Stabilizer PID controller GUI advanced tab

In figures 5.54 & 5.55 the parameters tuned for the limiting case of $\theta_{131} = 16°$ & $\theta_{132} = 16°$ can be seen. The output of the controller is limited to the maximum available torques of the servo motors and clamping type anti winding method has been implemented in order to prevent a burst of controller signals as a result of error accumulation. If actuator torques are saturated; due to "I" term of the PID controller, the error starts to build up. Therefore, the error accumulation due to saturation of the actuators has to be released.

The idea of the anti-winding method is keeping the integrated value below some specified point. Clamping method, which is the simplest and most popular anti-winding method, turns the integrator off whenever integrating is not desired anymore. Clamping method has two checks to do. In the first one, it compares the the output of the PID controller before and after the saturation check. If these values are equal, it means that no saturation takes place and this check gives the output 0. If not equal, then it means that saturation has been reached and this time output of this check becomes 1.

The second check is to compare the sign of the error with the sign of the output of the PID controller. If they are both positive, it means that integrator is still adding to the output to increase it. Similarly if they are both negative, the integrator is trying to make it more negative. By an "AND" block we both look at those outputs in order to understand if the controller is saturating and on top of that if the integrator is making everything worse. In other words, by adding these checks, the integrator branch becomes a conditional integrator. If the output is saturating and the sign of the error is the same sign with the controller output; then clamping decision is made and connection of this this "I" branch is removed temporarily. In the real time target version of the Simulink model, Simulink's built-in PID block cannot be used due to the requirement of external parameter feeding feature. Thus, custom PID block has been created and it can be seen that this logic has been implemented in that custom made block too.

When the system characteristics are adjusted by means of sliders at the top; the controller parameters are optimized by the help of additional graphs provided by the same block. These graphs have been included in figures 5.56 & 5.57 for illustration purposes.

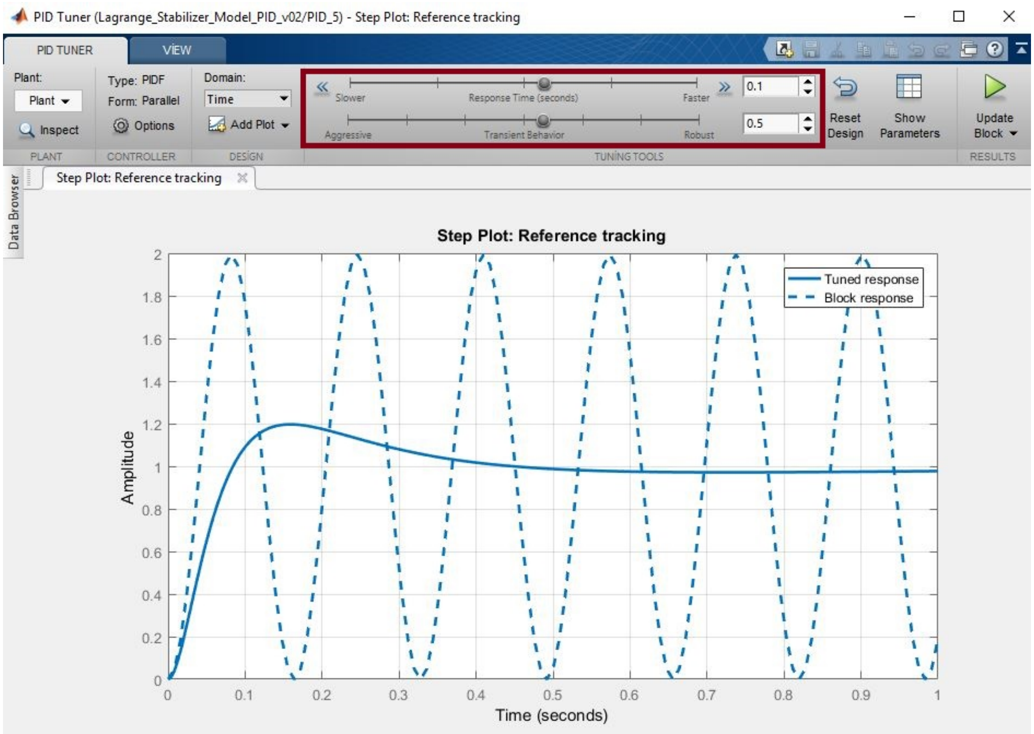


Figure 5.56: Stabilizer PID Tuner adjustment of system characteristics

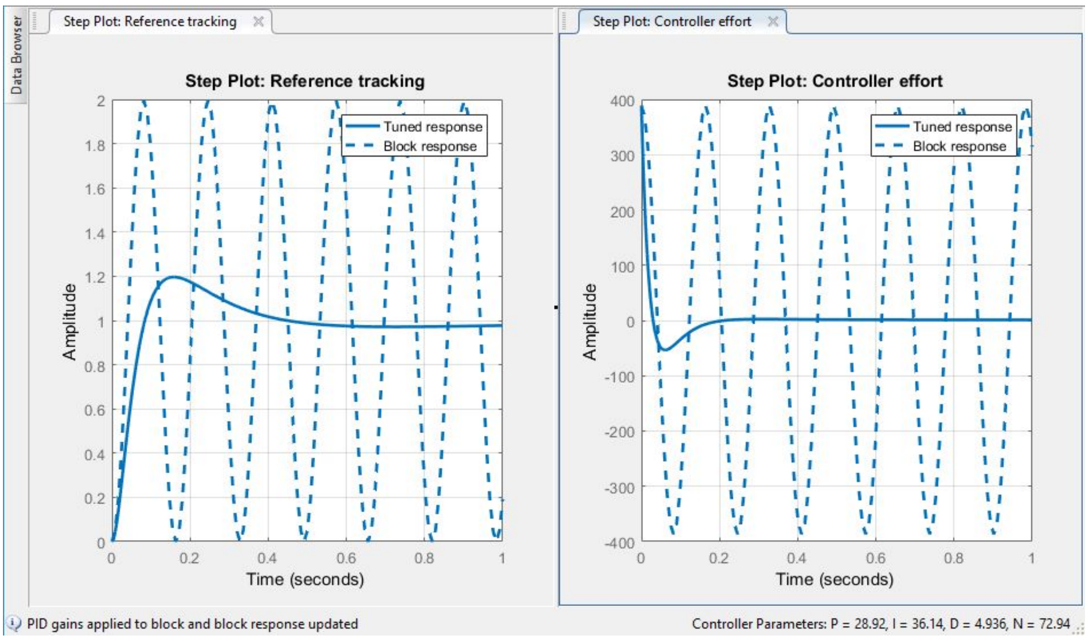


Figure 5.57: Stabilizer PID Tuner graphs of system characteristics

There is another difference between the PID controllers of the disturber and stabilizer. Although from the first time the PID controllers have done a decent job; some systematic manual tweaks have been made to achieve the best results. This requires a good command on the responsibilities of each of the *P*, *I* or *D* parameters. However, as these concepts are extensively studied by the control community; relying on the cumulative knowledge, better results have been achieved after this manual intervention. Nevertheless, if not the starting point of the parameter selection has been achieved by means of Simulink PID design block; it would have been a much more difficult task than it has been for our case.

Actually, this method of utilizing PID controller design block was not our first attempt. The first attempt was a more textbook type method, which was pole placement by examining on the Root Locus graph. This method has also worked out but optimization could not be achieved since pole placement is not much of an intuitive method. It is sometimes hard to guess which pole to move where for a desired characteristic change. The work devoted to this method, therefore is not included in the report since its success has been marginally overridden by the latter method.

5.3.3.2 Description of the Simulation Model

For simulation purposes two different Simulink models have been created. The only difference between them is the PID blocks. In conjuction with the Matlab script developed, the first simulation model is used to obtain the parameter library. On the other hand, the second simulation model uses that parameter library supplied gains for the controller. The second version is in a sense a preparation for the real time target model.

Since real time target we have used in our system is not supported by Matlab 2016b or later versions; PID blocks of the version we have used in our experimental set-up do not support external referencing. Nevertheless, according to the θ_{131} & θ_{132} angles sensed, taking corresponding gains from the library and supplying them to PID blocks is a must according to the architecture of the model. Therefore, custom PID blocks have been designed and utilized in PID real time model. Two simulation models are given in figures 5.58 & 5.59.

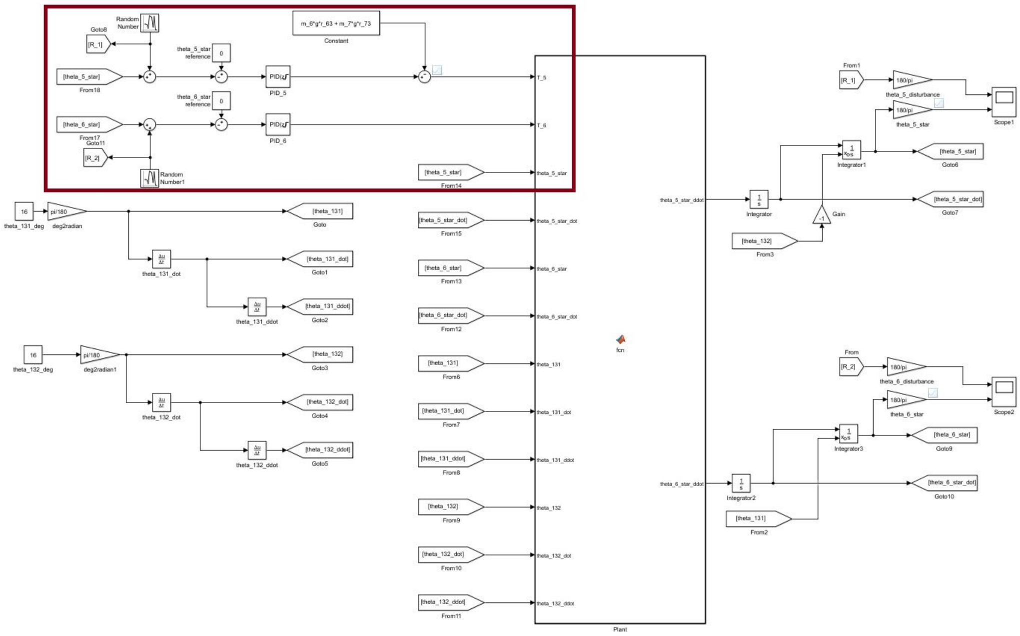


Figure 5.58: Simulation model of stabilizer with sliding mode PID controller first version

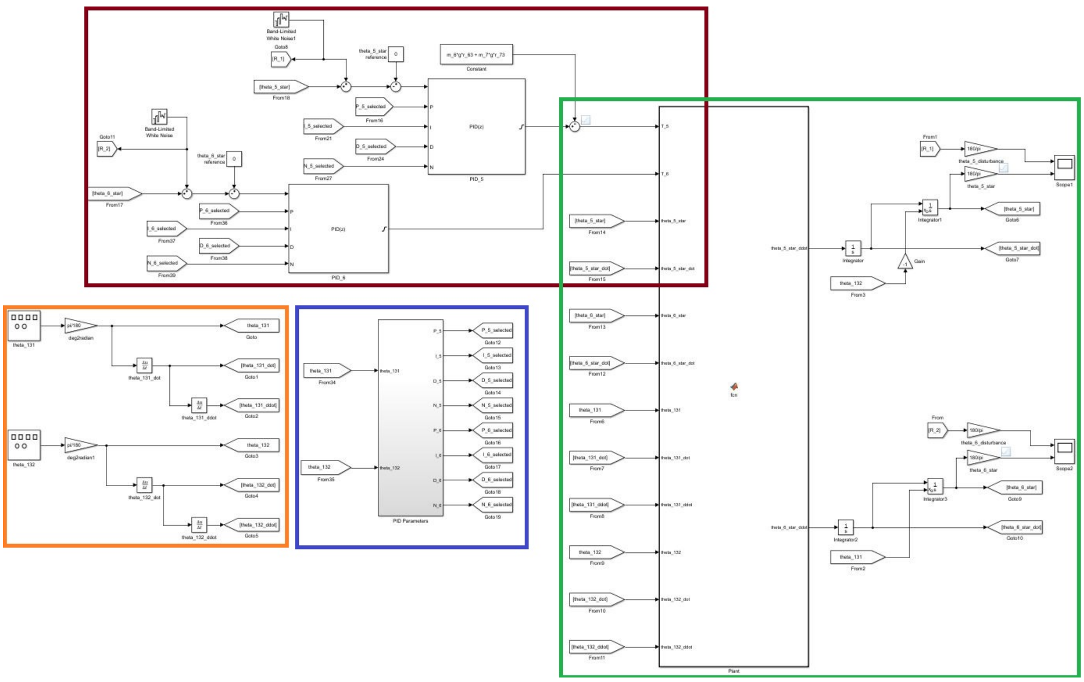


Figure 5.59: Simulation model of stabilizer with sliding mode PID controller second version

As mentioned before, only difference between these models is the region highlighted in red. The highlighted region in red of Figure 5.58 has been explained in section 5.3.3.1. Therefore, we will continue with explaining the highlighted regions of the second model shown in Figure 5.58. The first region we will talk about the controller and is squared in claret red. The magnified version is given in Figure 5.60.

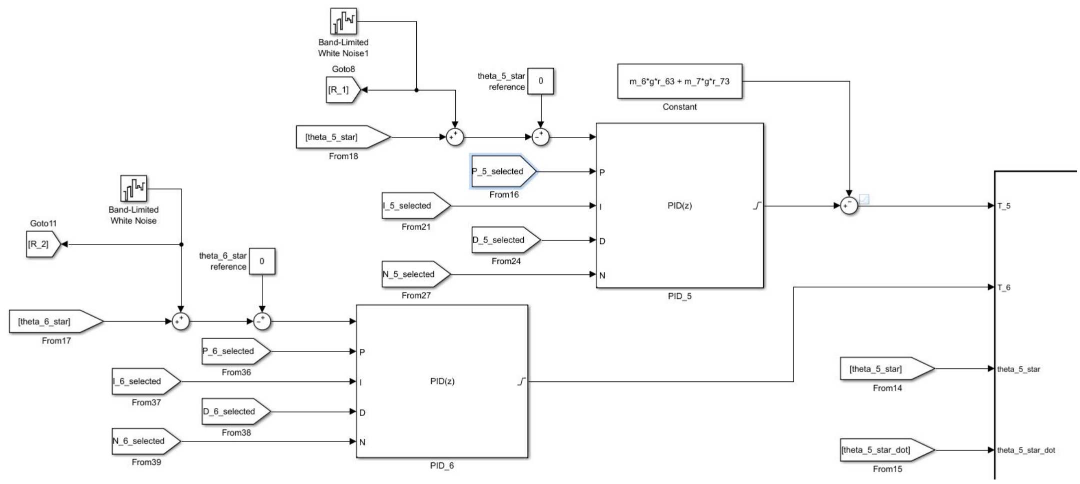


Figure 5.60: Simulation model of stabilizer with sliding mode PID controller - highlighted controller module

This part has nothing new; like done before PID controller takes in the error and generates the torque signals. The parameters of the controller are selected from the library and fed externally to the controllers. For simulation purposes, the output stabilization angles have been calculated by means of the plant model. Moreover, real life sensor noise has been imitated by means of band limited white noise block of Simulink.

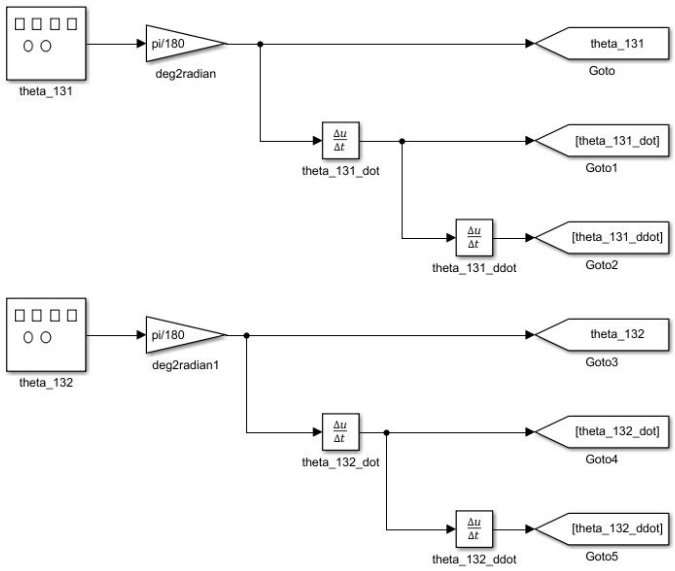


Figure 5.61: Simulation model of stabilizer with sliding mode PID controller - highlighted disturber module

Motion generating blocks are located on the left. Similar to the LQR model, first and second derivatives of the base(payload) angles θ_{131} & θ_{132} to be used in the plant model; whereas the angles itself are used for selecting convenient gains from the library.

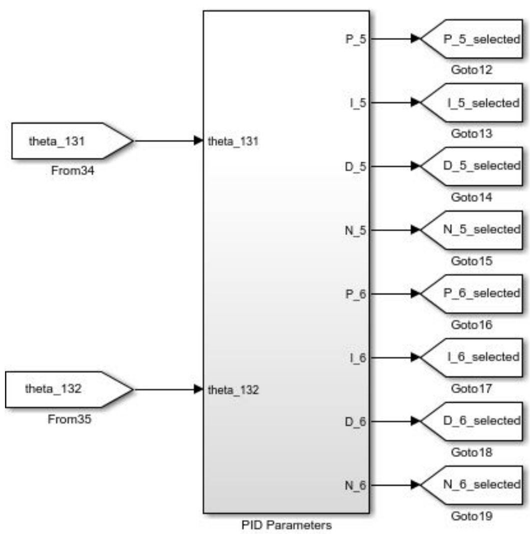


Figure 5.62: Simulation model of stabilizer with sliding mode PID controller - highlighted gain library module

Let's look under the mask of "PID Parameters" block to get a better understanding of how it works. The inside of this block is given in Figure 5.63.

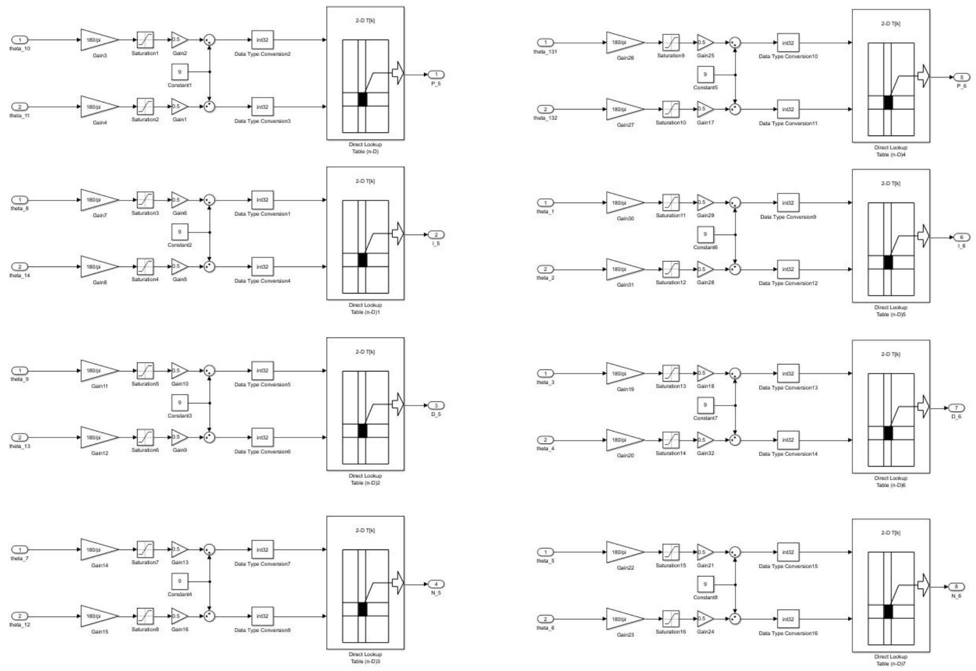


Figure 5.63: Simulation model of stabilizer with sliding mode PID controller - under the mask of PID parameters block

As it can be seen it is a repetition of the same logic for each PID parameter. Each controller has P, I, D and filter coefficient N components so this repetition takes place for 8 times. Since they are all the same, we can inspect on one of them to understand how it works.

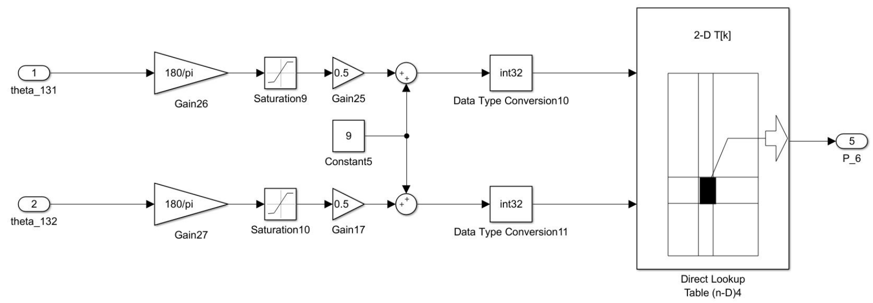


Figure 5.64: Simulation model of stabilizer with sliding mode PID controller - under the mask of PID parameters block, one sequence focused

Once the radians are converted into degrees they are saturated to the maximum available joint angles. The elements in the angle arrays have been spaced for every two degrees which makes 17 elements for $\pm 16^\circ$ for each angle of θ_{131} & θ_{132}. To find the corresponding array integer the saturated degree is divided by 2 and 9 is added to that value. When the outcome is converted to *int32* type and rounded to nearest, it takes the corresponding gain from the library. Lastly, we will talk about the plant module given in Figure 5.65.

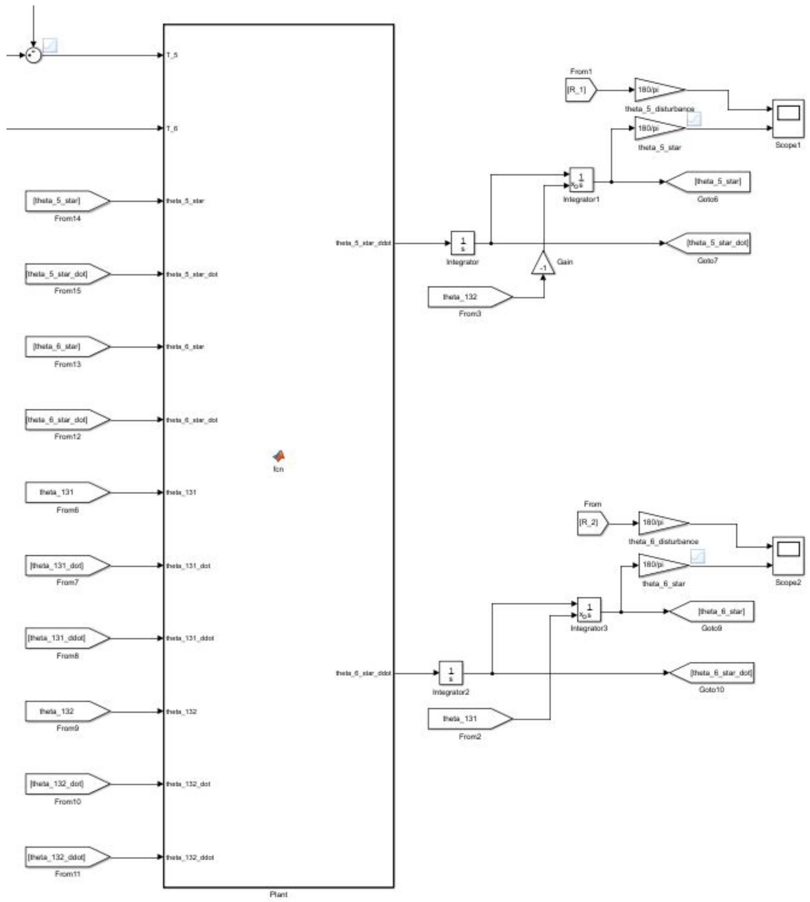


Figure 5.65: Simulation model of stabilizer with sliding mode PID controller - highlighted plant module

The plant module represents the system dynamics. It takes the generated torque values and calculates the corresponding acceleration values. The global platform angles (θ_5^* & θ_6^*) are obtained by integration the angular acceleration terms ($\ddot{\theta}_5^*$ & $\ddot{\theta}_6^*$) twice. These angles are then sent back to controller as a feedback as it can be seen in Figure 5.60.

5.3.3.3 Analysis and Experiment Results

Motion profiles of the disturber have been shown in sections 5.2.2.3 & 5.3.2.3, therefore it is not necessary to present them again.

Let's start with the sine profile. Analysis results of the results will be given in the same order with the previous sections.

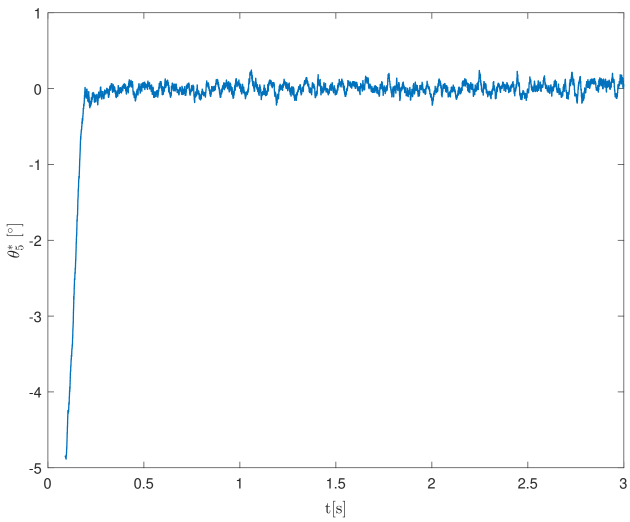


Figure 5.66: Step response of θ_5^* with sliding mode PID controller

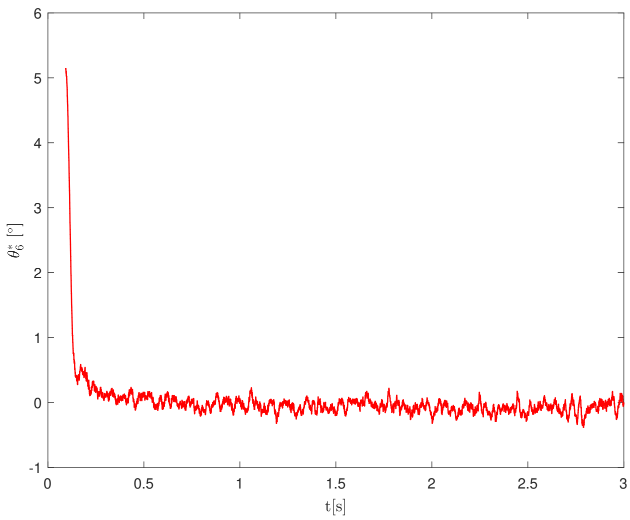


Figure 5.67: Step response of θ_6^* with sliding mode PID controller

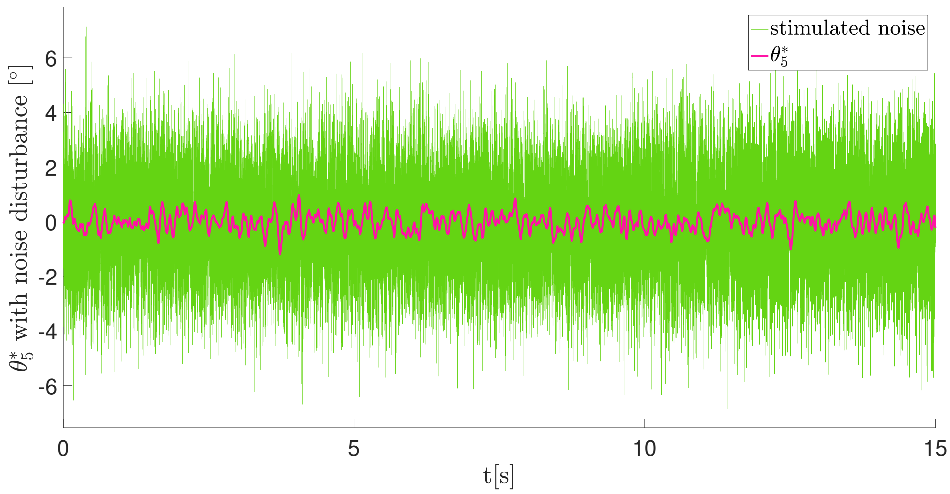


Figure 5.68: Simulation results of stabilizer with sliding mode PID controller and sensor noise under sine disturber profile - θ_5^* vs $t(s)$

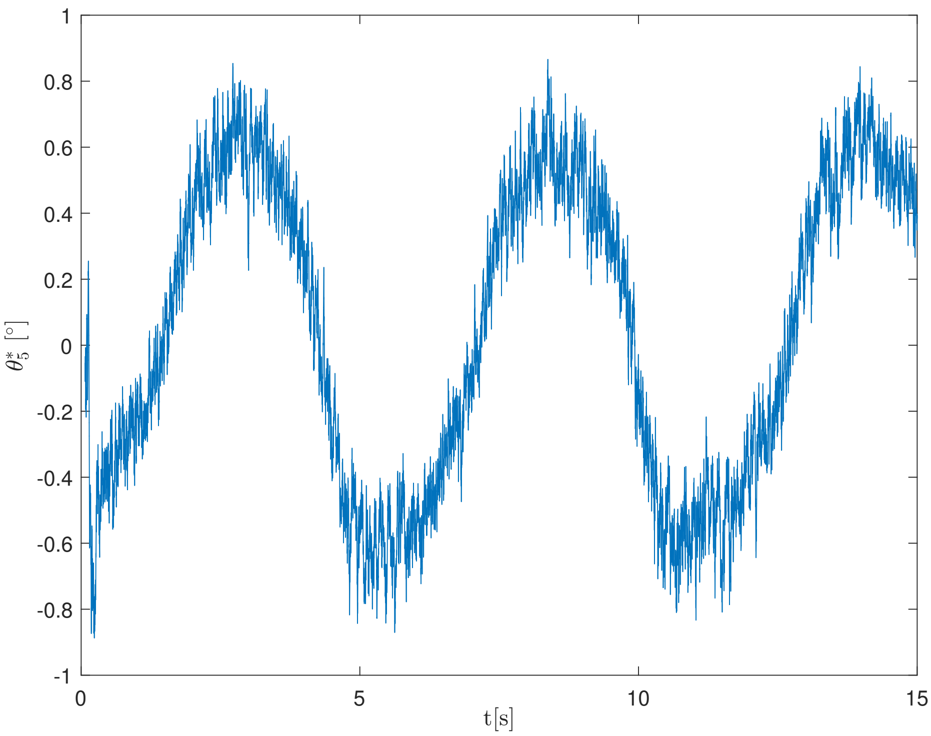


Figure 5.69: Test results of stabilizer with sliding mode PID controller and sensor noise under sine disturber profile - θ_5^* vs $t(s)$

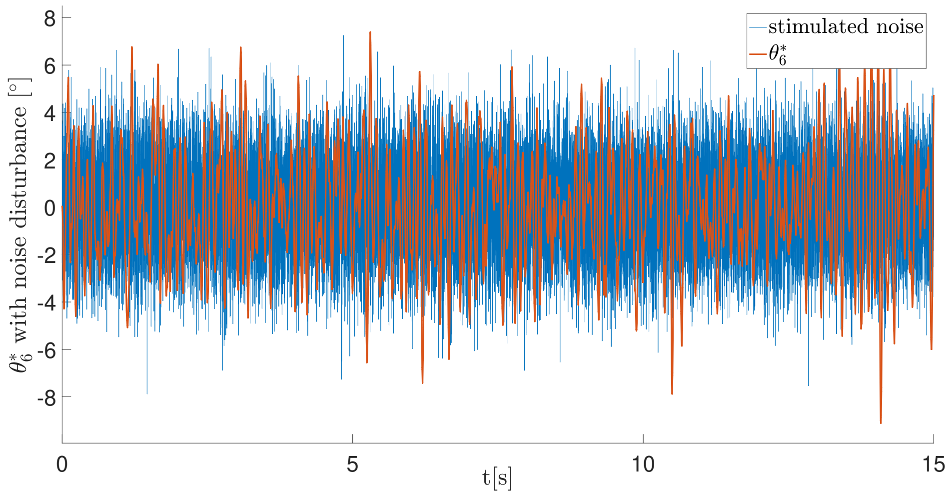


Figure 5.70: Simulation results of stabilizer with sliding mode PID controller and sensor noise under sine disturber profile - θ_6^* vs $t(s)$

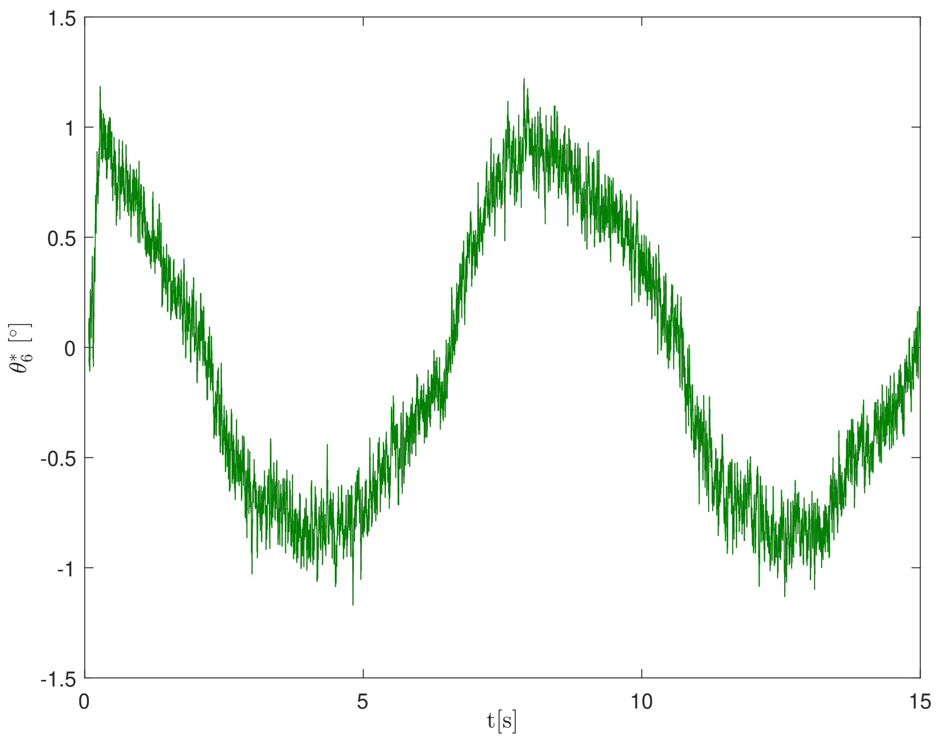


Figure 5.71: Test results of stabilizer with sliding mode PID controller under sine disturber profile - θ_6^* vs $t(s)$

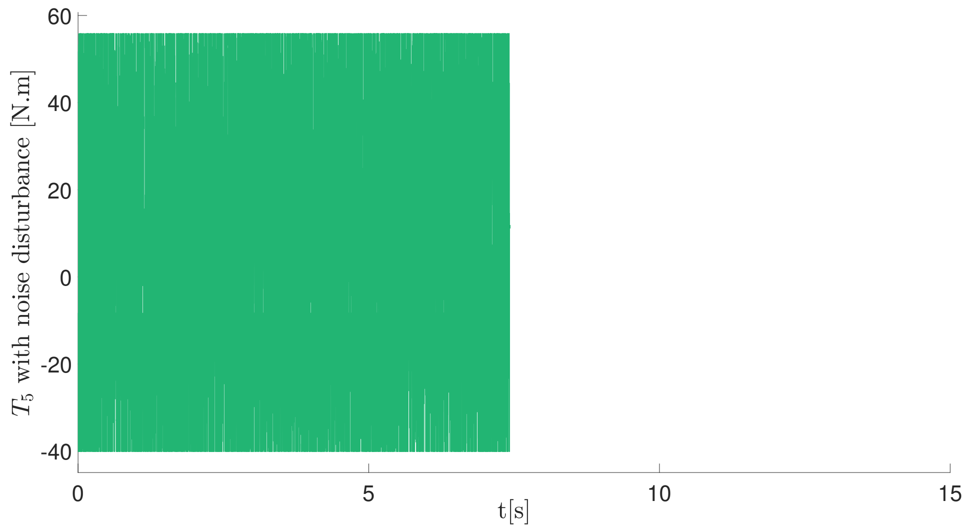


Figure 5.72: Simulation results of stabilizer with sliding mode PID controller and sensor noise under sine disturber profile - T_5 vs $t(s)$

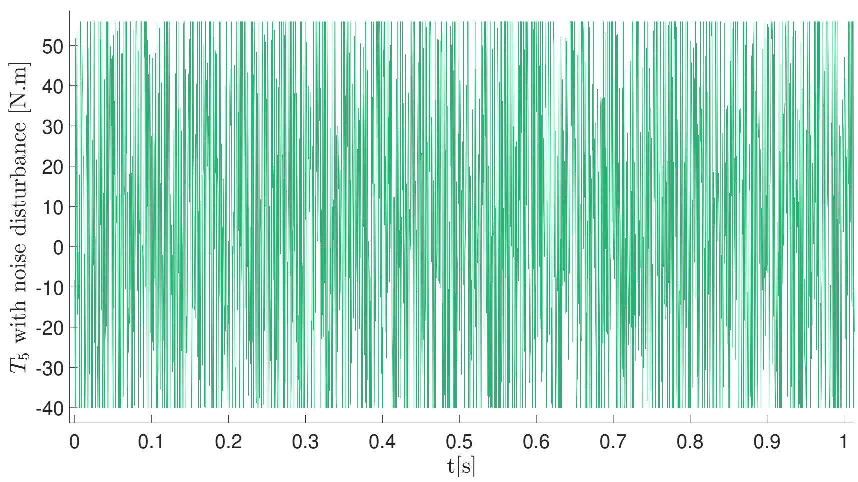


Figure 5.73: Simulation results of stabilizer with sliding mode PID controller and sensor noise under sine disturber profile - (T_5 zoomed into 0-1 sec.) vs $t(s)$

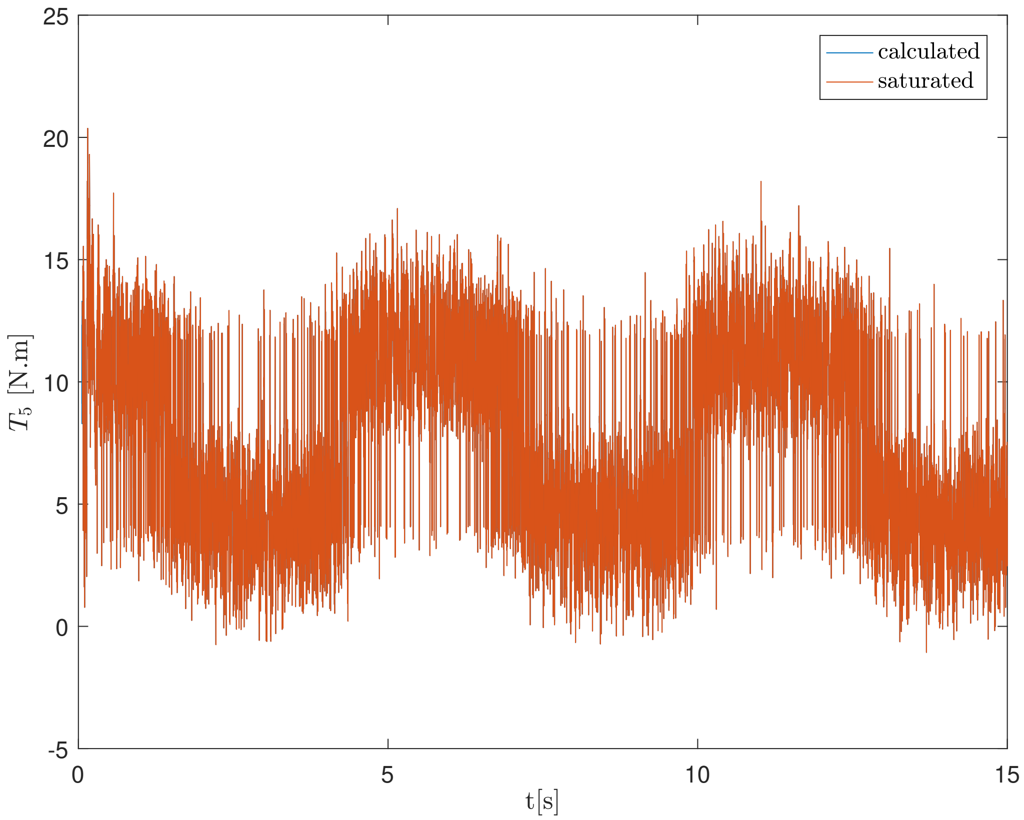


Figure 5.74: Test results of stabilizer with sliding mode PID controller under sine disturber profile - T_5 vs $t(s)$

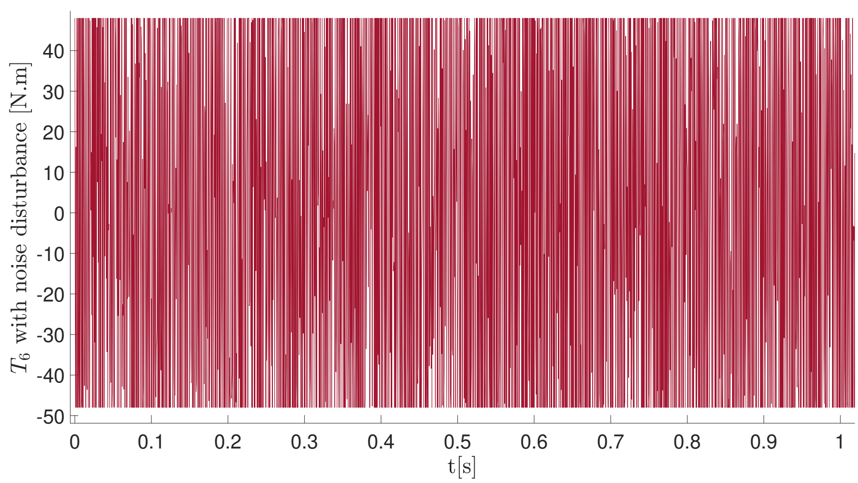


Figure 5.75: Simulation results of stabilizer with sliding mode PID controller and sensor noise under sine disturber profile - (T_6 zoomed into 0-1 sec.) vs $t(s)$

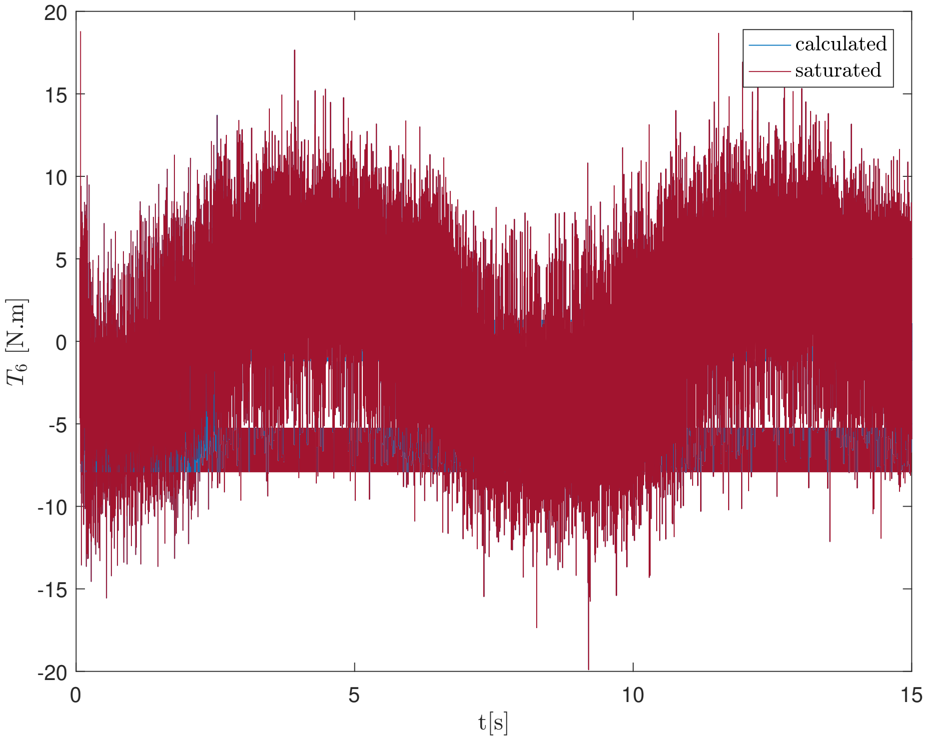


Figure 5.76: Test results of stabilizer with sliding mode PID controller under sine disturber profile - T_6 vs $t(s)$

The second profile is trapezoidal profile. Like we have done before, in order to prevent overloading of results; only test results will be presented for the remaining disturber profiles.

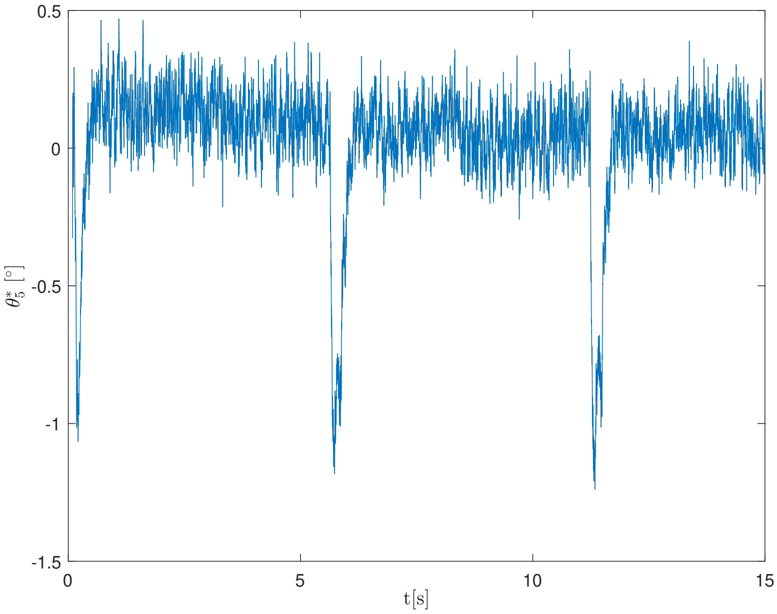


Figure 5.77: Test results of stabilizer with sliding mode PID controller under trapezoidal disturber profile - θ_5^* vs $t(s)$

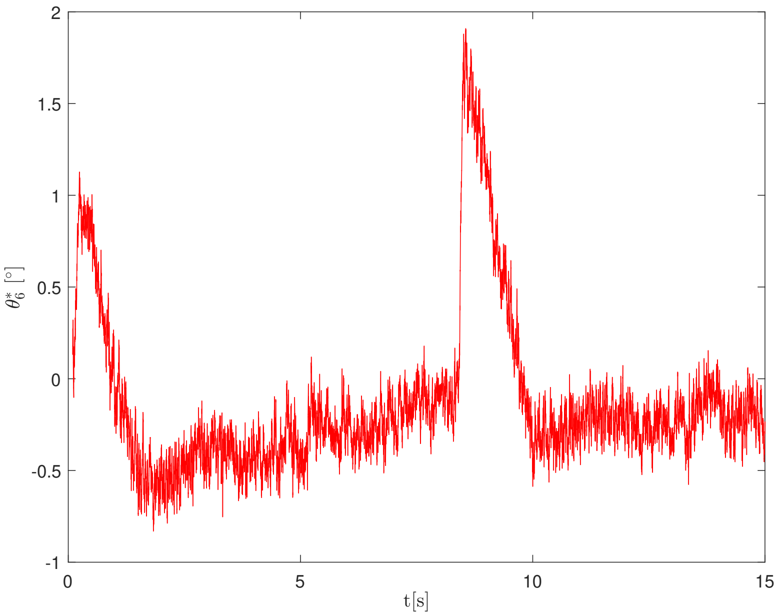


Figure 5.78: Test results of stabilizer with sliding mode PID controller under trapezoidal disturber profile - θ_6^* vs $t(s)$

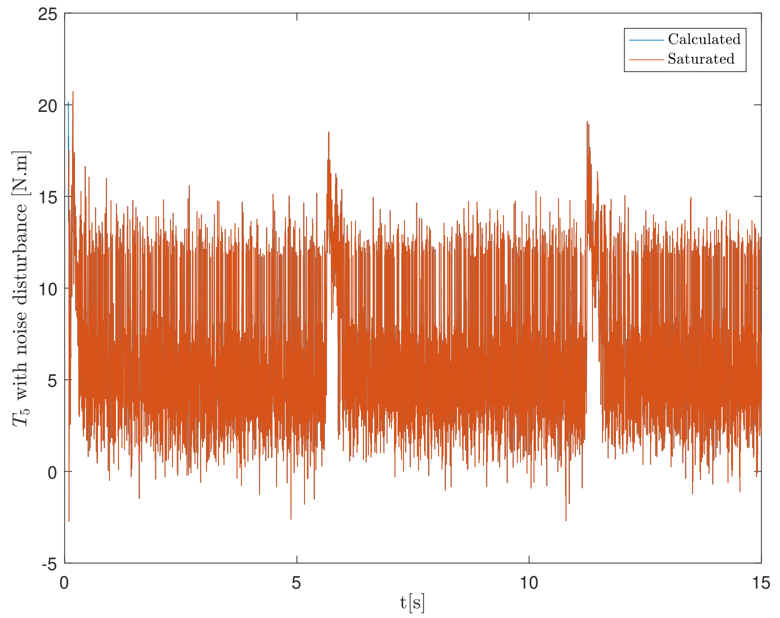


Figure 5.79: Test results of stabilizer with sliding mode PID controller and under trapezoidal disturber profile - T_5 vs $t(s)$

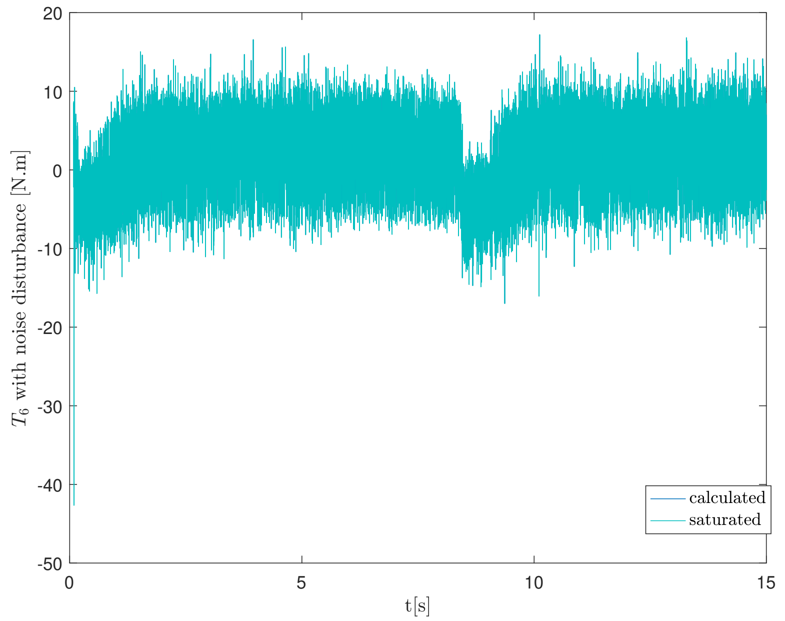


Figure 5.80: Test results of stabilizer with sliding mode PID controller under trapezoidal disturber profile - T_6 vs $t(s)$

Next, random profile will follow.

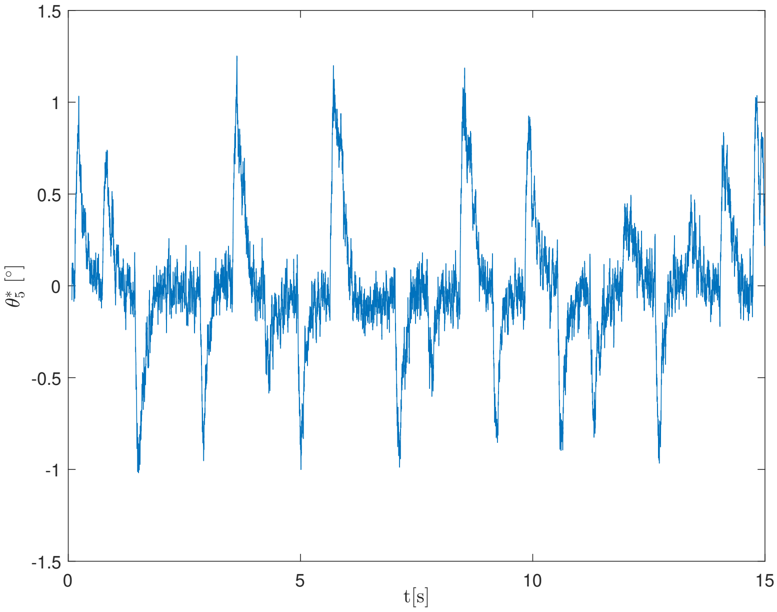


Figure 5.81: Test results of stabilizer with sliding mode PID controller under random disturber profile - θ_5^* vs $t(s)$

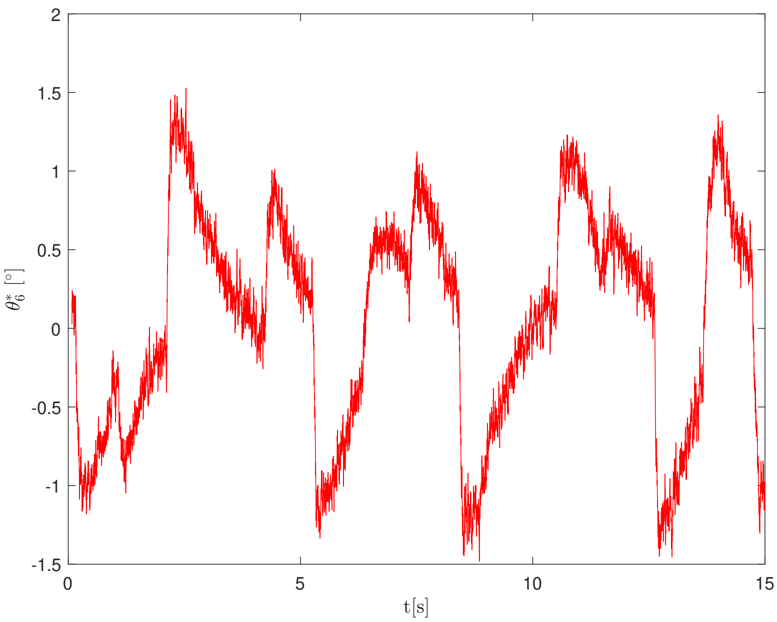


Figure 5.82: Test results of stabilizer with sliding mode PID controller under random disturber profile - θ_6^* vs $t(s)$

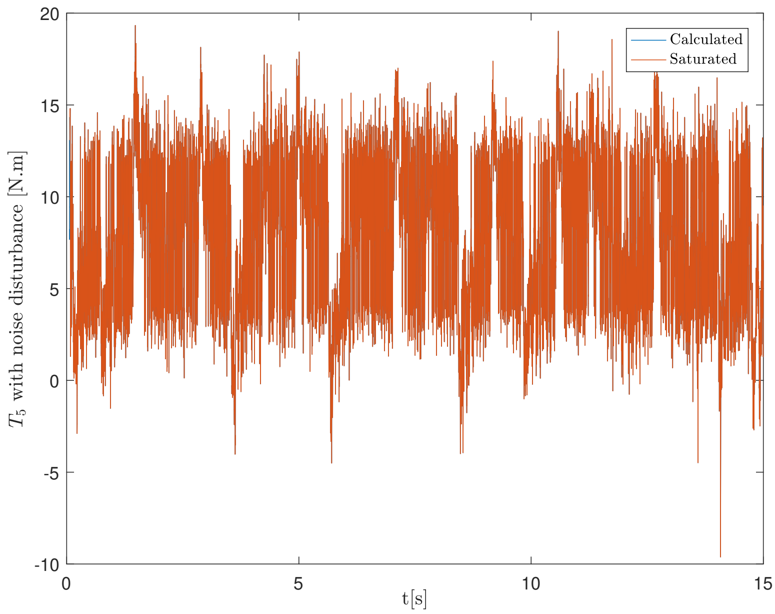


Figure 5.83: Test results of stabilizer with sliding mode PID controller and under random disturber profile - T_5 vs $t(s)$

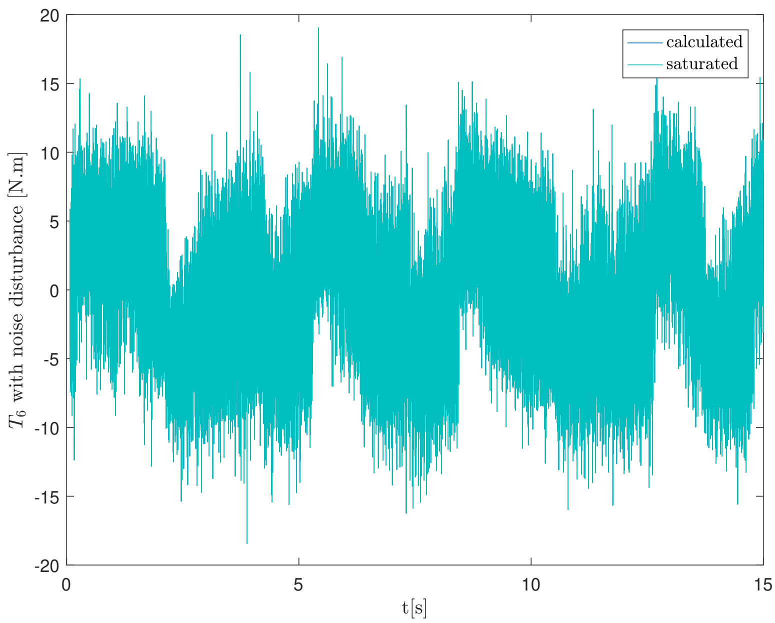


Figure 5.84: Test results of stabilizer with sliding mode PID controller under random disturber profile - T_6 vs $t(s)$

Finally, the results of Roketsan vehicle profile are as follows:

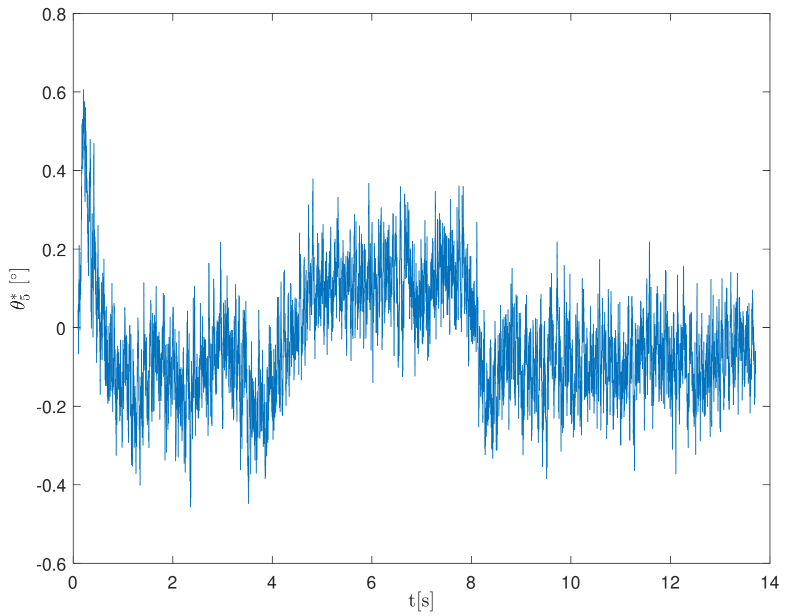


Figure 5.85: Test results of stabilizer with sliding mode PID controller under Roketsan vehicle disturber profile - θ_5^* vs $t(s)$

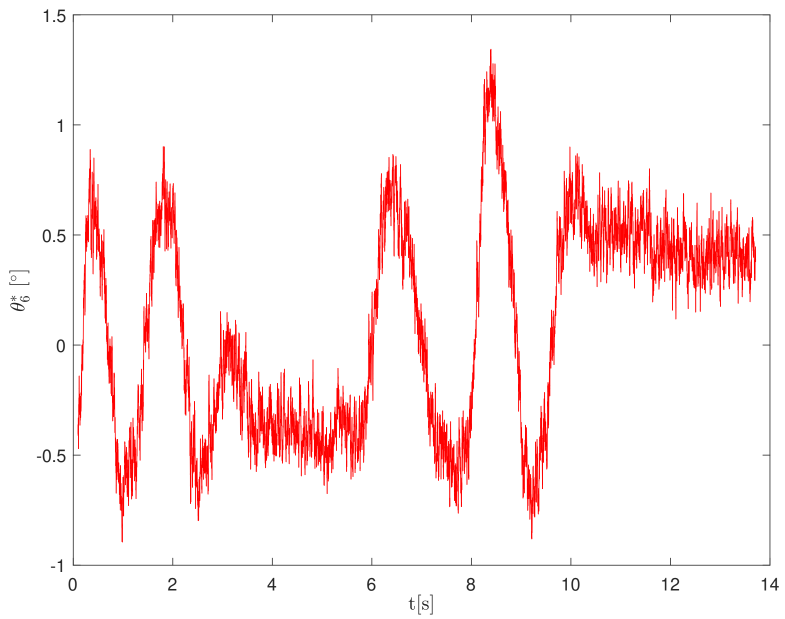


Figure 5.86: Test results of stabilizer with sliding mode PID controller under Roketsan vehicle disturber profile - θ_6^* vs $t(s)$

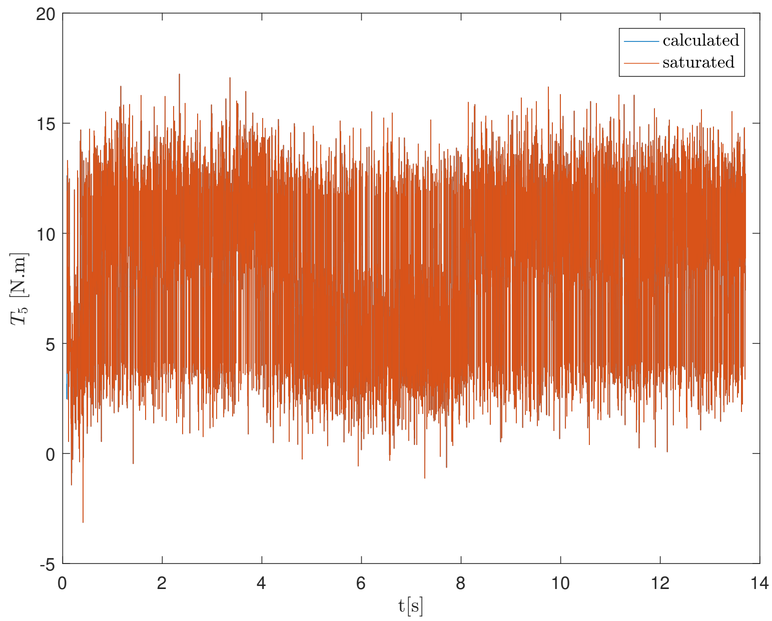


Figure 5.87: Test results of stabilizer with sliding mode PID controller and under Roketsan vehicle disturber profile - T_5 vs $t(s)$

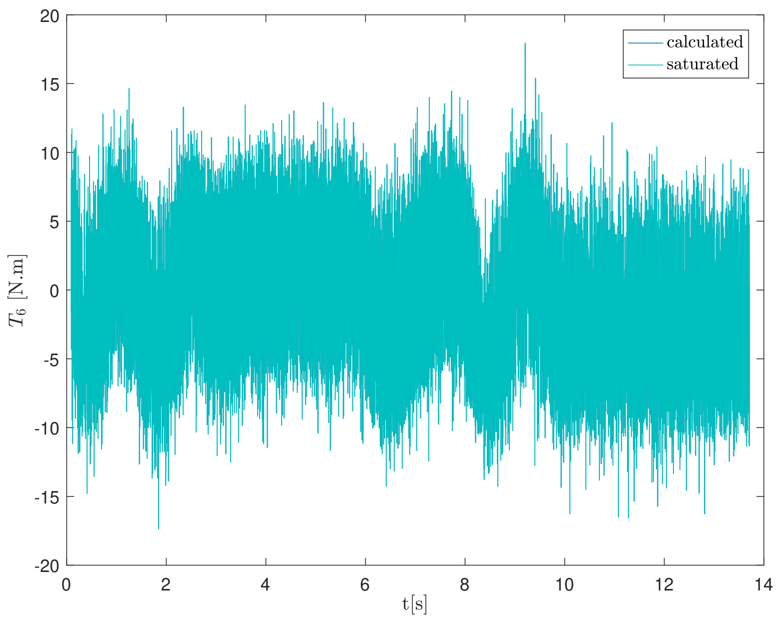


Figure 5.88: Test results of stabilizer with sliding mode PID controller under Roketsan vehicle disturber profile - T_6 vs $t(s)$

5.4 Description of Real Time Target Simulink Model

The performance and controller efforts should be examined before the real life tests. In order to do that, Simulink models explained so far have been created and results are observed. Note that there are some differences between every simulation model and real time target model. These differences emerge from the sources of inputs and the methodology for providing outputs. There is also a slight difference between each type of controller in real time models but they share many common blocks. Therefore, in order to avoid overloading; only PID real time controller will be discussed. In Figure 5.89, the general view of the model can be seen. The claret red represents the disturber region. The navy blue represents the controller region and green represents the feedback branch and output signal branch. All these regions will be explained in detail by expanding these blocks further until every detail is clear.

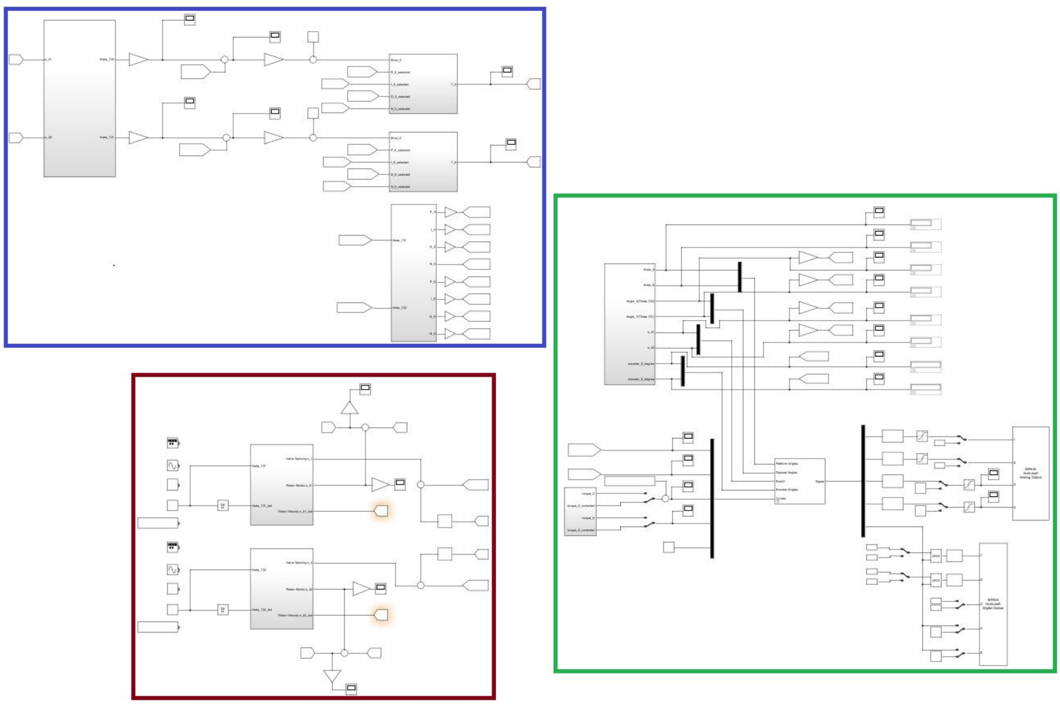


Figure 5.89: Simulink Real Time model of stabilizer with LQR controller and disturber

First let's start with the disturber region. In Figure 5.90 the claret red region alone, can be seen.

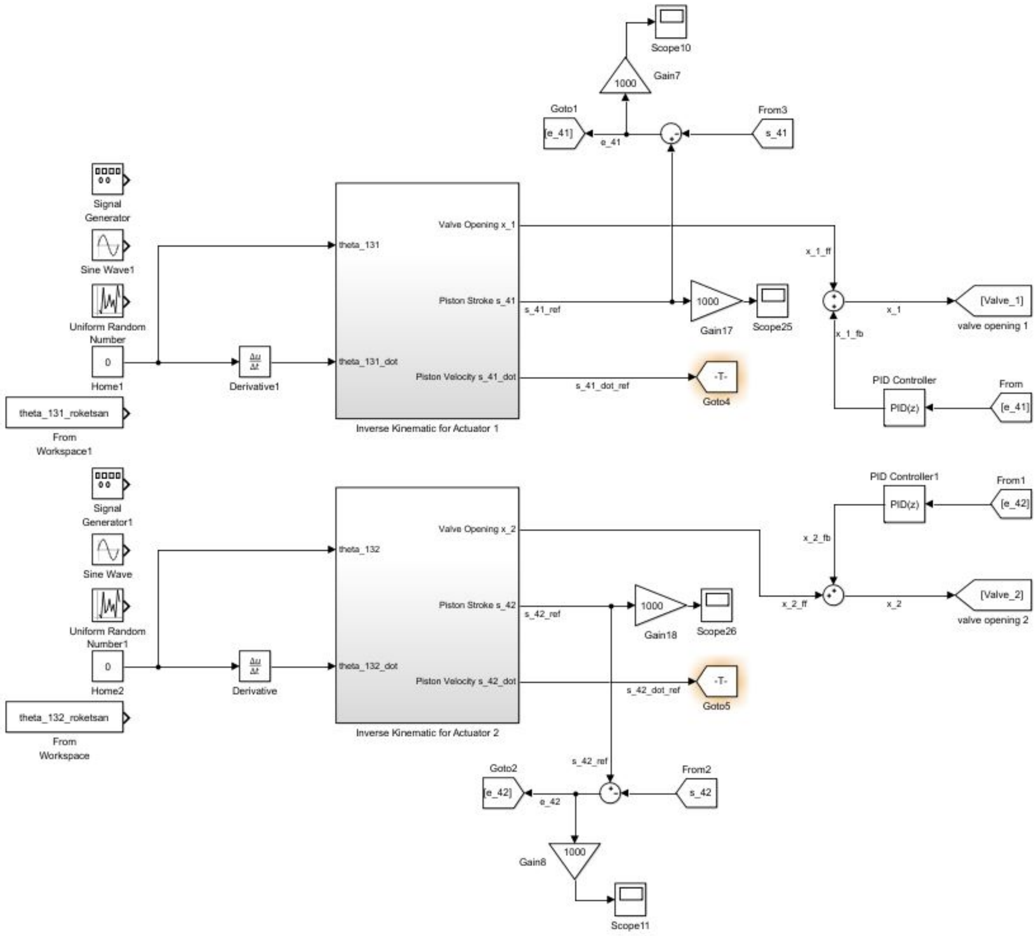


Figure 5.90: Simulink Real Time model of disturber

Please note that the inverse kinematic block has been explained in Figure 5.6. On the left, blocks for 4 different motion profiles are included. These blocks give the inputs required for the inverse kinematics. These inputs are the payload angles and their first derivatives. As a result, reference cylinder strokes and stroke velocities corresponding to these payload angles are calculated. The stroke velocity is a direct function of the flowrate which is our control parameter through the flowrate control valves. Actual strokes are measured by the linear transducers and its difference from the reference stroke is the input of the PID controller. Note that two different PID controllers are designed; each responsible for actuating a single cylinder.

Next, we'll proceed with the highlighted controller region which is given in Figure

5.91. But before we do that, we have to point out that there are several versions of the Simulink Real Time model. In the first several versions both payload and platform angles were sensed by the inclinometers. This set of sensor arrangement is called the first configuration.

Then, the second configuration has been created where we had to change the sensors and came up with an alternative. In this configuration, both inclinometers have been cancelled due to some problems. The reasons are explained in detail in Chapter 6. Therefore, the payload angles are derived from the cylinder strokes by means of the data sensed from the linear transducers mounted on the cylinders. In order to do that forward kinematics block have been used. However, in this section you'll see the traces of old configuration. For instance, you'll see that we are still getting measurements from the inclinometers. The reasons will again be explained in Chapter 6; but in short these measurements have only been used in calibration processes.

Currently gyroscopes are being installed to the current set-up which will bring us back to the initial configuration but this time gyroscopes will be used to determine the Euler angles instead of the inclinometers.

Please remind that in order to prevent overload on the report; only the model version which has been used to obtain the results presented in this study and only sliding mode PID controller version of it will be explained here. Note that since this is an evolutionary process; this is the 7^{th} version and previous 6 versions will not be explained.

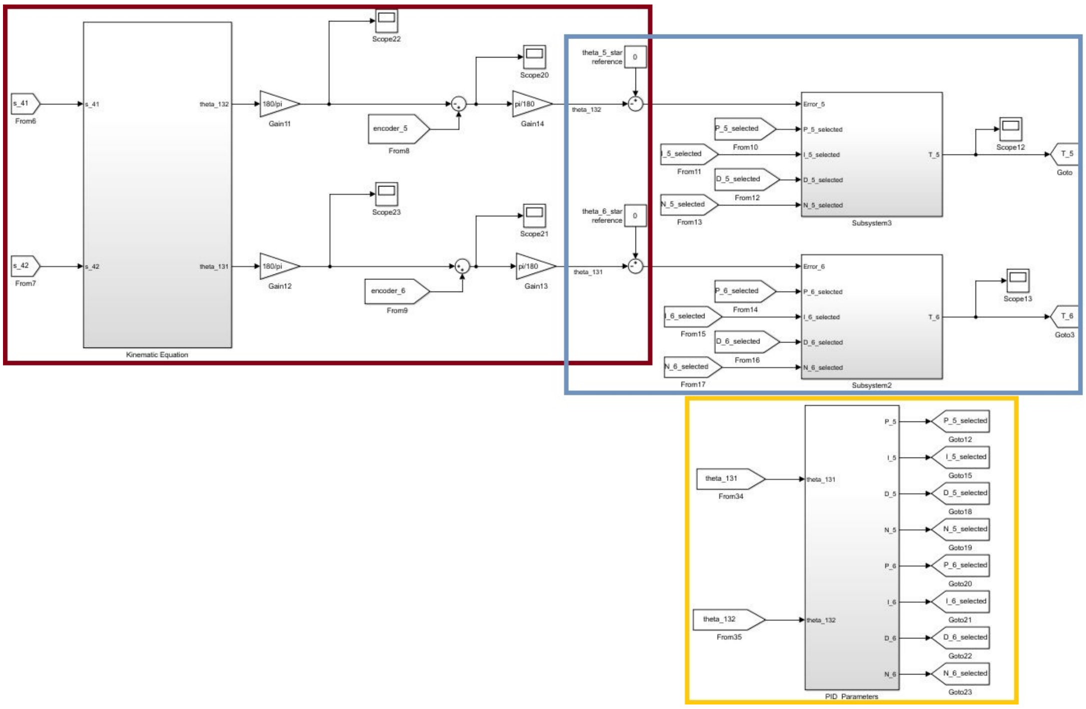


Figure 5.91: Simulink Real Time model of controller

This model will be investigated in three parts. The first part is highlighted with claret red region and is given in Figure 5.92.

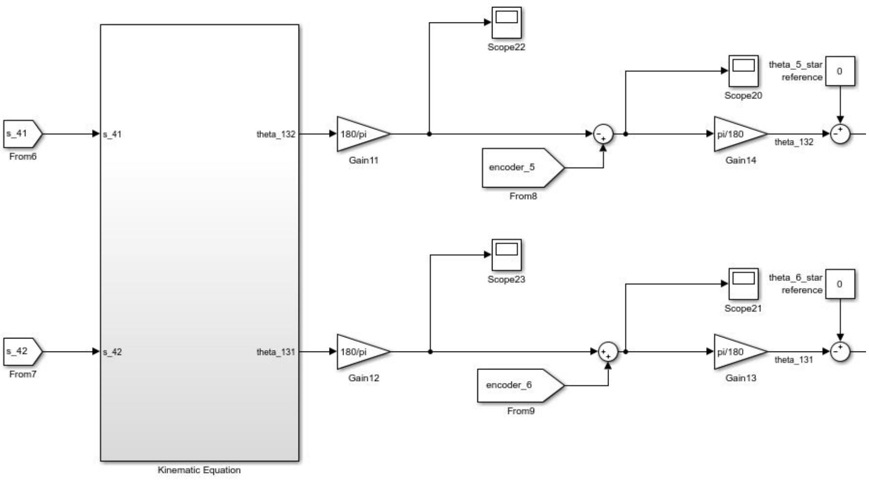


Figure 5.92: Simulink Real Time model of controller - obtaining global platform angles

What we aim to control here is the global platform angles defined with respect to the ground which are labelled as θ_5^* & θ_6^*. Unlike we have done in the first configuration where inclinometers are used to sense these angles directly; this time they cannot be sensed directly but has to be derived. What we have in hand in the second configuration is linear transducers which give the strokes of the cylinders and incremental encoders which sense the change of angle in their own DOF.

First of all, we will start by explaining how we have utilized the linear transducers. Forward kinematics block is used to predict the payload angles θ_{131} & θ_{132} by taking the sensed stroke data s_{41} & s_{42} from the linear transducers of the hydraulic cylinders as inputs. In order to do that we have used jam nuts below the platform base plate to calibrate with the data measured from the inclinometers. Please note that attentive and careful efforts have been dedicated to this procedure prevent any errors. Once we have made sure that both inclinometers and forward kinematics give identical results; we started using this method. When we look under the mask of forward kinematics, we will see the following model given in Figure 5.93.

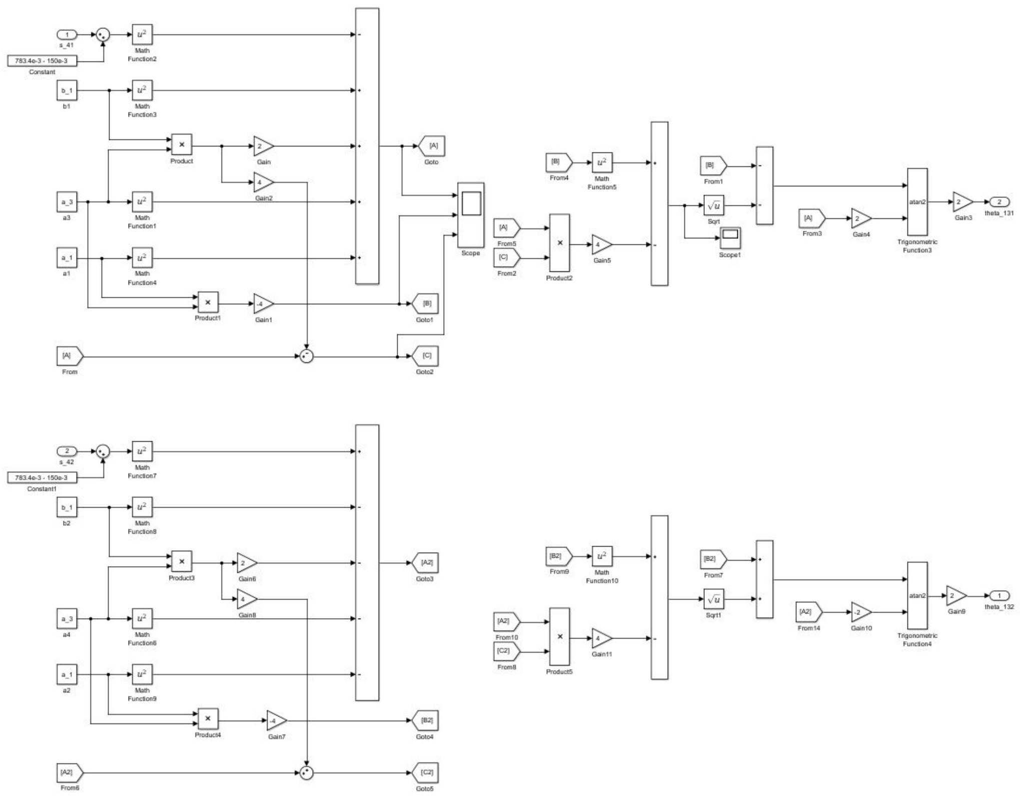


Figure 5.93: Forward kinematics module used in disturber model

Secondly, we will talk about how we have used the data obtained from the encoders. The encoders we have used are incremental type and thus takes the initial orientation before the motion begins as zero reference point. However, we have another tool we can use to make them act as absolute encoders. This tool is the inclinometer which has been cancelled and won't be useful in real time measurements. Nevertheless, before starting the experiments, the data sensed from these sensors are used to stabilize all the global platform angles. Therefore, encoders take the right reference for stabilization. In order to do that both N-E equations derived in section 4.8 or Lagrange equations derived in section 4.9 can be used. When using Lagrange equations, necessary variables are set to zero in order eliminate disturber related parameters.

At this point what we have in hand are payload angles θ_{131} & θ_{132} and local gimbal angles θ_5 & θ_6. But we know how to derive the global platform angles by using definition given in equations (5.2) & (5.3). After we have obtained θ_5^* & θ_6^*; we will give a target value for our controller; which is actually called the reference point. Since we want to keep the platform parallel to the ground this value is set to 0. If desired, the platform can also be kept at a defined orientation other than than zero by changing the reference points.

Next region that will be explained is the region that includes PID controller blocks and is given in Figure 5.94.

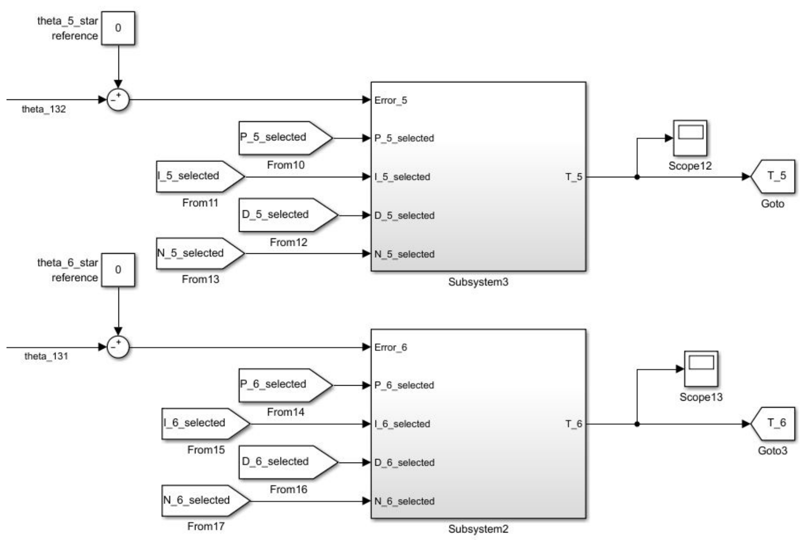


Figure 5.94: Simulink Real Time model of controller - PID controllers

This controller block does exactly what it is supposed to do which is taking in the error in the global orientation angles and generating the necessary torques. Please note that the controller parameters have to be externally fed into the PID blocks. However, this feature is not included in the Matlab version we are using but it is included in the latest version of Matlab. Therefore, we have created our own block which is given in Figure 5.95. Note that like all the PID controllers clamping type anti-winding mechanism has been implemented here too.

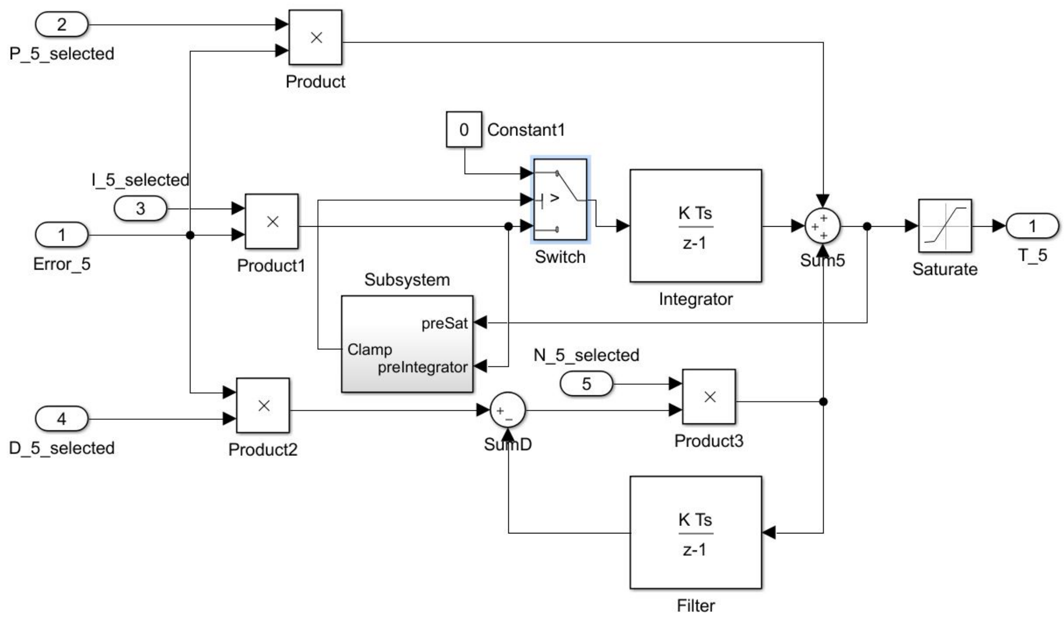


Figure 5.95: Simulink Real Time model of controller - custom made PID controller block

Lastly, in our discussion about the controller region highlighted in navy blue in Figure 5.89, we will talk about selecting convenient parameters from the previously formed parameter library. In order to do that, as seen in Figure 5.96, the library block is created and in Figure 5.97 we will look under the "PID Parameters" block.

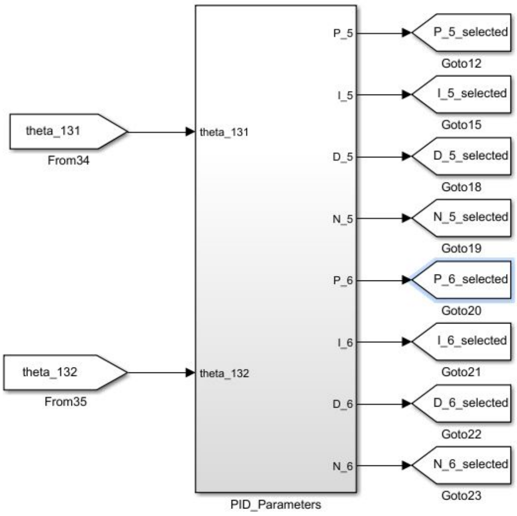


Figure 5.96: Simulink Real Time model of controller - PID Parameters library

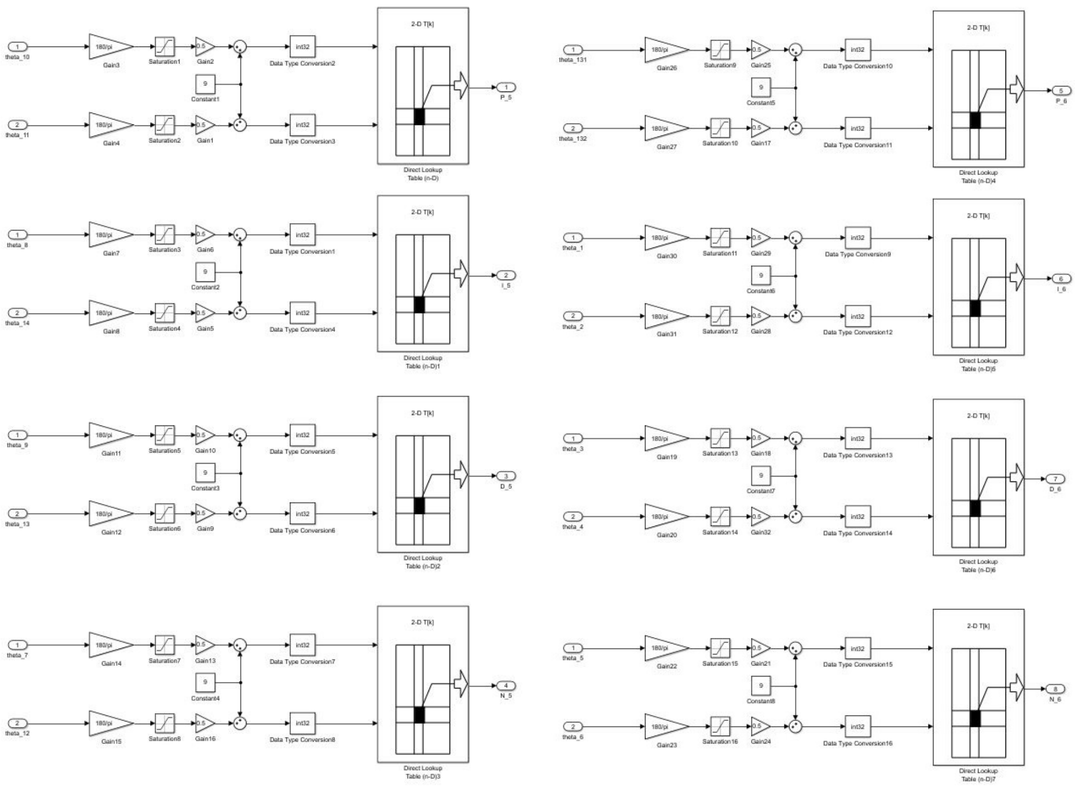


Figure 5.97: Simulink Real Time model of controller - PID Parameters library parameter selection algorithm

That brings us to the part where we will talk about the mechanisms of how we collect feedback data and how we implement the generated output torques and valve openings. This region has been highlighted in green in Figure 5.89 and magnified version is given in Figure 5.98.

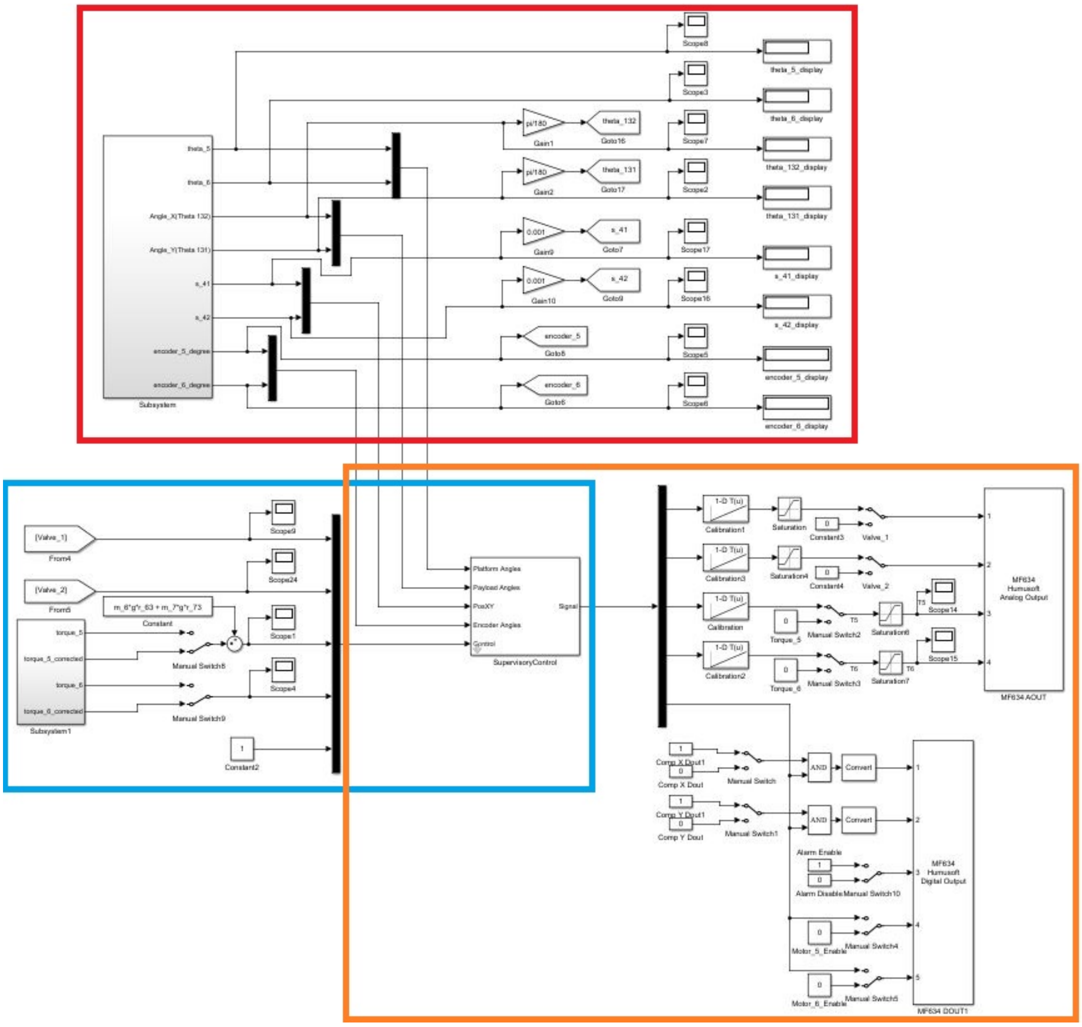


Figure 5.98: Simulink Real Time model of feedback and output implementation

We will start explaining this part from the inner red region. The region is responsible for generating feedback data. In Figure 5.99, the red region is given in detail.

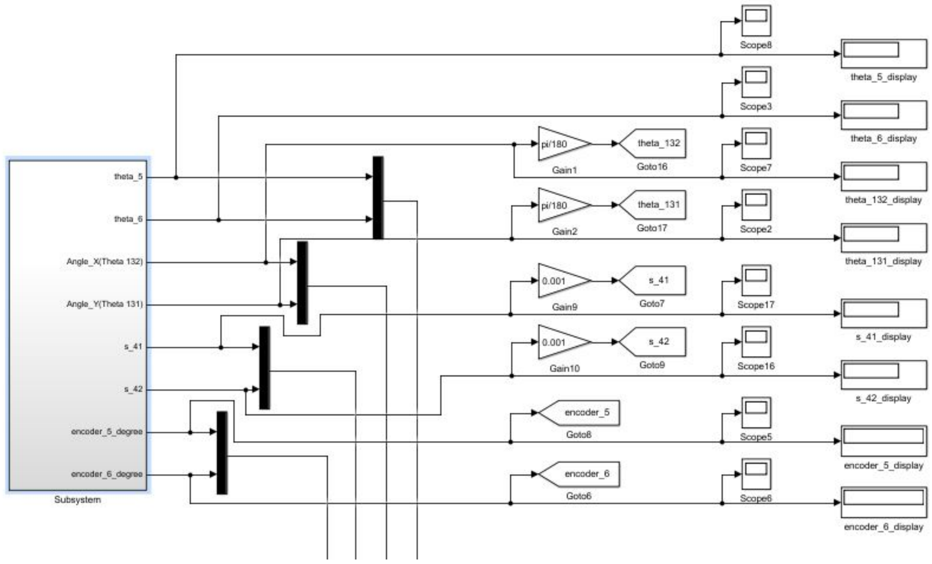


Figure 5.99: Simulink Real Time model of feedback and output implementation - feedback mechanism

If we look under the mask of the subsystem located in the left; we can see the following model given in Figure 5.100.

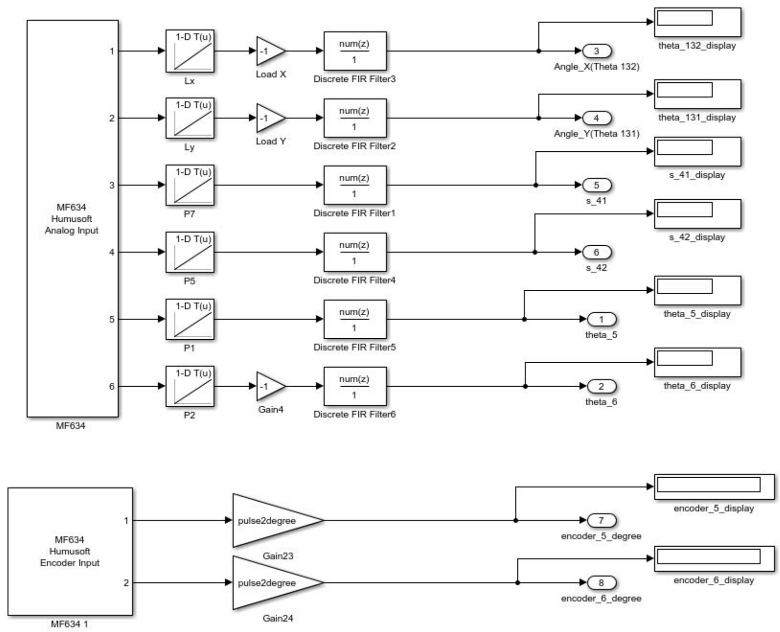


Figure 5.100: Simulink Real Time model of feedback and output implementation - feedback mechanism Humusoft analog input block

Note that signals of Humusoft Analog Input block corresponds to the feedback signals we are using in our controller. The signal orders are as it has been presented in Figure 3.25. First block right after the Humusoft block matches the voltage values with the corresponding minimum and maximum strokes or angles. Then averaging blocks are applied to suppress the sensor noise.

Now we will look at the blue region given in Figure in 5.98. The magnified version is given in Figure 5.101.

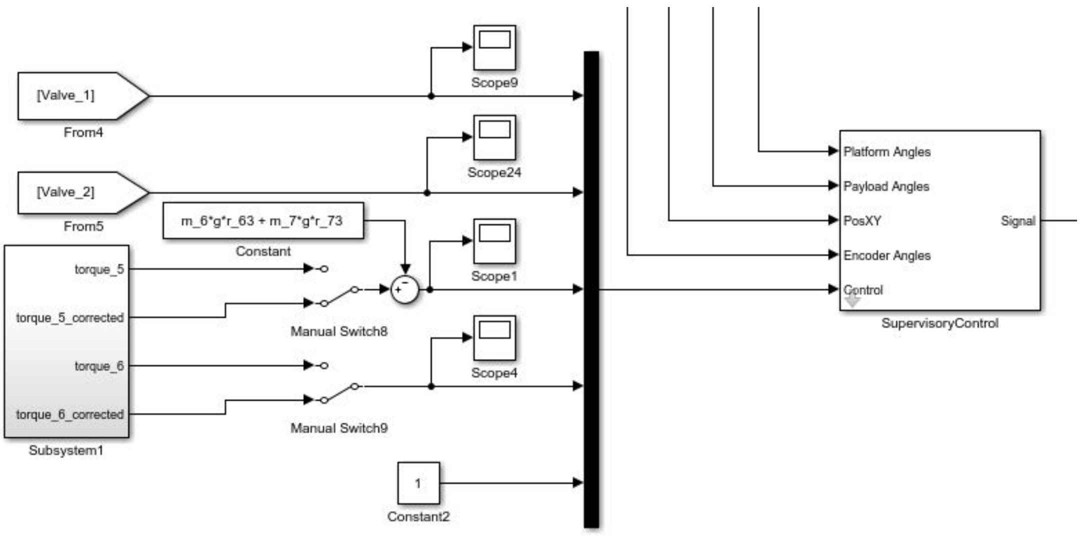


Figure 5.101: Simulink Real Time model of feedback and output implementation - friction compensation and supervisory control blocks

This region has two main tasks. On the left, friction compensation block and on the right, supervisory control block is given. Their look under the mask models are given in figures 5.102 & 5.103.

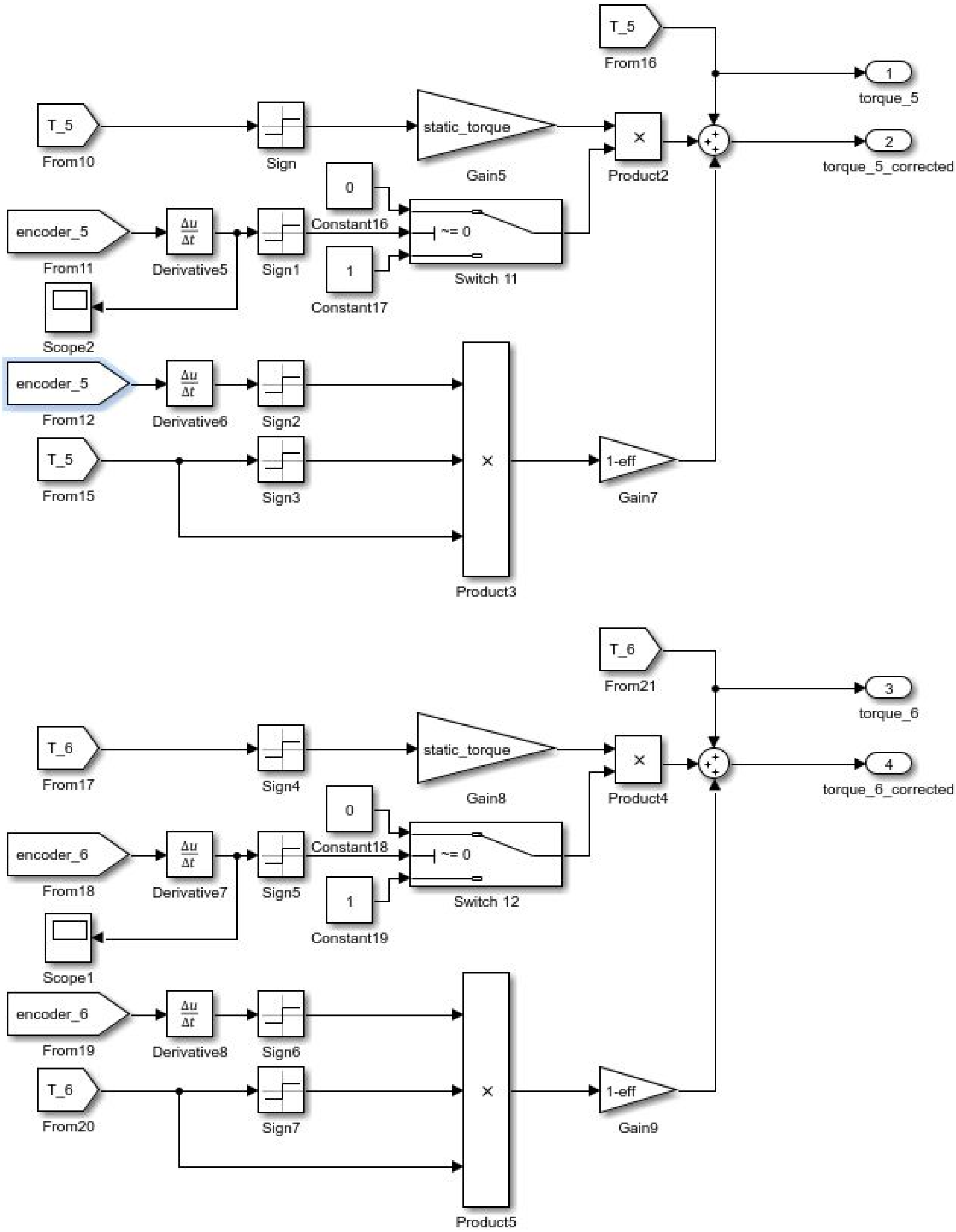


Figure 5.102: Simulink Real Time model of feedback and output implementation - friction compensation block

In the first experiments, we have seen that some of the torque commands which try to stabilize the platform with small interventions are dampened by the static friction coefficient of the system. At that point we have seen that a torque correction module should be created to prevent this from happening again.

This is how this block works. First it tries to understand if the body is trying to accelerate from rest or if it is already in motion. If it is starting its motion from the rest point; then it adds the static friction compensation term which has a static friction torque value found from the experiments.

Nevertheless, in either case this term should also be summed up with the torque loss because of the inefficiency due to the viscous friction. Again the efficiency term of the dynamic system has been found from the experiments. It compares the sign of the actual motion with the generated torque command. If they have the same sign; then necessary torque is added to the to the summation. If they have opposite sides; then this amount of torque is subtracted from the total sum. This is due to the fact that friction works in favour of us when we are trying to decelerate the body first and then change its direction of motion. However, it works against us when we are trying to accelerate the body in its already headed direction.

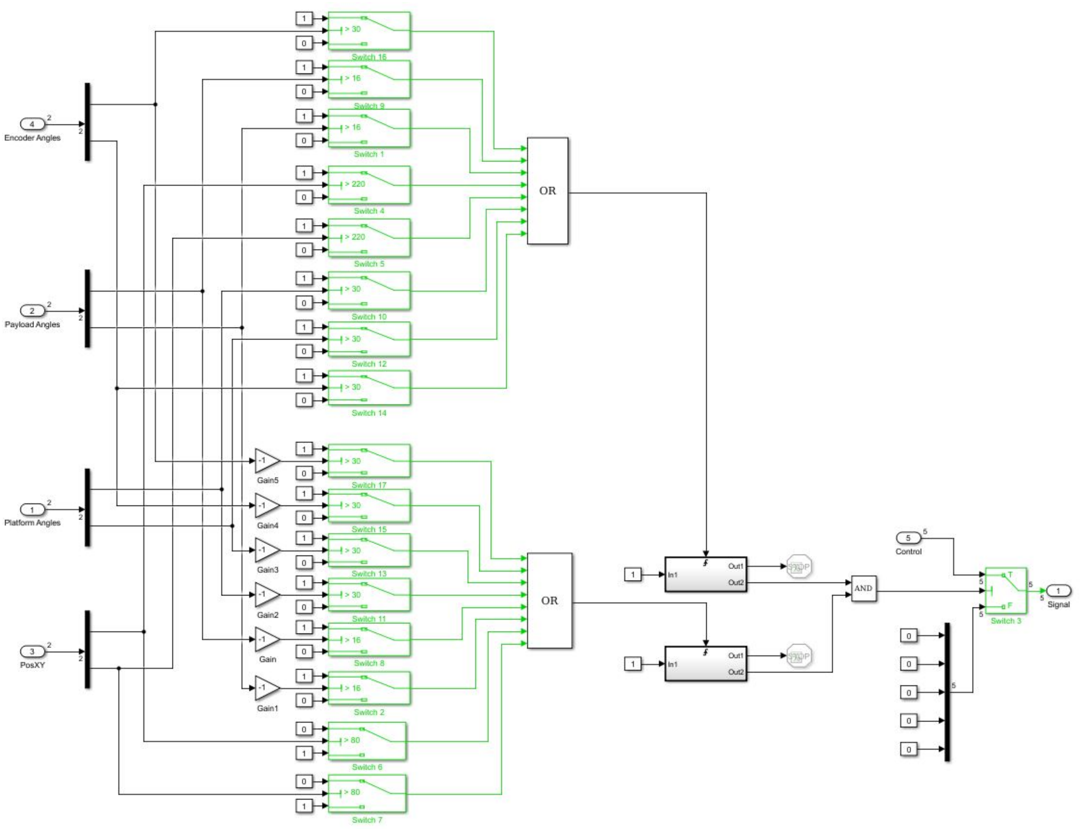


Figure 5.103: Simulink Real Time model of feedback and output implementation - supervisory control block

This block may seem a little complicated at first but it is just a repetition of same logic mechanism. As a whole it cuts off the power from the actuators when predefined safety limits are exceeded. These limits are written to the block itself as both minimum or maximum values. For safety reasons minimum and maximum payload and platform angles as well as minimum and maximum cylinder strokes are monitored in real time before we decide to actuate the motors and valves.

Since the block is a pattern of the same blocks; it will be enough to explain just one of them. For example let's look at one of the platform angles. Defined limits are $\pm 30°$. It takes this commanded value as an input and directs it into the switch blocks. The switch blocks generates 1 if the limit is exceeded. For example if $35°$ is sensed, the switch block compares it with the threshold value of 30 and generate 1 as an output since the limit is exceeded.

Another branch of it goes to another switch block for checking the minimum limit. This time this value is multiplied with -1 and then sent to another switch block. For example if the angle sensed is $-35°$, it will be multiplied with -1 and the switch block will take 35 as an input. Its comparison criteria is greater or equal to the threshold value. It will compare 35 with 30 and again generate 1 since the limit is exceeded again.

Both outputs of these two branches are then sent to an "OR" block. If either of them is 1; it means that the limits are exceeded and actuator commands must immediately be stopped to prevent an accident. This check is being done for all critical parameters. If one of these check signal an alarm, not only the corresponding actuator is stopped but all of the actuators are stopped.

The limits should be set to slightly lower values than the critical since it will take time for an inertial system to dampen all its energy and stop. This block has been very useful during the experiments and saved a lot of efforts and time.

Finally we will talk about the region orange region given in Figure in 5.98. The magnified version is given in Figure 5.104.

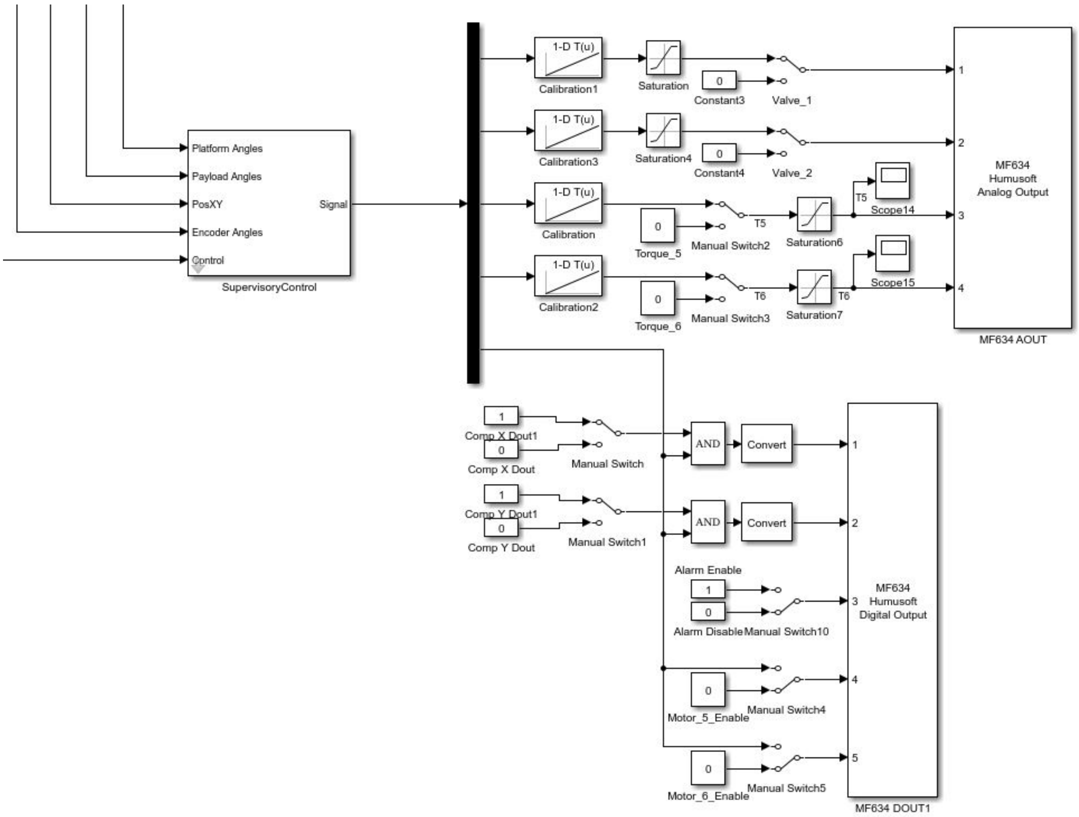


Figure 5.104: Simulink Real Time model of feedback and output implementation - implementation of actuator signals

This part of the model sends the output signals to the corresponding actuators. In order to do that, it has to take a "pass" check from the supervisory control block. Once this check is passed, corresponding voltage value to the generated the output torques or valve openings are calculated by the look up tables. If not, digital output block of Humusoft is not activated and no matter what we have given to the analog output block the actuators are off.

A look up table works simple; it matches minimum and maximum voltages to the minimum and maximum actuator parameters and interpolates if an intermediate value is desired. Since all the actuators behave almost linearly, this method works great.

Then the second block is saturation. If the generated commands are higher than the limits we have set; then these signals are saturated and only available amount of this signal is given to the actuator. Note that this parameter is adjustable by the designer and in the initial phase of the experiments; it has been observed to be a good practice to limit these values until you are sure of the success of your model.

Sometimes, we do not need control for calibration or assembly purposes. In these cases we can give as much as torque or flowrate as we want by simply using manual switches and typing in the desired values.

Finally, we have arrived at the analog output block of Humusoft. As the name suggests, now the generated commands can be given by means of Humusoft DAQ. Again the outputs are ordered according to the distribution given in Figure 3.25.

CHAPTER 6

CONCLUSION AND FUTURE WORK

The experimental set-up we have worked with has been very useful in terms of observing many critical aspects. These aspects can be investigated under different topics.

6.1 Success of the Mathematical Model

6.1.1 Disturber

The first method that has been attempted to obtain the governing equations was using Newton-Euler equations. In order to do that FBD's like the ones given in figures 6.1 & 6.2 are drawn.

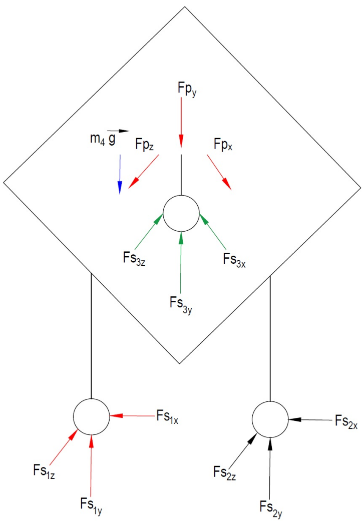


Figure 6.1: Free body diagram of the platform plate

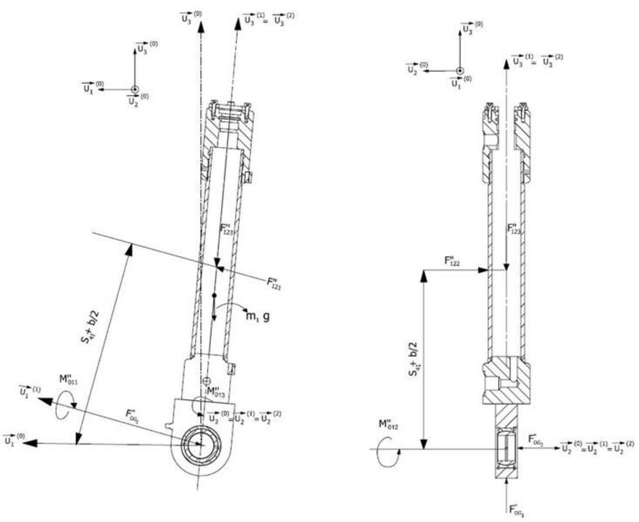


Figure 6.2: Free body diagram of the hydraulic cylinder

After all of the FBD's are drawn; 3 force and 3 moment equations are written for each body. In order to obtain the differential equations for the disturber, this process is repeated for 5 bodies. All these equations are represented in matrix form and all 30 unknowns are isolated by algebraic manipulations.

For control purposes, the inputs are piston acceleration terms and outputs are pressures of the pistons. This is called inverse dynamics.

At the same time, all structural forces can be found when these equations are solved. Nevertheless, they are only required once for design purposes in order to size the machine elements. Therefore, it is important to isolate only desired variables for computational purposes which is the method we have followed.

However, at that point we have faced some problems. One of them was the generated output of the controller was pressure but what we can control through the valves was the flowrate. It has been seen that an equation that will relate these two variables is required. Coupled with continuity equation Bernoulli equation seemed like a good candidate at first. However, it was very hard to reflect different characteristics of many different hydraulic elements on the line so the success we have achieved was limited. Moreover, it has been observed that we were considerably violating some of the major assumptions behind the Bernoulli equation.

We were stubborn to make these long equations work so another method was tried. That method was connecting pressure indicators on the hydraulic cylinders and try to make a correlation through curve fitting between the flowrate and pressure parameters. However, after some time it has been understood that this was another fail attempt since first a lot of noise has been observed on those pressure indicators and second when different scenarios are applied the results were considerably different such that no convincing correlation can be created.

Another problem we have faced is that we could that reflect the forces of the hydraulic hoses. At first it seemed that they could be ignored but then it has been understood that they had to be incorporated into the N-E equations. As it can be guessed it was very hard to model the hose force.

Last problem was prediction of the friction coefficients. Unlike other problems, this time a satisfactory solution could be obtained. The hydraulic cylinders have been extracted and retracted multiple times for an array of actuator speeds. These experiments have been repeated for a considerable number of times and when the end results are examined it has been seen that the variance of the results for each corresponding actuator speed was low. This friction coefficient library has been embedded into simulations for later use.

All in all, it has been understood that this method is not suitable for this type of problem so we had to change the methodology from scratch. When we think again in a white sheet what we need was just changing in the orientation of the payload and it was nothing but a geometry problem. Then, kinematics has been introduced to obtain the loop closure equations and their derivatives. This much simpler approach worked much better since only assumption we made was the rigid body assumption. In this case we do not need to derive complimentary equations, predict unknown forces or obtain a friction coefficient library. Moreover, in the end what the controller predicts can be sent to actuators without any need of change between the parameters. In this case the output was actuator speed and flowrate controlled by the control valve is directly proportional to it.

The conclusion from all this effort was one should always examine carefully what instruments he/she can use for control purposes and model the system accordingly. Another conclusion is one should always aim for the simplest approach that can work since in the end it becomes much easier to debug and optimize it and it generally works better.

There is another discussion about mathematical modelling of disturber. This mechanism has been synthesized so that both actuators work simultaneously all the time to change the orientation of the payload. However, when obtaining N-E or Lagrange equations of the stabilizer successive rotations methodology has been used. This is not exactly the counterpart of actual simultaneous rotation. However, when thinking about this detail; we have realized that both of these modelling and actual world can be synchronized by a simple approach. When using successive rotations approach, transformation matrices are multiplied in rotation order so one of the angles are always prioritized. This logic is applied to physical system through clearance arrangement carefully done during assembly process. In one of the base joints of hydraulic cylinders the clearances are arranged to be zero by carefully management of tolerances of the parts and ability of bronze rings to shrink. The other base joint on the other hand, the base joint of the other cylinder has a clearance about 0.1 - 0.2 mm. This is again done on purpose. By this way, when both of the cylinders are actuated; the first rotation occurs around the perpendicular axis of the rigid joint base. In other words, the actuator which has more clearance initiates the movement. The time gap between these two actuators are almost unrecognisable, in the order of milliseconds, and after initiation simultaneous movement occurs all the time. This small detail has provided us a fully matched actual movement and a successful modelling.

6.1.2 Stabilizer

First of all, we have seen that a decent mathematical model that can represent the actual dynamics of the system can be obtained successfully by using first principles. Although study done for Newton-Euler equations are not included in this thesis, since it has been more than a hundred pages of derivation alone, we have seen that the end results are identical. Some terms seem to be different at first but either they can be converted from one to another or when numerical values are inserted into the equations same result is obtained. Moreover, energy conservation of the system is calculated in real time and in idealized model we have seen that the change in total energy is almost negligible. A very insignificant amount of energy change is seen and that's certainly due to the numerical errors. In figures 6.3 - 6.6, the free motion of the gimbals and total energy of the system at two distinct times as they are moving, can be seen.

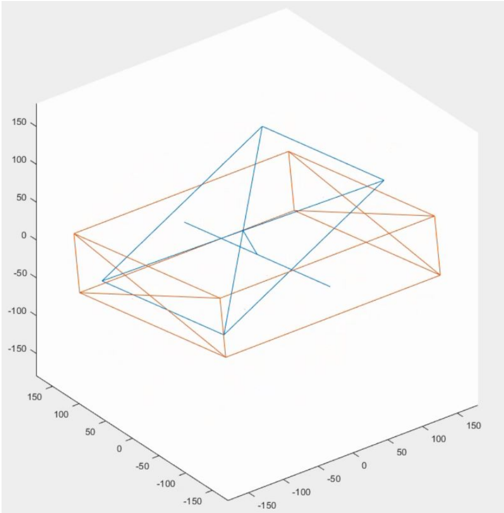


Figure 6.3: Motion of the idealized non-actuated system at t = 4 sec.

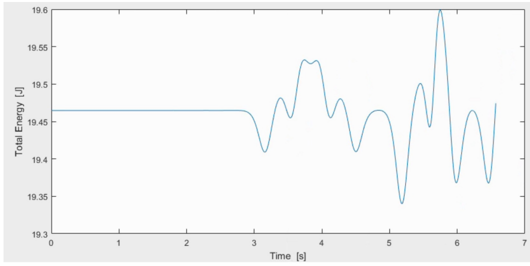


Figure 6.4: Total energy of the idealized non-actuated system at t = 4 sec.

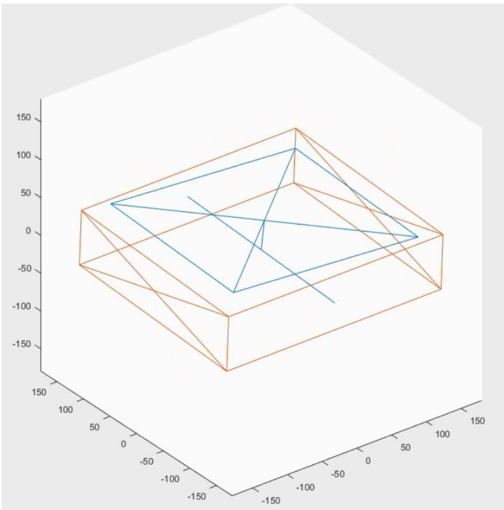


Figure 6.5: Motion of the idealized non-actuated system at t = 8 sec.

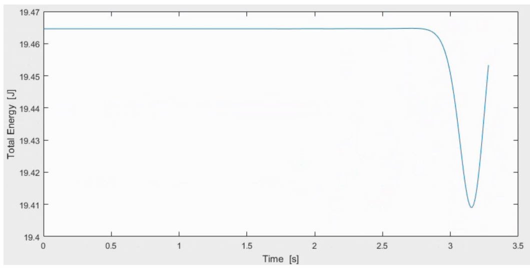


Figure 6.6: Total energy of the idealized non-actuated system at t = 8 sec.

Although all different approaches of first principles have given us the same result and we have observed that the total energy is unchanging; this result has to be consistent with the real life results. In order to do that we'll compare the results of the the simulation results with the test results. But before we do that we'll point out some critical aspects.

Note that the mathematical model derived is an idealized model and before it can be used for the controller it should be linearized so there is some amount of precision loss in that step. Moreover, it does not reflect the effects of sensor noise, backlash and friction.

All these problems will be investigated in different sections but at this point in order to make a statement by comparing test and simulation results, we can point out some important aspects shortly. In current configuration of sensors, stabilizer uses encoders so backlash effects are not included in data collected; however this effect can be seen visually.

For friction part, we have derived versions of the governing equations with frictions. However, friction is a function of velocity and a very hard parameter to guess and even though good approximations have been made through experiments, the non-linear plant model takes hours to be solved which cripples optimization process of the controllers. Therefore, we ended up optimizing the controllers with idealized plant and friction compensation module has been included in real time target version of the Simulink model.

Lastly, in order to measure true success of the simulation model; a sensor noise close enough to the one measured in test have been added as random or white noise blocks. In the light of all these aspects, the comparison of both results are given in Figure . Please note that these graphs have been presented in figures 5.31 & 5.32.

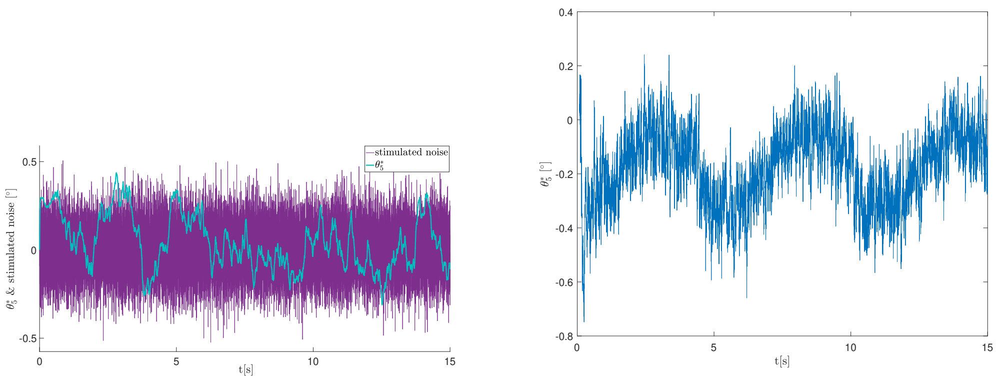


Figure 6.7: Comparison of test and simulation results of LQR

Noise patterns in test environment is not as always the same; they are random. Their frequencies and shapes change but after a series of tests you can have an idea of main frequencies and amplitudes. Therefore, of course it is impossible to reflect the same noise pattern in simulation environment. But when similar or a more noise than expected is given to the simulation; it can be seen that the behaviours of simulation and test results resemble and the plus minus bandwidth of the amplitudes are also in correlation. This behaviour has been observed in the majority of the tests independent of the disturber motion pattern.

When we look at the step responses of both gimbal axes, we can see that settling time is around 0.15-0.2 seconds. There is a steady state error and some noisy wobbling pattern and this detail is going to be discussed in this chapter. Moreover, disregarding a few exception points the stabilization error is generally stuck between in 0.2-0.3 degrees no matter the motion profile is except the short time intervals of impulsive motions. All the motion profiles the stabilizer is subjected is more challenging than the vehicle data sent by Roketsan. Furthermore, even the motion profiles presented in this study has been accelerated up to 5 times; it has been seen that the change in stabilization error is small and the errors are still more or less in this bandwidth. Therefore, it can be confidently said that mathematical modelling has been successful.

6.2 Problems Encountered and Solution Methods

6.2.1 Sensor Malfunctions

From the very beginning of the study, the stabilization logic and methodology that first came to our heads did not change very much. We have thought that using gyroscopes for feedback is the best choice for feedback. However, due to some problems, the schedule and budget of this project has been limited. When delivery times and cost of the IMU's with already embedded sensor fusion and Kalman filter seemed to be unaffordable; as an alternative we have utilized inclinometers for feedback purposes. As a remainder, the locations of the inclinometers have been highlighted in Figure 6.8. The inclinometers in the payload plate or platform base plate serves the purpose of sending the values of payload angles θ_{131} & θ_{132}. On the other hand, the top inclinometer which is located on the platform sends the platform angles θ_5^* & θ_6^* as a feedback.

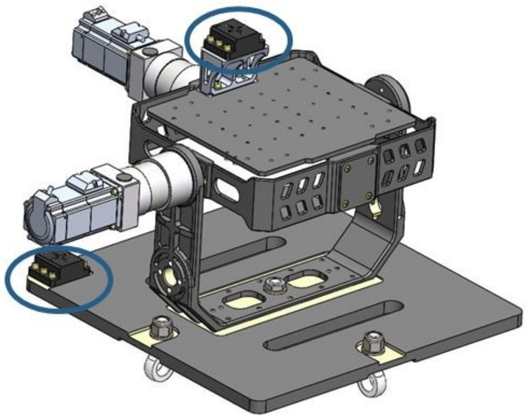


Figure 6.8: Positions of the inclinometers

However, after some time it has been understood that this was a mistake due to late response times and noisy data. Even though the catalogue data values for response time was a challenge, which was claimed to be less than 0.1 seconds, the response time of the actual product we have received came up to be 4 times slower. The response time of the inclinometers we have tried to use, which has also been approved by the manufacturer as a defect, can be seen from Figure 6.9.

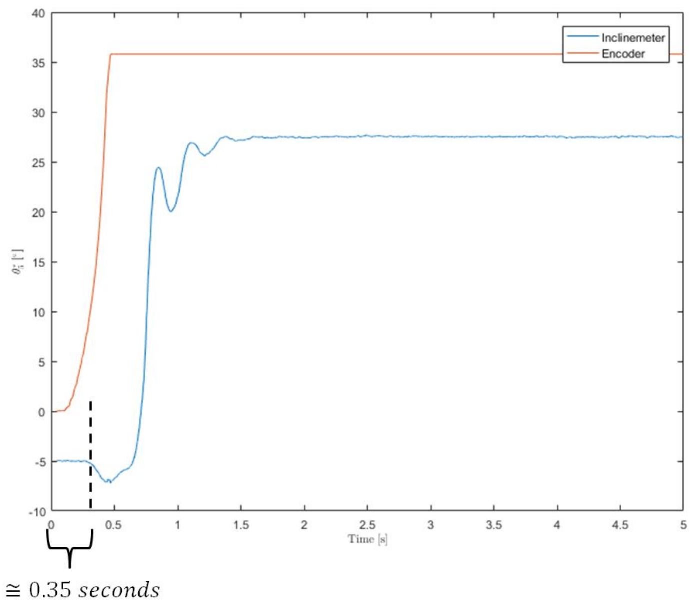


Figure 6.9: Response time of the inclinometer

As a back up plan, we have switched to this configuration. We have decided to use encoders for measuring the platform angles and using linear transducers for obtaining the payload angles. Please note that the angles we are receiving from the encoders are locally inertial, not globally inertial. In other words, the parameter we are measuring is θ_5 & θ_6 rather than the θ_5^* & θ_6^*. The transition between them has been done by means of summing or subtracting the payload angles θ_{131} & θ_{132} with them. Therefore, this data has become a derivative term rather than a directly measured data.

Similarly, when the bottom inclinometer is cancelled, the alternative for this data has been selected to be linear transducers of the hydraulic cylinders which feedback us the strokes. This time, by using forward kinematics payload angles θ_{131} & θ_{132} have been computed. Note that again this has been a derivative data rather than a direct measurement but a fine calibration between the strokes and payload angles have been done before switching to this configuration. As a result of this calibration, we have seen that almost no error exists between the angle measured and the angle calculated by the corresponding actuator strokes. After that this derivative approach has been used safely. Routine checks and recalibration have been made regularly.

Another important point is that although these inclinometers have been cancelled out; they are used before each experiment to globally stabilize the stabilizer platform slowly. This is done because encoders we have used are incremental and as a result of this the stabilizer tries to protect its last position. This approach has helped us to observe the platform to be stabilized globally.

To sum up this section; we have lost very precious time and efforts due to this wrong choice but in the end we have found an alternative solution and concluded this study. As you can guess all the results presented in Chapter 5 are obtained as a result of this configuration. Right now, the configuration is being switched back to the initial configuration with gyroscopes. In order to do that BMI-088 type gyroscopes have been purchased. Current study with them is to get meaningful data from them through mechanisation process with quaternions and sensor fusion and lastly through a Kalman filter.

6.2.2 Backlash and Structural Considerations

Backlash is a clearance issue resulting from the gaps of the mating parts in mechanisms and it results in plays and motion loss. The model of the gearboxes we have selected are called low backlash types and they were supposed to be permitting a backlash of maximum 0.2 degrees.

However, when we make the measurements we have seen that the outer axis can move about 0.6 degrees whereas the clearance in the inner axis is about 0.4 degrees. Please note that the motor brakes were on when we took these measurements. These measurements were also approved by the distributor company technical service and they said that we can return the gearboxes if we want since these gearboxes are defected. Again due to limited time, we had to carry on with these gearboxes to conclude this study.

In the first configuration of sensor feedback arrangement, this play has caused tremendous problems with data collection. In physical observation it has always caused undesired vibrations.

Since we are currently switching to gyroscope configuration, it will certainly have a diminishing effect on the quality of the sensor data. Therefore, before we switch to gyroscopes for feedback, we have to replace these with non-backlash types and also introduce a zero backlash mechanism which connects every parent and child body with appropriate preloaded springs. These springs can either be a simple tension or compression spring according to the design method or a torsion spring. Again these tension or compression springs can be arranged such that they can create moments in the mating part of the parent and child body. Alternatively, a combination of torsion springs can do the same job. Also, as another alternative tension springs can be directly mounted between the two parent and child bodies.

In the initial experiments when we were trying to stabilize the platform with inclinometers, we have observed a ladder effect on the collected data. This ladder effect can be seen in Figure 6.11 and it points out an interesting result worth to discuss about.

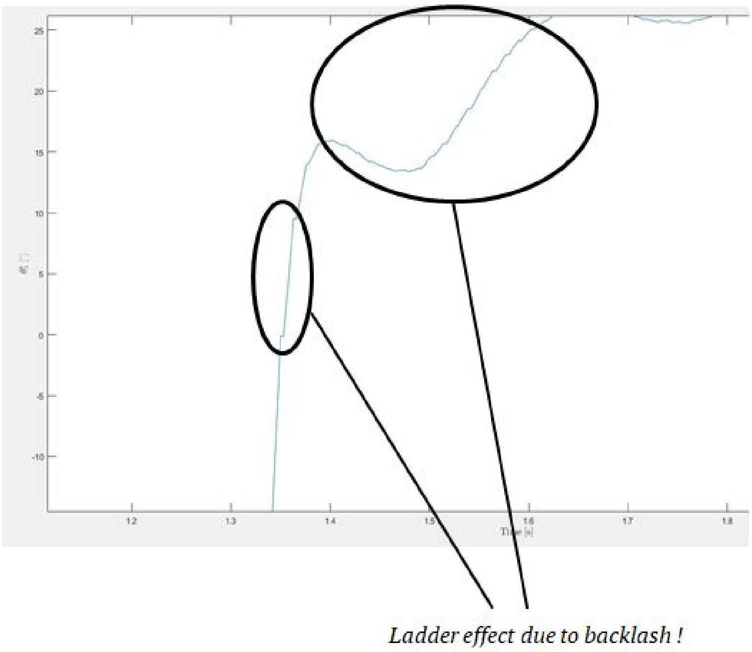


Figure 6.10: Demonstration of ladder effect on sensor readings due to vibration and backlash

When we inspect on the test result carefully, we can see that between each reading a step like pattern exists. In this study, it has been named as ladder effect. If we divide the jumps between each step to the time intervals; it can be predicted that the system resonates around 80 Hz. This is a critical vibrational mode caused both by the lack of stiffness of the mechanical design and extreme effect of unexpected backlash. This has led to a future work for us to both eliminate backlash and design a much stiffer value.

There is another aspect to this discussion. The center of gravity of the outer gimbal is not balanced on purpose to add extra challenge to the controllers and see if we can deal with this. The test results indicate that we have managed this effect decently; however physically observed vibration issue proves us the opposite. The offset between the geometric center of the platform and the center of gravity creates a beam effect and coupled with backlash it creates this vibratory motion. In addition to this, as it has been stated before, the stiffness of the structure seems to be not good enough to prevent this effect entirely or dampen it more than the current situation.

6.2.3 Sensor Noise

Noise is a random disturbance on a signal. When you are dealing with sensors it is unavoidable. The sources of the noise are environment its operating in, implementation and manufacturing defects. For the environmental source aspect, it can be naturally occurring noise or man made noise. It can exist at very specific frequencies or can spread out across the spectrum.

For some control laws, as long as the noise is low amplitude, it won't impact the system much. But that's not true for ideal PID controller because it has a pure derivative. Derivatives amplify high frequency signals and take tiny wiggles and amplify them to values that can impact the system. If we leave really high frequency noise in the system, the derivative path will see that noise even it is really small. One method to cope with this problem is to lower the amplitude of the high frequency noise since it produces a steeper slope at same amplitude compared to a low frequency noise. In order to do that, a filter that will block frequencies above a certain point from entering our derivative and causing problems has been designed. This certain point is called the cutoff frequency and where to place it is designer's choice. That's why, in our PID controller designs we have always used the filtered version and have introduced the N parameter.

Moreover, the since the mathematical model includes a variety of derivatives such angular velocities and accelerations, the derivative block has been used extensively. If a term is related to a sensor and its derivative is going to be taken, it causes very serious problems.

The noise problem has been observed in touchpad, linear transducers and inclinometers and we certainly expect noisy data from the gyroscopes we are currently switching into. This problem is non-existent with the encoders since we are counting digital pulses to calculate the rotation speed. In order to demonstrate the noisy data, Figure 6.11 has been presented as follows:

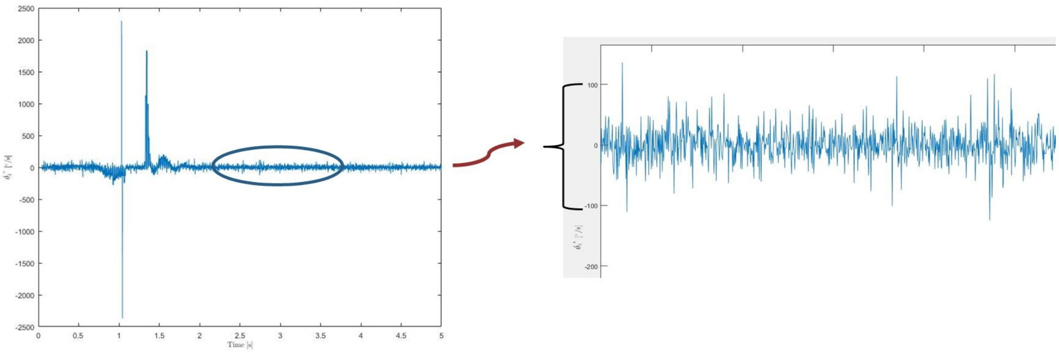


Figure 6.11: Demonstration of ladder effect on sensor readings due to vibration and backlash

In order to cope with this problem two methodologies are followed. First one is simple and effective but comes with a payback. It is called averaging and in our case it averages last 20 data points to reduce the variance. It has been very helpful but it causes some delay. However since the sampling time we have used is 5.10^{-4}; its delay effect has been negligible. However, it has been observed that due to this significantly small sampling time of the real time controller; these errors amplified due to nature of derivative operations.

The next method we have tried is filtering. Ideally, what we want to achieve is filtering all of the noise and perfectly pass the entire signal; however, this isn't something that can be achieved in practice. Based on the assumption that for a lot of applications the noise across the spectrum is relatively low amplitude or low power and the signal we are interested in keeping has comparatively large power and low frequency; a first order low pass filter design seems reasonable. In this case low frequency noise doesn't really impact the derivative much if the amplitude is small and therefore we can often remove all of the bothersome noise to our derivative simply by blocking or attenuating just the high frequency information. By the way, low pass filter is exactly what we applied within all of the PID controllers.

Low pass filter allow frequencies below the cut-off point to pass mostly unchanged. After that range it applies staged and increasing blocking as the frequencies of the signal increase. It does not remove the noise entirely, it just makes it smaller so that even after we amplify it through the derivative it won't impact our system much. The key aspect of the filter is deciding on where to put the cut-off frequency so that maximum high frequency noise is removed without impinging on the frequencies that are actually in the signal that gives us valuable information.

In order to do that, firstly Fourier transform of the data has been obtained and frequencies and their corresponding amplitudes have been examined. It has been seen that the noise is in the form of white noise and in order to diminish its effects; a filter block of Simulink has been included in the model. In the early stages of experimentation, when inclinometers were still not cancelled; it seemed reasonable to set the cut-off frequency about 80 Hz. By this way; it is aimed to remove unwanted signals around that region which is detected by Fourier transform and also diminish the effects of resonant data due to backlash and vibration. However, this filter has done more harm than good to the results. It has been seen that it has filtered some of the useful data we need in addition to the unwanted ones. When the cut-off frequency is lowered further; the noise seems to diminish but the payback was; we were losing more useful data and as a result of this trade-off we could not improve the stabilization results. More and more trial and errors have been implemented by looking at the Fourier transform data but the result was the same and in the end this method was abandoned. The conclusion from these efforts was it might be lack of technical skills in signal processing and filtering rather than the method itself that prevent us benefiting from this approach. This item has been also noted as a future work and it requires more time and efforts for best utilization.

There is one last topic worth to be discussed. The topic is about the vibratory wobbling results seen in step response graphs. This is also another indicator of the noise problem we are facing. Since only feedbacks are obtained from the encoders and linear transducers when these results are obtained; the main candidate of the noise is linear transducers. This is not surprising at all since we are aware of the fact that data obtained from the transducers is considerably noisy.

Another reason for this problem might be due to a systematic noise problem and in this case it will be much harder to solve. We have made simple checks such as whether the main plug is grounded or not or whether appropriate connections are grounded together or not. These simple check do not point a problem. Moreover, signals have been measured in the inputs and outputs of the DAQ too; and this is not the problem either. In addition to these all the cabling is done by high quality material and they are all shielded. The cable lengths are kept at minimum too. However, the problem might be much more technical. For instance, the layout of all the hardware must be carefully inspected and effects of each component on its surroundings has to be observed carefully. To do that, a careful inspection has to be done from scratch till the end and all rack cabinet layout might be doomed to be reshaped. This point gives birth to another future work item.

Finally, since we are currently switching to the gyroscope configuration now; this problem has to solved satisfactorily once and for all.

6.3 Future Work

Since the aim of this study is to design a family of stabilizers, this study is going to carry on. The simple design solution used in this study is an on purpose act to gather know how with as simple as possible equipment and design. The future work planned will be explained separately both for the existing design and the new prototype that will be designed. R & D studies will continue with the existing prototype after some improvements and at the same time a second prototype will be designed.

6.3.1 Existing Design

For the existing design, first of all both of the gearboxes will be replaced with low or zero backlash ones. Then, as a secondary precaution a zero-backlash mechanism will be introduced. The modal analysis will be applied and the stiffness of the frames will be increased with addition of modular parts to the existing bodies.

Nextly, the procedure of obtaining meaningful data from the newly purchased gyroscopes will be completed.

Moreover, the source of noises will be investigated carefully and necessary precautions will be taken to improve the performance. The precautions may lead up to a new layout design for the rack cabinet and replacement of necessary equipment and hardware.

In addition to these improvements; filter designs and signal processing will be studied extensively and successful filters will be designed and incorporated into existing models.

Existing controller designs will be optimized and improved and many other controller strategies will be developed. In order to have a good idea of the success of these improvements; experiments will be repeated after each step for comparison.

6.3.2 New Prototype

Since next prototype will act as a transition prototype to the mass production; it will be designed as close as to the actual product. This means that all of the design will change from scratch and second prototype will have very little resemblance or no resemblance at all with the current one.

The first change in the new design will be about actuation mechanism. Depending on the torque requirements, direct drive torque motors or frameless motors coupled with harmonic drives is considered to be used. Therefore, a no clearance design will be obtained provided the fact that other mating bodies can be obtained with no clearance if desired; by means of careful tolerancing, good knowledge on machining and coating. If any undesired clearance or play is existent, the design will be supported zero-backlash mechanisms.

Moreover, encoders will be located on the opposite side of actuating interface along with a torque meter to form a cascaded control infrastructure. Both structural and modal analysis studies will be carefully done in order to prevent any deflection issue which will hinder rigid body assumption and any undesired vibratory behaviour.

Since this prototype is going to be tested according to military standards, new generation design will have necessary precautions in terms of EMI / EMF considerations and sealing.

More importantly, extra care will be given to manage the center of mass locations and distribution of inertia terms in the inertia tensor. Additionally, the actuation axes will be designed such that its axis will be collinear with the center of mass of the child assembly. Thus, a naturally balanced, self stabilizing design can be created which will simplify the task of controllers significantly. Furthermore, careful management of center of mass and inertia tensors will greatly simplify the mathematical modelling.

The sensors and especially the gyroscopes will be selected to a high quality type compatible with military standards. If it has embedded proven filters, sensor fusion and Kalman filter, then it might have a higher probability to be used in our system.

Final revision is expected to be about the degree of freedoms. Procedure developed in 2 DOF design will be extended to 3 DOF design for better stabilization possibility.

6.4 Conclusion

When we compare the test results of approximately 0.2 - 0.3 degrees error bandwidth in without impulse motions and 0.2 seconds settling time with the design criteria given in section 1.3; in short we can say that the study is successful in spite of the very tight schedule, budget problems, unexpected problems encountered due to COTS items and mentioned in this report and mainly one man staff dedicated to this study. Please note that in very challenging motion profiles with impulsive motion patterns the stabilization error may peak up to the range in between 0.5 - 1 degrees for a very short time and comes back into the error bandwidth of 0.2 - 0.3 again in very short time intervals.

The aim of this study was to obtain the know-how from scratch to design a family of stabilizers. In addition to the design criteria met; the study has accomplished the task in that sense too. At this point we have a good idea of the sources of the problems and how to solve them. These details has been explained in section 6.3. This means that the next prototype is certainly going to be much more successful than the current prototype. This is a great thing considering the fact that even the results of this current prototype can compete with some of the rival equivalent products before the improvements planned to be done are applied. To sum up, this study has been very beneficial as a joint study of industry and academy and the future of the project is very promising.

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Appendix A

BALL AND BEAM THEORETICAL BACKGROUND

A.1 Ball on a Plate System

The schematic representation of a ball on a plate system can be found as follows:

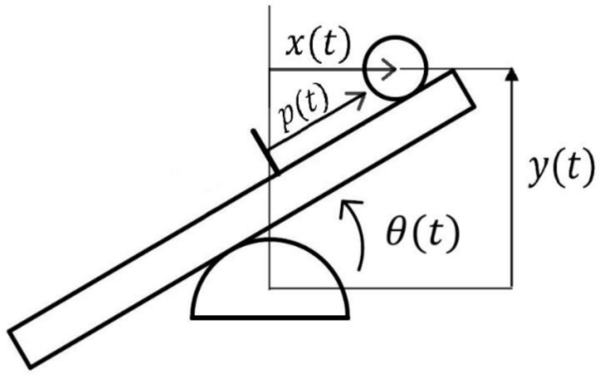


Figure A.6: Schematic representation of ball on a plate system

The ball rolls on the beam without slipping under the action of the force of gravity. The beam is tilted from an external torque to control the position of the ball on the beam. First, a set of generalized coordinates need to be defined in a way that the system could be fully described. The generalized coordinates are defined as:

$$q(t) = \begin{bmatrix} p(t) \\ \theta(t) \end{bmatrix} \tag{A.1}$$

The Lagrangian of the system is defined as

$$L = K - U \tag{A.2}$$

where K is the kinetic energy and U is the potential energy. The kinetic energy of the beam is:

$$K_1 = \frac{1}{2}J\dot{\theta}^2 \tag{A.3}$$

The kinetic energy of the ball is:

$$K_2 = \frac{1}{2}J_b\dot{\theta}_b^2 + m\vartheta_b^2 \tag{A.4}$$

where $\dot{\theta}_b$ and ϑ_b are the angular and linear velocities of the ball, respectively.

$$\dot{\theta}_b = \frac{\dot{\vartheta}_p}{r} \tag{A.5}$$

$$\vartheta_b^2 = \dot{x}^2 + \dot{y}^2 \tag{A.6}$$

$$x = pcos\theta \tag{A.7}$$

$$\dot{x} = \dot{p}cos\theta - p\dot{\theta}sin\theta \tag{A.8}$$

$$\dot{x}^2 = \dot{p}^2cos^2\theta - 2\dot{p}p\dot{\theta}sin\theta cos\theta + p^2\theta^2sin^2\theta \tag{A.9}$$

$$y = psin\theta \tag{A.10}$$

$$\dot{y} = \dot{p}sin\theta + p\dot{\theta}cos\theta \tag{A.11}$$

$$\dot{y}^2 = \dot{p}^2sin^2\theta + 2\dot{p}p\dot{\theta}\theta sin\theta cos\theta + p^2\dot{\theta}^2cos^2\theta \tag{A.12}$$

By substituting (A.9) and (A.12) into (A.6):

$$\vartheta_b^2 = \dot{p}^2cos^2\theta - 2\dot{p}p\dot{\theta}sin\theta cos\theta + p^2\dot{\theta}^2sin^2\theta + \dot{p}^2sin^2\theta \\ +2\dot{p}p\dot{\theta}sin\theta cos\theta + p^2\dot{\theta}^2cos^2\theta \tag{A.13}$$

which gives:

$$\vartheta_b^2 = \dot{p}^2 + p^2\dot{\theta}^2 \tag{A.14}$$

Using (A.4) for the kinetic energy,

$$K_2 = \frac{1}{2}J_b\frac{p^2}{r^2} + \frac{1}{2}m(\dot{p}^2 + p^2\dot{\theta}^2) \tag{A.15}$$

$$K_2 = \frac{1}{2}(\frac{J_b}{r^2} + m)\dot{p}^2 + \frac{1}{2}mp^2\dot{\theta}^2 \tag{A.16}$$

The potential energy of the system is given by:

$$U = mgpsin\theta \tag{A.17}$$

Substituting (A.3), (A.16) and (A.17) into (A.2) gives:

$$L = \frac{1}{2}(\frac{J_b}{r^2} + m)\dot{p}^2 + \frac{1}{2}(mp^2 + J_b)\dot{\theta} - mgpsin\theta \tag{A.18}$$

The first Lagrange equation is given by:

$$\frac{d}{dt} = \left(\frac{\partial L}{\partial \dot{p}}\right) = 0 \tag{A.19}$$

Now we proceed to compute this equation step by step:

$$\frac{\partial L}{\partial \dot{p}} = \left(\frac{J_b}{r^2} + m\right)\dot{p} \tag{A.20}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{p}} = \left(\frac{J_b}{r^2} + m\right)\ddot{p} \tag{A.21}$$

$$\frac{\partial L}{\partial p} = mp\dot{\theta}^2 - mgsin\theta \tag{A.22}$$

Substituting (A.20)-(A.22) into (A.19), the first equation of motion of the ball and beam system is derived:

$$\left(\frac{J_b}{r_2} + m\right)\ddot{p} + mgpsin\theta - mp\dot{\theta}^2 = 0 \tag{A.23}$$

The second Lagrange equation is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau \tag{A.24}$$

where τ is the external torque applied to the beam. We drive this equation using a similar approach.

$$\frac{\partial L}{\partial \dot{\theta}} = (mp^2 + J)\dot{\theta} \tag{A.25}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = (mp^2 + J)\dot{\theta} + 2mp\dot{p}\dot{\theta} \tag{A.26}$$

$$\frac{\partial L}{\partial \theta} = -mppcos\theta \tag{A.27}$$

Substituting (A.26) and (A.27) into (A.24), we can obtain the second equation of motion for the bell and beam:

$$(mp^2 + J)\ddot{\theta} + 2mp\dot{p}\dot{\theta} + mgpcos\theta = \tau \tag{A.28}$$

We will now proceed to derive the equations of motion for Bell and Beam system using Newtonian mechanics. X axis is defined to be parallel to the beam. Since the axis rotates with time, the derivatives of the unit vectors need to be also considered while calculating the velocities and accelerations. At first some theoretical background information will be given so that it would be easier to follow the derivations.

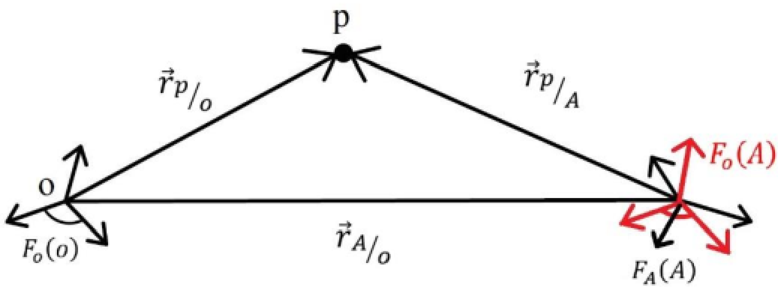


Figure A.7: Position of a point viewed from different coordinate frames

$F_o(A)$ is just translated without any rotation, while $F_a(A)$ is both translated and rotated at the same time.
$\vec{r}_{p/O}$: Position vector of the point p with respect to point O
Position Relationship:

$$\vec{r}_{p/O} = \vec{r}_{p/A} + \vec{r}_{A/O} \tag{A.29}$$

Velocity Relationship:
Differentiate in either F_0 or F_a say F_a:

$$D_0\vec{r}_{p/O} = D_0\vec{r}_{p/A} + D_0\vec{r}_{A/O} \tag{A.30}$$

Definition of Velocity:
$\vec{V}_{p/A/F_0}$: Velocity of p with respect to point A with respect to frame F_0.
This definition requires three items:

1. Point of interest, i.e. the point whose motion we are trying to obtain: p
2. Point of reference: A
3. Frame of differentiation: F_0

Therefore, using (A.30):

$$\vec{V}_{p/O/F_0} = \vec{V}_{p/A/F_0} + \vec{V}_{A/O/F_0} \tag{A.31}$$

$$D_0\vec{r}_{p/O} = D_a\vec{r}_{p/A} + \vec{\omega}_{a/O} \times \vec{r}_{p/A} \tag{A.32}$$

$$D_0\vec{r}_{p/O} = D_a\vec{r}_{p/A} + \vec{\omega}_{a/O} \times \vec{r}_{p/A} \tag{A.33}$$

Inserting (A.33) into (A.31):

$$\vec{V}_{p/O/F_0} = \vec{V}_{p/A/F_a} + \vec{\omega}_{a/O} \times \vec{r}_{p/A} + \vec{V}_{A/O/F_0} \tag{A.34}$$

Alternative Notation:

$\vec{V}_{p/A/F_0} = \vec{V}_{p/F_0(A)}$: Relative velocity of p w.r.t. the frame Fo(A)

$$\vec{V}_{p/F_0(O)} = \vec{V}_{p/F_a(A)} + \vec{\omega}_{a/O} \times \vec{r}_{p/A} + \vec{V}_{A/F_0(O)} \tag{A.35}$$

Acceleration Relationships:

$$D_0^2\vec{r}_{p/A} = \vec{a}_{p/A/F_0} = \vec{a}_{p/F_0(A)} \tag{A.36}$$

Differentiate the velocity equation first in F_0:

$$D_0(\vec{V}_{p/O/F_0}) = D_0(\vec{V}_{p/A/F_a}) + D_0(\vec{\omega}_{a/O} \times \vec{r}_{p/A}) + D_0(\vec{V}_{A/O/F_0}) \tag{A.37}$$

$$D_0(\vec{V}_{p/A/F_a}) = D_a(\vec{V}_{p/A/F_a}) + \vec{\omega}_{a/O} + \vec{V}_{p/A/F_a} \tag{A.38}$$

$$D_0(\vec{\omega}_{a/O} \times \vec{r}_{p/A}) = D_a(\vec{\omega}_{a/O} \times \vec{r}_{p/A}) + \vec{\omega}_{a/O} \times (\vec{\omega}_{a/O} \times \vec{r}_{p/A}) \tag{A.39}$$

$$D_a(\vec{\omega}_{a/O} \times \vec{r}_{p/A}) = \underbrace{D_a(\vec{\omega}_{a/O})}_{\vec{\alpha}_{a/O/F_a}} \times \vec{r}_{p/A} + \vec{\omega}_{a/O} \times \underbrace{D_a(\vec{r}_{p/A})}_{\vec{V}_{p/A/F_a}} \tag{A.40}$$

Insert (A.40) into (A.39) to get:

$$D_0(\vec{\omega}_{a/O} \times \vec{r}_{p/A}) = \vec{\alpha}_{a/O} + \vec{r}_{p/A} + \vec{\omega}_{a/O} \times \vec{V}_{p/A/F_a} + \vec{\omega}_{a/O} \times (\vec{\omega}_{a/O} \times \vec{r}_{p/A}) \tag{A.41}$$

And rewrite (A.10) to get:

$$D_0(\vec{V}_{p/A/F_a}) = \vec{a}_{p/A/F_a} + \vec{\omega}_{a/O} \times \vec{V}_{p/A/F_a} \tag{A.42}$$

Combining (A.9), (A.13) and (14) results:

$$\begin{aligned}\vec{a}_{p/O/F_0} = {} & \vec{a}_{p/A/F_a} + \vec{\omega}_{a/O} \times \vec{V}_{p/A/F_a} + \vec{\omega}_{a/O} \times \vec{r}_{p/A} + \vec{\omega}_{a/O} \times \vec{V}_{p/A/F_a} \\ & + \vec{\omega}_{a/O} \times (\vec{\omega}_{a/O} \times \vec{r}_{p/A} + \vec{a}_{A/O/F_0})\end{aligned} \tag{A.43}$$

$$\begin{aligned}\vec{a}_{p/O/F_0} = {} & \vec{a}_{p/A/F_a} + 2\,\vec{\omega}_{a/O} \times \vec{V}_{p/A/F_a} + \vec{\omega}_{a/O} \times \vec{r}_{p/A} + \vec{\omega}_{a/O} \times \vec{V}_{p/A/F_a} \\ & + \vec{\omega}_{a/O} \times (\vec{\omega}_{a/O} \times (\vec{\omega}_{a/O} \times \vec{r}_{p/A}))\end{aligned} \tag{A.44}$$

We can also write:

$$\vec{a}_p/F_0(O) = \vec{a}_p/F_a(A) + 2\,\vec{\omega}_{a/O} \times \vec{V}_p/F_a(A) + \vec{\alpha}_{a/O} \times \vec{r}_{p/A} + \vec{\omega}_{a/O} \times (\vec{\omega}_{a/O} \times \vec{r}_{p/A}) + \vec{a}_A/F_0(O) \quad \text{(A.45)}$$

$\vec{a}_p/F_0(O)$: Relative acceleration of p w.r.t. $F_0(O)$

$\vec{a}_p/F_a(A)$: Relative acceleration of p w.r.t. $F_a(A)$

Simplified rotation when there are only two reference frames is involved:

$\vec{r} = \vec{r}_{p/O}$	$\vec{r}' = \vec{r}'_{p/A}$	$\vec{r}^0 = \vec{r}^0_{A/O}$
$\vec{V} = \vec{V}_p/F_0(O)$,	$\vec{V}' = \vec{V}_p/F_a(A)$,	$\vec{V}^0 = \vec{V}_A/F_0(O)$
$\vec{a} = \vec{a}_p/F_0(O)$,	$\vec{a}' = \vec{a}_p/F_a(A)$,	$\vec{a}^0 = \vec{a}_A/F_0(O)$
$\vec{\omega} = \vec{\omega}_{a/O}$,	$\vec{\alpha} = \vec{\alpha}_{a/O}$	

Therefore,

$$\vec{r} = \vec{r}' + \vec{r}^o \quad \text{(A.46)}$$

$$\vec{V} = \vec{V}' + +\vec{\omega} \times \vec{r}' + \vec{V}^0 \quad \text{(A.47)}$$

$$\vec{a} = \vec{a}' + 2\vec{\omega} \times \vec{V}' + \vec{\alpha} \times \vec{r}' + \vec{\omega} \times (\vec{\omega}\times)\vec{r}') + \vec{a}^0 \quad \text{(A.48)}$$

Velocity-dependent relative acceleration terms:

$2\vec{\omega} \times \vec{V}'$: Coriolis Acceleration

$\vec{\omega} \times (\vec{\omega} \times \vec{r}')$: Centripetal Acceleration

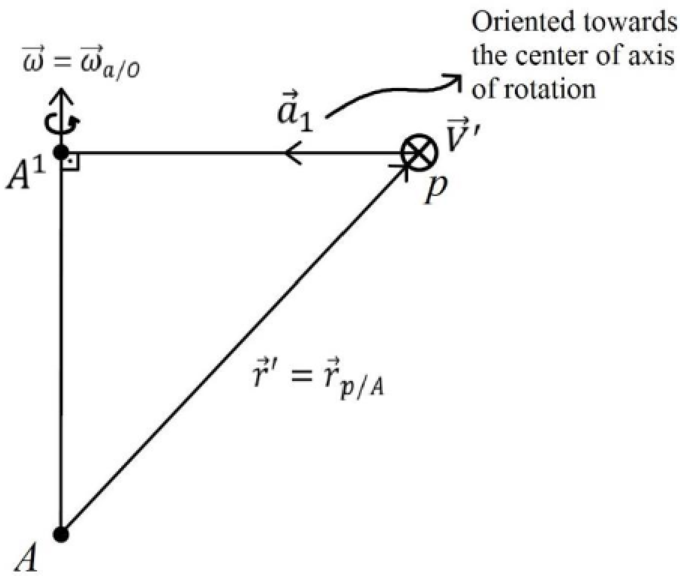


Figure A.8: Representation of centripetal acceleration

$\vec{V}_1 = \vec{\omega} \times \vec{r}^1$ First cross product

$\vec{a}_1 = \vec{\omega} \times (\vec{\omega} \times \vec{r}')$

$\vec{a}_1 = \vec{\omega} \times \vec{V}_1$ Centripetal Acceleration

Corresponding matrix equations are as follows:

Let $\hat{C} = \hat{C}^{(0,a)}$

$$\overline{r} = \overline{r}^{(0)}, \qquad \overline{V} = \overline{V}^{(0)}, \qquad \overline{a} = \overline{a}^{(0)}$$

$$\overline{r}^0 = \{\vec{r}^0\}^{(0)}, \qquad \overline{V}^0 = \{\vec{V}^0\}^{(0)}, \qquad \overline{a}^0 = \{\vec{a}^0\}^{(0)}$$

$$\overline{r}^0 = \{\vec{r}\}^{(a)}, \qquad \overline{V}' = \{\vec{V}\}^{(a)}, \qquad \overline{a}' = \{\vec{a}'\}^{(a)}$$

Therefore,

$$\overline{r} = \overline{C}\overline{r}' + \overline{r}^0 \tag{A.49}$$

$$\overline{V} = \hat{C}(\overline{V} + \tilde{\omega}\overline{r}) + \overline{V}^0 \tag{A.50}$$

$$\overline{a} = \hat{C}[\overline{a}' + 2\tilde{\omega}\overline{V}' + (\tilde{\alpha} + \tilde{\omega})\overline{r}'] + \overline{a}^0 \tag{A.51}$$

Translation or transport terms related to the motion of A w.r.t. o can be expressed as:

$$\overline{r}^{(0)} = \{\vec{r}_{A/O}\}^{(0)}, \qquad \overline{V}^{(0)} = \{\vec{V}_{A/O/F_0}\}^{(0)}, \qquad \overline{a}^{(0)} = \{\vec{a}_{A/O/F_0}\}^{(0)}$$

2^{nd} law in a non-inertial reference frame is illustrated in Figure 9.

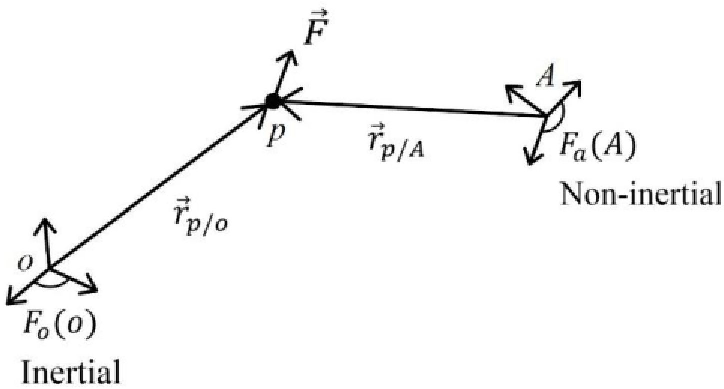


Figure A.9: A point viewed from inertial and non-inertial frames

2^{nd} law of Newton is expressed as:

$$\vec{F} = m\vec{a} \tag{A.1}$$

where

If 2^{nd} law is written in $F_a(A)$:

$$\vec{F} = m\vec{a} = \left[\vec{a}' + 2\vec{\omega} \times \vec{V}' + \vec{\alpha} \times \vec{r}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{a}^0\right] \tag{A.2}$$

$$\vec{a}' = \vec{a}_{P/F_{a(A)}}, \quad \vec{V}' = \vec{V}_{P/F_{a(A)}}, \quad \vec{r}' = \vec{r}_{P/A}, \quad \vec{a}^o = \vec{a}_{P/F_{a(o)}}$$
$$\vec{\omega} = \vec{\omega}_{a/o}, \quad \vec{\alpha} = \vec{\alpha}_{a/o}$$

If $\vec{F}'$ is considered as the apparent force felt in $F_a(A)$, then:

$$m\vec{a}' = \vec{F}' \tag{A.3}$$

$$\vec{F}' = \vec{F} + \vec{F}^* \tag{A.4}$$

Where,

$\vec{F}$ is the actual force acting on p. and

$\vec{F}^*$ is inertial force

$$\vec{F}^* = -m\left[2\vec{\omega} \times \vec{V}' + \vec{\alpha} \times \vec{r}' + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}'\right) + \vec{a}_o\right] \tag{A.5}$$

Note that:

$$\vec{F}^* = \vec{0} \text{ if } \vec{\omega} = \vec{0}.$$

$$\vec{\alpha} = \vec{0}, \text{ and}$$

$\vec{a}_o = \vec{O}$ because it is an inertial reference frame and does not translate.

At this point, special names given to the velocity-dependent terms.

$$\vec{F}^*_{cor} = -2m\vec{\omega} \times \vec{V}' : CoriolisForce$$

$$\vec{F}^*_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r} : CentrifugalForce$$

As it can be seen in Figure 10, Coriolis Force is normal to $\vec{V}'$

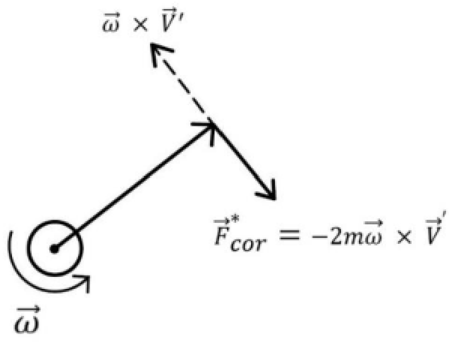


Figure A.10: Representation of the Coriolis force

$\vec{F}^*_{cf}$ is away from instantaneous axis of rotation described by $\vec{\omega}$

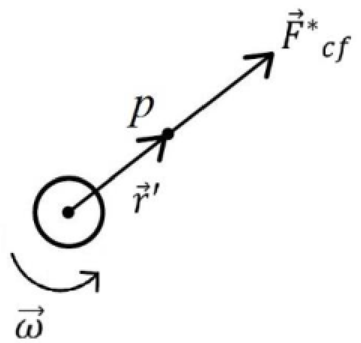


Figure A.11: Representation of instantaneous axis of rotation

We will now proceed to derive the equations of motion for Ball and Beam system using Newtonian mechanics. We define the x-axis to be parallel to the beam.Since this axis rotates with time, the time derivatives of the unit vectors must be considered when calculating velocities and accelerations.The absolute acceleration of a body is given by (Merriam & Kraig, 2002) as:

$$\vec{a}_a = \vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{V}_{rel} + \vec{a}_{rel} \tag{A.1}$$

Where,
$\vec{\omega}$:Angular velocity of the rotating axis
$\vec{r}$: Position vector

$\vec{V}_{rel}$:Velocity relative to the rotating axis $\vec{a}_{rel}$: Acceleration of the body relative to the rotating coordinate system Recall that:

$$\vec{a}_P/F_0(O) = \vec{a}_P/F_a(O)+2\vec{\omega}_{a/O}\times\vec{V}_{P/F_a}(A)+\vec{\alpha}_{a/O}\times\vec{r}_{P/A}+\vec{\omega}_a\times(\vec{\omega}_{a/O}\times\vec{r}_{P/A})+\vec{a}_{A/F_0}(O) \quad \text{(A.2)}$$

Consider the set of coordinate axes rotating with an angular velocity of ω as shown in the Figure 12:

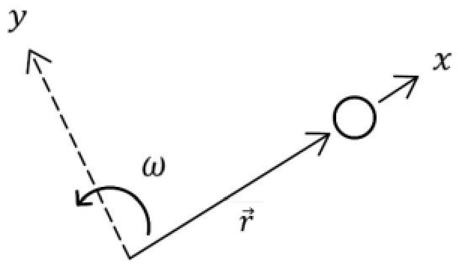


Figure A.12: Coordinate axes rotation

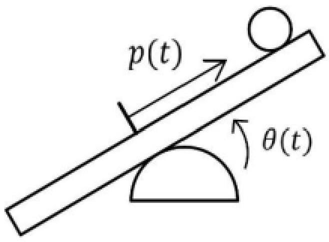


Figure A.13: Ball on a plate representation

$$\vec{r} = p\hat{i}, \qquad \vec{v}_{rei} = p\hat{i}, \qquad \vec{a}_{rei} = \ddot{p}\hat{i}$$
$$\vec{\omega} = \dot{\theta}\hat{k}$$

Now, the vector products needed to calculate the absolute acceleration can be performed.

$$\vec{\dot{\omega}} \times \vec{r} = \ddot{\theta}\hat{k} \times p\hat{i} = p\,\ddot{\theta}\hat{j} \quad \text{(A.3)}$$

$$\vec{\omega} \times \vec{r} = \dot{\theta}\hat{k} \times p\hat{i} = p\,\ddot{\theta}\hat{j} \quad \text{(A.4)}$$

$$\vec{\omega} \times (\omega \vec{\times} \vec{r}) = \dot{\theta}\hat{k} \times p\dot{\theta}\hat{j} = -p\dot{\theta}^2\hat{\imath} \quad \text{(A.5)}$$

$$2\vec{\omega} \times \vec{v}_{rei} = 2\dot{\theta}\hat{k} \times \dot{p}\hat{\imath} = 2\dot{p}\dot{\theta}\hat{j} \quad \text{(A.6)}$$

Inserting (A.3)-(A.6) into (A.1):

$$\vec{a}_a = p\ddot{\theta}\hat{j} - p\dot{\theta}^2\hat{1} + 2\dot{p}\dot{\theta}\hat{j} + \ddot{p}\hat{i} \quad \text{(A.7)}$$

$$\vec{a}_a = (\ddot{p} - p\dot{\theta}^2)\hat{\imath} + (p\ddot{\theta} + 2\dot{p}\dot{\theta})\hat{j} \quad \text{(A.8)}$$

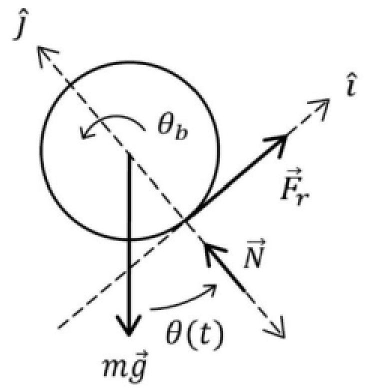


Figure A.14: Free body diagram of the ball

$$J_b\ddot{\theta}_b = \vec{F}_r.\vec{r} \quad \text{(A.9)}$$

Where J_b is the moment of inertia of the ball about its center. And,

$$\theta_b = -\frac{p}{r} \quad \text{(A.10)}$$

Where, r is the radius of the ball.

Substituting (A.10) into (A.9) and solving for F_r yields:

$$\dot{\theta}_b = -\frac{\dot{p}}{r}, \quad \ddot{\theta}_b = \frac{\ddot{p}}{r}$$

$$\frac{-J_{b\ddot{p}}}{r} = F_r.r$$

$$F_r = \frac{-J_{b.\ddot{p}}}{r^2} \tag{A.11}$$

We now proceed to sum forces acting on the ball in the i direction, using the i component of the acceleration vector (8) yields:

$$F_r - mgsin\theta = m(\ddot{p} - p\dot{\theta}^2) \tag{A.12}$$

Substituting (A.11) into (A.12), the first equation of motion is obtained:

$$(\frac{J_b}{r^2} + m)\ddot{p} + mgsin\theta - mp\dot{\theta}^2 = 0 \tag{A.13}$$

To compute the second equation of motion, we must first compute the normal force N. as shown in the previous FBD. Summing all forces in the j direction yields:

$$N - mgcos\theta = m(p\ddot{\theta} + 2\dot{p}\dot{\theta}) \tag{A.14}$$

$$N = m(p\ddot{\theta} + 2\dot{\theta}\theta p + mgcos\theta) \tag{A.15}$$

Summing torques acting on the beam yields:

$$\tau - N_p = J\ddot{\theta} \tag{A.16}$$

Where τ is the external applied torque and J is the inertia of the beam. Substituting (A.15) into (A.16) provides the second equation of motion.

$$(mp^2 + J)\ddot{\theta} + 2mp\dot{p}\dot{\theta} + 2mgpcos\theta = \tau \tag{A.17}$$

The term $p\dot{\theta}^2$ in (A.13)and $2mp\dot{p}\dot{\theta}$ in (A.17) would be missing if we do not take into consideration the effect of the rotating axis of the beam.

Appendix B

BALL AND BEAM SIMULATION RESULTS

B.1 Simulation Results

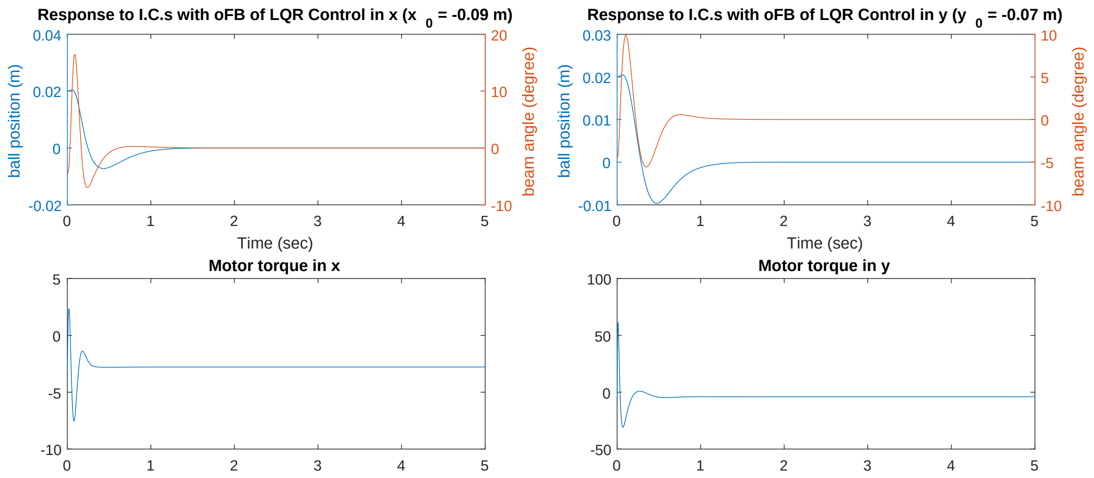


Figure B.1: Response of LQR to I.C.s

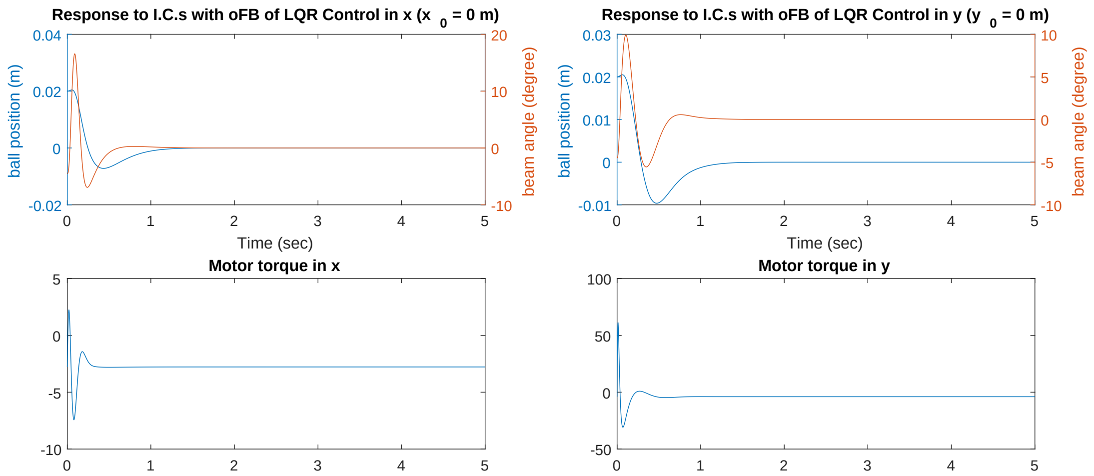


Figure B.2: Response of LQR to I.C.s

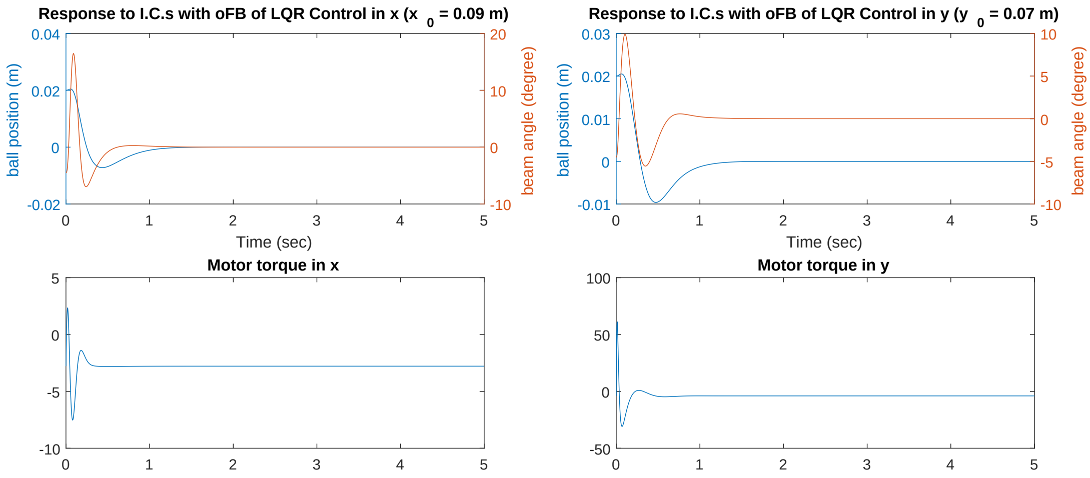


Figure B.3: Response of LQR to I.C.s

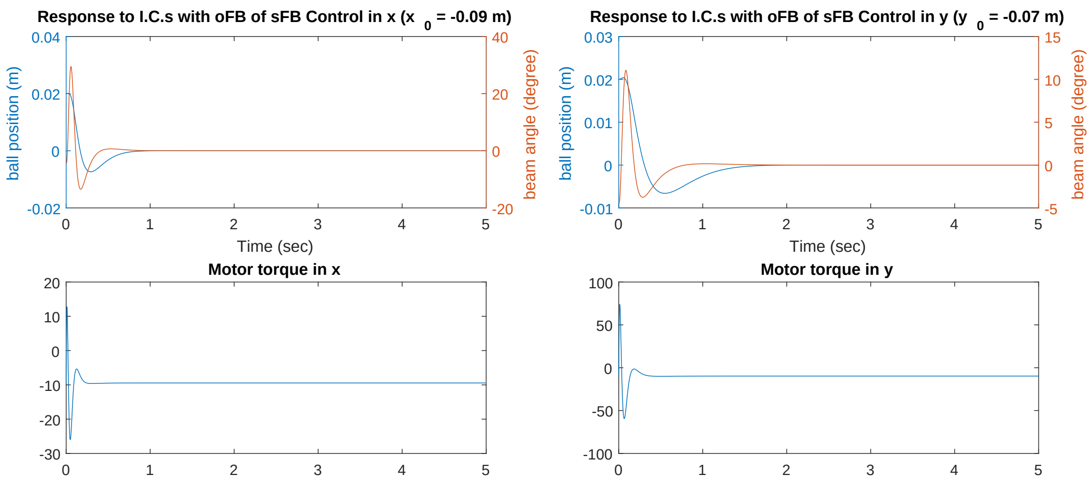


Figure B.4: Response of state feedback controller to I.C.s

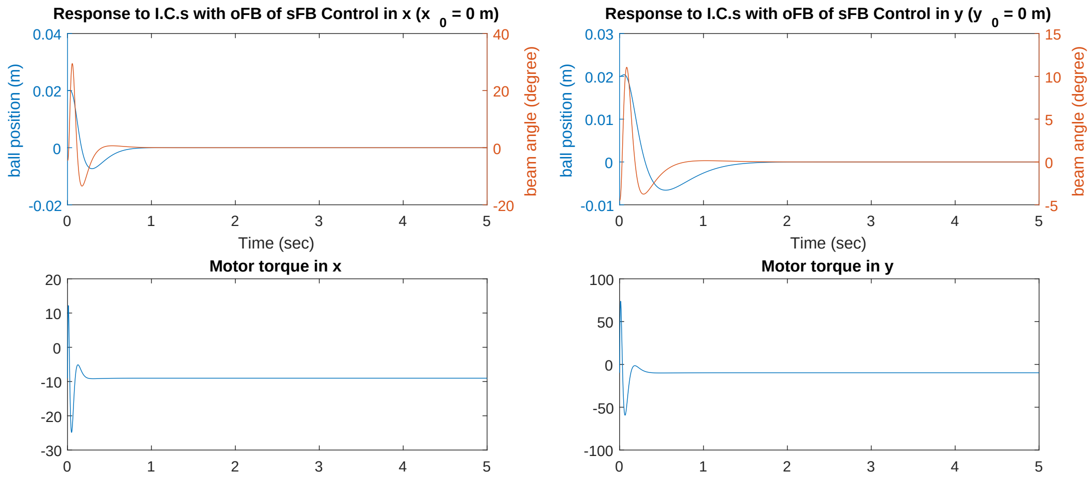


Figure B.5: Response of state feedback controller to I.C.s

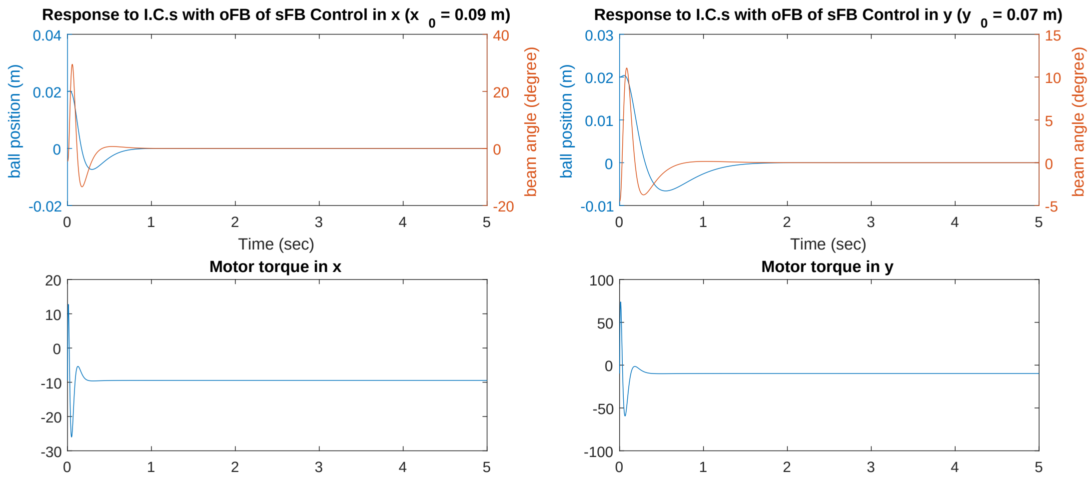


Figure B.6: Response of state feedback controller to I.C.s

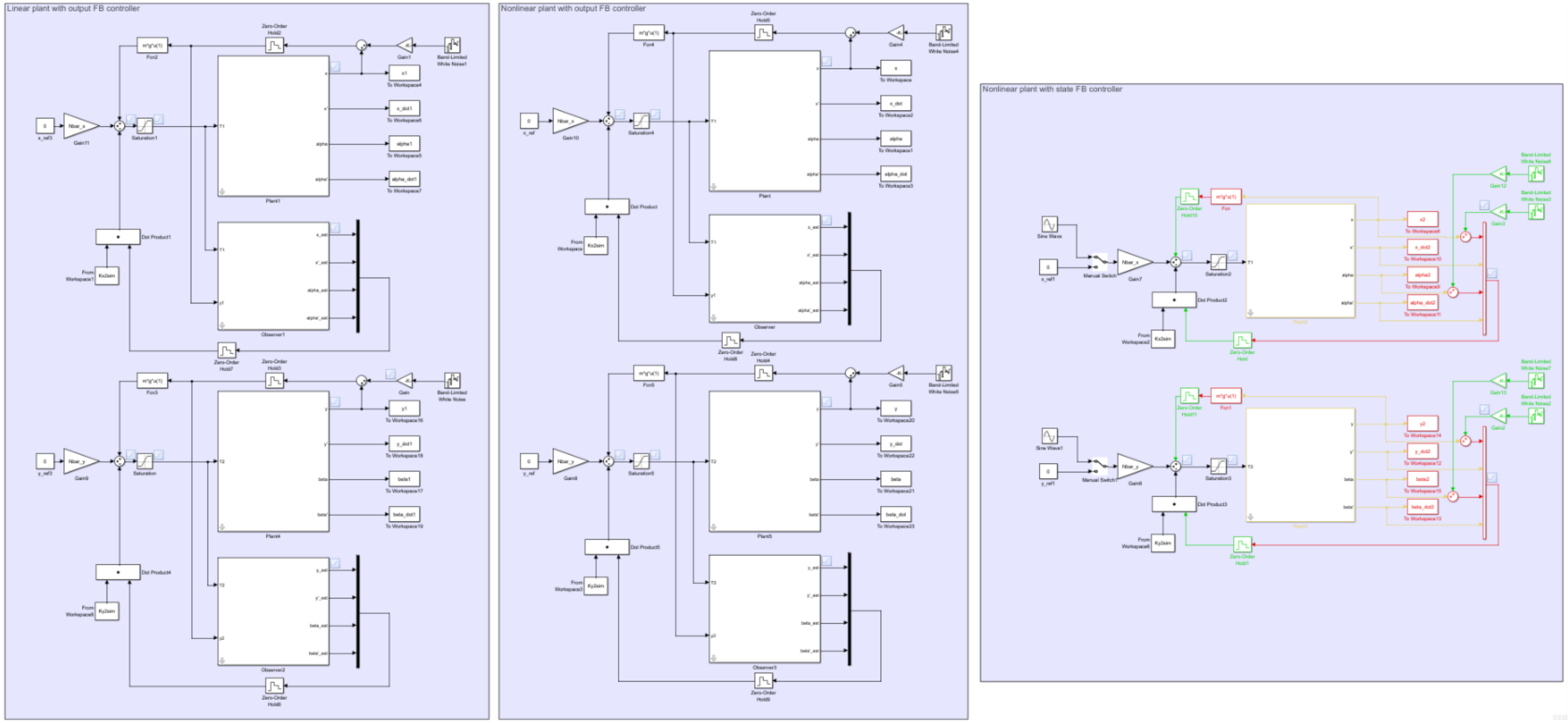
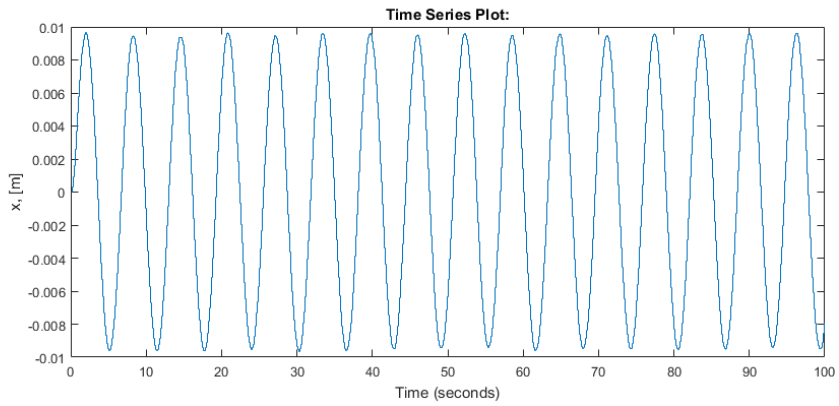


Figure B.7: Simulink model for LQR, state feedback and observer feedback controllers.



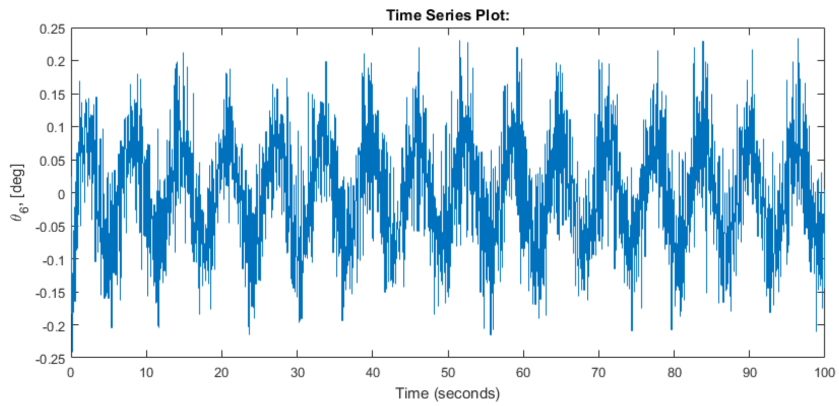


Figure B.8: Tracking a sinusoidal reference input for x position of the ball.

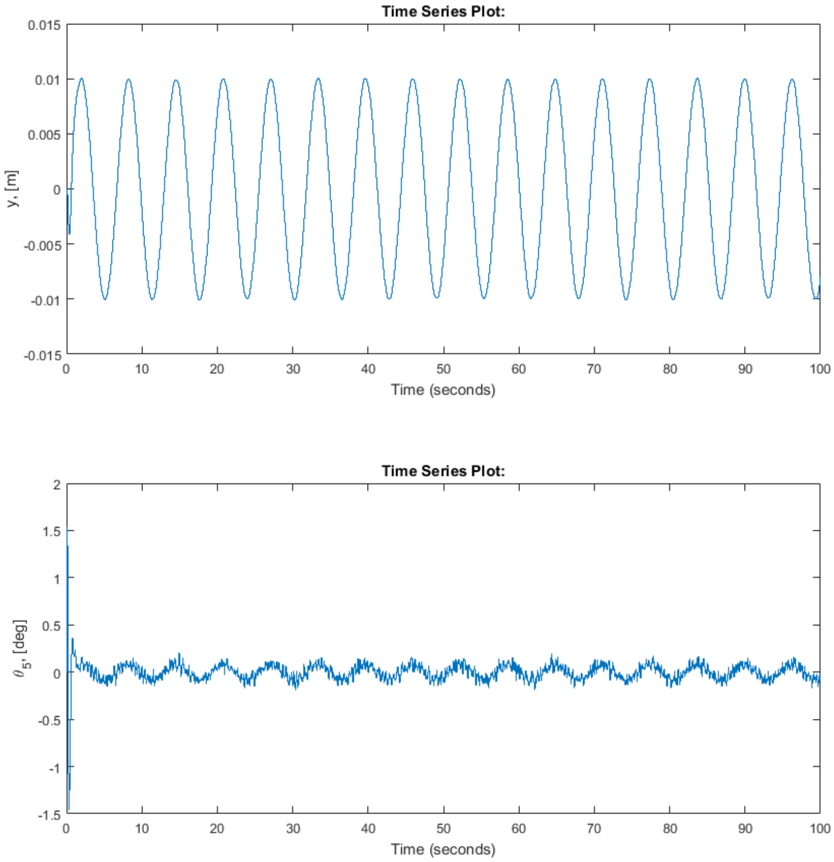


Figure B.9: Tracking a sinusoidal reference input for y position of the ball.

Appendix C

OBTAINING INERTIA AND MASS CENTERS

C.1 Ball and Beam

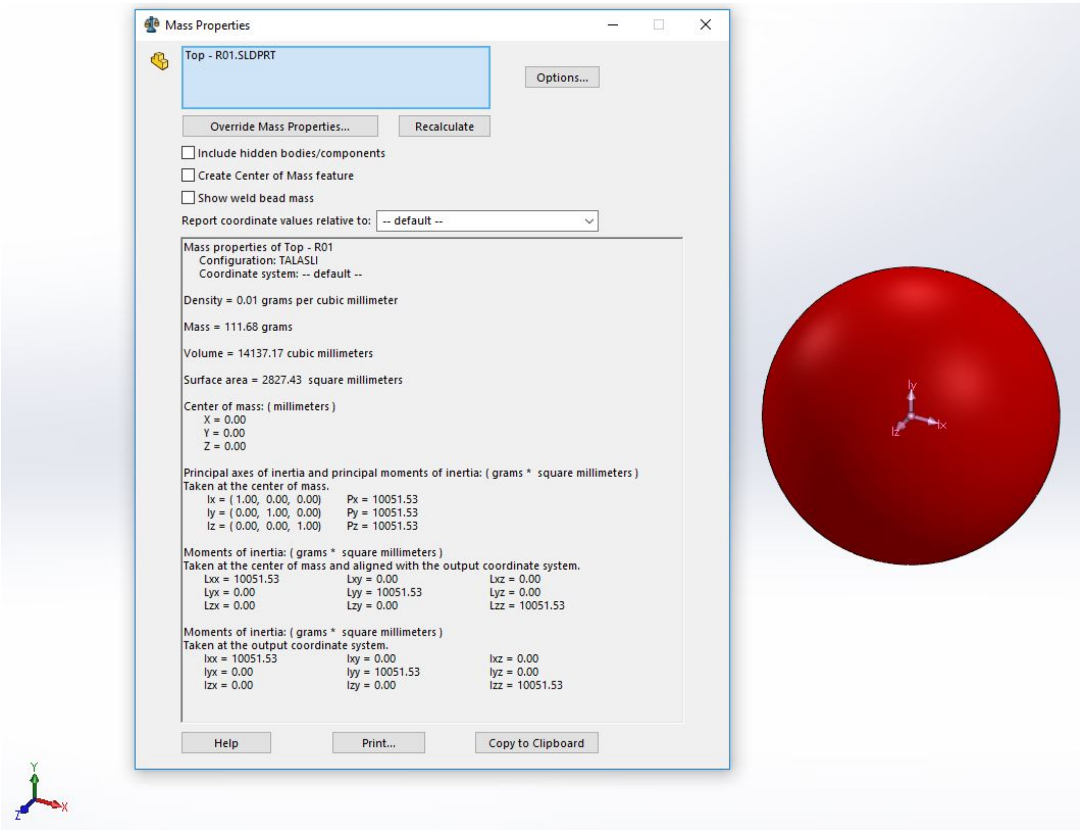


Figure C.1: Inertia and mass center of the ball

.

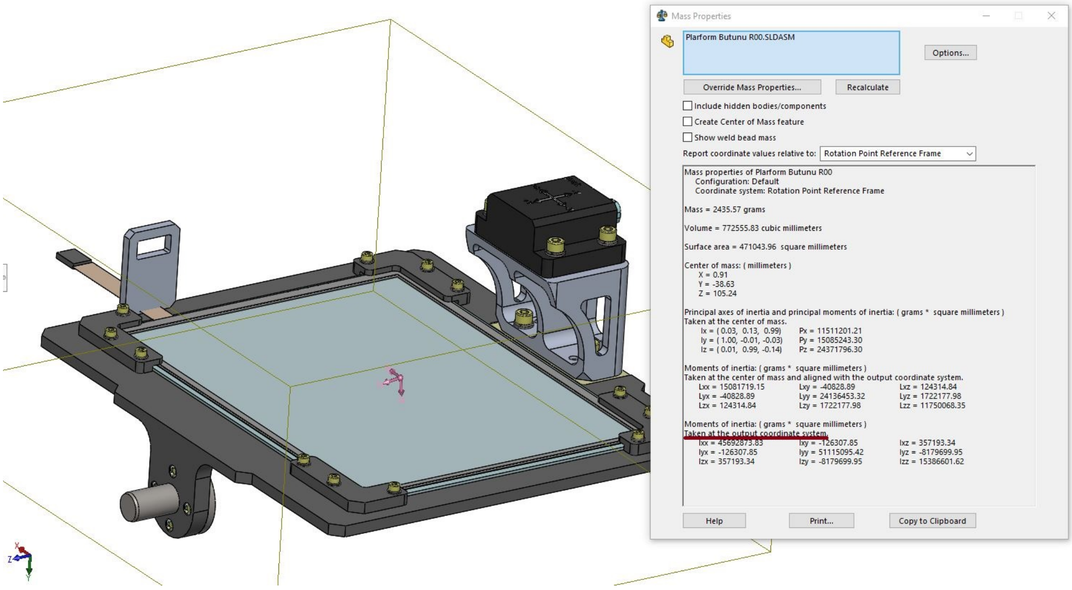


Figure C.2: Inertia and mass center of the inner axis

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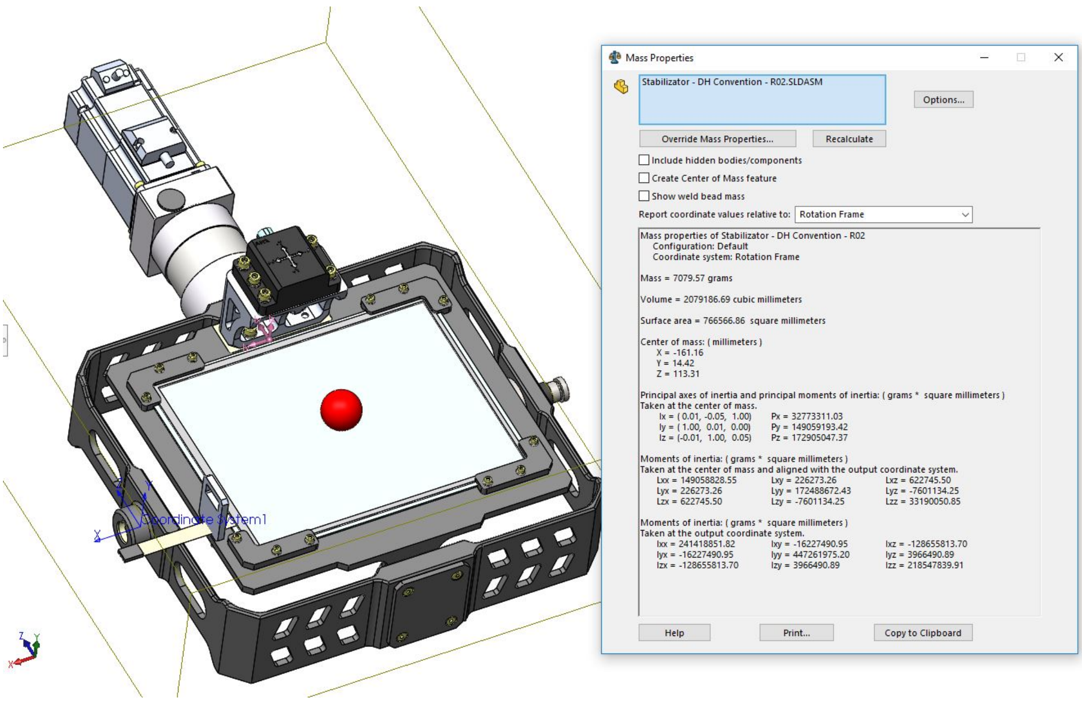

Figure C.3: Inertia and mass center of the outer axis

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C.2 Stabilizer

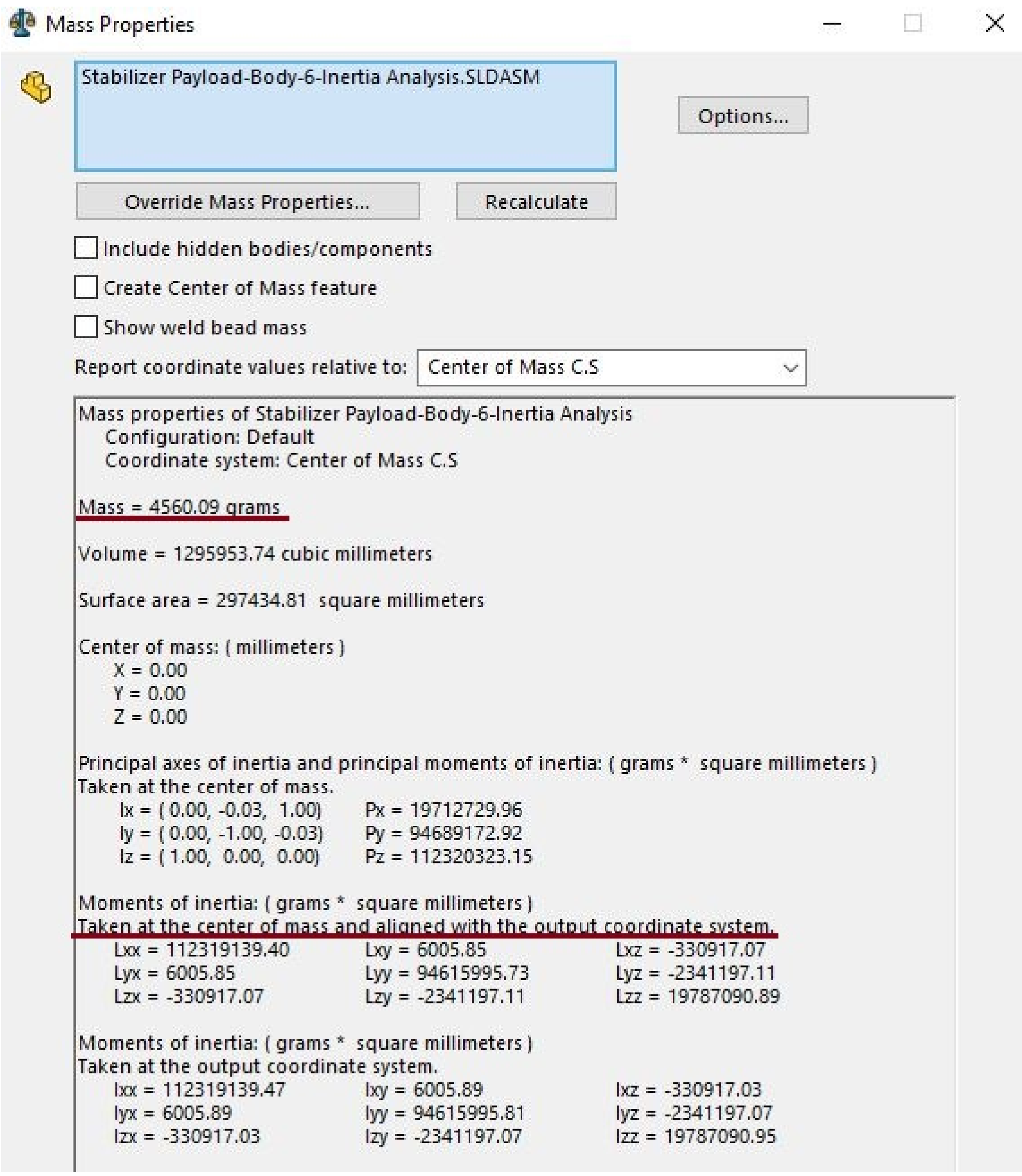

Figure C.4: Inertia and mass center of body-6 of the stabilizer

.

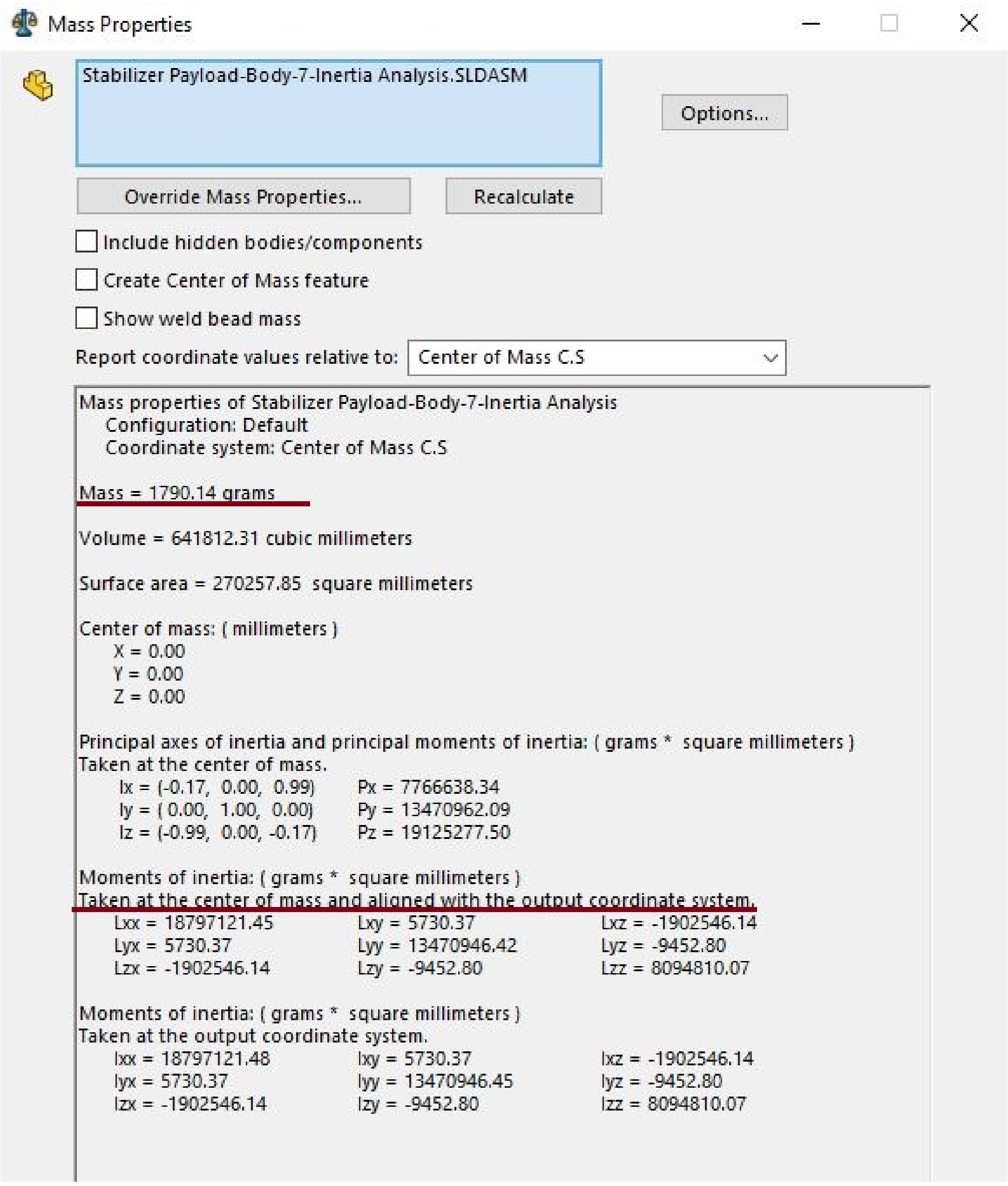

Figure C.5: Inertia and mass center of body-7 of the stabilizer

.

Appendix D

MATLAB CODES

D.1 Ball and Beam Codes

Listing D.1: Ball and Beam Various Controller Design

```
clear; clc; close all;

Ts = 5e 4;

eff = 0.8;

deltatheta = 36;   degree
deltapulse = 26831;   encer pulses
numberofpulsespermotorrevolution = deltapulse 10/25;   It is
    calculated as 10000
pulse2degree = 360/250000;

statictorque = 3.75;   N.m
zerovelocitytolerance = 0.02;   deg/sec

g = 9.81;                    gravitational acceleration
m b = 111.68e 3;             mass of the ball
m p 1 = 7.0796;              mass of the outer plate
m p 2 = 2.4356;              mass of the inner plate
r = 15e 3;                   radius of the ball
h = 53.8e 3;                 offset of the ball from the joint
J b = 10051.53e 9;           moment of inertia of the ball
J p 1 = 241418851.82e 9;     moment of inertia of the plate about u1/x
J p 2 = 15386601.62e 9;      moment of inertia of the plate about u2/y
a 1 = 14.42e 3;
b 1 = 113.31e 3;
a 2 = 38.63e 3;
b 2 = 0;
x 0 = 0;
y 0 = 0;

A x 2 3 =  m b g/(J b/r 2 + m b);
A x 4 1 =  m b g/(J p 2+m p 2 (a 2 2+b 2 2)+m b (x 0 2+h 2));
A x 4 3 = (m b g h+m p 2 g a 2)/(J p 2+m p 2 (a 2 2+b 2 2)+m b (x 0 2+h 2));

A x = [ 0       1      0      0
        0       0    A x 2 3    0
        0       0      0      1
      A x 4 1    0    A x 4 3    0];
```

```
B x = [0;0;0;1/(J p 2+m p 2  (a2 2+b2 2)+m b  (x0 2+h 2))];
C x = [1 0 0 0;
       0 0 1 0];
D x = [0; 0];

A y 23 =  m b g/(J b/r 2 + m b);
A y 41 =  m b g/(J p 1+m p 1  (a1 2+b1 2)+m b  (y0 2+h 2));
A y 43 = (m b g h+m p 1 g a1)/(J p 1+m p 1  (a1 2+b1 2)+m b  (y0 2+h 2));

A y = [ 0      1     0      0
        0      0   A y 23   0
        0      0     0      1
      A y 41   0   A y 43   0];

B y = [0;0;0;1/(J p 1+m p 1  (a1 2+b1 2)+m b  (y0 2+h 2))];
C y = [1 0 0 0;
       0 0 1 0];
D y = [0; 0];

x0max = 0.08;
xddotmax = A x 23 sind(5);
lambdax = sqrt( xddotmax/x0max);     Equation 2.36 page 93 (Meirovitch
     Fundamentals of Vibrations)
betax = 0.05;

  lambdax = 2.5;    Paper daki degerler
  betax = 0.35;     Paper daki degerler

lambday = 2.5;
betay = 0.35;

   Touchpad Parameters

X = [0  89  89;
     0  60  59];

V = [2.534 3.981 3.915;
     2.541 1.382 3.540;
     1     1     1    ];

A = X/V;


x0 = [0.090 0 0.090];
y0 = [0.070 0 0.070];


Q x = zeros(4,4,3);
Q x(1,1,:) = [20000 20000 20000];
Q x(2,2,:) = [ 1000  1000  1000];
Q x(3,3,:) = [ 1000  1000  1000];

Q y = zeros(4,4,3);
Q y(1,1,:) = [40000 40000 40000];
Q y(2,2,:) = [ 2000  2000  2000];
Q y(3,3,:) = [ 2000  2000  2000];


POx = [0.001; 0.001; 0.001];   Percent overshoot as ratio
```

```
Tsx = [0.5  ; 0.5  ; 0.5  ];   Settling time in seconds
zetax = sqrt(log(POx) . 2 ./ (pi 2 + log(POx) . 2));
thetax = acos(zetax);
sigmax = 3.9./Tsx;

px(:,1) =  sigmax + 1i sigmax . tan(thetax);
px(:,2) =  sigmax   1i sigmax . tan(thetax);
px(:,3) =  6 sigmax;
px(:,4) =  6.1 sigmax;

POy = [0.001; 0.001; 0.001];   Percent overshoot as ratio
Tsy = [1.0  ; 1.0  ; 1.0  ];   Settling time in seconds
zetay = sqrt(log(POy) . 2 ./ (pi 2 + log(POy) . 2));
thetay = acos(zetay);
sigmay = 3.9./Tsy;

py(:,1) =  sigmay + 1i sigmay . tan(thetay);
py(:,2) =  sigmay   1i sigmay . tan(thetay);
py(:,3) =  6 sigmay;
py(:,4) =  6.1 sigmay;


for i = 1:3
    [NbarxLQR(i), NbaryLQR(i), KxLQR(i,:), KyLQR(i,:), LxLQR(:,:,
        i), LyLQR(:,:,i)] = getControllerParameters(x0(i), y0(i),  LQR
         , Qx(:,:,i), Qy(:,:,i), px(i,:), py(i,:), Ts);
    saveas(gcf,[ C: Users MusabCagri Desktop ornektemplate10062019sg
        figures AppendixC response2ICLQR , num2str(i)], epsc )
    [NbarxsFB(i), NbarysFB(i), KxsFB(i,:), KysFB(i,:), LxsFB(:,:,
        i), LysFB(:,:,i)] = getControllerParameters(x0(i), y0(i),  sFB
         , Qx(:,:,i), Qy(:,:,i), px(i,:), py(i,:), Ts);
    saveas(gcf,[ C: Users MusabCagri Desktop ornektemplate10062019sg
        figures AppendixC response2ICsFB , num2str(i)], epsc )
end
close all

Lx = LxLQR(:,:,2);
Ly = LyLQR(:,:,2);
```

Listing D.2: Touchpad output model

```
clc; clear;
syms x y V1 V2 positive
syms Rx Ry R1 R2 Rtouch Vcc positive
Rx = 394; Ry = 493; R1 = 103.6; R2 = 159.1; Rtouch = 681.5; Vcc = 5;


  Req = simplify(x y Rx Ry/(x Rx+y Ry) + ((1 x) Rx+R1) ((1 y) Ry+R2)/((1 x
    ) Rx+R1+(1 y) Ry+R2));
  itot = simplify(Vcc/Req);

  r1 = ((1 y) Ry+R2)/((1 x) Rx+R1+(1 y) Ry+R2);
  r2 = ((1 x) Rx+R1)/((1 x) Rx+R1+(1 y) Ry+R2);

  i1 = itot r1;
  i2 = itot r2;

    xtest = 0.72;
    ytest = 0.65;

```

```
    V 1 = R1   double(subs(i1 , x,y  , xtest,ytest ))
    V 2 = R2   double(subs(i2 , x,y  , xtest,ytest ))


  S = solve([V1 == i1 R1,V2 == i2 R2], [x,y]);


  simplify(S.x)
  simplify(S.y)
  simplify(i1 R1)
  simplify(i2 R2)

    V 1 = 0.968;
    V 2 = 1.035;
    double(subs(S.x , V1, V2  , V 1 ,  V 2 ))
    double(subs(S.y , V1, V2  , V 1 ,  V 2 ))


A = [ 1  1  1  0  0  0;
      1  0  0  1  0  1;
      0  0  1  1  1  0;
0  (1 y) Ry+R2 0 0 y Ry 0;
 (1 x) Rx+R1 0 0 x Rx 0 0;
0 0 0 x Rx  y Ry  Rtouch];

B = [0; 0; 0; Vcc; Vcc; 0];

i = A B;

i1 = simplify(i(1));
i2 = simplify(i(2));

S = solve([V1 == i1 R1,V2 == i2 R2], [x,y]);

i1 = subs(i1,  Rx, Ry, R1, R2, Rtouch, Vcc ,  394 , 493, 98.8, 159.1, 681.5,
     5 ) ;
i2 = subs(i2,  Rx, Ry, R1, R2, Rtouch, Vcc ,  394 , 493, 98.8, 159.1, 681.5,
     5 ) ;

xtest = 0.72;
ytest = 0.65;

V 1 = R 1   double(subs(i1 , x, y  , xtest, ytest ))
V 2 = R 2   double(subs(i2 , x, y  , xtest, ytest ))

double(subs(S.x , V1, V2, Rx, Ry, R1, R2, Rtouch, Vcc  , V 1 , V 2 , 394, 493,
    98.8, 159.1, 681.5, 5 ))
double(subs(S.y , V1, V2, Rx, Ry, R1, R2, Rtouch, Vcc  , V 1 , V 2 , 394, 493,
    98.8, 159.1, 681.5, 5 ))
```

Listing D.3: Touchpad calibration parameters

```
Ts = 5e 4;

X = [0  89  89  93;
     0  60  59  57];

V = [2.534 3.981 3.915 1.245;
     2.541 1.382 3.540 1.699;
     1     1     1     1    ];
```

```
A = X/V;
```

D.2 Disturber Codes

Listing D.4: Newton-Euler equations for the disturber

```
clear
clc

syms J111 J112 J113 J121 J122 J123 J131 J132 J133 ...
     J211 J212 J213 J221 J222 J223 J231 J232 J233 ...
     J311 J312 J313 J321 J322 J323 J331 J332 J333 ...
     J411 J412 J413 J421 J422 J423 J431 J432 J433 ...
     J511 J512 J513 J521 J522 J523 J531 J532 J533 real


syms F011p F012p F013p M011p M012p M013p ...
     F031p F032p F033p M031p M032p M033p ...
     F051p F052p F053p M051p M052p M053p ...
     F251p F252p F253p M251p M252p M253p ...
     F451p F452p F453p M451p M452p M453p ...


syms F011dp F012dp F013dp M011dp M012dp M013dp ...
     F031dp F032dp F033dp M031dp M032dp M033dp ...
     F121dp F122dp F123dp M121dp M122dp M123dp ...
     F341dp F342dp F343dp M341dp M342dp M343dp ...
     F251dp F252dp F253dp M251dp M252dp M253dp ...
     F451dp F452dp F453dp M451dp M452dp M453dp ...


syms Fp1tp Fp2tp Fp3tp Mp1tp Mp2tp Mp3tp ...
     F051tp F052tp F053tp M051tp M052tp M053tp ...
     F251tp F252tp F253tp M251tp M252tp M253tp ...
     F451tp F452tp F453tp M451tp M452tp M453tp ...

syms theta131(t) theta132(t) theta141(t) theta142(t) s41(t) s42(t)

syms m1 m2 m3 m4 m5 g

syms r11 r12 r13 r21 r22 r23 r31 r32 r33 r41 r42 r43

syms r101 r102 r103 r121 r122 r123 r211 r212 r213 r251 r252 r253...
     r301 r302 r303 r341 r342 r343 r431 r432 r433 r451 r452 r453...
     r521 r522 r523 r541 r542 r543 r501 r502 r503 rp1 rp2 rp3

syms da1 da2 da3

  Basic Column Matrices

u1 = [1; 0; 0];
u2 = [0; 1; 0];
u3 = [0; 0; 1];

  Relations with Derivatives
```

```
theta131dot = diff(theta131);

theta131doubledot = diff(theta131dot);

theta132dot = diff(theta132);

theta132doubledot = diff(theta132dot);

theta141dot = diff(theta141);

theta141doubledot = diff(theta141dot);

theta142dot = diff(theta142);

theta142doubledot = diff(theta142dot);

 Inertia matrices

J1 = [J111 J112 J113; J121 J122 J123; J131 J132 J133];  Ineria dyadic resolved in Reference Frame 1

J2 = [J211 J212 J213; J221 J222 J223; J231 J232 J233];  Ineria dyadic resolved in Reference Frame 2

J3 = [J311 J312 J313; J321 J322 J323; J331 J332 J333];  Ineria dyadic resolved in Reference Frame 3

J4 = [J411 J412 J413; J421 J422 J423; J431 J432 J433];  Ineria dyadic resolved in Reference Frame 4

J5 = [J511 J512 J513; J521 J522 J523; J531 J532 J533];  Ineria dyadic resolved in Reference Frame 5

 Transformation Matrices

theta01 = theta141 pi/2;

theta03 = theta142 pi/2;

T 01 12 = simplify(BasicRotationMatrix(2,theta01));  C(0,1)

T 03 12 = simplify(BasicRotationMatrix(1,theta03));  C(0,3)

T 25 12 = T 01 12;  C(0,1)

T 25 13 = simplify(BasicRotationMatrix(2,theta131) BasicRotationMatrix(1,theta132) BasicRotationMatrix(2, sym(pi)/2));  C(0,5)

T 25 23 = simplify(T 01 12 T 25 13);  C(1,0) C(0,5)

T 45 12 = T 03 12;  C(0,3)

T 45 13 = T 25 13;  C(0,5)

T 45 23 = simplify(T 03 12 T 25 13);  C(3,0) C(0,5)

T 05 13 = T 25 13;  C(0,5)

 Force Definitions
```

```
F01dp = [F011dp; F012dp; F013dp];

F03dp = [F031dp; F032dp; F033dp];

F12dp = [F121dp; F122dp; F123dp];

F34dp = [F341dp; F342dp; F343dp];

F25dp = [F251dp; F252dp; F253dp];

F45dp = [F451dp; F452dp; F453dp];

F05tp = [F051tp; F052tp; F053tp];

Fp = [Fp1tp; Fp2tp; Fp3tp];

 Force Tranformations

F01p = T0112 F01dp;

F03p = T0312 F03dp;

F25p = T2512 F25dp;

F25tp = T2523 F25dp;

F45p = T4512 F45dp;

F45tp = T4523 F45dp;

F05p = T0513 F05tp;

 Force Components

Fbody = formula(F01p); F011p = Fbody(1); F012p = Fbody(2); F013p
    = Fbody(3);

Fbody = formula(F03p); F031p = Fbody(1); F032p = Fbody(2); F033p
    = Fbody(3);

Fbody = formula(F25p); F251p = Fbody(1); F252p = Fbody(2); F253p
    = Fbody(3);

Fbody = formula(F25tp); F251tp = Fbody(1); F252tp = Fbody(2);
   F253tp = Fbody(3);

Fbody = formula(F45p); F451p = Fbody(1); F452p = Fbody(2); F453p
    = Fbody(3);

Fbody = formula(F45tp); F451tp = Fbody(1); F452tp = Fbody(2);
   F453tp = Fbody(3);

Fbody = formula(F05p); F051p = Fbody(1); F052p = Fbody(2); F053p
    = Fbody(3);

 Moment Definitions

M01dp = [M011dp; M012dp; M013dp];

```

```
M 0 3 dp = [M 0 3 1 dp; M 0 3 2 dp; M 0 3 3 dp];

M 1 2 dp = [M 1 2 1 dp; M 1 2 2 dp; M 1 2 3 dp];

M 3 4 dp = [M 3 4 1 dp; M 3 4 2 dp; M 3 4 3 dp];

M 2 5 dp = [M 2 5 1 dp; M 2 5 2 dp; M 2 5 3 dp];

M 4 5 dp = [M 4 5 1 dp; M 4 5 2 dp; M 4 5 3 dp];

M 0 5 tp = [M 0 5 1 tp; M 0 5 2 tp; M 0 5 3 tp];

  Moment Transformations

M 0 1 p = T 0 1 1 2  M 0 1 dp;

M 0 3 p = T 0 3 1 2  M 0 3 dp;

M 2 5 p = T 2 5 1 2  M 2 5 dp;

M 2 5 tp = T 2 5 2 3  M 2 5 dp;

M 4 5 p = T 4 5 1 2  M 4 5 dp;

M 4 5 tp = T 4 5 2 3  M 4 5 dp;

M 0 5 p = T 0 5 1 3  M 0 5 tp;

  Moment Components

Mbody = formula(M 0 1 p); M 0 1 1 p = Mbody(1); M 0 1 2 p = Mbody(2); M 0 1 3 p
     = Mbody(3);

Mbody = formula(M 0 3 p); M 0 3 1 p = Mbody(1); M 0 3 2 p = Mbody(2); M 0 3 3 p
     = Mbody(3);

Mbody = formula(M 2 5 p); M 2 5 1 p = Mbody(1); M 2 5 2 p = Mbody(2); M 2 5 3 p
     = Mbody(3);

Mbody = formula(M 2 5 tp); M 2 5 1 tp = Mbody(1); M 2 5 2 tp = Mbody(2);
    M 2 5 3 tp = Mbody(3);

Mbody = formula(M 4 5 p); M 4 5 1 p = Mbody(1); M 4 5 2 p = Mbody(2); M 4 5 3 p
     = Mbody(3);

Mbody = formula(M 4 5 tp); M 4 5 1 tp = Mbody(1); M 4 5 2 tp = Mbody(2);
    M 4 5 3 tp = Mbody(3);

Mbody = formula(M 0 5 p); M 0 5 1 p = Mbody(1); M 0 5 2 p = Mbody(2); M 0 5 3 p
     = Mbody(3);



  Newton  Euler Equations

  Body 1

W 1 = simplify( m 1  g ( T 0 1 1 2  u 3));         Weight vector resolved in
    Reference Frame 1

```

```
 Definition of motion vectors

w 1 = theta141dot u 2;                    Angular velocity vector resolved in Reference Frame 1

w 1 tilda = SkewSymmetricMatrix(w 1);     Skew Symmetric Matrix of the angular velocity matrix

alfa1 = theta141doubledot u 2;            Angular acceleration vector resolved in Reference Frame 1

r 1 = [r 11; r 12; r 13];                 Position vector of C 1 resolved in Reference Frame 1

V 1 = diff(r 1,t) + w 1 tilda r 1;        Velocity vector of C 1 resolved in Reference Frame 1

a 1 = diff(V 1) + w 1 tilda V 1;          Acceleration vector of C 1 resolved in Reference Frame 1

 Force equations

F E 1 = F 01 dp   F 12 dp + W 1   m 1 a 1;

F E 1 body = formula(F E 1);

F E 11 = F E 1 body(1); F E 12 = F E 1 body(2); F E 13 = F E 1 body(3);

 Definition of Moment Arms

r 10 = [r101 0 r103];

r 10 tilda = SkewSymmetricMatrix(r 10);

r 12 = [r121 0 r123];

r 12 tilda = SkewSymmetricMatrix(r 12);

 Moment equations

M E 1 = r10tilda F 01 dp   r12tilda F 12 dp + M 01 dp   M 12 dp   (DyadicDotProduct(J 1,alfa1) + w1tilda DyadicDotProduct(J 1,w 1));

M E 1 body = formula(M E 1);

M E 11 = M E 1 body(1); M E 12 = M E 1 body(2); M E 13 = M E 1 body(3);



 Body 2

W 2 = simplify( m 2 g (T 01 12 u 3));      Weight vector resolved in Reference Frame 2

 Definition of motion vectors

w 2 = theta141dot u 2;                     Angular velocity vector resolved in Reference Frame 2

```

```
w2tilda = SkewSymmetricMatrix(w2);         Skew Symmetric Matrix of the
     angular velocity matrix

alfa2 = theta141doubledot u2;              Angular acceleration vector
    resolved in Reference Frame 2

r23 = s41   r213;                          Relation of r23 with s41

r2 = [r21; 0; r23];                        Position vector of C2
    resolved in Reference Frame 2

V2 = diff(r2,t) + w2tilda r2;              Velocity vector of C2
    resolved in Reference Frame 2

a2 = simplify(diff(V2) + w2tilda V2);      Acceleration vector of C2
    resolved in Reference Frame 2

  Force equations

FE2 = F12dp   F25dp + W2   m2 a2;

FE2body = formula(FE2);

FE21 = FE2body(1); FE22 = FE2body(2); FE23 = FE2body(3);

  Definition of Moment Arms

r21 = [r211 0 r213];

r21tilda = SkewSymmetricMatrix(r21);

r25 = [r251 0 r253];

r25tilda = SkewSymmetricMatrix(r25);

  Moment equations

ME2 =  r25tilda F25dp + r21tilda F12dp + M12dp   (
    DyadicDotProduct(J2,alfa2) + w2tilda DyadicDotProduct(J2,w2))
    ;

ME2body = formula(ME2);

ME21 = ME2body(1); ME22 = ME2body(2); ME23 = ME2body(3);



  Body 3

W3 = simplify( m3 g (T0312 u3));           Weight vector resolved in
    Reference Frame 3

  Definition of motion vectors

w3 = theta142dot u1;                       Angular velocity vector
    resolved in Reference Frame 3

w3tilda = SkewSymmetricMatrix(w3);         Skew Symmetric Matrix of the
    angular velocity matrix

```

```
alfa3 = theta142doubledot u1;          Angular acceleration vector
    resolved in Reference Frame 3

r3 = [0; r32; r33];                    Position vector of C3
    resolved in Reference Frame 3

V3 = diff(r3,t) + w3tilda r3;          Velocity vector of C3
    resolved in Reference Frame 3

a3 = diff(V3) + w3tilda V3;            Acceleration vector of C3
    resolved in Reference Frame 3

 Force equations

FE3 = F03dp  F34dp + W3  m3 a3;

FE3body = formula(FE3);

FE31 = FE3body(1); FE32 = FE3body(2); FE33 = FE3body(3);

 Definition of Moment Arms

r30 = [0 r302 r303];

r30tilda = SkewSymmetricMatrix(r30);

r34 = [0 r342 r343];

r34tilda = SkewSymmetricMatrix(r34);

 Moment equations

ME3 =  r34tilda F34dp + r30tilda F03dp + M03dp  M34dp  (
    DyadicDotProduct(J3,alfa3) + w3tilda DyadicDotProduct(J3,w3))
    ;

ME3body = formula(ME3);

ME31 = ME3body(1); ME32 = ME3body(2); ME33 = ME3body(3);



 Body 4

W4 = simplify( m4 g (T0312 u3));       Weight vector resolved in
    Reference Frame 4

 Definition of motion vectors

w4 = theta142dot u1;                   Angular velocity vector
    resolved in Reference Frame 4

w4tilda = SkewSymmetricMatrix(w4);     Skew Symmetric Matrix of the
    angular velocity matrix

alfa4 = theta142doubledot u1;          Angular acceleration vector
    resolved in Reference Frame 4

r43 = s42  r433;                       Relation of r43 with s42

```

```
r 4 = [0; r 42; r 43];                         Position vector of C 4
    resolved in Reference Frame 4

V 4 = diff(r 4 ,t) + w 4 t i l d a  r 4;           Velocity vector of C 4
    resolved in Reference Frame 4

a 4 = simplify(diff(V 4) + w 4 t i l d a  V 4);    Acceleration vector of C 4
    resolved in Reference Frame 4

  Force equations

F E 4 = F 3 4 dp   F 4 5 dp + W 4   m 4  a 4;

F E 4 body = formula(F E 4);

F E 4 1 = F E 4 body(1); F E 4 2 = F E 4 body(2); F E 4 3 = F E 4 body(3);

  Definition of Moment Arms

r 4 3 = [0 r 4 32 r 4 33];

r 4 3 tilda = SkewSymmetricMatrix(r 4 3);

r 4 5 = [0 r 4 52 r 4 53];

r 4 5 tilda = SkewSymmetricMatrix(r 4 5);

  Moment equations

M E 4 = r 4 3 tilda F 3 4 dp   r 4 5 tilda F 4 5 dp + M 3 4 dp   (
    DyadicDotProduct(J 4 ,alfa 4) + w 4 t i l d a DyadicDotProduct(J 4 ,w 4))
    ;

M E 4 body = formula(M E 4);

M E 4 1 = M E 4 body(1); M E 4 2 = M E 4 body(2); M E 4 3 = M E 4 body(3);



  Body 5

W 5 = simplify( m 5  g ( T 2 5 1 3  u 3));
    Weight vector resolved in Reference Frame 5

  Definition of motion vectors

w 5 = simplify(T 2 5 1 3 (theta 1 3 2 dot  u 1 + theta 1 3 1 dot  u 2));
    Angular velocity vector resolved in Reference Frame 5

w 5 t i l d a = SkewSymmetricMatrix(w 5);                            Skew
    Symmetric Matrix of the angular velocity matrix

alfa 5 = diff(w 5 ,t);
    Angular acceleration vector resolved in Reference Frame 5

r j = simplify(d a 2 ( T 2 5 1 3  u 3));
    Definition of r j

r 5 = r j   r p1  u 1   r p2  u 2   r p3  u 3;
    Position vector of C 5 resolved in Reference Frame 5
```

```
V 5 = simplify(diff(r 5 ,t) + w 5 tilda  r 5);
    Velocity vector of  C 5  resolved in Reference Frame 5

a 5  = simplify(diff(V 5) + w 5 tilda  V 5);
    Acceleration vector of  C 5  resolved in Reference Frame 5

  Force equations

F E 5  = F 25 tp + F 45 tp + F 05 tp + F p + W 5    m 5  a 5;

F E 5 body = formula(F E 5);

F E 51  = F E 5 body(1); F E 52  = F E 5 body(2); F E 53  = F E 5 body(3);

  Definition of Moment Arms

r 52  = [r 521  r 522  r 523];

r 52 tilda  = SkewSymmetricMatrix(r 52);

r 54  = [r 541  r 542  r 543];

r 54 tilda  = SkewSymmetricMatrix(r 54);

r 50  = [r 501  r 502  r 503];

r 50 tilda  = SkewSymmetricMatrix(r 50);

r p  = [r p1  r p2  r p3];

r ptilda  = SkewSymmetricMatrix(r p);

  Moment equations

M E 5  = r 52 tilda  F 25 tp + r 54 tilda  F 45 tp + r 50 tilda  F 05 tp +
    r ptilda  F p    (DyadicDotProduct(J 5 ,alfa5) + w 5 tilda
    DyadicDotProduct(J 5 , w 5));

M E 5 body = formula(M E 5);

M E 51  = M E 5 body(1); M E 52  = M E 5 body(2); M E 53  = M E 5 body(3);
```

D.3 Stabilizer Codes

Listing D.5: Matlab function to calculate basic rotation matrices

```
function R = BasicRotationMatrix(u,theta)

    if u == 1
        R = [1         0                 0;
             0    cos(theta)      sin(theta);
             0    sin(theta)    cos(theta)];

    elseif u == 2
        R = [cos(theta)      0     sin(theta);
                 0           1          0;
             sin(theta)      0     cos(theta)];
```

```

    elseif u == 3
        R = [cos(theta)    sin(theta)    0;
             sin(theta)    cos(theta)    0;
              0              0           1];
    end

end
```

Listing D.6: Matlab function to calculate partial derivative for symbolic expressions

```
function fout = deriv(f, g)

  deriv differentiates f with respect to g = g(t)
  the variable g = g(t) is a function of time

syms t x dx

lg =  diff(g,t), g ;
lx =  dx, x ;
f1 = subs(f, lg, lx);
f2 = diff(f1,x);
fout = subs(f2, lx, lg);

end
```

Listing D.7: Parameters loaded before the experiments started

```
clear; clc;
load( D : Google Drive Zafer Thesis MATLAB Simulink Stabilizer Workspace
    Kayitlari PIDParametersv00.mat )
  Global variables

Ts = 5e 4;

g = 9.81;

g 0 = [0; 0;  g];

  Masses

m 6 = 1e 3 4530.59;       kg

m 7 = 1e 3 2557.47;       kg

  Inertia Terms

J Motor GB = 8786000e 9;     kg.m 2  Inertia of rotating parts at the output
     of gearbox

J 6 1 1 = 19787090.89e 9;         J 6 1 2 = 330917.07e 9;
                        J 6 1 3 = 2341197.11e 9;              kg.m 2
J 6 2 1 = 330917.07e 9;           J 6 2 2 = 112319139.40e 9 + J Motor GB;
          J 6 2 3 = 6005.85e 9;                   kg.m 2
J 6 3 1 = 2341197.11e 9;          J 6 3 2 = 6005.85e 9;
                           J 6 3 3 = 94615995.73e 9;              kg.m 2

J 7 1 1 = 18797121.45e 9;         J 7 1 2 = 5730.37e 9;
                           J 7 1 3 = 1902546.14e 9;              kg.m 2
```

```
J721 = 5730.37e 9;              J722 = 13470946.42e 9;              J723 = 9452.80e 9;                  kg.m 2
J731 = 1902546.14e 9;          J732 = 9452.80e 9;              J733 = 8094810.07e 9 + JMotorGB;   kg.m 2

J6 = [J611 J612 J613; J621 J622 J623; J631 J632 J633];   Ineria dyadic resolved in Reference Frame 6   kg.m 2

J7 = [J711 J712 J713; J721 J722 J723; J731 J732 J733];   Ineria dyadic resolved in Reference Frame 7   kg.m 2

  Center of Mass

r61 = 0.43;            r62 = 3.09;    r63 = 1e 3  (166.93);   Defined in Frame 6

r71 = 1e 3  38.33;     r72 = 0;        r73 = 1e 3  (19.40);   Defined in Frame 7

rGO1 = 1e 3  266.1;    rGO2 = 0;       rGO3 = 0;   Defined in Frame 5

  Controller Parameters

Q(1,1) = 3.2828e+05;
Q(2,2) = 0.2;
Q(3,3) = 3.2828e+05;
Q(4,4) = 0.2;

p = 1;

R = p [1 0; 0 1];


  Ts = 2e 4; Previous sample time

Ts = 5e 4;

a1 = 783.4e 3;

a3 = 193e 3;

b1 = 193e 3;

s410 = 783.4e 3;

s420 = 783.4e 3;

dpiston = 40e 3;

reqstroke = 300e 3;

Apiston = pi dpiston 2/4;

convltmin2m3s = 1/1000/60;

  Qmax1 = 33 convltmin2m3s;   m3/s

  Qmax2 = 24 convltmin2m3s;   m3/s

```

```
 Intermediate Calculations

nmotor = 1500;   1/s

Fullstrokevolume = 376991.12e 9;   m 3

tfast = 21.4;   s

tslow = 22.1;   s

Qpump = nmotor  9.8e 3;   m 3/s

QQfast = Fullstrokevolume/tfast;   m 3/s

Q max 1 lt = QQfast  1000  60  tfast/tslow;   liter/min

Q max 2 lt = QQfast  1000  60;   liter/min

Q max 1 = Q max 1 lt  conv lt min 2 m3 s;   m 3/s

Q max 2 = Q max 2 lt  conv lt min 2 m3 s;   m 3/s

 Intermediate Calculations

s 41 max = Q max 1/Apiston;

s 42 max = Q max 2/Apiston;

t 1 min = 2 reqstroke/s 41 max;    Min coefficient must be 1.6 while working with sine (Now coef=3)

t 2 min = 2 reqstroke/s 42 max;    Min coefficient must be 1.6 while working with sine

theta131amplitute = pi/30;

theta132amplitute = pi/30;
```

Listing D.8: Checking the Lagrange equations using energy conservation method

```
clear
clc

syms t

syms J511 J512 J513 J521 J522 J523 J531 J532 J533 ...
     J611 J612 J613 J621 J622 J623 J631 J632 J633 ...
     J711 J712 J713 J721 J722 J723 J731 J732 J733 real ...

syms theta5(t) theta6(t) theta131(t) theta132(t)

syms m5 m6 m7 g

syms r51 r52 r53 r61 r62 r63 r71 r72 r73

syms rGO1

syms theta131132dot theta5131dot theta5132dot theta6131dot theta6132dot theta56dot

```

```
syms T5 T6

syms theta5state theta5dotstate theta6state theta6dotstate
    theta5ddotstate theta6ddotstate

syms theta5star(t) theta6star(t)

syms theta5starddotn theta6starddotn

  Definition of Angular Velocities

theta131dot = diff(theta131,t);

theta132dot = diff(theta132,t);

theta5dot = diff(theta5,t);

theta6dot = diff(theta6,t);

theta131ddot = diff(theta131,t,2);

theta132ddot = diff(theta132,t,2);

theta5ddot = diff(theta5,t,2);

theta6ddot = diff(theta6,t,2);

  Definition of State Variable Angular Velocities

theta5stardot = diff(theta5star,t);

theta6stardot = diff(theta6star,t);

theta5starddot = diff(theta5star,t,2);

theta6starddot = diff(theta6star,t,2);

  Simplified T5

T5manuelsimplified = ...
 ...
(theta5starddotn)  ...
( J622 + m6 r63 2 + m7 r73 2 ...
+ J711 sin(theta6(t)) 2 ...
+ J712 sin(2 theta6(t)) ...
+ (J722 + m7 r71 2) cos(theta6(t)) 2) ...
...
  diff(theta132(t), t, t)  ...
( J711 cos(theta131(t)) sin(theta6(t)) 2 ...
+ J712 cos(theta131(t)) sin(2 theta6(t)) ...
  J712 sin(theta131(t)) cos(2 theta6(t)) cos(theta5(t)   theta132(t))
    ...
+ (J622 + m6 r63 2 + m7 r73 2) cos(theta131(t)) ...
+ (J722 + m7 r71 2) cos(theta131(t)) cos(theta6(t)) 2 ...
  (J722   J711 + m7 r71 2)/2 sin(theta131(t)) sin(2 theta6(t)) cos(
    theta5(t)   theta132(t)) ...
  (m6 r63 rGO1 + m7 r73 rGO1) cos(theta131(t)) sin(theta5(t)) ...
+ m7 r71 r73 sin(theta131(t)) sin(theta6(t)) sin(theta5(t)
    theta132(t)) ...
+ m7 r71 rGO1 cos(theta131(t)) cos(theta5(t)) cos(theta6(t)) ...
```

```
  J622 + m6 r63 2 + m7 r73 2 ...
  J711 sin(theta6(t)) 2 ...
  J712 sin(2 theta6(t)) ...
  (J722 + m7 r71 2) cos(theta6(t)) 2) ...
...
+ m7 r71 r73 sin(theta6(t)) (theta6starddotn) ...
...
+ diff(theta131(t), t, t) ...
( J712 cos(2 theta6(t)) sin(theta5(t)   theta132(t)) ...
+ m7 r71 r73 sin(theta6(t)) cos(theta5(t)   theta132(t)) ...
+ (J722   J711 + m7 r71 2) sin(2 theta6(t)) sin(theta5(t)   theta132
    (t))/2 ...
  m7 r71 r73 sin(theta6(t))) ...
...
+ diff(theta131(t), t) 2 ...
( (m6 r63 rGO1 + m7 r73 rGO1) cos(theta132(t)) cos(theta5(t)
    theta132(t)) ...
+ (J733 + J633   J611   m7 r73 2   m6 r63 2) sin(2 (theta5(t)
    theta132(t)))/2 ...
+ (m7 r71 2   J711) cos(theta6(t)) 2 sin(2 (theta5(t)   theta132(t)))
    /2 ...
+ m7 r71 r73 cos(theta6(t)) cos(2 (theta5(t)   theta132(t))) ...
+ m7 r71 rGO1 cos(theta132(t)) cos(theta6(t)) sin(theta5(t)
    theta132(t)) ...
  J613 cos(2 (theta5(t)   theta132(t))) ...
+ J712 sin(2 theta6(t)) sin(2 (theta5(t)   theta132(t)))/2 ...
  J722 sin(theta6(t)) 2 sin(2 (theta5(t)   theta132(t)))/2) ...
...
+ diff(theta132(t), t) 2 ...
( J712 sin(theta5(t)   theta132(t)) (cos(theta131(t))   1) sin(
    theta131(t)) cos(2 theta6(t)) ...
  J712 sin(theta5(t)   theta132(t)) sin(theta131(t)) 2 sin(2 theta6(t)
    ) cos(theta5(t)   theta132(t)) ...
+ J613 sin(theta131(t)) 2 cos(2 (theta5(t)   theta132(t))) ...
+ (J611   J633   J733 + m7 r73 2 + m6 r63 2) sin(theta131(t)) 2 sin
    (2 (theta5(t)   theta132(t)))/2 ...
+ (J711   m7 r71 2) sin(theta131(t)) 2 cos(theta6(t)) 2 sin(2 (theta5
    (t)   theta132(t)))/2 ...
+ J722 sin(theta131(t)) 2 sin(theta6(t)) 2 sin(2 (theta5(t)   theta132
    (t)))/2 ...
+ (m6 r63 rGO1 + m7 r73 rGO1) cos(theta131(t)) 2 cos(theta5(t)) ...
  (m6 r63 rGO1 + m7 r73 rGO1) sin(theta131(t)) 2 sin(theta132(t))
    sin(theta5(t)   theta132(t)) ...
  m7 r71 r73 sin(theta131(t)) 2 cos(theta6(t)) cos(2 (theta5(t)
    theta132(t))) ...
  m7 r71 r73 (cos(theta131(t))   1) sin(theta131(t)) sin(theta6(t))
    cos(theta5(t)   theta132(t)) ...
+ m7 r71 rGO1 cos(theta131(t)) 2 sin(theta5(t)) cos(theta6(t)) ...
+ m7 r71 rGO1 sin(theta131(t)) 2 sin(theta132(t)) cos(theta6(t)) cos(
    theta5(t)   theta132(t)) ...
+ (J711   J722   m7 r71 2) (cos(theta131(t))   1) sin(theta131(t))
    sin(2 theta6(t)) sin(theta5(t)   theta132(t))/2) ...
...
  diff(theta5(t), t) diff(theta6(t), t) ...
( (J722   J711 + m7 r71 2) sin(2 theta6(t)) ...
  2 J712 cos(2 theta6(t))) ...
...
+ diff(theta6(t), t) diff(theta131(t), t) ...
( (J722   J711) cos(2 theta6(t)) sin(theta5(t)   theta132(t)) ...
+ J733 sin(theta5(t)   theta132(t)) ...
```

```
+ 2 J712 sin(2 theta6(t)) sin(theta5(t)   theta132(t)) ...
+ 2 m7 r71 2 cos(theta6(t)) 2 sin(theta5(t)   theta132(t)) ...
+ 2 m7 r71 r73 cos(theta6(t)) cos(theta5(t)   theta132(t))) ...
...
+ diff(theta6(t), t) diff(theta132(t), t)  ...
( 2 J712 cos(theta131(t)) cos(2 theta6(t)) ...
+ 2 J712 sin(theta131(t)) sin(2 theta6(t)) cos(theta5(t)   theta132(t)) ...
+ (J722   J711 + m7 r71 2) cos(theta131(t)) sin(2 theta6(t)) ...
+ (J722   J711) sin(theta131(t)) cos(2 theta6(t)) cos(theta5(t)   theta132(t)) ...
+ J733 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ 2 m7 r71 2 sin(theta131(t)) cos(theta6(t)) 2 cos(theta5(t)   theta132(t)) ...
  2 m7 r71 r73 sin(theta131(t)) cos(theta6(t)) sin(theta5(t)   theta132(t))) ...
...
+ diff(theta131(t), t) diff(theta132(t), t)  ...
( J622 sin(theta131(t)) ...
+ J711 sin(theta131(t)) sin(theta6(t)) 2 ...
  J711 sin(theta131(t)) cos(theta6(t)) 2 cos(2 (theta5(t)   theta132(t))) ...
+ (J722   J711 + m7 r71 2) (2 cos(theta131(t))   1) sin(2 theta6(t)) cos(theta5(t)   theta132(t))/2 ...
  2 (m6 r63 rGO1 + m7 r73 rGO1) sin(theta131(t)) cos(theta132(t)) sin(theta5(t)   theta132(t)) ...
+ 2 (m6 r63 2 + m7 r73 2) sin(theta131(t)) sin(theta5(t)   theta132(t)) 2 ...
+ J712 (1   2 cos(theta131(t))) cos(2 theta6(t)) cos(theta5(t)   theta132(t)) ...
+ J712 sin(theta131(t)) sin(2 theta6(t)) (1 + cos(2 (theta5(t)   theta132(t)))) ...
+ (J733   J611 + J633) sin(theta131(t)) cos(2 (theta5(t)   theta132(t))) ...
+ J722 sin(theta131(t)) cos(theta6(t)) 2 ...
  J722 sin(theta131(t)) sin(theta6(t)) 2 cos(2 (theta5(t)   theta132(t))) ...
+ 2 J613 sin(theta131(t)) sin(2 (theta5(t)   theta132(t))) ...
+ 2 m7 r71 2 sin(theta131(t)) cos(theta6(t)) 2 cos(theta5(t)   theta132(t)) 2 ...
  m7 r71 r73 (2 cos(theta131(t))   1) sin(theta6(t)) sin(theta5(t)   theta132(t)) ...
  2 m7 r71 r73 sin(theta131(t)) cos(theta6(t)) sin(2 (theta5(t)   theta132(t))) ...
+ 2 m7 r71 rGO1 sin(theta131(t)) cos(theta132(t)) cos(theta6(t)) cos(theta5(t)   theta132(t))) ...
...
+ m7 r71 r73 cos(theta6(t)) diff(theta6(t), t) 2 ...
...
  (m6 r63 + m7 r73) g cos(theta131(t)) cos(theta5(t)   theta132(t)) ...
  g m7 r71 cos(theta131(t)) cos(theta6(t)) sin(theta5(t)   theta132(t));

  Simplified T 6
disp ( Simplified T 6 )

T6manuelsimplified = ...
...
diff(theta5(t), t) 2 ...
```

```
...
((m 7  r 71 2   J 711 + J 722) sin (2 theta 6(t))/2 ...   1 + 3 + 4
  J 712 cos (2 theta 6(t))) ...          2
...
 diff(theta 132(t), t, t) ...
...
( m 7  r 71  r 73 sin(theta 6(t)) ...
+ m 7  r 71 2 sin(theta 131(t)) sin(theta 5(t)   theta 132(t)) ...
                              5
+ m 7  r 71  r 73 cos(theta 131(t)) sin(theta 6(t)) ...
                                              7
+ m 7  r 71  r 73 sin(theta 131(t)) cos(theta 6(t)) cos(theta 5(t)
   theta 132(t)) ...    8
+ J 733 sin(theta 131(t)) sin(theta 5(t)   theta 132(t))...
                                       9
  m 7  r 71  r GO1 cos(theta 131(t)) sin(theta 5(t)) sin(theta 6(t)) ...
                         6
  m 7  r 71  r GO1 sin(theta 131(t)) sin(theta 132(t)) cos(theta 6(t))) ...
                      1 0
...
+ m 7  r 71  r 73 sin(theta 6(t)) ( theta 5 star ddot n) ...
...
+ diff(theta 131(t), t, t) ...
...
( (m 7  r 71 2 + J 733) ...
+ (J 733  + m 7  r 71 2) cos(theta 5(t)   theta 132(t)) ...
                                        3 4
  m 7  r 71  r 73 cos(theta 6(t)) sin(theta 5(t)   theta 132(t)) ...
                             3 5
+ m 7  r 71  r GO1 cos(theta 132(t))) cos(theta 6(t)) ...
                                           3 6
...
+ (theta 6 star ddot n) ( m 7  r 71 2 + J 733)...
...
+ diff(theta 132(t), t) 2 ...
...
( (J 733 + m 7  r 71 2) sin(theta 131(t)) cos(theta 5(t)   theta 132(t)) ...
                                                       1 1 + 14
  J 712 cos(theta 131(t)) 2 cos (2 theta 6(t)) ...

   1 2
+ J 712 sin(theta 131(t)) 2 cos (2 theta 6(t)) cos(theta 5(t)   theta 132(t)
   ) 2 ...                                           1 3
+ J 712 sin (2 theta 131(t)) sin (2 theta 6(t)) cos(theta 5(t)   theta 132(t)
   )...                                              2 9
+ (m 7  r 71 2   J 711 + J 722) cos(theta 131(t)) 2 sin(theta 6(t)) cos(
   theta 6(t)) ...                                      15 + 16 + 21
+ (J 722   J 711) sin(theta 131(t)) cos(theta 131(t)) cos (2 theta 6(t)) cos
   (theta 5(t)   theta 132(t)) ...              2 0 + 23 + 19 + 22
  (m 7  r 71 2   J 711 + J 722) sin(theta 131(t)) 2 sin(theta 6(t)) cos(
   theta 6(t)) cos(theta 5(t)   theta 132(t)) 2 ... 17 + 18 + 25
+ m 7  r 71 2 sin(theta 131(t)) cos(theta 131(t)) cos (2 theta 6(t)) cos(
   theta 5(t)   theta 132(t)) ...                     2 8
  m 7  r 71  r 73 sin(theta 131(t)) (1 + cos(theta 131(t))) cos(theta 6(t))
   sin(theta 5(t)   theta 132(t)) ...            2 6 + 31
+ m 7  r 71  r 73 sin(theta 131(t)) 2 sin(theta 6(t)) sin(theta 5(t)
   theta 132(t)) cos(theta 5(t)   theta 132(t))...    3 2
+ m 7  r 71  r GO1 cos(theta 131(t)) 2 cos(theta 5(t)) sin(theta 6(t)) ...
                                                          2 4
  m 7  r 71  r GO1 sin(theta 131(t)) 2 sin(theta 132(t)) sin(theta 6(t)) sin(
```

```
     theta5(t)   theta132(t)) ...                30
+ m7  r71  rGO1 sin(theta131(t)) (1 + cos(theta131(t))) cos(theta132(t)
     ) cos(theta6(t))) ...                          27 + 33
...
+ diff(theta131(t), t) 2 ...
...
( J712 cos(2 theta6(t)) sin(theta5(t)   theta132(t)) 2 ...
                                                 37
+ (J711   J722   m7  r71 2)/2 sin(2 theta6(t)) sin(theta5(t)
     theta132(t)) 2 ...              38 + 39 + 40
  m7  r71  r73 sin(theta6(t)) sin(theta5(t)   theta132(t)) cos(theta5(
     t)   theta132(t)) ...   41
+ m7  r71  rGO1 cos(theta132(t)) sin(theta6(t)) cos(theta5(t)
     theta132(t))) ...              42
...
+ diff(theta131(t), t) diff(theta132(t), t) ...
...
( (J733 + m7  r71 2) (1   cos(theta131(t))) sin(theta5(t)   theta132(t
     )) ...                                              43 + 44 + 45
     + 48
  (J711   J722   m7  r71 2) cos(theta131(t)) cos(2 theta6(t)) sin(
     theta5(t)   theta132(t)) ...                             46 + 47 +
      51
+ (J711   J722   m7  r71 2)/2 sin(theta131(t)) sin(2 theta6(t)) sin
     (2(theta5(t)   theta132(t))) ...                        56 + 57 +
     60
+ m7  r71  r73 sin(theta131(t)) sin(theta6(t)) (1   cos(2(theta5(t)
     theta132(t)))) ...                                  50 + 58
+ 2 J712 cos(theta131(t)) sin(2 theta6(t)) sin(theta5(t)   theta132(t)
     ) ...                                               52
+ J712 sin(theta131(t)) cos(2 theta6(t)) sin(2(theta5(t)   theta132(t
     ))) ...                                             53
+ m7  r71  r73 cos(theta6(t)) cos(theta5(t)   theta132(t)) ...
                                                                 54
  m7  r71  rGO1 sin(theta132(t)) cos(theta6(t)) ...

      49
  2 m7  r71  rGO1 sin(theta131(t)) cos(theta132(t)) sin(theta6(t)) sin(
     theta5(t)   theta132(t))) ...                       55 + 59
...
  diff(theta5(t), t) diff(theta132(t), t) ...
...
( (J733 + m7  r71 2) sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
                              61 + 67
  (J711   J722   m7  r71 2) cos(theta131(t)) sin(2 theta6(t)) ...
                                 63 + 64 + 68
  (J711   J722   m7  r71 2) sin(theta131(t)) cos(2 theta6(t)) cos(
     theta5(t)   theta132(t)) ...   65 + 66 + 69
  2 J712 cos(theta131(t)) cos(2 theta6(t)) ...
                                                     62
+ 2 J712 sin(theta131(t)) sin(2 theta6(t)) cos(theta5(t)   theta132(t)
     ) ...                        70
  2 m7  r71  r73 sin(theta131(t)) cos(theta6(t)) sin(theta5(t)
     theta132(t))) ...                 71
...
  diff(theta5(t), t) diff(theta131(t), t)  ...
...
( (J733 + m7  r71 2) sin(theta5(t)   theta132(t)) ...
                            74
  (J711   J722   m7  r71 2) cos(2 theta6(t)) sin(theta5(t)   theta132
```

```
    (t)) ... 72 + 73 + 75
+ 2 J712 sin(2 theta6(t)) sin(theta5(t)   theta132(t)) ...
                        76
+ 2 m7  r71  r73 cos(theta6(t)) cos(theta5(t)   theta132(t))) ...
                77
...
  g m7  r71 sin(theta131(t)) cos(theta6(t)) ...
  g m7  r71 cos(theta131(t)) sin(theta6(t)) cos(theta5(t)   theta132(t
    ));

  Rearrange equations of motions according to state variables
disp( Rearrange equations of motions according to state variables )

variableoriginals =  theta5(t), theta6(t), theta5dot, theta6dot ;

variablesubs =  theta5star + theta132(t), theta6star   theta131(t),
    theta5stardot + theta132dot, theta6stardot   theta131dot ;

T5manuelsimplified = subs(T5manuelsimplified, variableoriginals,
    variablesubs);

T6manuelsimplified = subs(T6manuelsimplified, variableoriginals,
    variablesubs);

T5manuelsimplified = simplify(T5manuelsimplified);

T6manuelsimplified = simplify(T6manuelsimplified);

  Lagrange Equations
disp( Lagrange Equations )

Lagrange1 = T5 == T5manuelsimplified;

Lagrange2 = T6 == T6manuelsimplified;

[K,L] = equationsToMatrix([Lagrange1, Lagrange2], [theta5starddotn,
    theta6starddotn]);

X = linsolve(K,L);

X = simplify(X);

X1 = X(1);   X2 = X(2);

multipliedangvel =  theta131dot theta132dot, theta5dot
    theta131dot, theta5dot theta132dot, ...
                   theta6dot theta131dot, theta6dot theta132dot,
                        theta5dot theta6dot ;

shortenedangvel =  theta131132dot, theta5131dot, theta5132dot,
    theta6131dot, theta6132dot, theta56dot ;

X1 = subs(X1, multipliedangvel, shortenedangvel);

X2 = subs(X2, multipliedangvel, shortenedangvel);

X1 = collect(X1, [theta131dot2, theta132dot2, theta131132dot,
    T5, T6, ...
             theta5131dot, theta5132dot, theta6131dot,
                 theta6132dot, theta56dot, ...
```

```
                theta5dot2, theta6dot2, theta131ddot,
                   theta132ddot, theta5ddot, theta6ddot]);

X2 = collect(X2, [theta131dot2, theta132dot2, theta131132dot,
    T5, T6, ...
                theta5131dot, theta5132dot, theta6131dot,
                   theta6132dot, theta56dot, ...
                theta5dot2, theta6dot2, theta131ddot,
                   theta132ddot, theta5ddot, theta6ddot]);

X1 = subs(X1, shortenedangvel, multipliedangvel);

X2 = subs(X2, shortenedangvel, multipliedangvel);

variablenames =  g, m5, m6, m7, J511, J512, J513, J521, J522,
    J523, J531, J532, J533, ...
                                J611, J612, J613, J621, J622,
                                   J623, J631, J632, J633, ...
                                J711, J712, J713, J721, J722,
                                   J723, J731, J732, J733, ...
                                r51,  r52,  r53,  r61,  r62,  r63
                                   ,  r71,  r72,  r73, rGO1 ;

variablevalues =   9.81 , 1e 3 15325.23 , 1e 3 4530.59 , 1e 3 2557.47 ,...
      g   Masses
                   595993867.8e 9 , 751458.22e 9 , 5144079.54e 9 ,...
                         Inertia Terms J511 J512 J513
                  751458.22e 9 , 641588537.25e 9 , 145338017.74e 9 ,...
                        Inertia Terms J521 J522 J523
                  5144079.54e 9 , 145338017.74e 9 , 336549983.33e 9 ,...
                       Inertia Terms J531 J532 J533
                   18898872.13e 9 , 328799.77e 9 , 1485177.89e 9 ,...
                         Inertia Terms J611 J612 J613
                  328799.77e 9 , 110603524.64e 9 , 3815.25e 9 ,...
                           Inertia Terms J621 J622 J623
                  1485177.89e 9 , 3815.25e 9 , 93783786.61e 9 ,...
                           Inertia Terms J631 J632 J633
                   11824640.51e 9 , 1679850.64e 9 , 137920.31e 9 ,...
                          Inertia Terms J711 J712 J713
                  1679850.64e 9 , 24252622.46e 9 , 37299.29e 9 ,...
                          Inertia Terms J721 J722 J723
                   137920.31e 9 , 37299.29e 9 , 15203370.83e 9 ,...
                            Inertia Terms J731 J732 J733
                   1e 3  98.45 , 0 , 1e 3  (60.66) ,...
                                   Center of Mass r5
                   0 , 0 , 1e 3  (168.02) ,...
                                          Center of Mass r6
                   1e 3  39.19 , 0 , 1e 3  (16.28) ,...
                                   Center of Mass r7
                   1e 3  266.1;
                                                                 Center
                      of Mass rGO

X1subs = subs(X1, variablenames, variablevalues)
X2subs = subs(X2, variablenames, variablevalues)


 GlobalVariablesv1
theta5 = theta5starsim.Data;
theta6 = theta6starsim.Data;
```

```
theta5dot = theta5stardotsim.Data;
theta6dot = theta6stardotsim.Data;

time = theta5starsim.Time;

theta5 = deg2rad(theta5);
theta6 = deg2rad(theta6);

b6x = 37; b6y = 154.5; b6z = 138.5;

b7x = 38.8; b7y = 134; b7z = 115.5;

body6 = [b6x  b6y  b6z;   0
         b6x  b6y  b6z;   1
         b6x  b6y  b6z;   2
         b6x  b6y  b6z;   3
         b6x  b6y  b6z;   4
         b6x  b6y  b6z;   5
         b6x  b6y  b6z;   6
         b6x  b6y  b6z;   7
         b6x  b6y  b6z;   8
         b6x  b6y  b6z;   9
         b6x  b6y  b6z;   10
         b6x  b6y  b6z;   11
         b6x  b6y  b6z;   12
         b6x  b6y  b6z;   13
         b6x  b6y  b6z;   14
         b6x  b6y  b6z;   15
         b6x  b6y  b6z];  16

body6 = body6 ;

body7 = [b7x  b7x  b7x  b7x  b7x  b7x  b7x  b7x  b7x    0    0    0;
         b7y  b7y  b7y  b7y  b7y  b7y  b7y  b7y    0    0    0    0;
         b7z  b7z  b7z  b7z  b7z  b7z  b7z  b7z    0    0  b6z  b6z];


  Basic Column Matrices

u1 = [1; 0; 0];
u2 = [0; 1; 0];
u3 = [0; 0; 1];

rGO = [rGO1; 0; 0];       Offset distance between Point O   Ground

figure(1)
h1 = gcf;
h1.Units =  normalized ;
h1.Position = [0.0 0.1 0.5 0.8];

figure(2)
h2 = gcf;
h2.Units =  normalized ;
h2.Position = [0.5 0.3 0.5 0.4];


for i = 1:length(theta5)
    i
    T5612(:,:,i) = BasicRotationMatrix(3,theta5(i))
       BasicRotationMatrix(1,pi/2);   C(5,6)
```

```
    T 6 7 1 2 (:,:,i) = BasicRotationMatrix(3,theta 6 (i));   C(6,7)
    T 5 7 1 2 (:,:,i) = T 5 6 1 2 (:,:,i)   T 6 7 1 2 (:,:,i) ;      C(5,7)
    T 2 5 1 3 = BasicRotationMatrix(2,0) BasicRotationMatrix(1,0) BasicRotationMatrix(2, pi/2);   C(0,5)

    body 6 (:,:,i+1) = T 2 5 1 3   T 5 6 1 2 (:,:,i)   body 6 (:,:,1);
    body 7 (:,:,i+1) = T 2 5 1 3   T 5 7 1 2 (:,:,i)   body 7 (:,:,1);

    figure(1)
    plot3(body 7 (1,:,i+1), body 7 (2,:,i+1), body 7 (3,:,i+1))
    hold on
    plot3(body 6 (1,:,i+1), body 6 (2,:,i+1), body 6 (3,:,i+1))
    axis equal
    xlim([180 180])
    ylim([180 180])
    zlim([180 180])
    hold off

    F1(i) = getframe(gcf);

    w 6 (:,i) = [  0;
                   theta 5 dot (i);
                   0];

    w 7 (:,i) = [  theta 5 dot (i) sin(theta 6 (i));
                   theta 5 dot (i) cos(theta 6 (i));
                   theta 6 dot (i)];

      Definition of Translational Velocities

    v 6 (:,i) = [  theta 5 dot (i) r 63; ...
                   0; ...
                   0];

    v 7 (:,i) = [  r 73  theta 5 dot (i) cos(theta 6 (i));
                   r 73  theta 5 dot (i) sin(theta 6 (i)) + r 71  theta 6 dot (i) ; ...
                   r 71  theta 5 dot (i) cos(theta 6 (i))];

    U 6 (i) =  g m 6 ( r 63 sin(theta 5 (i))   r GO1);
    T 6 (i) = 1/2   (m 6  v 6 (:,i)  v 6 (:,i) + w 6 (:,i)  J 6  w 6 (:,i));

    U 7 (i) =  g m 7 ( r 73 sin(theta 5 (i))   r GO1   r 71 cos(theta 6 (i)) cos(theta 5 (i)));
    T 7 (i) = 1/2   (m 7  v 7 (:,i)  v 7 (:,i) + w 7 (:,i)  J 7  w 7 (:,i));

    Etotal(i) = U 6 (i) + T 6 (i) + U 7 (i) + T 7 (i);

    figure(2)
    plot(time(1:i),Etotal(1:i))
    xlabel( Time, [s] )
    ylabel( Total Energy, [J] )

    F2(i) = getframe(gcf);
    drawnow

end
```

```
  Recording the video
video1 = VideoWriter( StabilizerMotion.mp4 ,  MPEG 4 );
video1.FrameRate = 100;   setting the Frame Rate
open(video1)
writeVideo(video1, F1);
close(video1)

video2 = VideoWriter( TotalEnergy.mp4 ,  MPEG 4 );
video2.FrameRate = 100;   setting the Frame Rate
open(video2)
writeVideo(video2, F2);
close(video2)
```

Listing D.9: Linearization of the non-linear Lagrange equations

```
clear
clc

syms t

syms J511 J512 J513 J521 J522 J523 J531 J532 J533 ...
     J611 J612 J613 J621 J622 J623 J631 J632 J633 ...
     J711 J712 J713 J721 J722 J723 J731 J732 J733 real ...

syms theta5(t) theta6(t) theta131(t) theta132(t)

syms m5 m6 m7 g

syms r51 r52 r53 r61 r62 r63 r71 r72 r73

syms rGO1

syms theta131132dot theta5131dot theta5132dot theta6131dot
    theta6132dot theta56dot

syms T5 T6

syms theta5state theta5dotstate theta6state theta6dotstate
    theta5ddotstate theta6ddotstate

syms theta5star(t) theta6star(t)

syms theta5starddotn theta6starddotn

 Definition of Angular Velocities

theta131dot = diff(theta131,t);

theta132dot = diff(theta132,t);

theta5dot = diff(theta5,t);

theta6dot = diff(theta6,t);

theta131ddot = diff(theta131,t,2);

theta132ddot = diff(theta132,t,2);

theta5ddot = diff(theta5,t,2);

```

```
theta6ddot = diff(theta6,t,2);

 Definition of State Variable Angular Velocities

theta5stardot = diff(theta5star,t);

theta6stardot = diff(theta6star,t);

theta5starddot = diff(theta5star,t,2);

theta6starddot = diff(theta6star,t,2);

 Simplified T5

T5manuelsimplified = ...
 ...
(theta5starddotn) ...
( J622 + m6 r63 2 + m7 r73 2 ...
+ J711 sin(theta6(t)) 2 ...
+ J712 sin(2 theta6(t)) ...
+ (J722 + m7 r71 2) cos(theta6(t)) 2) ...
...
 diff(theta132(t), t, t) ...
( J711 cos(theta131(t)) sin(theta6(t)) 2 ...
+ J712 cos(theta131(t)) sin(2 theta6(t)) ...
 J712 sin(theta131(t)) cos(2 theta6(t)) cos(theta5(t)  theta132(t)) ...
+ (J622 + m6 r63 2 + m7 r73 2) cos(theta131(t)) ...
+ (J722 + m7 r71 2) cos(theta131(t)) cos(theta6(t)) 2 ...
 (J722  J711 + m7 r71 2)/2 sin(theta131(t)) sin(2 theta6(t)) cos(theta5(t)  theta132(t)) ...
 (m6 r63 rGO1 + m7 r73 rGO1) cos(theta131(t)) sin(theta5(t)) ...
+ m7 r71 r73 sin(theta131(t)) sin(theta6(t)) sin(theta5(t)  theta132(t)) ...
+ m7 r71 rGO1 cos(theta131(t)) cos(theta5(t)) cos(theta6(t)) ...
 J622 + m6 r63 2 + m7 r73 2 ...
 J711 sin(theta6(t)) 2 ...
 J712 sin(2 theta6(t)) ...
 (J722 + m7 r71 2) cos(theta6(t)) 2) ...
...
+ m7 r71 r73 sin(theta6(t)) (theta6starddotn) ...
...
+ diff(theta131(t), t, t) ...
( J712 cos(2 theta6(t)) sin(theta5(t)  theta132(t)) ...
+ m7 r71 r73 sin(theta6(t)) cos(theta5(t)  theta132(t)) ...
+ (J722  J711 + m7 r71 2) sin(2 theta6(t)) sin(theta5(t)  theta132(t))/2 ...
 m7 r71 r73 sin(theta6(t))) ...
...
+ diff(theta131(t), t) 2 ...
( (m6 r63 rGO1 + m7 r73 rGO1) cos(theta132(t)) cos(theta5(t)  theta132(t)) ...
+ (J733 + J633  J611  m7 r73 2  m6 r63 2) sin(2 (theta5(t)  theta132(t)))/2 ...
+ (m7 r71 2  J711) cos(theta6(t)) 2 sin(2 (theta5(t)  theta132(t)))/2 ...
+ m7 r71 r73 cos(theta6(t)) cos(2 (theta5(t)  theta132(t))) ...
+ m7 r71 rGO1 cos(theta132(t)) cos(theta6(t)) sin(theta5(t)  theta132(t)) ...
 J613 cos(2 (theta5(t)  theta132(t))) ...
```

```
+ J712 sin(2 theta6(t)) sin(2 (theta5(t)   theta132(t)))/2 ...
  J722 sin(theta6(t)) 2 sin(2 (theta5(t)   theta132(t)))/2) ...
...
+ diff(theta132(t), t) 2  ...
( J712 sin(theta5(t)   theta132(t)) (cos(theta131(t))   1) sin(
    theta131(t)) cos(2 theta6(t)) ...
  J712 sin(theta5(t)   theta132(t)) sin(theta131(t)) 2 sin(2 theta6(t)
    ) cos(theta5(t)   theta132(t)) ...
+ J613 sin(theta131(t)) 2 cos(2 (theta5(t)   theta132(t))) ...
+ (J611   J633   J733 + m7 r73 2 + m6 r63 2) sin(theta131(t)) 2 sin
    (2 (theta5(t)   theta132(t)))/2 ...
+ (J711   m7 r71 2) sin(theta131(t)) 2 cos(theta6(t)) 2 sin(2 (theta5
    (t)   theta132(t)))/2 ...
+ J722 sin(theta131(t)) 2 sin(theta6(t)) 2 sin(2 (theta5(t)   theta132
    (t)))/2 ...
+ (m6 r63 rGO1 + m7 r73 rGO1) cos(theta131(t)) 2 cos(theta5(t)) ...
  (m6 r63 rGO1 + m7 r73 rGO1) sin(theta131(t)) 2 sin(theta132(t))
    sin(theta5(t)   theta132(t)) ...
  m7 r71 r73 sin(theta131(t)) 2 cos(theta6(t)) cos(2 (theta5(t)
    theta132(t))) ...
  m7 r71 r73 (cos(theta131(t))   1) sin(theta131(t)) sin(theta6(t))
    cos(theta5(t)   theta132(t)) ...
+ m7 r71 rGO1 cos(theta131(t)) 2 sin(theta5(t)) cos(theta6(t)) ...
+ m7 r71 rGO1 sin(theta131(t)) 2 sin(theta132(t)) cos(theta6(t)) cos(
    theta5(t)   theta132(t)) ...
+ (J711   J722   m7 r71 2) (cos(theta131(t))   1) sin(theta131(t))
    sin(2 theta6(t)) sin(theta5(t)   theta132(t))/2) ...
...
  diff(theta5(t), t) diff(theta6(t), t)  ...
( (J722   J711 + m7 r71 2) sin(2 theta6(t)) ...
  2 J712 cos(2 theta6(t))) ...
...
+ diff(theta6(t), t) diff(theta131(t), t)  ...
( (J722   J711) cos(2 theta6(t)) sin(theta5(t)   theta132(t)) ...
+ J733 sin(theta5(t)   theta132(t)) ...
+ 2 J712 sin(2 theta6(t)) sin(theta5(t)   theta132(t)) ...
+ 2 m7 r71 2 cos(theta6(t)) 2 sin(theta5(t)   theta132(t)) ...
+ 2 m7 r71 r73 cos(theta6(t)) cos(theta5(t)   theta132(t))) ...
...
+ diff(theta6(t), t) diff(theta132(t), t)  ...
( 2 J712 cos(theta131(t)) cos(2 theta6(t)) ...
+ 2 J712 sin(theta131(t)) sin(2 theta6(t)) cos(theta5(t)   theta132(t)
    ) ...
+ (J722   J711 + m7 r71 2) cos(theta131(t)) sin(2 theta6(t)) ...
+ (J722   J711) sin(theta131(t)) cos(2 theta6(t)) cos(theta5(t)
    theta132(t)) ...
+ J733 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ 2 m7 r71 2 sin(theta131(t)) cos(theta6(t)) 2 cos(theta5(t)
    theta132(t)) ...
  2 m7 r71 r73 sin(theta131(t)) cos(theta6(t)) sin(theta5(t)
    theta132(t))) ...
...
+ diff(theta131(t), t) diff(theta132(t), t)  ...
( J622 sin(theta131(t)) ...
+ J711 sin(theta131(t)) sin(theta6(t)) 2 ...
  J711 sin(theta131(t)) cos(theta6(t)) 2 cos(2 (theta5(t)   theta132(t
    ))) ...
+ (J722   J711 + m7 r71 2) (2 cos(theta131(t))   1) sin(2 theta6(t))
    cos(theta5(t)   theta132(t))/2 ...
  2 (m6 r63 rGO1 + m7 r73 rGO1) sin(theta131(t)) cos(theta132(t))
```

```
    sin(theta5(t)   theta132(t)) ...
+ 2 (m6 r63 2 + m7 r73 2) sin(theta131(t)) sin(theta5(t)   theta132(
    t)) 2 ...
+ J712 (1   2 cos(theta131(t))) cos(2 theta6(t)) cos(theta5(t)
    theta132(t)) ...
+ J712 sin(theta131(t)) sin(2 theta6(t)) (1 + cos(2 (theta5(t)
    theta132(t)))) ...
+ (J733   J611 + J633) sin(theta131(t)) cos(2 (theta5(t)   theta132(t
    ))) ...
+ J722 sin(theta131(t)) cos(theta6(t)) 2 ...
  J722 sin(theta131(t)) sin(theta6(t)) 2 cos(2 (theta5(t)   theta132(t
    ))) ...
+ 2 J613 sin(theta131(t)) sin(2 (theta5(t)   theta132(t))) ...
+ 2 m7 r71 2 sin(theta131(t)) cos(theta6(t)) 2 cos(theta5(t)
    theta132(t)) 2 ...
  m7 r71 r73 (2 cos(theta131(t))   1) sin(theta6(t)) sin(theta5(t)
    theta132(t)) ...
  2 m7 r71 r73 sin(theta131(t)) cos(theta6(t)) sin(2 (theta5(t)
    theta132(t))) ...
+ 2 m7 r71 rGO1 sin(theta131(t)) cos(theta132(t)) cos(theta6(t)) cos(
    theta5(t)   theta132(t))) ...
...
+ m7 r71 r73 cos(theta6(t)) diff(theta6(t), t) 2 ...
...
  (m6 r63 + m7 r73) g cos(theta131(t)) cos(theta5(t)   theta132(t))
    ...
  g m7 r71 cos(theta131(t)) cos(theta6(t)) sin(theta5(t)   theta132(t
    ));

  Simplified T6
disp( Simplified T6 )

T6manuelsimplified = ...
...
diff(theta5(t), t) 2 ...
...
((m7 r71 2   J711 + J722) sin(2 theta6(t))/2 ...   1 + 3 + 4
  J712 cos(2 theta6(t))) ...          2
...
  diff(theta132(t), t, t) ...
...
( m7 r71 r73 sin(theta6(t)) ...
+ m7 r71 2 sin(theta131(t)) sin(theta5(t)   theta132(t)) ...
                          5
+ m7 r71 r73 cos(theta131(t)) sin(theta6(t)) ...
                                    7
+ m7 r71 r73 sin(theta131(t)) cos(theta6(t)) cos(theta5(t)
    theta132(t)) ...    8
+ J733 sin(theta131(t)) sin(theta5(t)   theta132(t))...
                               9
  m7 r71 rGO1 cos(theta131(t)) sin(theta5(t)) sin(theta6(t)) ...
                       6
  m7 r71 rGO1 sin(theta131(t)) sin(theta132(t)) cos(theta6(t))) ...
                    10
...
+ m7 r71 r73 sin(theta6(t)) (theta5starddotn) ...
...
+ diff(theta131(t), t, t) ...
...
( (m7 r71 2 + J733) ...
```

```
+ (J733  + m7 r71 2) cos(theta5(t)   theta132(t)) ...
                                    34
  m7 r71 r73 cos(theta6(t)) sin(theta5(t)   theta132(t)) ...
                          35
+ m7 r71 rGO1 cos(theta132(t))) cos(theta6(t)) ...
                                       36
...
+ (theta6starddotn) (m7 r71 2 + J733)...
...
+ diff(theta132(t), t) 2 ...
...
( (J733 + m7 r71 2) sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
                                                   11 + 14
  J712 cos(theta131(t)) 2 cos(2 theta6(t)) ...

   12
+ J712 sin(theta131(t)) 2 cos(2 theta6(t)) cos(theta5(t)   theta132(t)
   ) 2 ...                                         13
+ J712 sin(2 theta131(t)) sin(2 theta6(t)) cos(theta5(t)   theta132(t)
   )...                                            29
+ (m7 r71 2   J711 + J722) cos(theta131(t)) 2 sin(theta6(t)) cos(
   theta6(t)) ...                                     15 + 16 + 21
+ (J722   J711) sin(theta131(t)) cos(theta131(t)) cos(2 theta6(t)) cos
   (theta5(t)   theta132(t)) ...                   20 + 23 + 19 + 22
  (m7 r71 2   J711 + J722) sin(theta131(t)) 2 sin(theta6(t)) cos(
   theta6(t)) cos(theta5(t)   theta132(t)) 2 ...  17 + 18 + 25
+ m7 r71 2 sin(theta131(t)) cos(theta131(t)) cos(2 theta6(t)) cos(
   theta5(t)   theta132(t)) ...                       28
  m7 r71 r73 sin(theta131(t)) (1 + cos(theta131(t))) cos(theta6(t))
   sin(theta5(t)   theta132(t)) ...             26 + 31
+ m7 r71 r73 sin(theta131(t)) 2 sin(theta6(t)) sin(theta5(t)
   theta132(t)) cos(theta5(t)   theta132(t))...    32
+ m7 r71 rGO1 cos(theta131(t)) 2 cos(theta5(t)) sin(theta6(t)) ...
                                                    24
  m7 r71 rGO1 sin(theta131(t)) 2 sin(theta132(t)) sin(theta6(t)) sin(
   theta5(t)   theta132(t)) ...            30
+ m7 r71 rGO1 sin(theta131(t)) (1 + cos(theta131(t))) cos(theta132(t)
   ) cos(theta6(t))) ...                           27 + 33
...
+ diff(theta131(t), t) 2 ...
...
( J712 cos(2 theta6(t)) sin(theta5(t)   theta132(t)) 2 ...
                                          37
+ (J711   J722   m7 r71 2)/2 sin(2 theta6(t)) sin(theta5(t)
   theta132(t)) 2 ...              38 + 39 + 40
  m7 r71 r73 sin(theta6(t)) sin(theta5(t)   theta132(t)) cos(theta5(
   t)   theta132(t)) ...   41
+ m7 r71 rGO1 cos(theta132(t)) sin(theta6(t)) cos(theta5(t)
   theta132(t))) ...               42
...
+ diff(theta131(t), t) diff(theta132(t), t) ...
...
( (J733 + m7 r71 2) (1   cos(theta131(t))) sin(theta5(t)   theta132(t
   )) ...                                                    43 + 44 + 45
   + 48
  (J711   J722   m7 r71 2) cos(theta131(t)) cos(2 theta6(t)) sin(
   theta5(t)   theta132(t)) ...                                46 + 47 +
    51
+ (J711   J722   m7 r71 2)/2 sin(theta131(t)) sin(2 theta6(t)) sin
   (2 (theta5(t)   theta132(t))) ...                         56 + 57 +
```

```
    60
+ m7 r71 r73 sin(theta131(t)) sin(theta6(t)) (1   cos(2(theta5(t)
    theta132(t)))) ...                                  50 + 58
+ 2 J712 cos(theta131(t)) sin(2 theta6(t)) sin(theta5(t)   theta132(t)
    ) ...                                            52
+ J712 sin(theta131(t)) cos(2 theta6(t)) sin(2(theta5(t)   theta132(t
    ))) ...                                             53
+ m7 r71 r73 cos(theta6(t)) cos(theta5(t)   theta132(t)) ...
                                                              54
  m7 r71 rGO1 sin(theta132(t)) cos(theta6(t)) ...

    49
  2 m7 r71 rGO1 sin(theta131(t)) cos(theta132(t)) sin(theta6(t)) sin(
    theta5(t)   theta132(t))) ...                       55 + 59
...
  diff(theta5(t), t) diff(theta132(t), t) ...
...
( (J733 + m7 r71 2) sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
                               61 + 67
  (J711   J722   m7 r71 2) cos(theta131(t)) sin(2 theta6(t)) ...
                                 63 + 64 + 68
  (J711   J722   m7 r71 2) sin(theta131(t)) cos(2 theta6(t)) cos(
    theta5(t)   theta132(t)) ...   65 + 66 + 69
  2 J712 cos(theta131(t)) cos(2 theta6(t)) ...
                                                      62
+ 2 J712 sin(theta131(t)) sin(2 theta6(t)) cos(theta5(t)   theta132(t)
    ) ...                     70
  2 m7 r71 r73 sin(theta131(t)) cos(theta6(t)) sin(theta5(t)
    theta132(t))) ...                  71
...
  diff(theta5(t), t) diff(theta131(t), t)  ...
...
( (J733 + m7 r71 2) sin(theta5(t)   theta132(t)) ...
                             74
  (J711   J722   m7 r71 2) cos(2 theta6(t)) sin(theta5(t)   theta132
    (t)) ... 72 + 73 + 75
+ 2 J712 sin(2 theta6(t)) sin(theta5(t)   theta132(t)) ...
                          76
+ 2 m7 r71 r73 cos(theta6(t)) cos(theta5(t)   theta132(t))) ...
                      77
...
  g m7 r71 sin(theta131(t)) cos(theta6(t)) ...
  g m7 r71 cos(theta131(t)) sin(theta6(t)) cos(theta5(t)   theta132(t
    ));

  Rearrange equations of motions according to state variables
disp( Rearrange equations of motions according to state variables )

variableoriginals =  theta5(t), theta5dot, theta6(t), theta6dot ;

variablesubs =  theta5star + theta132(t), theta5stardot +
    theta132dot, theta6star   theta131(t), theta6stardot
    theta131dot ;

T5manuelsimplified = subs(T5manuelsimplified, variableoriginals,
    variablesubs);

T6manuelsimplified = subs(T6manuelsimplified, variableoriginals,
    variablesubs);
```

```
T5manuelsimplified = simplify(T5manuelsimplified);

T6manuelsimplified = simplify(T6manuelsimplified);

  Lagrange Equations
disp ( Lagrange Equations )

Lagrange1 = T5 == T5manuelsimplified;

Lagrange2 = T6 == T6manuelsimplified;

[K,L] = equationsToMatrix([Lagrange1, Lagrange2], [theta5starddotn,
    theta6starddotn]);

X = linsolve(K,L);

X = simplify(X);

X1 = X(1);   X2 = X(2);

multipliedangvel =  theta131dot theta132dot, theta5stardot
    theta131dot, theta5stardot theta132dot, ...
                      theta6stardot theta131dot, theta6stardot
                          theta132dot, theta5stardot theta6stardot
                          ;

shortenedangvel =  theta131132dot, theta5131dot, theta5132dot,
    theta6131dot, theta6132dot, theta56dot ;

X1 = subs(X1, multipliedangvel, shortenedangvel);

X2 = subs(X2, multipliedangvel, shortenedangvel);

X1 = collect(X1, [theta131dot2, theta132dot2, theta131132dot,
    T5, T6, ...
                theta5131dot, theta5132dot, theta6131dot,
                    theta6132dot, theta56dot, ...
                theta5stardot2, theta6stardot2, theta131ddot,
                    theta132ddot, theta5ddot, theta6ddot]);

X2 = collect(X2, [theta131dot2, theta132dot2, theta131132dot,
    T5, T6, ...
                theta5131dot, theta5132dot, theta6131dot,
                    theta6132dot, theta56dot, ...
                theta5stardot2, theta6stardot2, theta131ddot,
                    theta132ddot, theta5ddot, theta6ddot]);

X1 = subs(X1, shortenedangvel, multipliedangvel);

X2 = subs(X2, shortenedangvel, multipliedangvel);

  Formation of State variable Matrices

statevariables =  theta5star, theta5stardot, theta6star,
    theta6stardot ;

f =  theta5stardot, X1, theta6stardot, X2 ;

  Definition of Jacobian matrices
disp ( Definition of Jacobian matrices )
```

```
A = sym(zeros(length(statevariables),length(statevariables)));

B = sym(zeros(length(statevariables),2));

for i = 1:length(statevariables)

    for j = 1:length(statevariables)

        A(i,j) = deriv(f i , statevariables j );

    end

    B(i,1) = simplify(deriv(f i , T 5));

    B(i,2) = simplify(deriv(f i , T 6));

end

  Equilibrium point
disp ( Equilibrium point )

equilibriumvariables =  theta5star, theta5stardot, theta6star,
    theta6stardot, theta5starddotn, theta6starddotn ;

equilibriumvalues = 0 , 0, 0, 0, 0, 0 ;

 T 5 0 = subs(T5manuelsimplified, equilibriumvariables,
    equilibriumvalues);

 T 6 0 = subs(T6manuelsimplified, equilibriumvariables,
    equilibriumvalues);

statevariablesjacobian =  theta5star, theta5stardot, theta6star,
    theta6stardot, T5 , T6 ;

equilibriumpointsjacobian = 0 , 0, 0, 0, T 5 0 , T 6 0 ;

A = subs(A, statevariablesjacobian, equilibriumpointsjacobian);

B = subs(B, statevariablesjacobian, equilibriumpointsjacobian);

A = simplify(A);   B = simplify(B);

A = subs(A, multipliedangvel, shortenedangvel);

A = collect(A, [theta131dot 2 , theta132dot 2 , theta131132dot,...
                theta5131dot , theta5132dot , theta6131dot ,
                    theta6132dot , theta56dot , ...
                theta5dot 2 , theta6dot 2 , theta131ddot ,
                    theta132ddot , theta5ddot , theta6ddot]);

A = subs(A, shortenedangvel, multipliedangvel);


B = subs(B, multipliedangvel, shortenedangvel);

B = collect(B, [theta131dot 2 , theta132dot 2 , theta131132dot,...
                theta5131dot , theta5132dot , theta6131dot ,
                    theta6132dot , theta56dot , ...
```

```
                theta5dot 2, theta6dot 2, theta131ddot,
                    theta132ddot, theta5ddot, theta6ddot]);

B = subs(B, shortenedangvel, multipliedangvel);
```

Listing D.10: Lagrange equations for the stabilizer

```
clear
clc

syms J511 J512 J513 J521 J522 J523 J531 J532 J533 ...
     J611 J612 J613 J621 J622 J623 J631 J632 J633 ...
     J711 J712 J713 J721 J722 J723 J731 J732 J733 real ...

syms theta5(t) theta6(t) theta131(t) theta132(t)

syms m5 m6 m7 g

syms r51 r52 r53 r61 r62 r63 r71 r72 r73

syms rGO1

syms theta131132dot theta5131dot theta5132dot theta6131dot
    theta6132dot theta56dot

syms T5 T6

  Basic Column Matrices

u1 = [1; 0; 0];
u2 = [0; 1; 0];
u3 = [0; 0; 1];

  Gravity Matrix

gvec = [0; 0;  g];

  Transformation Matrices

C56 = simplify(BasicRotationMatrix(3,theta5) BasicRotationMatrix(1,
    sym(pi)/2));   C(5,6)

C67 = simplify(BasicRotationMatrix(3,theta6));   C(6,7)

C57 = C56   C67 ;   C(5,7)

C05 = simplify(BasicRotationMatrix(2,theta131) BasicRotationMatrix(1,
    theta132) BasicRotationMatrix(2,sym(pi)/2));   C(0,5)

C06 = C05   C56;   C(0,6)

C07 = C05   C57;   C(0,7)

  Inertia matrices

J5 = [J511   0   J513;   0   J522 J523; J513 J523 J533];   Ineria
    dyadic resolved in Reference Frame 5

J6 = [J611   0   J613;   0   J622   0  ; J613   0   J633];   Ineria
    dyadic resolved in Reference Frame 6
```

```
J7 = [J711 J712   0  ; J712 J722   0  ;   0     0   J733];   Ineria dyadic resolved in Reference Frame 7

 Definition of Angular Velocities

theta131dot = diff(theta131,t);

theta132dot = diff(theta132,t);

theta5dot = diff(theta5,t);

theta6dot = diff(theta6,t);

theta131ddot = diff(theta131,t,2);

theta132ddot = diff(theta132,t,2);

theta5ddot = diff(theta5,t,2);

theta6ddot = diff(theta6,t,2);

w5 = [theta132dot sin(theta131) cos(theta132)   theta131dot sin(theta132); ...
      theta132dot sin(theta131) sin(theta132) + theta131dot cos(theta132); ...
      theta132dot cos(theta131)];

w6 = [theta131dot sin(theta5   theta132) + theta132dot sin(theta131) cos(theta5   theta132); ...
      theta5dot + theta132dot cos(theta131); ...
      theta131dot cos(theta5   theta132)   theta132dot sin(theta131) sin(theta5   theta132)];

w7 = [theta131dot sin(theta5   theta132) cos(theta6) + theta132dot sin(theta131) cos(theta5   theta132) cos(theta6)...
      theta5dot sin(theta6) + theta132dot cos(theta131) sin(theta6); ...
      ...
      theta131dot sin(theta5   theta132) sin(theta6)   theta132dot sin(theta131) cos(theta5   theta132) sin(theta6) ...
      theta5dot cos(theta6) + theta132dot cos(theta131) cos(theta6);...
      ...
      theta131dot cos(theta5   theta132)   theta132dot sin(theta131) sin(theta5   theta132) + theta6dot];

 Definition of Translational Velocities

V5 = [(theta132dot sin(theta131) sin(theta132) + theta131dot cos(theta132)) r53; ...
       theta132dot cos(theta131) r51   theta132dot sin(theta131) cos(theta132) r53 + theta131dot sin(theta132) r53; ...
      (theta132dot sin(theta131) sin(theta132) + theta131dot cos(theta132)) r51];

V6 = [theta132dot cos(theta131) (r63   rGO1 sin(theta5)) theta5dot r63; ...
      theta131dot (rGO1 cos(theta132)   r63 sin(theta5   theta132)) + theta132dot sin(theta131) (rGO1 sin(theta132)   r63 cos
```

```
        (theta5 theta132)); ...
    theta132dot cos(theta131) cos(theta5) rGO1];

V7 = [theta131dot sin(theta6) ((rGO1 cos(theta132))   r73 sin(theta5 theta132))...
   + theta132dot (rGO1 sin(theta131) sin(theta132) sin(theta6) rGO1 sin(theta5) cos(theta131) cos(theta6)...
    r73 sin(theta131) sin(theta6) cos(theta5 theta132)+r73 cos(theta131) cos(theta6))  r73 theta5dot cos(theta6); ...
   ...
   ...
    theta131dot (rGO1 cos(theta132) cos(theta6) + r71 cos(theta5 theta132)   r73 sin(theta5   theta132) cos(theta6))...
   + theta132dot (rGO1 sin(theta131) sin(theta132) cos(theta6) r71 sin(theta131) sin(theta5   theta132)...
   + rGO1 sin(theta5) sin(theta6) cos(theta131)   r73 sin(theta131) cos(theta5   theta132) cos(theta6)   r73 sin(theta6) cos(theta131)) ...
   + r73 theta5dot sin(theta6) + r71 theta6dot; ...
   ...
   ...
    theta131dot r71 sin(theta5   theta132) sin(theta6) + r71 theta5dot cos(theta6)...
   + theta132dot ( rGO1 cos(theta131) cos(theta5) + r71 sin(theta131) cos(theta5   theta132) sin(theta6) r71 cos(theta131) cos(theta6))];

 Derivatives

 Independent variable : theta5



 Body 6

q1 = theta5;

 Derivative of translational kinetic energy of Body 6 w.r.t theta5dot and time

K6t = 0.5 m6 (V6 .  V6);

K6tq1dot = deriv(K6t, theta5dot);       Partial derivative of translational kinetic energy of Body 6 w.r.t theta5dot

K6tq1dott = deriv(K6tq1dot, t);         Derivative w.r.t time

 Derivative of rotational kinetic energy of Body 6 w.r.t theta5dot and time

K6r = 0.5(w6 .  J6  w6);

K6rq1dot = deriv(K6r, theta5dot);       Partial derivative of rotational kinetic energy of Body 6 w.r.t theta5dot

K6rq1dott = deriv(K6rq1dot, t);         Derivative w.r.t time

 Derivative of total kinetic energy of Body 6 w.r.t theta5dot and time

K6q1dott = K6tq1dott + K6rq1dott;
```

```
disp ( Derivative of total kinetic energy of Body 6 w.r.t theta5dot and time )



  Derivative of translational kinetic energy of Body 6 w.r.t theta5

V6q1 = deriv(V6, theta5);                     Partial derivative of translational velocity of Body 6 w.r.t theta5

K6tq1 = m6 V6 . V6q1;                         Partial derivative of translational kinetic energy of Body 6 w.r.t theta5

K6tq1 = simplify(K6tq1);

K6tq1 = expand(K6tq1 , ArithmeticOnly ,true);


multipliedangvel =  theta131dot theta132dot, theta5dot theta131dot, theta5dot theta132dot, ...
                     theta6dot theta131dot, theta6dot theta132dot, theta5dot theta6dot ;

shortenedangvel =  theta131132dot, theta5131dot, theta5132dot, theta6131dot, theta6132dot, theta56dot ;

K6tq1 = subs(K6tq1 , multipliedangvel, shortenedangvel);


K6tq1 = collect(K6tq1 , [theta131dot 2 , theta132dot 2 , theta131132dot, theta5132dot]);

K6tq1 = subs(K6tq1 , shortenedangvel, multipliedangvel);



  Derivative of rotational kinetic energy of Body 6 w.r.t theta5

w6q1 = deriv(w6, theta5);                     Partial derivative of angular velocity of Body 6 w.r.t theta5

K6rq1 = w6 . J6 w6q1;                         Partial derivative of rotational kinetic energy of Body 6 w.r.t theta5

K6rq1 = simplify(K6rq1);

K6rq1 = expand(K6rq1 , ArithmeticOnly ,true);


K6rq1 = subs(K6rq1 , multipliedangvel, shortenedangvel);


K6rq1 = collect(K6rq1 , [theta131dot 2 , theta132dot 2 , theta131132dot, theta5132dot]);

K6rq1 = subs(K6rq1 , shortenedangvel, multipliedangvel);

  Partial derivative of total kinetic energy of Body 6 w.r.t theta5

```

```
K 6 q1 = K 6 t q1 + K 6 r q1;



 Potential Energy of Body 6

r GO = [r GO1; 0; 0];                                        Offset distance between Point O   Ground

r 6 = simplify(C 56  r GO + r 63  u 3);                        Position vector of C 6 resolved in Reference Frame 6

r 6 0 = C 06   r 6;

r 6 0 body = formula(r 6 0);

U 6 =  m 6 dot(gvec , r 6 0 body . );

U 6 = simplify(expand(U 6));

 Derivative of potential energy of Body 6 w.r.t theta 5

U 6 q1 = deriv(U 6 , theta5);



 Body 7

 Derivative of translational kinetic energy of Body 7 w.r.t theta 5 dot and time

K 7 t = 0.5 m 7 ( V 7 .  V 7);

K 7 t q1 dot = deriv(K 7 t , theta 5 dot);       Partial derivative of translational kinetic energy of Body 7 w.r.t theta 5 dot

K 7 t q1 dot t = deriv(K 7 t q1 dot , t);        Derivative w.r.t time

 Derivative of rotational kinetic energy of Body 7 w.r.t theta 5 dot and time

K 7 r = 0.5 ( w 7 .  J 7  w 7);

K 7 r q1 dot = deriv(K 7 r , theta 5 dot);       Partial derivative of rotational kinetic energy of Body 7 w.r.t theta 5 dot

K 7 r q1 dot t = deriv(K 7 r q1 dot , t);        Derivative w.r.t time

 Derivative of total kinetic energy of Body 7 w.r.t theta 5 dot and time

K 7 q1 dot t = K 7 t q1 dot t + K 7 r q1 dot t;



 Derivative of Translational Velocity of Body 7 w.r.t theta 5

V 7 q1 = deriv(V 7 , theta5);                    Partial derivative of translational velocity of Body 7 w.r.t theta 5

V 7 q1 = simplify(expand(V 7 q1 , ArithmeticOnly ,true));
```

```
V 7 q1 = collect(V 7 q1 , [theta131dot, theta132dot]);

  V 71 q1

V 7body = formula(V 7);

V 7 q1body = formula(V 7 q1);

V 71 q1 = V 7body(1) V 7 q1body(1);

V 71 q1 = simplify(V 71 q1);

V 71 q1 = expand(V 71 q1 , ArithmeticOnly ,true);

V 71 q1 = subs(V 71 q1 , multipliedangvel, shortenedangvel);

V 71 q1 = collect(V 71 q1 , [theta131dot 2 , theta132dot 2 ,
    theta131132dot, theta5131dot, ...
                             theta5132dot,  theta6131dot,
                                 theta6132dot]);

V 71 q1 = subs(V 71 q1 , shortenedangvel, multipliedangvel);

  V 72 q1

V 72 q1 = V 7body(2) V 7 q1body(2);

V 72 q1 = simplify(V 72 q1);

V 72 q1 = expand(V 72 q1 , ArithmeticOnly ,true);

V 72 q1 = subs(V 72 q1 , multipliedangvel, shortenedangvel);

V 72 q1 = collect(V 72 q1 , [theta131dot 2 , theta132dot 2 ,
    theta131132dot, theta5131dot, ...
                             theta5132dot,  theta6131dot,
                                 theta6132dot]);

V 72 q1 = subs(V 72 q1 , shortenedangvel, multipliedangvel);

  V 73 q1

V 73 q1 = V 7body(3) V 7 q1body(3);

V 73 q1 = simplify(V 73 q1);

V 73 q1 = expand(V 73 q1 , ArithmeticOnly ,true);

V 73 q1 = subs(V 73 q1 , multipliedangvel, shortenedangvel);

V 73 q1 = collect(V 73 q1 , [theta131dot 2 , theta132dot 2 ,
    theta131132dot, theta5131dot, ...
                             theta5132dot,  theta6131dot,
                                 theta6132dot]);

V 73 q1 = subs(V 73 q1 , shortenedangvel, multipliedangvel);
```

```
K 7 t q1 = m 7  V 7 .  V 7 q1;                    Partial derivative of
    translational kinetic energy of Body 7  w.r.t theta 5

K 7 t q1 = simplify(K 7 t q1);

K 7 t q1 = expand(K 7 t q1 , ArithmeticOnly ,true);

K 7 t q1 = subs(K 7 t q1 , multipliedangvel, shortenedangvel);

K 7 t q1 = collect(K 7 t q1 , [theta 5 dot 2 , theta 6 dot 2 , theta 131 dot
     2 , theta 132 dot 2 , theta 131132 dot , theta 5 131 dot ,  ...
                              theta 5 132 dot ,  theta 6 131 dot ,
                                  theta 6 132 dot , theta 5 6 dot]);

K 7 t q1 = subs(K 7 t q1 , shortenedangvel, multipliedangvel);



  Derivative of Angular Velocity of Body 7  w.r.t theta 5

w 7 q1 = deriv(w 7 , theta 5);                    Partial derivative of
    angular velocity of Body 7  w.r.t theta 5

w 7 q1 = simplify(expand(w 7 q1 , ArithmeticOnly ,true));

w 7 q1 = collect(w 7 q1 , [theta 131 dot, theta 132 dot]);


K 7 r q1 = w 7 .  J 7  w 7 q1;                    Partial derivative of
    rotational kinetic energy of Body 7  w.r.t theta 5

K 7 r q1 = simplify(K 7 r q1);

K 7 r q1 = expand(K 7 r q1 , ArithmeticOnly ,true);


K 7 r q1 = subs(K 7 r q1 , multipliedangvel, shortenedangvel);

K 7 r q1 = collect(K 7 r q1 , [theta 5 dot 2 , theta 6 dot 2 , theta 131 dot
     2 , theta 132 dot 2 , theta 131132 dot , theta 5 131 dot ,  ...
                              theta 5 132 dot ,  theta 6 131 dot ,
                                  theta 6 132 dot , theta 5 6 dot]);

K 7 r q1 = subs(K 7 r q1 , shortenedangvel, multipliedangvel);

  Partial derivative of total kinetic energy of Body 7  w.r.t theta 5

K 7 q1 = K 7 t q1 + K 7 r q1;

   Potential Energy of Body 7

r 7 = simplify(C 57  r GO + r 71  u 1 + r 73  u 3);
    Position vector of C 7  resolved in Reference Frame 7

r 7 0 = C 07   r 7;

r 7 0 body = formula(r 7 0);

U 7 =  m 7 dot(gvec , r 7 0 body . );

```

```
U 7 = simplify(expand(U 7));

U 7 = m 7 g ( r GO1 cos(theta131) cos(theta132)+r 71 cos(theta131) cos(
    theta5 theta132) cos(theta6)...
           r 71 sin(theta131) sin(theta6) r 73 sin(theta5 theta132)
              cos(theta131));

  Derivative of potential energy of Body 7 w.r.t theta5

U 7 q1 = simplify(deriv(U 7, theta5));

  Derivative of total potential energy w.r.t theta5

U total q 1 = U 6 q1 + U 7 q1;



  Independent variable : theta6



  Body 6

q 2 = theta6;

  Derivative of translational kinetic energy of Body 6 w.r.t theta6dot
    and time

K 6 t q2 dot = deriv(K 6 t , theta6dot);        Partial derivative of
    translational kinetic energy of Body 6 w.r.t theta6dot

K 6 t q2 dot t = deriv(K 6 t q2 dot , t);        Derivative w.r.t time

  Derivative of Rotational Velocity of Body 6 w.r.t theta5dot and time

K 6 r q2 dot = deriv(K 6 r , theta6dot);        Partial derivative of
    rotational kinetic energy of Body 6 w.r.t theta5dot

K 6 r q2 dot t = deriv(K 6 r q2 dot , t);        Derivative w.r.t time

  Derivative of total kinetic energy of Body 6 w.r.t theta6dot and time

K 6 q2 dot t = K 6 t q2 dot t + K 6 r q2 dot t;



  Derivative of Translational Velocity of Body 6 w.r.t theta6

V 6 q2 = deriv(V 6, theta6);                    Partial derivative of
    translational velocity of Body 6 w.r.t theta6

K 6 t q2 = m 6 V 6 . V 6 q2;                    Partial derivative of
    translational kinetic energy of Body 6 w.r.t theta6

K 6 t q2 = simplify(K 6 t q2);

K 6 t q2 = expand(K 6 t q2 , ArithmeticOnly ,true);

K 6 t q2 = subs(K 6 t q2 , multipliedangvel, shortenedangvel);
```

```
K 6 t q2 = collect(K 6 t q2 , [theta131dot 2 , theta132dot 2 ,
    theta131132dot , theta5132dot]);

K 6 t q2 = subs(K 6 t q2 , shortenedangvel , multipliedangvel);



 Derivative of Angular Velocity of Body 6 w.r.t theta6

w 6 q2 = deriv(w 6 , theta6);                    Partial derivative of
    angular velocity of Body 6 w.r.t theta6

K 6 r q2 = w 6 .  J 6  w 6 q2;                   Partial derivative of
    rotational kinetic energy of Body 6 w.r.t theta6

K 6 r q2 = simplify(K 6 r q2);

K 6 r q2 = expand(K 6 r q2 , ArithmeticOnly ,true);

K 6 r q2 = subs(K 6 r q2 , multipliedangvel , shortenedangvel);

K 6 r q2 = collect(K 6 r q2 , [theta131dot 2 , theta132dot 2 ,
    theta131132dot , theta5132dot]);

K 6 r q2 = subs(K 6 r q2 , shortenedangvel , multipliedangvel);

 Partial derivative of total kinetic energy of Body 6 w.r.t theta6

K 6 q2 = K 6 t q2 + K 6 r q2;

 Derivative of potential energy of Body 6 w.r.t theta6

U 6 q2 = deriv(U 6 , theta6);



 Body 7

 Derivative of Translational Velocity of Body 7 w.r.t theta6dot and time

K 7 t q2 dot = deriv(K 7 t , theta6dot);         Partial derivative of
    translational kinetic energy of Body 7 w.r.t theta6dot

K 7 t q2 dot t = deriv(K 7 t q2 dot , t);        Derivative w.r.t time

 Derivative of Rotational Velocity of Body 7 w.r.t theta6dot and time

K 7 r = 0.5 (w 7 .  J7  w7);

K 7 r q2 dot = deriv(K 7 r , theta6dot);         Partial derivative of
    rotational kinetic energy of Body 7 w.r.t theta6dot

K 7 r q2 dot t = deriv(K 7 r q2 dot , t);        Derivative w.r.t time

 Derivative of total kinetic energy of Body 7 w.r.t theta6dot and time

K 7 q2 dot t = K 7 t q2 dot t + K 7 r q2 dot t;
```

```
 Derivative of Translational Velocity of Body 7 w.r.t theta6

V7q2 = deriv(V7, theta6);                    Partial derivative of translational velocity of Body 7 w.r.t theta6

V7q2 = simplify(expand(V7q2, ArithmeticOnly,true));

V7q2 = collect(V7q2, [theta131dot, theta132dot, theta5dot]);


K7tq2 = m7 V7. V7q2;                    Partial derivative of translational kinetic energy of Body 7 w.r.t theta6

K7tq2 = simplify(K7tq2);

K7tq2 = expand(K7tq2, ArithmeticOnly,true);

K7tq2 = subs(K7tq2, multipliedangvel, shortenedangvel);

K7tq2 = collect(K7tq2, [theta5dot2, theta6dot2, theta131dot2, theta132dot2, theta131132dot, theta5131dot, ...
                              theta5132dot, theta6131dot, theta6132dot, theta56dot]);

K7tq2 = subs(K7tq2, shortenedangvel, multipliedangvel);



 Derivative of Angular Velocity of Body 7 w.r.t theta6

w7q2 = deriv(w7, theta6);                    Partial derivative of angular velocity of Body 7 w.r.t theta6

w7q2 = simplify(expand(w7q2, ArithmeticOnly,true));

w7q2 = collect(w7q2, [theta131dot, theta132dot, theta5dot]);


K7rq2 = w7. J7 w7q2;                    Partial derivative of rotational kinetic energy of Body 7 w.r.t theta6

K7rq2 = simplify(K7rq2);

K7rq2 = expand(K7rq2, ArithmeticOnly,true);


K7rq2 = subs(K7rq2, multipliedangvel, shortenedangvel);

K7rq2 = collect(K7rq2, [theta5dot2, theta6dot2, theta131dot2, theta132dot2, theta131132dot, theta5131dot, ...
                              theta5132dot, theta6131dot, theta6132dot, theta56dot]);

K7rq2 = subs(K7rq2, shortenedangvel, multipliedangvel);

 Partial derivative of kinetic energy of Body 7 w.r.t theta6

K7q2 = K7tq2 + K7rq2;

 Derivative of potential energy of Body 7 w.r.t theta6
```

```
U 7 q2 = simplify(deriv(U 7 , theta 6));

 Derivative of total potential energy w.r.t theta 6

U t o t a l q 2 = U 6 q 2 + U 7 q 2;



 Lagrange equations

 For q1 = theta 5

 Langrange1 = (K 6 q1 dot t + K 7 q1 dot t)   (K 6 q1 + K 7 q1) +
    U t o t a l q 1   T 5;

T 5 original = (K 6 q1 dot t + K 7 q1 dot t)   (K 6 q1 + K 7 q1) +
    U t o t a l q 1;

T 5 original = simplify(expand(T 5 original));

T 5 original = subs(T 5 original, multipliedangvel, shortenedangvel);

T 5 original = collect(T 5 original, [theta 131 dot 2, theta 132 dot 2,
    theta 131132 dot, theta 5131 dot, theta 5132 dot, theta 6131 dot,
    theta 6132 dot]);

 T 5 = subs(T 5, shortenedangvel, multipliedangvel);

 For q2 = theta 6

 Langrange2 = (K 6 q2 dot t + K 7 q2 dot t)   (K 6 q2 + K 7 q2) +
    U t o t a l q 2   T 6;

T 6 original = (K 6 q2 dot t + K 7 q2 dot t)   (K 6 q2 + K 7 q2) +
    U t o t a l q 2;

T 6 original = simplify(expand(T 6 original));

T 6 original = subs(T 6 original, multipliedangvel, shortenedangvel);

T 6 original = collect(T 6 original, [theta 131 dot 2, theta 132 dot 2,
    theta 131132 dot, theta 5131 dot, theta 5132 dot, theta 6131 dot,
    theta 6132 dot]);

 T 6 = subs(T 6, shortenedangvel, multipliedangvel);

variablenames =  g, m 5, m 6, m 7, J511, J512, J513, J521, J522,
    J523, J531, J532, J533, ...
                                J611, J612, J613, J621, J622,
                                   J623, J631, J632, J633, ...
                                J711, J712, J713, J721, J722,
                                   J723, J731, J732, J733, ...
                                r51,  r52,  r53,  r61,  r62,  r63
                                   ,  r71,  r72,  r73, rGO1, ...
                                theta131dot, theta132dot,
                                   theta131132dot, theta5131dot,
                                    ...
                                theta5132dot, theta6131dot,
                                   theta6132dot, theta56dot,
```

```
                                        ...;
                                  theta5(t), theta6(t), theta131(t),
                                      theta132(t) ;

variablevalues =   9.81 , 1e315325.23 , 1e34530.59 , 1e32557.47 ,...
       g   Masses
                     595993867.8e9 , 751458.22e9 , 5144079.54e9 ,...
                             Inertia Terms J511 J512 J513
                    751458.22e9 , 641588537.25e9 , 145338017.74e9 ,...
                           Inertia Terms J521 J522 J523
                    5144079.54e9 , 145338017.74e9 , 336549983.33e9 ,...
                          Inertia Terms J531 J532 J533
                     18898872.13e9 , 328799.77e9 , 1485177.89e9 ,...
                             Inertia Terms J611 J612 J613
                    328799.77e9 , 110603524.64e9 , 3815.25e9 ,...
                                Inertia Terms J621 J622 J623
                    1485177.89e9 , 3815.25e9 , 93783786.61e9 ,...
                                Inertia Terms J631 J632 J633
                     11824640.51e9 , 1679850.64e9 , 137920.31e9 ,...
                              Inertia Terms J711 J712 J713
                    1679850.64e9 , 24252622.46e9 , 37299.29e9 ,...
                               Inertia Terms J721 J722 J723
                     137920.31e9 , 37299.29e9 , 15203370.83e9 ,...
                                 Inertia Terms J731 J732 J733
                     1e3  98.45 , 0 , 1e3  (60.66) ,...
                                         Center of Mass r5
                     0 , 0 , 1e3  (168.02) ,...
                                                  Center of Mass r6
                     1e3  39.19 , 0 , 1e3  (16.28) ,...
                                         Center of Mass r7
                     1e3  266.1, ...
                                                             Center of
                         Mass rGO
                     21, 29, 43, 25, 35, 15, 78, 18, ...
                     12, 22, 16, 9   ;

T5original = subs(T5original, variablenames, variablevalues);

T6original = subs(T6original, variablenames, variablevalues);

T5original = simplify(T5original);

T5doubleorj = double(T5original);

T6original = simplify(T6original);

T6doubleorj = double(T6original);
```

Listing D.11: Lagrange equations for the stabilizer simplified by hand calculations

```
clear
clc

syms J511 J512 J513 J521 J522 J523 J531 J532 J533 ...
     J611 J612 J613 J621 J622 J623 J631 J632 J633 ...
     J711 J712 J713 J721 J722 J723 J731 J732 J733 real ...

syms theta5(t) theta6(t) theta131(t) theta132(t)

syms m5 m6 m7 g
```

```
syms r51 r52 r53 r61 r62 r63 r71 r72 r73

syms rGO1

syms theta131132dot theta5131dot theta5132dot theta6131dot
    theta6132dot theta56dot

syms T5 T6

  Basic Column Matrices

u1 = [1; 0; 0];
u2 = [0; 1; 0];
u3 = [0; 0; 1];

  Gravity Matrix

gvec = [0; 0;  g];

  Transformation Matrices

C56 = simplify(BasicRotationMatrix(3,theta5) BasicRotationMatrix(1,
    sym(pi)/2));   C(5,6)

C67 = simplify(BasicRotationMatrix(3,theta6));   C(6,7)

C57 = C56   C67 ;   C(5,7)

C05 = simplify(BasicRotationMatrix(2,theta131) BasicRotationMatrix(1,
    theta132) BasicRotationMatrix(2,sym(pi)/2));   C(0,5)

C06 = C05   C56;   C(0,6)

C07 = C05   C57;   C(0,7)

  Inertia matrices

J5 = [J511   0   J513;   0   J522 J523; J513 J523 J533];   Ineria
    dyadic resolved in Reference Frame 5

J6 = [J611   0   J613;   0   J622   0  ; J613   0   J633];   Ineria
    dyadic resolved in Reference Frame 6

J7 = [J711 J712   0  ; J712 J722   0  ;   0     0   J733];   Ineria
    dyadic resolved in Reference Frame 7

  Definition of Angular Velocities

theta131dot = diff(theta131,t);

theta132dot = diff(theta132,t);

theta5dot = diff(theta5,t);

theta6dot = diff(theta6,t);

theta131ddot = diff(theta131,t,2);

theta132ddot = diff(theta132,t,2);
```

```
theta5ddot = diff(theta5,t,2);

theta6ddot = diff(theta6,t,2);

w5 = [theta132dot sin(theta131) cos(theta132)  theta131dot sin(theta132); ...
      theta132dot sin(theta131) sin(theta132) + theta131dot cos(theta132); ...
      theta132dot cos(theta131)];

w6 = [theta131dot sin(theta5  theta132) + theta132dot sin(theta131) cos(theta5  theta132); ...
      theta5dot + theta132dot cos(theta131); ...
      theta131dot cos(theta5  theta132)  theta132dot sin(theta131) sin(theta5  theta132)];

w7 = [theta131dot sin(theta5  theta132) cos(theta6) + theta132dot sin(theta131) cos(theta5  theta132) cos(theta6)...
      theta5dot sin(theta6) + theta132dot cos(theta131) sin(theta6); ...
      ...
      theta131dot sin(theta5  theta132) sin(theta6)  theta132dot sin(theta131) cos(theta5  theta132) sin(theta6) ...
      theta5dot cos(theta6) + theta132dot cos(theta131) cos(theta6);...
      ...
      theta131dot cos(theta5  theta132)  theta132dot sin(theta131) sin(theta5  theta132) + theta6dot];

 Definition of Translational Velocities

V5 = [(theta132dot sin(theta131) sin(theta132) + theta131dot cos(theta132)) r53; ...
       theta132dot cos(theta131) r51  theta132dot sin(theta131) cos(theta132) r53 + theta131dot sin(theta132) r53; ...
      (theta132dot sin(theta131) sin(theta132) + theta131dot cos(theta132)) r51];

V6 = [theta132dot cos(theta131) (r63  rGO1 sin(theta5)) theta5dot r63; ...
      theta131dot (rGO1 cos(theta132)  r63 sin(theta5  theta132)) + theta132dot sin(theta131) (rGO1 sin(theta132)  r63 cos(theta5 theta132)); ...
      theta132dot cos(theta131) cos(theta5) rGO1];

V7 = [theta131dot sin(theta6) ((rGO1 cos(theta132))  r73 sin(theta5 theta132))...
     + theta132dot (rGO1 sin(theta131) sin(theta132) sin(theta6) rGO1 sin(theta5) cos(theta131) cos(theta6)...
      r73 sin(theta131) sin(theta6) cos(theta5 theta132)+r73 cos(theta131) cos(theta6))  r73 theta5dot cos(theta6); ...
     ...
     ...
      theta131dot (rGO1 cos(theta132) cos(theta6) + r71 cos(theta5 theta132)  r73 sin(theta5  theta132) cos(theta6))...
     + theta132dot (rGO1 sin(theta131) sin(theta132) cos(theta6) r71 sin(theta131) sin(theta5  theta132)...
     + rGO1 sin(theta5) sin(theta6) cos(theta131)  r73 sin(theta131) cos(theta5  theta132) cos(theta6)  r73 sin(theta6) cos(
```

```
        theta131)) ...
    + r73 theta5dot sin(theta6) + r71 theta6dot; ...
     ...
     ...
      theta131dot r71 sin(theta5   theta132) sin(theta6) + r71
          theta5dot cos(theta6)...
    + theta132dot ( rGO1 cos(theta131) cos(theta5) + r71 sin(
        theta131) cos(theta5   theta132) sin(theta6) r71 cos(
        theta131) cos(theta6))];

  Derivatives

 Independent variable : theta5



 Body 6

q1 = theta5;

 Derivative of translational kinetic energy of Body 6 w.r.t theta5dot
   and time

K6t = 0.5m6 (V6 .  V6);

K6tq1dot = deriv(K6t, theta5dot);        Partial derivative of
   translational kinetic energy of Body 6 w.r.t theta5dot

K6tq1dott = deriv(K6tq1dot, t);          Derivative w.r.t time

 Derivative of rotational kinetic energy of Body 6 w.r.t theta5dot and
   time

K6r = 0.5(w6 .  J6  w6);

K6rq1dot = deriv(K6r, theta5dot);        Partial derivative of
   rotational kinetic energy of Body 6 w.r.t theta5dot

K6rq1dott = deriv(K6rq1dot, t);          Derivative w.r.t time

 Derivative of total kinetic energy of Body 6 w.r.t theta5dot and time

K6q1dott = K6tq1dott + K6rq1dott;



 Derivative of translational kinetic energy of Body 6 w.r.t theta5

V6q1 = deriv(V6, theta5);                Partial derivative of
   translational velocity of Body 6 w.r.t theta5

K6tq1 = m6 V6 .  V6q1;                   Partial derivative of
   translational kinetic energy of Body 6 w.r.t theta5

K6tq1 = simplify(K6tq1);

K6tq1 = expand(K6tq1, ArithmeticOnly,true);


multipliedangvel =  theta131dot theta132dot, theta5dot
```

```
    theta131dot, theta5dot theta132dot, ...
                    theta6dot theta131dot, theta6dot theta132dot,
                         theta5dot theta6dot ;

shortenedangvel =  theta131132dot, theta5131dot, theta5132dot,
    theta6131dot, theta6132dot, theta56dot ;

K6tq1 = subs(K6tq1, multipliedangvel, shortenedangvel);


K6tq1 = collect(K6tq1, [theta131dot2, theta132dot2,
    theta131132dot, theta5132dot]);

K6tq1 = subs(K6tq1, shortenedangvel, multipliedangvel);



  Derivative of rotational kinetic energy of Body 6 w.r.t theta5

w6q1 = deriv(w6, theta5);                     Partial derivative of
    angular velocity of Body 6 w.r.t theta5

K6rq1 = w6 .  J6  w6q1;                       Partial derivative of
    rotational kinetic energy of Body 6 w.r.t theta5

K6rq1 = simplify(K6rq1);

K6rq1 = expand(K6rq1 , ArithmeticOnly ,true);


K6rq1 = subs(K6rq1, multipliedangvel, shortenedangvel);


K6rq1 = collect(K6rq1, [theta131dot2, theta132dot2,
    theta131132dot, theta5132dot]);

K6rq1 = subs(K6rq1, shortenedangvel, multipliedangvel);

  Partial derivative of total kinetic energy of Body 6 w.r.t theta5

K6q1 = K6tq1 + K6rq1;



  Potential Energy of Body 6

rGO = [rGO1; 0; 0];                                          Offset
    distance between Point O   Ground

r6 = simplify(C56 rGO + r63 u3);                             Position
     vector of C6 resolved in Reference Frame 6

r60 = C06   r6;

r60body = formula(r60);

U6 =  m6 dot(gvec , r60body.);

U6 = simplify(expand(U6));

```

```
 Derivative of potential energy of Body 6 w.r.t theta5

U6q1 = deriv(U6, theta5);



 Body 7

 Derivative of translational kinetic energy of Body 7 w.r.t theta5dot
   and time

K7t = 0.5m7 (V7 .  V7);

K7tq1dot = deriv(K7t, theta5dot);        Partial derivative of
   translational kinetic energy of Body 7 w.r.t theta5dot

K7tq1dott = deriv(K7tq1dot, t);         Derivative w.r.t time

K7tq1dott = m7 ((r71 2 cos(theta6(t)) 2 + r73 2) diff(theta5(t),
   t, t) ...
 ...
 diff(theta132(t), t, t)  ...
( cos(theta131(t)) r71 2 cos(theta6(t)) 2 ...
 sin(theta6(t)) sin(theta131(t)) cos(theta5(t)   theta132(t)) r71 2
   cos(theta6(t)) ...
+ sin(theta5(t)   theta132(t)) sin(theta6(t)) sin(theta131(t)) r71
   r73 ...
+ rGO1 cos(theta5(t)) cos(theta131(t)) r71 cos(theta6(t)) ...
+ cos(theta131(t)) r73 2 ...
 rGO1 cos(theta131(t)) sin(theta5(t)) r73) ...
 ...
+ diff(theta132(t), t) 2  ...
( sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t)) sin(
   theta131(t)) r71 2 ...
+ r73 sin(theta6(t)) sin(theta131(t)) cos(theta5(t)   theta132(t))
   r71) ...
 ...
+ diff(theta131(t), t, t)  ...
( r71 2 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t)) ...
+ r73 sin(theta6(t)) cos(theta5(t)   theta132(t)) r71) ...
 ...
+ diff(theta131(t), t) diff(theta132(t), t)  ...
( r73 2 sin(theta131(t)) ...
+ r71 2 cos(theta6(t)) 2 sin(theta131(t)) ...
+ r71 r73 sin(theta5(t)   theta132(t)) sin(theta6(t)) ...
 r71 2 cos(theta6(t)) sin(theta6(t)) cos(theta5(t)   theta132(t)) ...
 r73 rGO1 sin(theta5(t)) sin(theta131(t)) ...
+ r71 2 cos(theta6(t)) cos(theta131(t)) sin(theta6(t)) cos(theta5(t)
    theta132(t)) ...
+ r71 rGO1 cos(theta5(t)) cos(theta6(t)) sin(theta131(t)) ...
 r71 r73 sin(theta5(t)   theta132(t)) cos(theta131(t)) sin(theta6(t)
   )) ...
 ...
 diff(theta5(t), t) diff(theta131(t), t)  ...
( cos(theta6(t)) sin(theta6(t)) cos(theta5(t)   theta132(t)) r71 2 ...
+ r73 sin(theta5(t)   theta132(t)) sin(theta6(t)) r71) ...
 ...
+ diff(theta6(t), t) diff(theta131(t), t)  ...
( 2 r71 2 sin(theta5(t)   theta132(t)) cos(theta6(t)) 2 ...
 sin(theta5(t)   theta132(t)) r71 2 ...
```

```
+ r71 r73 cos(theta5(t)   theta132(t)) cos(theta6(t))) ...
 ...
+ diff(theta6(t), t) diff(theta132(t), t)  ...
( r71 2 cos(theta131(t)) sin(2 theta6(t)) ...
  r71 2 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ 2 r71 2 cos(theta6(t)) 2 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ r71 rGO1 cos(theta5(t)) cos(theta131(t)) sin(theta6(t)) ...
  r71 r73 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta131(t))) ...
 ...
+ diff(theta5(t), t) diff(theta132(t), t)  ...
( r73 rGO1 cos(theta5(t)) cos(theta131(t)) ...
+ r71 rGO1 cos(theta6(t)) cos(theta131(t)) sin(theta5(t)) ...
  r71 2 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t)) sin(theta131(t)) ...
  r71 r73 sin(theta6(t)) sin(theta131(t)) cos(theta5(t)   theta132(t))) ...
 ...
+ r71 r73 cos(theta6(t)) diff(theta6(t), t) 2 ...
 ...
+ r71 r73 sin(theta6(t)) diff(theta6(t), t, t) ...
 ...
  r71 2 sin(2 theta6(t)) diff(theta5(t), t) diff(theta6(t), t));

 Derivative of rotational kinetic energy of Body 7 w.r.t theta5dot and time

K7r = 0.5(w7 .  J7 w7);

K7rq1dot = deriv(K7r, theta5dot);       Partial derivative of rotational kinetic energy of Body 7 w.r.t theta5dot

K7rq1dott = deriv(K7rq1dot, t);        Derivative w.r.t time

K7rq1dott = diff(theta5(t), t, t)  ...
( J711 ...
  J711 cos(theta6(t)) 2 ...
+ J722 cos(theta6(t)) 2 ...
+ J712 sin(2 theta6(t))) ...
 ...
+ diff(theta131(t), t, t)  ...
( J712 sin(theta5(t)   theta132(t)) ...
  2 J712 sin(theta5(t)   theta132(t)) cos(theta6(t)) 2 ...
  J711 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t)) ...
+ J722 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t))) ...
 ...
 diff(theta132(t), t, t)  ...
( J711 cos(theta131(t)) ...
  J711 cos(theta6(t)) 2 cos(theta131(t)) ...
+ J722 cos(theta6(t)) 2 cos(theta131(t)) ...
  J712 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ 2 J712 cos(theta6(t)) cos(theta131(t)) sin(theta6(t)) ...
+ 2 J712 cos(theta6(t)) 2 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ J711 cos(theta6(t)) sin(theta6(t)) sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
  J722 cos(theta6(t)) sin(theta6(t)) sin(theta131(t)) cos(theta5(t)   theta132(t))) ...
 ...
```

```
+ diff(theta132(t), t) 2 ...
( J712 sin(theta5(t)   theta132(t)) sin(theta131(t)) ...
  2 J712 sin(theta5(t)   theta132(t)) cos(theta6(t)) 2 sin(theta131(t)) ...
  J711 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t)) sin(theta131(t)) ...
+ J722 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t)) sin(theta131(t))) ...
...
+ diff(theta5(t), t) diff(theta6(t), t) ...
( 4 J712 cos(theta6(t)) 2 ...
  2 J712 ...
+ J711 sin(2 theta6(t)) ...
  J722 sin(2 theta6(t))) ...
...
+ diff(theta6(t), t) diff(theta132(t), t) ...
( 2 J712 cos(theta131(t)) ...
  4 J712 cos(theta6(t)) 2 cos(theta131(t)) ...
  J711 cos(theta131(t)) sin(2 theta6(t)) ...
+ J722 cos(theta131(t)) sin(2 theta6(t)) ...
+ J711 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
  J722 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
  2 J711 cos(theta6(t)) 2 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ 2 J722 cos(theta6(t)) 2 sin(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ 2 J712 sin(theta131(t)) sin(2 theta6(t)) cos(theta5(t)   theta132(t))) ...
...
+ diff(theta5(t), t) diff(theta132(t), t) ...
( 2 J712 sin(theta5(t)   theta132(t)) cos(theta6(t)) 2 sin(theta131(t)) ...
  J712 sin(theta5(t)   theta132(t)) sin(theta131(t)) ...
+ J711 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t)) sin(theta131(t)) ...
  J722 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t)) sin(theta131(t))) ...
...
+ diff(theta131(t), t) diff(theta132(t), t) ...
( J711 sin(theta131(t)) ...
  J712 cos(theta5(t)   theta132(t)) ...
+ 2 J712 cos(theta6(t)) 2 cos(theta5(t)   theta132(t)) ...
  J711 cos(theta6(t)) 2 sin(theta131(t)) ...
+ J722 cos(theta6(t)) 2 sin(theta131(t)) ...
+ J712 cos(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ J711 cos(theta6(t)) sin(theta6(t)) cos(theta5(t)   theta132(t)) ...
  J722 cos(theta6(t)) sin(theta6(t)) cos(theta5(t)   theta132(t)) ...
  2 J712 cos(theta6(t)) 2 cos(theta131(t)) cos(theta5(t)   theta132(t)) ...
+ 2 J712 cos(theta6(t)) sin(theta6(t)) sin(theta131(t)) ...
  J711 cos(theta6(t)) cos(theta131(t)) sin(theta6(t)) cos(theta5(t)   theta132(t)) ...
+ J722 cos(theta6(t)) cos(theta131(t)) sin(theta6(t)) cos(theta5(t)   theta132(t))) ...
...
+ diff(theta5(t), t) diff(theta131(t), t) ...
( J712 cos(theta5(t)   theta132(t)) ...
  2 J712 cos(theta6(t)) 2 cos(theta5(t)   theta132(t)) ...
  J711 cos(theta6(t)) sin(theta6(t)) cos(theta5(t)   theta132(t)) ...
+ J722 cos(theta6(t)) sin(theta6(t)) cos(theta5(t)   theta132(t))) ...
```

```
 ...
+ diff(theta6(t), t) diff(theta131(t), t)  ...
( J711 sin(theta5(t)   theta132(t)) ...
  J722 sin(theta5(t)   theta132(t)) ...
  2 J711 sin(theta5(t)   theta132(t)) cos(theta6(t)) 2 ...
+ 2 J722 sin(theta5(t)   theta132(t)) cos(theta6(t)) 2 ...
+ 2 J712 sin(theta5(t)   theta132(t)) sin(2 theta6(t)));

  Derivative of total kinetic energy of Body 7 w.r.t theta5dot and time

K7q1dott = K7tq1dott + K7rq1dott;



  Derivative of Translational Velocity of Body 7 w.r.t theta5

V7q1 = deriv(V7, theta5);                    Partial derivative of
    translational velocity of Body 7 w.r.t theta5

V7q1 = simplify(expand(V7q1, ArithmeticOnly,true));

V7q1 = collect(V7q1, [theta131dot, theta132dot]);

  V71q1

V7body = formula(V7);

V7q1body = formula(V7q1);

V71q1 = V7body(1) V7q1body(1);

V71q1 = simplify(V71q1);

V71q1 = expand(V71q1, ArithmeticOnly,true);

V71q1 = subs(V71q1, multipliedangvel, shortenedangvel);

V71q1 = collect(V71q1, [theta131dot2, theta132dot2,
    theta131132dot, theta5131dot, ...
                            theta5132dot, theta6131dot,
                                theta6132dot]);

V71q1 = subs(V71q1, shortenedangvel, multipliedangvel);

  V72q1

V72q1 = V7body(2) V7q1body(2);

V72q1 = simplify(V72q1);

V72q1 = expand(V72q1, ArithmeticOnly,true);

V72q1 = subs(V72q1, multipliedangvel, shortenedangvel);

V72q1 = collect(V72q1, [theta131dot2, theta132dot2,
    theta131132dot, theta5131dot, ...
                            theta5132dot, theta6131dot,
                                theta6132dot]);

V72q1 = subs(V72q1, shortenedangvel, multipliedangvel);
```

```
 V 7 3 q1

V 7 3 q1 = V 7 body (3) V 7 q1 body (3);

V 7 3 q1 = simplify(V 7 3 q1);

V 7 3 q1 = expand(V 7 3 q1 , ArithmeticOnly ,true);

V 7 3 q1 = subs(V 7 3 q1 , multipliedangvel, shortenedangvel);

V 7 3 q1 = collect(V 7 3 q1 , [theta 131 dot 2 , theta 132 dot 2 ,
    theta 131 132 dot, theta 5 131 dot,  ...
                              theta 5 132 dot,  theta 6 131 dot,
                                  theta 6 132 dot]);

V 7 3 q1 = subs(V 7 3 q1 , shortenedangvel, multipliedangvel);



K 7 t q1 = m 7  V 7 .  V 7 q1;                     Partial derivative of
    translational kinetic energy of Body 7  w.r.t theta 5

K 7 t q1 = simplify(K 7 t q1);

K 7 t q1 = expand(K 7 t q1 , ArithmeticOnly ,true);

K 7 t q1 = subs(K 7 t q1 , multipliedangvel, shortenedangvel);

K 7 t q1 = collect(K 7 t q1 , [theta 5 dot 2 , theta 6 dot 2 , theta 131 dot
    2 , theta 132 dot 2 , theta 131 132 dot, theta 5 131 dot,  ...
                              theta 5 132 dot,  theta 6 131 dot,
                                  theta 6 132 dot, theta 5 6 dot]);

K 7 t q1 = subs(K 7 t q1 , shortenedangvel, multipliedangvel);

 Manually simplified terms

K 7 t q1 = m 7 (...
...
diff(theta 6 (t), t) diff(theta132 (t), t) ...
...
(r 71  r 73  sin(theta 5 (t)   theta132 (t)) cos(theta 6 (t)) sin(theta131 (t))
     ...
 r 71  2  sin(theta131 (t)) cos(theta 5 (t)   theta132 (t)) ...
+ r 71  r GO1  cos(theta 5 (t)) cos(theta131 (t)) sin(theta 6 (t))) ...
 ...
+ diff(theta132 (t), t) 2  ...
 ...
( r 71  2  sin(theta 5 (t) theta132 (t)) (sin(theta131 (t))) 2  cos(theta 5 (t)
    theta132 (t)) (cos(theta 6 (t))) 2  ...     1 + 3
+ r 71  2  sin(theta 5 (t)   theta132 (t)) cos(theta 6 (t)) cos(theta131 (t))
    sin(theta 6 (t)) sin(theta131 (t)) ...      2
+ r 71  r 73  (sin(theta131 (t))) 2  cos(theta 6 (t)) cos (2 (theta 5 (t)
    theta132 (t))) ...                                4 + 5
+ r 71  r 73  cos(theta131 (t)) sin(theta 6 (t)) sin(theta131 (t)) cos(theta 5
    (t)   theta132 (t)) ...                   6
 r 71  r GO1  sin(theta 5 (t)) cos(theta 6 (t)) cos(theta131 (t)) 2  ...
                                                      7
 r 71  r GO1  sin(theta132 (t)) cos(theta 6 (t)) sin(theta131 (t)) 2  cos(
```

```
theta5(t)   theta132(t)) ...                 8
r73 2 sin(theta5(t) theta132(t)) (sin(theta131(t))) 2 cos(theta5(t)
  theta132(t))...                        9 + 10
+ r73 rGO1 sin(theta132(t)) sin(theta5(t) theta132(t)) (sin(theta131(
  t))) 2 ...                             11 + 12
r73 rGO1 cos(theta5(t)) (cos(theta131(t))) 2 ...
                                                                  13 + 14
+ rGO1 2 sin(theta5(t)) cos(theta5(t)) (cos(theta131(t))) 2 ...
                                                    15 + 16
rGO1 2 cos(theta5(t)) sin(theta5(t)) cos(theta131(t)) 2) ...
                                                     17
...
+ diff(theta131(t), t) 2 ...
...
(   r71 2 sin(theta5(t) theta132(t)) cos(theta5(t) theta132(t)) (cos(
  theta6(t))) 2 ...
r71 r73 cos(theta6(t)) cos(2(theta5(t) theta132(t))) ...
rGO1 cos(theta132(t)) r71 sin(theta5(t)   theta132(t)) cos(theta6(t
  )) ...
+ r73 2 sin(theta5(t) theta132(t)) cos(theta5(t) theta132(t)) ...
rGO1 r73 cos(theta132(t)) cos(theta5(t)   theta132(t)))...
...
+ (r71 2 cos(theta6(t)) sin(theta6(t)) cos(theta5(t)   theta132(t))
  ...
r71 r73 sin(theta5(t)   theta132(t)) sin(theta6(t))) diff(theta5(t)
  , t) diff(theta131(t), t) ...
...
diff(theta6(t), t) diff(theta131(t), t) ...
...
(r71 2 sin(theta5(t)   theta132(t)) ...
+ r71 r73 cos(theta6(t)) cos(theta5(t)   theta132(t))) ...
...
+ diff(theta5(t), t) diff(theta132(t), t) ...
( r71 rGO1 cos(theta131(t)) sin(theta5(t)) cos(theta6(t)) ...
r71 2 sin(theta5(t)   theta132(t)) sin(theta131(t)) cos(theta6(t))
  sin(theta6(t)) ...
r71 r73 sin(theta131(t)) cos(theta5(t)   theta132(t)) sin(theta6(t)
  ) ...
+ r73 rGO1 cos(theta131(t)) cos(theta5(t))) ...
...
+ diff(theta131(t), t) diff(theta132(t), t)  ...
( r71 2 sin(theta131(t)) (sin(theta5(t)   theta132(t))) 2 (cos(theta6(
  t))) 2 ...           36 + 37
r71 2 cos(theta131(t)) cos(theta6(t)) sin(theta6(t)) cos(theta5(t)
   theta132(t)) ...    38
r71 2 sin(theta131(t)) (cos(theta5(t)   theta132(t))) 2 (cos(theta6(
  t))) 2 ...           39 + 40
+ 2 r71 r73 sin(theta131(t)) sin(2(theta5(t) theta132(t))) cos(
  theta6(t)) ...             41
+ r71 r73 sin(theta5(t)   theta132(t)) sin(theta6(t)) cos(theta131(t)
  ) ...                42
r71 rGO1 sin(theta131(t)) cos(theta6(t)) cos(2 theta132(t) theta5(t
  )) ...               43 + 44
+ r73 2 sin(theta131(t)) (cos(2(theta5(t) theta132(t)))) ...
                                   45 + 46 + 47 + 48
+ r73 rGO1 sin(theta131(t)) sin(theta5(t) 2 theta132(t))));
                                    49 + 50 + 51 +52
```

```
 Derivative of Angular Velocity of Body 7 w.r.t theta5

w 7 q1 = deriv(w 7 , theta5);                    Partial derivative of
    angular velocity of Body 7 w.r.t theta5

w 7 q1 = simplify(expand(w 7 q1 , ArithmeticOnly ,true));

w 7 q1 = collect(w 7 q1 , [theta131dot, theta132dot]);


K 7 r q1 = w 7 .  J 7  w 7 q1;                    Partial derivative of
    rotational kinetic energy of Body 7 w.r.t theta5

K 7 r q1 = simplify(K 7 r q1);

K 7 r q1 = expand(K 7 r q1 , ArithmeticOnly ,true);


K 7 r q1 = subs(K 7 r q1 , multipliedangvel, shortenedangvel);

K 7 r q1 = collect(K 7 r q1 , [theta5dot 2 , theta6dot 2 , theta131dot
    2 , theta132dot 2 , theta131132dot, theta5131dot,  ...
                              theta5132dot,  theta6131dot,
                                  theta6132dot, theta56dot]);

K 7 r q1 = subs(K 7 r q1 , shortenedangvel, multipliedangvel);

K 7 r q1(t) = ...
...
diff(theta132(t), t) 2 ...
( J733 sin(theta5(t)   theta132(t)) sin(theta131(t)) 2 cos(theta5(t)
     theta132(t)) ...                                          1
  J711 sin(theta5(t)   theta132(t)) cos(theta6(t)) 2 sin(theta131(t))
     2 cos(theta5(t)   theta132(t))...                           2
  J712 sin(theta131(t)) sin(theta5(t)   theta132(t)) cos(theta131(t))
    cos(2 theta6(t)) ...                                       3 + 5
  J722 sin(theta5(t)   theta132(t)) sin(theta6(t)) 2 sin(theta131(t))
     2 cos(theta5(t)   theta132(t))...                           4
+ 2 J712 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t))
    sin(theta131(t)) 2 cos(theta5(t)   theta132(t))...         6
+ (J722   J711) sin(theta131(t)) sin(theta5(t)   theta132(t)) sin(
    theta6(t)) cos(theta131(t)) cos(theta6(t))) ...              7 + 8
...
+ diff(theta131(t), t) 2 ...
(J711 sin(theta5(t)   theta132(t)) cos(theta5(t)   theta132(t)) cos(
    theta6(t)) 2 ...
  2 J712 sin(theta5(t)   theta132(t)) cos(theta5(t)   theta132(t)) cos
    (theta6(t)) sin(theta6(t)) ...
+ J722 sin(theta5(t)   theta132(t)) cos(theta5(t)   theta132(t)) sin(
    theta6(t)) 2 ...
  J733 sin(theta5(t)   theta132(t)) cos(theta5(t)   theta132(t))) ...
...
+ diff(theta5(t), t) diff(theta132(t), t) ...
 ( J712 sin(theta131(t)) sin(theta5(t)   theta132(t)) cos(2 theta6(t))
      ...                                   9 + 10
 +(J711   J722) sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(
     theta6(t)) sin(theta131(t)))...        11 + 12
...
+ diff(theta5(t), t) diff(theta131(t), t) ...
(  J712 cos(theta5(t)   theta132(t)) cos(2 theta6(t))...
```

```
                                                   13 + 14
+ (J722   J711) sin(theta6(t)) cos(theta6(t)) cos(theta5(t)
    theta132(t)))...                         15 + 16
...
+ diff(theta131(t), t) diff(theta132(t), t) ...
 ( J733 sin(theta131(t)) cos(2(theta5(t)   theta132(t)))...
                                                17 + 18
+ J711 sin(theta131(t)) cos(theta6(t)) 2 cos(2(theta5(t)   theta132(t
    ))) ...                           19 + 20
+ J722 sin(theta131(t)) sin(theta6(t)) 2 cos(2(theta5(t)   theta132(t
    )))...                            21 + 22
+ J712 cos(theta131(t)) cos(theta5(t)   theta132(t)) cos(2 theta6(t))
    ...                                23 + 24
+ (J711   J722) sin(theta6(t)) cos(theta131(t)) cos(theta6(t)) cos(
    theta5(t)   theta132(t))...         26 + 27
   J712 sin(theta131(t)) sin(2 theta6(t)) cos(2(theta5(t)   theta132(
    t))))...                          25 + 28
...
  J733 sin(theta5(t)   theta132(t)) diff(theta6(t), t) diff(theta131(t
    ), t)   ...                       29
...
  J733 sin(theta131(t)) cos(theta5(t)   theta132(t)) diff(theta6(t), t
    ) diff(theta132(t), t);

  Partial derivative of total kinetic energy of Body 7 w.r.t theta5

K7q1 = K7tq1 + K7rq1;

  Potential Energy of Body 7

r7 = simplify(C57 rGO + r71 u1 + r73 u3);
    Position vector of C7 resolved in Reference Frame 7

r70 = C07   r7;

r70body = formula(r70);

U7 =  m7 dot(gvec , r70body.);

U7 = simplify(expand(U7));

U7 = m7 g (rGO1 cos(theta131) cos(theta132)+r71 cos(theta131) cos(
    theta5 theta132) cos(theta6)...
             r71 sin(theta131) sin(theta6) r73 sin(theta5 theta132)
               cos(theta131));

  Derivative of potential energy of Body 7 w.r.t theta5

U7q1 = simplify(deriv(U7, theta5));

  Derivative of total potential energy w.r.t theta5

Utotalq1 = U6q1 + U7q1;



  Independent variable : theta6
```

```
 Body 6

q 2 = theta 6;

 Derivative of translational kinetic energy of Body 6 w.r.t theta 6 dot
   and time

K 6 t q2 dot = deriv(K 6 t , theta 6 dot);       Partial derivative of
   translational kinetic energy of Body 6 w.r.t theta 6 dot

K 6 t q2 dot t = deriv(K 6 t q2 dot , t);        Derivative w.r.t time

 Derivative of Rotational Velocity of Body 6 w.r.t theta 5 dot and time

K 6 r q2 dot = deriv(K 6 r , theta 6 dot);       Partial derivative of
   rotational kinetic energy of Body 6 w.r.t theta 5 dot

K 6 r q2 dot t = deriv(K 6 r q2 dot , t);        Derivative w.r.t time

 Derivative of total kinetic energy of Body 6 w.r.t theta 6 dot and time

K 6 q2 dot t = K 6 t q2 dot t + K 6 r q2 dot t;



 Derivative of Translational Velocity of Body 6 w.r.t theta 6

V 6 q2 = deriv(V 6 , theta 6);                   Partial derivative of
   translational velocity of Body 6 w.r.t theta 6

K 6 t q2 = m 6  V 6 .  V 6 q2;                   Partial derivative of
   translational kinetic energy of Body 6 w.r.t theta 6

K 6 t q2 = simplify(K 6 t q2 );

K 6 t q2 = expand(K 6 t q2 , ArithmeticOnly ,true);

K 6 t q2 = subs(K 6 t q2 , multipliedangvel, shortenedangvel);

K 6 t q2 = collect(K 6 t q2 , [theta 131 dot 2 , theta 132 dot 2 ,
   theta 131 132 dot, theta 5 132 dot]);

K 6 t q2 = subs(K 6 t q2 , shortenedangvel, multipliedangvel);



 Derivative of Angular Velocity of Body 6 w.r.t theta 6

w 6 q2 = deriv(w 6 , theta 6);                   Partial derivative of
   angular velocity of Body 6 w.r.t theta 6

K 6 r q2 = w 6 .  J 6  w 6 q2;                   Partial derivative of
   rotational kinetic energy of Body 6 w.r.t theta 6

K 6 r q2 = simplify(K 6 r q2 );

K 6 r q2 = expand(K 6 r q2 , ArithmeticOnly ,true);

K 6 r q2 = subs(K 6 r q2 , multipliedangvel, shortenedangvel);
```

```
K 6 r q2 = collect(K 6 r q2 , [theta131dot 2 , theta132dot 2 ,
    theta131132dot , theta5132dot]);

K 6 r q2 = subs(K 6 r q2 , shortenedangvel, multipliedangvel);

 Partial derivative of total kinetic energy of Body 6 w.r.t theta6

K 6 q2 = K 6 t q2 + K 6 r q2;

 Derivative of potential energy of Body 6 w.r.t theta6

U 6 q2 = deriv(U 6 , theta6);



 Body 7

 Derivative of Translational Velocity of Body 7 w.r.t theta6dot and time

K 7 t q2 dot = deriv(K 7 t , theta6dot);       Partial derivative of
    translational kinetic energy of Body 7 w.r.t theta6dot

K 7 t q2 dot t = deriv(K 7 t q2 dot , t);        Derivative w.r.t time

K 7 t q2 dot t = diff(theta132(t), t) 2 ...
( m 7 r 71 2 sin(theta131(t)) cos(theta5(t)  theta132(t)) ...
  m 7 r 71 r 73 sin(theta5(t)  theta132(t)) cos(theta6(t)) sin(
    theta131(t)) ...
+ m 7 r 71 rGO1 cos(theta6(t)) cos(theta132(t)) sin(theta131(t))) ...
 ...
 diff(theta132(t), t, t) ...
( m 7 r 71 r 73 cos(theta131(t)) sin(theta6(t)) ...
  m 7 r 71 rGO1 cos(theta131(t)) sin(theta5(t)) sin(theta6(t)) ...
+ m 7 r 71 r 73 cos(theta6(t)) sin(theta131(t)) cos(theta5(t)
    theta132(t)) ...
+ m 7 r 71 2 sin(theta5(t)  theta132(t)) sin(theta131(t)) ...
  m 7 r 71 rGO1 cos(theta6(t)) sin(theta131(t)) sin(theta132(t))) ...
 ...
+ diff(theta131(t), t, t) ...
( m 7 r 71 2 cos(theta5(t)  theta132(t)) ...
  m 7 r 71 r 73 sin(theta5(t)  theta132(t)) cos(theta6(t)) ...
+ m 7 r 71 rGO1 cos(theta6(t)) cos(theta132(t))) ...
 ...
+ diff(theta131(t), t) diff(theta132(t), t) ...
( m 7 r 71 2 sin(theta5(t)  theta132(t)) ...
  m 7 r 71 2 sin(theta5(t)  theta132(t)) cos(theta131(t)) ...
  m 7 r 71 rGO1 cos(theta6(t)) sin(theta132(t)) ...
+ m 7 r 71 r 73 sin(theta6(t)) sin(theta131(t)) ...
+ m 7 r 71 r 73 cos(theta6(t)) cos(theta5(t)  theta132(t)) ...
  m 7 r 71 r 73 cos(theta6(t)) cos(theta131(t)) cos(theta5(t)
    theta132(t)) ...
+ m 7 r 71 rGO1 cos(theta6(t)) cos(theta131(t)) sin(theta132(t)) ...
  m 7 r 71 rGO1 sin(theta5(t)) sin(theta6(t)) sin(theta131(t))) ...
 ...
+ diff(theta5(t), t) diff(theta132(t), t) ...
( m 7 r 71 r 73 sin(theta5(t)  theta132(t)) cos(theta6(t)) sin(
    theta131(t)) ...
  m 7 r 71 2 sin(theta131(t)) cos(theta5(t)  theta132(t)) ...
+ m 7 r 71 rGO1 cos(theta5(t)) cos(theta131(t)) sin(theta6(t))) ...
 ...
```

```
+ diff(theta6(t), t) diff(theta132(t), t)  ...
( m7  r71  r73 sin(theta6(t)) sin(theta131(t)) cos(theta5(t) theta132(t)) ...
  m7  r71  r73 cos(theta6(t)) cos(theta131(t)) ...
+ m7  r71  rGO1 cos(theta6(t)) cos(theta131(t)) sin(theta5(t)) ...
  m7  r71  rGO1 sin(theta6(t)) sin(theta131(t)) sin(theta132(t))) ...
 ...
+ diff(theta6(t), t) diff(theta131(t), t)  ...
( m7  r71  r73 sin(theta5(t)   theta132(t)) sin(theta6(t)) ...
  m7  r71  rGO1 cos(theta132(t)) sin(theta6(t))) ...
 ...
 diff(theta5(t), t) diff(theta131(t), t)  ...
( m7 sin(theta5(t)   theta132(t)) r71 2 ...
+ m7  r73 cos(theta6(t)) cos(theta5(t)   theta132(t)) r71) ...
 ...
+ m7  r71 2 diff(theta6(t), t, t) ...
 ...
+ m7  r71  r73 sin(theta6(t)) diff(theta5(t), t, t) ...
 ...
+ m7  r71  r73 cos(theta6(t)) diff(theta5(t), t) diff(theta6(t), t);

 Derivative of Rotational Velocity of Body 7  w.r.t theta6dot and time

K7r = 0.5(w7 .  J7  w7);

K7rq2dot = deriv(K7r, theta6dot);        Partial derivative of rotational kinetic energy of Body 7 w.r.t theta6dot

K7rq2dott = deriv(K7rq2dot, t);         Derivative w.r.t time

K7rq2dott = J733 diff(theta6(t), t, t) ...
+ J733 cos(theta5(t)   theta132(t)) diff(theta131(t), t, t) ...
 ...
+ diff(theta131(t), t) diff(theta132(t), t)  ...
( J733 sin(theta5(t)   theta132(t)) ...
  J733 sin(theta5(t)   theta132(t)) cos(theta131(t))) ...
 ...
+ J733 sin(theta131(t)) cos(theta5(t)   theta132(t)) diff(theta132(t), t) 2 ...
 ...
  J733 sin(theta5(t)   theta132(t)) diff(theta5(t), t) diff(theta131(t), t) ...
 ...
  J733 sin(theta5(t)   theta132(t)) sin(theta131(t)) diff(theta132(t), t, t) ...
 ...
  J733 sin(theta131(t)) cos(theta5(t)   theta132(t)) diff(theta5(t), t) diff(theta132(t), t);

 Derivative of total kinetic energy of Body 7  w.r.t theta6dot and time

K7q2dott = K7tq2dott + K7rq2dott;



 Derivative of Translational Velocity of Body 7  w.r.t theta6

V7q2 = deriv(V7, theta6);                   Partial derivative of translational velocity of Body 7  w.r.t theta6

```

```
V 7 q2 = simplify(expand(V 7 q2 , ArithmeticOnly ,true));

V 7 q2 = collect(V 7 q2 , [theta131dot, theta132dot, theta5dot]);


K 7 t q2 = m 7  V 7 .  V 7 q2;                    Partial derivative of
    translational kinetic energy of Body 7  w.r.t theta6

K 7 t q2 = simplify(K 7 t q2);

K 7 t q2 = expand(K 7 t q2 , ArithmeticOnly ,true);

K 7 t q2 = subs(K 7 t q2 , multipliedangvel, shortenedangvel);

K 7 t q2 = collect(K 7 t q2 , [theta5dot 2 , theta6dot 2 , theta131dot
     2 , theta132dot 2 , theta131132dot, theta5131dot,  ...
                              theta5132dot,  theta6131dot,
                                  theta6132dot, theta56dot]);

K 7 t q2 = subs(K 7 t q2 , shortenedangvel, multipliedangvel);

K 7 t q2 = m 7 (...
...
diff(theta131(t), t) 2  ...
 ...
( r71 2 cos(theta6(t)) sin(theta6(t)) sin(theta5(t)   theta132(t)) 2
    ...
+ r73 r71 sin(theta6(t)) cos(theta5(t)   theta132(t)) sin(theta5(t)
     theta132(t)) ...
  rGO1 r71 cos(theta132(t)) sin(theta6(t)) cos(theta5(t)   theta132(t
    ))) ...
 ...
+ diff(theta132(t), t) 2 ...
...
( r71 rGO1 cos(theta5(t)) cos(theta131(t)) 2 sin(theta6(t)) ...
                                                       1
+ r71 2 cos(theta6(t)) sin(theta6(t)) sin(theta131(t)) 2 cos(theta5(t)
      theta132(t)) 2 ...                        2
  r71 2 cos(theta6(t)) cos(theta131(t)) 2 sin(theta6(t)) ...
                                                           3
  r71 2 sin(theta131(t)) cos(theta131(t)) cos(theta5(t)   theta132(t))
     cos(2 theta6(t)) ...                     4 + 5
+ r71 rGO1 sin(theta5(t)   theta132(t)) sin(theta6(t)) sin(theta131(t
    )) 2 sin(theta132(t)) ...                   6
+ r71 r73 sin(theta5(t)   theta132(t)) cos(theta6(t)) cos(theta131(t)
    ) sin(theta131(t)) ...                     7
  r71 r73 sin(theta5(t)   theta132(t)) sin(theta6(t)) sin(theta131(t)
    ) 2 cos(theta5(t)   theta132(t)) ...     8
  r71 rGO1 sin(theta131(t)) cos(theta131(t)) cos(theta132(t)) cos(
    theta6(t))) ...                                9 + 10
 ...
+ diff(theta5(t), t) diff(theta132(t), t)  ...
 ...
( r71 2 sin(theta131(t)) cos(theta5(t)   theta132(t)) cos(2 theta6(t))
    ...         11 + 13
+  2 r71 2 cos(theta131(t)) cos(theta6(t)) sin(theta6(t))...
                                  12
  r71 r73 sin(theta5(t)   theta132(t)) sin(theta131(t)) cos(theta6(t)
    )...       14
+ r71 rGO1 cos(theta5(t)) cos(theta131(t)) sin(theta6(t)))...
```

```
                        15
...
+ diff(theta6(t), t) diff(theta132(t), t) ...
...
( r71 r73 sin(theta6(t)) sin(theta131(t)) cos(theta5(t)   theta132(t)
  )...
 r71 r73 cos(theta6(t)) cos(theta131(t)) ...
+ r71 rGO1 cos(theta6(t)) cos(theta131(t)) sin(theta5(t))...
 r71 rGO1 sin(theta6(t)) sin(theta131(t)) sin(theta132(t))) ...
...
+ diff(theta131(t), t) diff(theta132(t), t)  ...
...
( r71 2 sin(theta5(t)   theta132(t)) cos(theta131(t)) cos(2 theta6(t))
  ...                                                  16 + 17
 r71 r73 cos(theta6(t)) cos(theta131(t)) cos(theta5(t)   theta132(t)
  )...                                                 18
+ r71 r73 sin(theta6(t)) sin(theta131(t)) cos(2 (theta5(t)   theta132
  (t)))...                                             19 + 20
+ r71 rGO1 sin(theta132(t)) cos(theta131(t)) cos(theta6(t)) ...
                                                             21 + 22
+ 2 r71 2 sin(theta5(t)   theta132(t)) cos(theta6(t)) sin(theta6(t))
  sin(theta131(t)) cos(theta5(t)   theta132(t))...    23
+ r71 rGO1 sin(theta131(t)) sin(theta6(t)) sin(theta5(t)   2 theta132
  (t)))...                                             24 + 25
...
 + diff(theta6(t), t) diff(theta131(t), t) ...
...
( r71 r73 sin(theta5(t)   theta132(t)) sin(theta6(t)) ...
 r71 rGO1 cos(theta132(t)) sin(theta6(t))) ...
...
+ diff(theta5(t), t) diff(theta131(t), t) ...
...
( r71 2 sin(theta5(t)   theta132(t)) cos(2 theta6(t))...          26
  + 27
+ r71 r73 cos(theta5(t)   theta132(t)) cos(theta6(t)))...
...
 r71 2 cos(theta6(t)) sin(theta6(t)) diff(theta5(t), t) 2 ...
...
+ r71 r73 cos(theta6(t)) diff(theta5(t), t) diff(theta6(t), t));



 Derivative of Angular Velocity of Body 7 w.r.t theta6

w7q2 = deriv(w7, theta6);                   Partial derivative of
  angular velocity of Body 7 w.r.t theta6

w7q2 = simplify(expand(w7q2, ArithmeticOnly,true));

w7q2 = collect(w7q2, [theta131dot, theta132dot, theta5dot]);


K7rq2 = w7. J7 w7q2;                        Partial derivative of
  rotational kinetic energy of Body 7 w.r.t theta6

K7rq2 = simplify(K7rq2);

K7rq2 = expand(K7rq2, ArithmeticOnly,true);
```

```
K 7 r q2 = subs(K 7 r q2 , multipliedangvel, shortenedangvel);

K 7 r q2 = collect(K 7 r q2 , [theta 5 dot 2 , theta 6 dot 2 , theta 131 dot 2 , theta 132 dot 2 , theta 131 132 dot , theta 5 131 dot , ...
                              theta 5 132 dot ,  theta 6 131 dot , theta 6 132 dot , theta 5 6 dot ]);

K 7 r q2 = subs(K 7 r q2 , shortenedangvel, multipliedangvel);

K 7 r q2 = diff(theta 5 (t), t) 2 . . .
...
( J712 cos (2 theta 6 (t)) ...
+ (J711   J722) sin(theta 6 (t)) cos(theta 6 (t))) ...
...
+ diff(theta 132 (t), t) 2 . . .
...
( J712 cos(theta 131 (t)) 2 cos (2 theta 6 (t)) ...
                                                                    4 +
    5
  J712 sin(theta 131 (t)) 2 cos(theta 5 (t)   theta 132 (t)) 2 cos (2 theta 6 ( t)) ...           6 + 7
+ (J711   J722) sin(theta 6 (t)) cos(theta 6 (t)) cos(theta 131 (t)) 2 ...
                                                   8 + 9
+ (J722   J711) sin(theta 6 (t)) sin(theta 131 (t)) 2 cos(theta 6 (t)) cos( theta 5 (t)   theta 132 (t)) 2 ...          10 + 11
+ (J711   J722) sin(theta 131 (t)) cos(theta 131 (t)) cos(theta 6 (t)) 2 cos (theta 5 (t)   theta 132 (t)) ...          12 + 13
+ (J722   J711) sin(theta 131 (t)) sin(theta 6 (t)) 2 cos(theta 131 (t)) cos (theta 5 (t)   theta 132 (t))...           14 + 15
  4 J712 cos(theta 6 (t)) cos(theta 131 (t)) sin(theta 6 (t)) sin(theta 131 (t )) cos(theta 5 (t)   theta 132 (t))) ...   16
...
+ diff(theta 131 (t), t) 2 . . .
...
(  J712 sin(theta 5 (t)   theta 132 (t)) 2 cos (2 theta 6 (t)) ...
                                             17 + 18
+ (J722   J711) sin(theta 5 (t)   theta 132 (t)) 2 sin(theta 6 (t)) cos( theta 6 (t))) ...           19 + 20
...
+ diff(theta 5 (t), t) diff(theta 131 (t), t) . . .
...
((J722   J711) sin(theta 5 (t)   theta 132 (t)) cos (2 theta 6 (t))...
                                        21 + 22 + 23 +24
+  4 J712 sin(theta 5 (t)   theta 132 (t)) cos(theta 6 (t)) sin(theta 6 (t))) ...                    25
...
+ diff(theta 5 (t), t) diff(theta 132 (t), t) . . .
...
 ( 2 J712 cos(theta 131 (t)) cos (2 theta 6 (t))...
                                                        26
+ 2 (J722   J711) cos(theta 6 (t)) cos(theta 131 (t)) sin(theta 6 (t))...
                                     27 + 28
+ (J722   J711) sin(theta 131 (t)) cos(theta 5 (t)   theta 132 (t)) cos (2 theta 6 (t)) ...          29 + 30 + 31 + 32
+  4 J712 cos(theta 6 (t)) sin(theta 6 (t)) sin(theta 131 (t)) cos(theta 5 (t) theta 132 (t)))...    33
...
...
+ diff(theta 131 (t), t) diff(theta 132 (t), t) . . .
((J711 J722) sin(theta 5 (t)   theta 132 (t)) cos(theta 131 (t))  cos (2
```

```
    theta6(t))...                                          34 + 35 + 36 + 37
   4 J712 sin(theta5(t)   theta132(t)) cos(theta6(t)) cos(theta131(t))
    sin(theta6(t))...                                          38
   2 J712 sin(theta131(t)) sin(theta5(t)   theta132(t)) cos(theta5(t)
     theta132(t)) cos(2 theta6(t))...                          39 + 40
+ 2 (J722  J711) sin(theta131(t)) sin(theta5(t)   theta132(t)) sin(
    theta6(t)) cos(theta5(t)   theta132(t)) cos(theta6(t)));   41+42

  Partial derivative of kinetic energy of Body 7 w.r.t theta6

K7q2 = K7tq2 + K7rq2;

  Derivative of potential energy of Body 7 w.r.t theta6

U7q2 = simplify(deriv(U7, theta6));

  Derivative of total potential energy w.r.t theta6

Utotalq2 = U6q2 + U7q2;



  Lagrange equations

  For q1 = theta5

  Langrange1 = (K6q1dott + K7q1dott)   (K6q1 + K7q1) +
    Utotalq1   T5;

T5manuel = (K6q1dott + K7q1dott)   (K6q1 + K7q1) +
    Utotalq1;

T5manuel = simplify(T5manuel);

T5manuel = subs(T5manuel, multipliedangvel, shortenedangvel);

T5manuel = collect(T5manuel, [theta131dot2, theta132dot2,
    theta131132dot,...
                    theta5131dot, theta5132dot, theta6131dot,
                       theta6132dot, theta56dot, ...
                    theta5dot2, theta6dot2, theta131ddot,
                       theta132ddot, theta5ddot, theta6ddot]);

T5manuel = subs(T5manuel, shortenedangvel, multipliedangvel);

  For q2 = theta6

  Langrange2 = (K6q2dott + K7q2dott)   (K6q2 + K7q2) +
    Utotalq2   T6;

T6manuel = (K6q2dott + K7q2dott)   (K6q2 + K7q2) +
    Utotalq2;

T6manuel = simplify(T6manuel);

T6manuel = subs(T6manuel, multipliedangvel, shortenedangvel);

T6manuel = collect(T6manuel, [theta131dot2, theta132dot2,
    theta131132dot,...
                    theta5131dot, theta5132dot, theta6131dot,
```

```
                    theta6132dot, theta56dot, ...
                  theta5dot2, theta6dot2, theta131ddot,
                    theta132ddot, theta5ddot, theta6ddot]);

T6manuel = subs(T6manuel, shortenedangvel, multipliedangvel);

variablenames =  g, m5, m6, m7, J511, J512, J513, J521, J522,
    J523, J531, J532, J533, ...
                                J611, J612, J613, J621, J622,
                                  J623, J631, J632, J633, ...
                                J711, J712, J713, J721, J722,
                                  J723, J731, J732, J733, ...
                                r51,  r52,  r53,  r61,  r62,  r63
                                  ,  r71,  r72,  r73, rGO1, ...
                                theta131dot, theta132dot,
                                  theta131132dot, theta5131dot,
                                   ...
                                theta5132dot, theta6131dot,
                                  theta6132dot, theta56dot,
                                  ...;
                                theta5(t), theta6(t), theta131(t),
                                  theta132(t) ;

variablevalues =   9.81 , 1e315325.23 , 1e34530.59 , 1e32557.47 ,...
       g   Masses
                   595993867.8e 9 , 751458.22e 9 , 5144079.54e 9 ,...
                           Inertia Terms J511 J512 J513
                  751458.22e 9 , 641588537.25e 9 , 145338017.74e 9 ,...
                         Inertia Terms J521 J522 J523
                  5144079.54e 9 , 145338017.74e 9 , 336549983.33e 9 ,...
                        Inertia Terms J531 J532 J533
                   18898872.13e 9 , 328799.77e 9 , 1485177.89e 9 ,...
                           Inertia Terms J611 J612 J613
                  328799.77e 9 , 110603524.64e 9 , 3815.25e 9 ,...
                              Inertia Terms J621 J622 J623
                  1485177.89e 9 , 3815.25e 9 , 93783786.61e 9 ,...
                              Inertia Terms J631 J632 J633
                   11824640.51e 9 , 1679850.64e 9 , 137920.31e 9 ,...
                            Inertia Terms J711 J712 J713
                  1679850.64e 9 , 24252622.46e 9 , 37299.29e 9 ,...
                             Inertia Terms J721 J722 J723
                   137920.31e 9 , 37299.29e 9 , 15203370.83e 9 ,...
                               Inertia Terms J731 J732 J733
                   1e 3  98.45 , 0 , 1e 3  (60.66) ,...
                                          Center of Mass r5
                   0 , 0 , 1e 3  (168.02) ,...
                                                  Center of Mass r6
                   1e 3  39.19 , 0 , 1e 3  (16.28) ,...
                                          Center of Mass r7
                   1e 3  266.1, ...
                                                             Center of
                       Mass rGO
                   21, 29, 43, 25, 35, 15, 78, 18, ...
                   12, 22, 16, 9   ;

T5manueldouble = double(subs(T5manuelsimplified, variablenames,
    variablevalues))
T6manueldouble = double(subs(T6manuelsimplified, variablenames,
    variablevalues))
```

Listing D.12: Matlab script to visualize the simulated motion of the stabilizer

```
clear; clc;
  GlobalVariables

theta5 = importfile( SolidworksSimulationData theta5solidworksv01.txt )
    ;
theta6 = importfile( SolidworksSimulationData theta6solidworksv01.txt )
    ;

theta5 = deg2rad(theta5);
theta6 = deg2rad(theta6);

b6x = 37; b6y = 154.5; b6z = 138.5;

b7x = 38.8; b7y = 134; b7z = 115.5;

body6 = [b6x  b6y  b6z;   0
          b6x  b6y  b6z;   1
          b6x  b6y  b6z;   2
          b6x  b6y  b6z;   3
          b6x  b6y  b6z;   4
          b6x  b6y  b6z;   5
          b6x  b6y  b6z;   6
          b6x  b6y  b6z;   7
          b6x  b6y  b6z;   8
          b6x  b6y  b6z;   9
          b6x  b6y  b6z;   10
          b6x  b6y  b6z;   11
          b6x  b6y  b6z;   12
          b6x  b6y  b6z;   13
          b6x  b6y  b6z;   14
          b6x  b6y  b6z;   15
          b6x  b6y  b6z];  16
body6 = body6 ;

body7 = [b7x  b7x  b7x  b7x  b7x  b7x  b7x  b7x  b7x    0    0    0;
          b7y  b7y  b7y  b7y  b7y  b7y  b7y  b7y    0    0    0    0;
          b7z  b7z  b7z  b7z  b7z  b7z  b7z  b7z    0    0  b6z  b6z];
figure
for i = 1:length(theta5)
    T5612(:,:,i) = BasicRotationMatrix(3,theta5(i))
        BasicRotationMatrix(1,pi/2);   C(5,6)
    T6712(:,:,i) = BasicRotationMatrix(3,theta6(i));   C(6,7)
    T5712(:,:,i) = T5612(:,:,i)   T6712(:,:,i) ;   C(5,7)
    T2513 = BasicRotationMatrix(2,0) BasicRotationMatrix(1,0)
        BasicRotationMatrix(2,pi/2);   C(0,5)


    body6(:,:,i+1) = T2513   T5612(:,:,i)   body6(:,:,1);
    body7(:,:,i+1) = T2513   T5712(:,:,i)   body7(:,:,1);
      plot3( body7(3,:,i+1), body7(2,:,i+1), body7(1,:,i+1))
      hold on
      plot3( body6(3,:,i+1), body6(2,:,i+1), body6(1,:,i+1))
      axis equal
    plot3(body7(1,:,i+1), body7(2,:,i+1), body7(3,:,i+1))
    hold on
    plot3(body6(1,:,i+1), body6(2,:,i+1), body6(3,:,i+1))
    axis equal
    xlim([180 180])
```

```
    ylim([180 180])
    zlim([180 180])
    hold off
    pause(0.05)
end
```

Listing D.13: Newton-Euler equations for the stabilizer equation elimination

```
clear
clc

syms J511 J512 J513 J521 J522 J523 J531 J532 J533 ...
     J611 J612 J613 J621 J622 J623 J631 J632 J633 ...
     J711 J712 J713 J721 J722 J723 J731 J732 J733 real ...

syms F561dp F562dp F563dp M561dp M562dp M563dp ...
     F671dp F672dp F673dp M671dp M672dp M673dp ...

syms Fg1 Fg2 Fg3 Mg1 Mg2 Mg3 ...

syms theta5 theta6 theta131 theta132

syms theta5dot theta6dot theta5ddot theta6ddot...
     theta131dot theta132dot theta131ddot theta132ddot

syms m5 m6 m7 g

syms r51 r52 r53 r61 r62 r63 r71 r72 r73

syms r561 r562 r563 r651 r652 r653 r671 r672 r673 ...
     rGO1 rGO2 rGO3 r761 r762 r763

  Construction of X matrix

X = [theta5ddot; theta6ddot; F561dp; F562dp; F563dp; M561dp;
    M563dp; F671dp;...
     F672dp; F673dp; M671dp; M672dp; Fg1; Fg2; Fg3; Mg1;
         Mg2; Mg3];

  Construction of A matrix

A = sym(zeros(18,18));

  Row 1

A(1,3) =  cos(theta5); A(1,5) = sin(theta5); A(1,13) = 1;

  Row 2

A(2,3) =  sin(theta5); A(2,5) =  cos(theta5); A(2,14) = 1;

  Row 3

A(3,4) = 1; A(3,15) = 1;

  Row 4

A(4,3) = r563 sin(theta5); A(4,5) = r563 cos(theta5); A(4,6) =  cos(
    theta5);

```

```
A(4,7) = sin(theta5); A(4,14) = r53;  A(4,16) = 1;

  Row 5

A(5,3) =  r563 cos(theta5); A(5,4) =  r561; A(5,5) = r563 sin(theta5);
     A(5,6) =  sin(theta5);

A(5,7) =  cos(theta5); A(5,13) =  r53; A(5,15) = r51; A(5,17) = 1;

  Row 6

A(6,3) =  r561 sin(theta5); A(6,5) =   r561 cos(theta5);

A(6,14) =  r51; A(6,18) = 1;

  Row 7

A(7,1) = m6 r63; A(7,3) = 1; A(7,8) =  cos(theta6); A(7,9) = sin(theta6
    );

  Row 8

A(8,4) = 1; A(8,8) =  sin(theta6); A(8,9) =  cos(theta6);

  Row 9

A(9,5) = 1; A(9,10) =  1;

  Row 10

A(10,4) =  r653; A(10,5) = r652; A(10,6) = 1; A(10,8) = r673 sin(theta6
    );

A(10,9) = r673 cos(theta6); A(10,11) =  cos(theta6); A(10,12) = sin(
    theta6);

  Row 11

A(11,1) = J622; A(11,3) = r653;  A(11,8) =  r673 cos(theta6);

A(11,9) = r673 sin(theta6);  A(11,11) =  sin(theta6); A(11,12) =  cos(
    theta6);

  Row 12

A(12,3) =  r652;  A(12,7) = 1;

  Row 13

A(13,1) = m7 r73 cos(theta6); A(13,8) = 1;

  Row 14

A(14,1) =  m7 r73 sin(theta6); A(14,2) =  m7 r71; A(14,9) = 1;

  Row 15

A(15,1) =  m7 r71 cos(theta6);  A(15,10) = 1;

  Row 16
```

```
A(16,1) = J711 sin(theta6) + J712 cos(theta6);

A(16,9) =  r763;  A(16,11) = 1;

  Row 17

A(17,1) = J722 cos(theta6) + J712 sin(theta6);

A(17,8) = r763; A(17,10) =  r761; A(17,12) = 1;

  Row 18

A(18,2) =  J733;       A(18,9) = r761;

  Construction of B matrix

B = sym(zeros(18,1));

B(1) = (cos(theta131) 2 theta132dot 2 ...
+ cos(theta132) 2 theta131dot 2 ...
+ sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
+ 2 cos(theta132) sin(theta131) sin(theta132) theta132dot
    theta131dot) m5 r51 ...
...
+ (sin(theta132) theta131dot theta132dot ...
  cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta132) theta131ddot ...
  sin(theta131) sin(theta132) theta132ddot ...
+ cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
  2 cos(theta131) sin(theta132) theta132dot theta131dot) m5 r53 ...
...
+ ( g cos(theta131) cos(theta132)) m5;

B(2) = ((sin(2 theta132) theta131dot 2)/2 ...
  (sin(2 theta132) theta132dot 2)/2  ...
+ cos(theta131) theta132ddot  ...
+ cos(theta131) 2 cos(theta132) sin(theta132) theta132dot 2  ...
  2 cos(theta132) 2 sin(theta131) theta131dot theta132dot) m5 r51
    ...
 ...
+ (cos(theta132) sin(theta131) theta132ddot  ...
  cos(theta132) theta132dot theta131dot  ...
  sin(theta131) sin(theta132) theta132dot 2  ...
  sin(theta132) theta131ddot  ...
+ 2 cos(theta131) cos(theta132) theta131dot theta132dot ...
+ cos(theta131) sin(theta131) sin(theta132) theta132dot 2) m5 r53
    ...
 ...
+ ( g cos(theta131) sin(theta132)) m5;

B(3) =  (cos(theta132) theta131ddot  ...
+ cos(theta132) sin(theta131) theta132dot 2  ...
  sin(theta132) theta131dot theta132dot  ...
+ sin(theta131) sin(theta132) theta132ddot  ...
+ cos(theta131) cos(theta132) sin(theta131) theta132dot 2) m5 r51
    ...
 ...
+ (theta131dot 2 + theta132dot 2  ...
  cos(theta131) 2 theta132dot 2) m5 r53 ;
```

```
B(4) = (sin(theta132) theta131ddot  ...
+ cos(theta132) theta131dot theta132dot ...
+ sin(theta131) sin(theta132) theta132dot 2  ...
  cos(theta132) sin(theta131) theta132ddot  ...
  cos(theta131) cos(theta132) theta132dot theta131dot) J511  ...
 ...
+ (cos(theta131) theta132ddot  ...
  cos(theta132) sin(theta131) 2 sin(theta132) theta132dot 2  ...
+ cos(theta132) sin(theta132) theta131dot 2  ...
  sin(theta131) theta131dot theta132dot  ...
  cos(theta132) 2 sin(theta131) theta132dot theta131dot  ...
+ sin(theta131) sin(theta132) 2 theta132dot theta131dot) J513  ...
 ...
+ (  cos(theta131) sin(theta131) sin(theta132) theta132dot 2  ...
  cos(theta131) cos(theta132) theta132dot theta131dot) J522  ...
 ...
+ (cos(theta131) 2 theta132dot 2  ...
  cos(theta132) 2 theta131dot 2  ...
  sin(theta131) 2 sin(theta132) 2 theta132dot 2  ...
   2 cos(theta132) sin(theta131) sin(theta132) theta132dot theta131dot) J523 ...
 ...
+ (cos(theta131) sin(theta131) sin(theta132) theta132dot 2  ...
+ cos(theta131) cos(theta132) theta132dot theta131dot) J533 ;

B(5) = (cos(theta131) cos(theta132) sin(theta131) theta132dot 2  ...
  cos(theta131) sin(theta132) theta132dot theta131dot) J511  ...
 ...
+ (cos(theta132) 2 sin(theta131) 2 theta132dot 2  ...
  cos(theta131) 2 theta132dot 2  ...
+ sin(theta132) 2 theta131dot 2  ...
   2 cos(theta132) sin(theta131) sin(theta132) theta132dot theta131dot) J513 ...
 ...
+ (sin(theta132) theta132dot theta131dot  ...
  cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta132) theta131ddot  ...
  sin(theta131) sin(theta132) theta132ddot ...
  cos(theta131) sin(theta132) theta132dot theta131dot) J522  ...
 ...
+ (cos(theta131) theta132ddot  ...
+ cos(theta132) sin(theta131) 2 sin(theta132) theta132dot 2  ...
  cos(theta132) sin(theta132) theta131dot 2  ...
  sin(theta131) theta131dot theta132dot  ...
+ cos(theta132) 2 sin(theta131) theta132dot theta131dot  ...
  sin(theta131) sin(theta132) 2 theta132dot theta131dot) J523  ...
 ...
+ (cos(theta131) sin(theta132) theta132dot theta131dot ...
  cos(theta131) cos(theta132) sin(theta131) theta132dot 2) J533  ;


B(6) = (cos(theta132) 2 sin(theta131) theta131dot theta132dot  ...
+ cos(theta132) sin(theta131) 2 sin(theta132) theta132dot 2  ...
  cos(theta132) sin(theta132) theta131dot 2  ...
  sin(theta131) sin(theta132) 2 theta131dot theta132dot) J511  ...
 ...
+ (sin(theta132) theta131ddot  ...
+ cos(theta132) theta132dot theta131dot  ...
+ sin(theta131) sin(theta132) theta132dot 2  ...
```

```
 cos(theta132) sin(theta131) theta132ddot ...
 2 cos(theta131) cos(theta132) theta131dot theta132dot ...
 cos(theta131) sin(theta131) sin(theta132) theta132dot 2) J513 ...
...
+ (cos(theta132) sin(theta132) theta131dot 2 ...
 cos(theta132) sin(theta131) 2 sin(theta132) theta132dot 2 ...
 cos(theta132) 2 sin(theta131) theta131dot theta132dot ...
+ sin(theta131) sin(theta132) 2 theta131dot theta132dot) J522 ...
...
+ (sin(theta132) theta131dot theta132dot ...
 cos(theta132) sin(theta131) theta132dot 2 ...
 cos(theta132) theta131ddot ...
 sin(theta131) sin(theta132) theta132ddot ...
+ cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
 2 cos(theta131) sin(theta132) theta132dot theta131dot) J523 ...
...
+ (cos(theta131) theta132ddot ...
 sin(theta131) theta132dot theta131dot) J533;


B(7) = ( cos(theta131) theta132ddot ...
+ sin(theta131) theta132dot theta131dot ...
 cos(theta5) cos(theta132) 2 sin(theta5) theta131dot 2 ...
+ cos(theta5) 2 cos(theta132) sin(theta132) theta131dot 2 ...
+ cos(theta5) sin(theta5) sin(theta132) 2 theta131dot 2 ...
 cos(theta132) sin(theta5) 2 sin(theta132) theta131dot 2 ...
+ cos(theta5) cos(theta132) 2 sin(theta5) sin(theta131) 2 theta132dot 2 ...
 cos(theta5) 2 cos(theta132) sin(theta131) 2 sin(theta132) theta132dot 2 ...
 cos(theta5) 2 cos(theta132) 2 sin(theta131) theta131dot theta132dot ...
 cos(theta5) sin(theta5) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
+ cos(theta132) sin(theta5) 2 sin(theta131) 2 sin(theta132) theta132dot 2 ...
+ cos(theta5) 2 sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
+ cos(theta132) 2 sin(theta5) 2 sin(theta131) theta132dot theta131dot ...
 sin(theta5) 2 sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
 4 cos(theta5) cos(theta132) sin(theta5) sin(theta131) sin(theta132) theta132dot theta131dot) m6 r63 ...
...
+ (sin(theta5) cos(theta131) theta132ddot ...
+ cos(theta5) cos(theta131) 2 theta132dot 2 ...
+ cos(theta5) cos(theta132) 2 theta131dot 2 ...
 sin(theta5) cos(theta132) sin(theta131) 2 sin(theta132) theta132dot 2 ...
+ sin(theta5) cos(theta132) sin(theta132) theta131dot 2 ...
+ cos(theta5) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
+ sin(theta5) sin(theta131) sin(theta132) 2 theta131dot theta132dot ...
 sin(theta5) sin(theta131) theta131dot theta132dot ...
 sin(theta5) cos(theta132) 2 sin(theta131) theta132dot theta131dot ...
+ 2 cos(theta5) cos(theta132) sin(theta131) sin(theta132) theta132dot theta131dot) m6 rGO1 ...
...
```

```
+ ( g cos(theta131) cos(theta5 ...
  theta132)) m6 ;

B(8) = (cos(theta132) sin(theta5) theta131ddot ...
  cos(theta5) sin(theta132) theta131ddot ...
+ 2 cos(theta5) cos(theta132) theta131dot theta5dot ...
  cos(theta5) cos(theta132) theta132dot theta131dot ...
  cos(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
+ cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
+ cos(theta5) cos(theta132) sin(theta131) theta132ddot ...
+ 2 sin(theta5) sin(theta132) theta5dot theta131dot ...
  sin(theta5) sin(theta132) theta132dot theta131dot ...
+ sin(theta5) sin(theta131) sin(theta132) theta132ddot ...
  cos(theta5) cos(theta131) sin(theta131) sin(theta132) theta132dot 2
    ...
+ cos(theta131) cos(theta132) sin(theta5) sin(theta131) theta132dot 2
    ...
+ 2 cos(theta5) sin(theta131) sin(theta132) theta5dot theta132dot
    ...
  2 cos(theta132) sin(theta5) sin(theta131) theta132dot theta5dot)
    m6 r63 ...
 ...
+ (sin(theta132) theta131dot theta132dot ...
  cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta132) theta131ddot ...
  sin(theta131) sin(theta132) theta132ddot ...
  cos(theta131) cos(theta132) sin(theta131) theta132dot 2) m6 rGO1
    ...
 ...
+ (g sin(theta131)) m6 ;

B(9) = (cos(theta5) 2 cos(theta132) 2 sin(theta131) 2 theta132dot 2
    ...
+ cos(theta5) 2 sin(theta132) 2 theta131dot 2 ...
+ 2 cos(theta5) cos(theta132) sin(theta5) sin(theta131) 2 sin(theta132
    ) theta132dot 2 ...
  2 cos(theta5) cos(theta132) sin(theta5) sin(theta132) theta131dot 2
    ...
+ cos(theta131) 2 theta132dot 2 ...
+ cos(theta132) 2 sin(theta5) 2 theta131dot 2 ...
+ sin(theta5) 2 sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
+ theta5dot 2 ...
  2 cos(theta5) 2 cos(theta132) sin(theta131) sin(theta132)
    theta132dot theta131dot ...
+ 2 cos(theta5) cos(theta132) 2 sin(theta5) sin(theta131) theta132dot
    theta131dot ...
  2 cos(theta5) sin(theta5) sin(theta131) sin(theta132) 2 theta132dot
    theta131dot ...
  2 cos(theta131) theta132dot theta5dot ...
+ 2 cos(theta132) sin(theta5) 2 sin(theta131) sin(theta132)
    theta132dot theta131dot) m6 r63 ...
 ...
+ (cos(theta5) cos(theta131) theta132ddot ...
  sin(theta5) cos(theta131) 2 theta132dot 2 ...
  sin(theta5) cos(theta132) 2 theta131dot 2 ...
  cos(theta5) cos(theta132) sin(theta131) 2 sin(theta132) theta132dot
    2 ...
+ cos(theta5) cos(theta132) sin(theta132) theta131dot 2 ...
  sin(theta5) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
+ cos(theta5) sin(theta131) sin(theta132) 2 theta131dot theta132dot
```

```
...
  cos(theta5) sin(theta131) theta131dot theta132dot  ...
  cos(theta5) cos(theta132) 2 sin(theta131) theta132dot theta131dot ...
   2 sin(theta5) cos(theta132) sin(theta131) sin(theta132) theta132dot theta131dot) m6 rGO1  ...
 ...
+ (g sin(theta5   theta132) cos(theta131)) m6  ;

B(10) = (cos(theta5) sin(theta132) theta131ddot  ...
  cos(theta132) sin(theta5) theta131ddot  ...
  cos(theta5) cos(theta132) theta131dot theta5dot  ...
+ cos(theta5) cos(theta132) theta131dot theta132dot  ...
+ cos(theta5) sin(theta131) sin(theta132) theta132dot 2  ...
  cos(theta132) sin(theta5) sin(theta131) theta132dot 2  ...
  cos(theta5) cos(theta132) sin(theta131) theta132ddot  ...
  sin(theta5) sin(theta132) theta5dot theta131dot ...
+ sin(theta5) sin(theta132) theta132dot theta131dot ...
  sin(theta5) sin(theta131) sin(theta132) theta132ddot  ...
  cos(theta5) cos(theta131) cos(theta132) theta132dot theta131dot ...
  cos(theta5) sin(theta131) sin(theta132) theta132dot theta5dot  ...
+ cos(theta132) sin(theta5) sin(theta131) theta132dot theta5dot  ...
  cos(theta131) sin(theta5) sin(theta132) theta132dot theta131dot) J611  ...
 ...
+  (2 cos(theta132) sin(theta5) theta131dot theta5dot ...
  sin(theta5) sin(theta132) theta131ddot  ...
  cos(theta5) cos(theta132) sin(theta131) theta132dot 2  ...
   2 cos(theta5) sin(theta132) theta131dot theta5dot  ...
  cos(theta5) cos(theta132) theta131ddot ...
+ cos(theta5) sin(theta132) theta131dot theta132dot  ...
  cos(theta132) sin(theta5) theta131dot theta132dot  ...
  sin(theta5) sin(theta131) sin(theta132) theta132dot 2  ...
  cos(theta5) sin(theta131) sin(theta132) theta132ddot  ...
+ cos(theta132) sin(theta5) sin(theta131) theta132ddot  ...
  cos(theta5) cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
+  2 cos(theta5) cos(theta132) sin(theta131) theta132dot theta5dot ...
  cos(theta131) sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
+  2 sin(theta5) sin(theta131) sin(theta132) theta132dot theta5dot) J613  ...
 ...
+ (cos(theta5) cos(theta131) cos(theta132) theta131dot theta132dot ...
  sin(theta5) sin(theta132) theta131dot theta5dot  ...
  cos(theta5) cos(theta132) theta131dot theta5dot  ...
+ cos(theta5) cos(theta131) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta131) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
  cos(theta5) sin(theta131) sin(theta132) theta132dot theta5dot  ...
+ cos(theta132) sin(theta5) sin(theta131) theta5dot theta132dot  ...
+ cos(theta131) sin(theta5) sin(theta132) theta132dot theta131dot) J622  ...
 ...
+ (cos(theta5) cos(theta132) theta131dot theta5dot ...
+ sin(theta5) sin(theta132) theta131dot theta5dot ...
```

```
  cos(theta5) cos(theta131) cos(theta132) theta131dot theta132dot ...
  cos(theta5) cos(theta131) sin(theta131) sin(theta132) theta132dot 2 ...
+ cos(theta131) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
+ cos(theta5) sin(theta131) sin(theta132) theta132dot theta5dot ...
  cos(theta132) sin(theta5) sin(theta131) theta5dot theta132dot ...
  cos(theta131) sin(theta5) sin(theta132) theta132dot theta131dot) J633 ;

B(11) = ((cos(theta131  theta5 + theta132) sin(theta5 + theta131  theta132) theta132dot 2)/4 ...
  (cos(theta5 + theta131  theta132) sin(theta131  theta5 + theta132) theta132dot 2)/4 ...
  (cos(theta5 + theta131  theta132) sin(theta5 + theta131  theta132) theta132dot 2)/4 ...
+ (cos(theta131  theta5 + theta132) sin(theta131  theta5 + theta132) theta132dot 2)/4 ...
  sin(theta5  theta132) cos(theta5  theta132) theta131dot 2 ...
  (sin(theta5  theta132) cos(theta5 + theta131  theta132) theta131dot theta132dot)/2 ...
+ (sin(theta5  theta132) cos(theta131  theta5 + theta132) theta131dot theta132dot)/2 ...
  (sin(theta5 + theta131  theta132) cos(theta5  theta132) theta131dot theta132dot)/2 ...
  (sin(theta131  theta5 + theta132) cos(theta5  theta132) theta131dot theta132dot)/2) J611 ...
 ...
+ ((sin(theta5 + theta131  theta132) 2 theta132dot 2)/4 ...
  (cos(theta131  theta5 + theta132) 2 theta132dot 2)/4 ...
  (cos(theta5 + theta131  theta132) 2 theta132dot 2)/4 ...
+ (sin(theta131  theta5 + theta132) 2 theta132dot 2)/4 ...
  cos(theta5  theta132) 2 theta131dot 2 ...
+ sin(theta5  theta132) 2 theta131dot 2 ...
+ (cos(theta5 + theta131  theta132) cos(theta131  theta5 + theta132) theta132dot 2)/2 ...
+ (sin(theta5 + theta131  theta132) sin(theta131  theta5 + theta132) theta132dot 2)/2 ...
  cos(theta5 + theta131  theta132) cos(theta5  theta132) theta131dot theta132dot ...
+ cos(theta131  theta5 + theta132) cos(theta5  theta132) theta131dot theta132dot ...
+ sin(theta5  theta132) sin(theta5 + theta131  theta132) theta131dot theta132dot ...
+ sin(theta5  theta132) sin(theta131  theta5 + theta132) theta131dot theta132dot) J613 ...
 ...
+ ( cos(theta131) theta132ddot ...
+ sin(theta131) theta132dot theta131dot) J622 ...
 ...
+ ((cos(theta5 + theta131  theta132) sin(theta5 + theta131  theta132) theta132dot 2)/4 ...
+ (cos(theta5 + theta131  theta132) sin(theta131  theta5 + theta132) theta132dot 2)/4 ...
  (cos(theta131  theta5 + theta132) sin(theta5 + theta131  theta132) theta132dot 2)/4 ...
  (cos(theta131  theta5 + theta132) sin(theta131  theta5 + theta132) theta132dot 2)/4 ...
+ sin(theta5  theta132) cos(theta5  theta132) theta131dot 2 ...
```

```
+ (sin(theta5   theta132) cos(theta5 + theta131   theta132) theta131dot theta132dot)/2 ...
  (sin(theta5   theta132) cos(theta131   theta5 + theta132) theta131dot theta132dot)/2 ...
+ (sin(theta5 + theta131   theta132) cos(theta5   theta132) theta131dot theta132dot)/2 ...
+ (sin(theta131   theta5 + theta132) cos(theta5   theta132) theta131dot theta132dot)/2) J633 ...
 ...
+ M562dp ;

B(12) = (cos(theta5) sin(theta132) theta131dot theta5dot ...
  cos(theta132) sin(theta5) theta5dot theta131dot ...
+ cos(theta5) cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta5) cos(theta132) sin(theta131) theta5dot theta132dot ...
  cos(theta5) cos(theta131) sin(theta132) theta132dot theta131dot ...
+ cos(theta131) cos(theta132) sin(theta5) theta131dot theta132dot ...
+ cos(theta131) sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
  sin(theta5) sin(theta131) sin(theta132) theta132dot theta5dot) J611 ...
 ...
+ (cos(theta5) sin(theta132) theta131ddot ...
  cos(theta132) sin(theta5) theta131ddot ...
   2 cos(theta5) cos(theta132) theta131dot theta5dot ...
+ cos(theta5) cos(theta132) theta132dot theta131dot ...
+ cos(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
  cos(theta5) cos(theta132) sin(theta131) theta132ddot ...
   2 sin(theta5) sin(theta132) theta5dot theta131dot ...
+ sin(theta5) sin(theta132) theta132dot theta131dot ...
  sin(theta5) sin(theta131) sin(theta132) theta132ddot ...
+ cos(theta5) cos(theta131) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta131) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
   2 cos(theta5) sin(theta131) sin(theta132) theta5dot theta132dot ...
+  2 cos(theta132) sin(theta5) sin(theta131) theta132dot theta5dot) J613 ...
 ...
+ (cos(theta132) sin(theta5) theta5dot theta131dot ...
  cos(theta5) sin(theta132) theta131dot theta5dot ...
  cos(theta5) cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
+ cos(theta5) cos(theta132) sin(theta131) theta5dot theta132dot ...
+ cos(theta5) cos(theta131) sin(theta132) theta132dot theta131dot ...
  cos(theta131) cos(theta132) sin(theta5) theta131dot theta132dot ...
  cos(theta131) sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
+ sin(theta5) sin(theta131) sin(theta132) theta132dot theta5dot) J622 ...
 ...
+ (cos(theta132) sin(theta5) theta5dot theta131dot ...
  sin(theta5) sin(theta132) theta131ddot ...
```

```
  cos(theta5) cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta5) sin(theta132) theta131dot theta5dot ...
  cos(theta5) cos(theta132) theta131ddot ...
+ cos(theta5) sin(theta132) theta132dot theta131dot ...
  cos(theta132) sin(theta5) theta131dot theta132dot ...
  sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta5) sin(theta131) sin(theta132) theta132ddot ...
+ cos(theta132) sin(theta5) sin(theta131) theta132ddot ...
+ cos(theta5) cos(theta132) sin(theta131) theta132dot theta5dot ...
  cos(theta5) cos(theta131) sin(theta132) theta131dot theta132dot ...
+ cos(theta131) cos(theta132) sin(theta5) theta132dot theta131dot ...
+ sin(theta5) sin(theta131) sin(theta132) theta132dot theta5dot) J633 ...
 ...
  M673dp ;

B(13) = (theta6dot 2 ...
+ theta131dot 2 ...
+ theta132dot 2 ...
+ cos(theta6) 2 theta5dot 2 ...
  cos(theta6) 2 theta132dot 2 ...
  cos(theta131) 2 theta132dot 2 ...
  cos(theta5) 2 cos(theta6) 2 theta131dot 2 ...
+ cos(theta5) 2 cos(theta6) 2 theta132dot 2 ...
+ 2 cos(theta6) 2 cos(theta131) 2 theta132dot 2 ...
  cos(theta6) 2 cos(theta132) 2 theta131dot 2 ...
+ cos(theta6) 2 cos(theta132) 2 theta132dot 2 ...
+ 2 cos(theta5) cos(theta132) theta131dot theta6dot ...
  cos(theta5) 2 cos(theta6) 2 cos(theta131) 2 theta132dot 2  ...
+ 2 cos(theta5) 2 cos(theta6) 2 cos(theta132) 2 theta131dot 2 ...
  2 cos(theta5) 2 cos(theta6) 2 cos(theta132) 2 theta132dot 2 ...
  cos(theta6) 2 cos(theta131) 2 cos(theta132) 2 theta132dot 2 ...
+ 2 sin(theta5) sin(theta132) theta131dot theta6dot ...
  2 cos(theta6) 2 cos(theta131) theta132dot theta5dot ...
+ 2 cos(theta5) 2 cos(theta6) 2 cos(theta131) 2 cos(theta132) 2 theta132dot 2 ...
+ 2 cos(theta5) sin(theta131) sin(theta132) theta6dot theta132dot ...
  2 cos(theta132) sin(theta5) sin(theta131) theta6dot theta132dot ...
  2 cos(theta5) cos(theta6) sin(theta6) sin(theta132) theta5dot theta131dot ...
+ 2 cos(theta6) cos(theta132) sin(theta5) sin(theta6) theta5dot theta131dot ...
+ 2 cos(theta5) cos(theta6) 2 cos(theta132) sin(theta5) sin(theta132) theta131dot 2 ...
  2 cos(theta5) cos(theta6) 2 cos(theta132) sin(theta5) sin(theta132) theta132dot 2 ...
+ 2 cos(theta5) cos(theta6) 2 sin(theta5) sin(theta131) theta131dot theta132dot ...
  2 cos(theta6) 2 cos(theta132) sin(theta131) sin(theta132) theta132dot theta131dot ...
+ 2 cos(theta5) cos(theta6) 2 cos(theta131) 2 cos(theta132) sin(theta5) sin(theta132) theta132dot 2 ...
  4 cos(theta5) cos(theta6) 2 cos(theta132) 2 sin(theta5) sin(theta131) theta132dot theta131dot ...
+ 4 cos(theta5) 2 cos(theta6) 2 cos(theta132) sin(theta131) sin(theta132) theta132dot theta131dot ...
```

```
 2 cos(theta5) cos(theta6) cos(theta131) cos(theta132) sin(theta6)
   sin(theta131) theta132dot 2 ...
+ 2 cos(theta5) cos(theta6) cos(theta132) sin(theta6) sin(theta131)
   theta5dot theta132dot ...
+ 2 cos(theta5) cos(theta6) cos(theta131) sin(theta6) sin(theta132)
   theta131dot theta132dot ...
 2 cos(theta6) cos(theta131) cos(theta132) sin(theta5) sin(theta6)
   theta131dot theta132dot ...
 2 cos(theta6) cos(theta131) sin(theta5) sin(theta6) sin(theta131)
   sin(theta132) theta132dot 2 ...
+ 2 cos(theta6) sin(theta5) sin(theta6) sin(theta131) sin(theta132)
   theta132dot theta5dot) m7 r71 ...
...
+ ( cos(theta6) cos(theta131) theta132ddot ...
+ cos(theta5) cos(theta6) sin(theta5) theta131dot 2 ...
 cos(theta5) cos(theta6) sin(theta5) theta132dot 2 ...
 cos(theta6) cos(theta132) sin(theta132) theta131dot 2 ...
+ cos(theta6) cos(theta132) sin(theta132) theta132dot 2 ...
 cos(theta5) sin(theta6) sin(theta132) theta131ddot ...
+ cos(theta132) sin(theta5) sin(theta6) theta131ddot ...
+ sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132ddot
   ...
+ cos(theta5) cos(theta6) cos(theta131) 2 sin(theta5) theta132dot 2
   ...
 2 cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) theta131dot 2
   ...
+ 2 cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) theta132dot 2
   ...
+ 2 cos(theta5) 2 cos(theta6) cos(theta132) sin(theta132) theta131dot
   2 ...
 2 cos(theta5) 2 cos(theta6) cos(theta132) sin(theta132) theta132dot
   2 ...
 cos(theta6) cos(theta131) 2 cos(theta132) sin(theta132) theta132dot
   2 ...
+ 2 cos(theta5) 2 cos(theta6) sin(theta131) theta131dot theta132dot
   ...
+ 2 cos(theta6) cos(theta132) 2 sin(theta131) theta131dot
   theta132dot ...
+ 2 cos(theta5) cos(theta132) sin(theta6) theta131dot theta5dot ...
 cos(theta5) cos(theta132) sin(theta6) theta132dot theta131dot ...
 cos(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot 2
   ...
+ cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot 2
   ...
+ cos(theta5) cos(theta132) sin(theta6) sin(theta131) theta132ddot
   ...
+ 2 sin(theta5) sin(theta6) sin(theta132) theta5dot theta131dot ...
 sin(theta5) sin(theta6) sin(theta132) theta131dot theta132dot ...
 cos(theta5) cos(theta131) sin(theta6) sin(theta131) sin(theta132)
   theta132dot 2 ...
+ cos(theta131) cos(theta132) sin(theta5) sin(theta6) sin(theta131)
   theta132dot 2 ...
+ 2 cos(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot
   theta5dot ...
 2 cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta5dot
   theta132dot ...
 2 cos(theta5) cos(theta6) cos(theta131) 2 cos(theta132) 2 sin(theta5
   ) theta132dot 2 ...
+ 2 cos(theta5) 2 cos(theta6) cos(theta131) 2 cos(theta132) sin(
   theta132) theta132dot 2 ...
```

```
  4 cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta131) theta132dot theta131dot ...
  4 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta131) sin(theta132) theta132dot theta131dot) m7 r73 ...
 ...
+ (cos(theta5) cos(theta6) theta132dot 2   cos(theta132) sin(theta6) theta131ddot ...
  cos(theta132) sin(theta6) sin(theta131) theta132dot 2 ...
+ cos(theta6) cos(theta131) sin(theta5) theta132ddot ...
+ sin(theta6) sin(theta132) theta132dot theta131dot ...
  sin(theta6) sin(theta131) sin(theta132) theta132ddot ...
+ cos(theta5) cos(theta6) cos(theta132) 2 theta131dot 2 ...
  cos(theta5) cos(theta6) cos(theta132) 2 theta132dot 2 ...
+ cos(theta5) cos(theta6) cos(theta131) 2 cos(theta132) 2 theta132dot 2 ...
+ cos(theta6) cos(theta132) sin(theta5) sin(theta132) theta131dot 2 ...
  cos(theta6) cos(theta132) sin(theta5) sin(theta132) theta132dot 2 ...
  cos(theta131) cos(theta132) sin(theta6) sin(theta131) theta132dot 2 ...
+ cos(theta6) cos(theta131) 2 cos(theta132) sin(theta5) sin(theta132) theta132dot 2 ...
  2 cos(theta6) cos(theta132) 2 sin(theta5) sin(theta131) theta132dot theta131dot ...
+ 2 cos(theta5) cos(theta6) cos(theta132) sin(theta131) sin(theta132) theta131dot theta132dot) m7 rGO1 ...
 ...
+ (g sin(theta6) sin(theta131) ...
  g cos(theta6) cos(theta131) sin(theta5) sin(theta132) ...
  g cos(theta5) cos(theta6) cos(theta131) cos(theta132)) m7 ;

B(14) = ((sin(2 theta6) theta132dot 2)/2 ...
  (sin(2 theta6) theta5dot 2)/2 ...
  cos(theta5) cos(theta132) theta131ddot ...
  sin(theta5) sin(theta132) theta131ddot ...
  cos(theta5) cos(theta132) sin(theta131) theta132dot 2 ...
+ cos(theta5) sin(theta132) theta131dot theta132dot ...
  cos(theta132) sin(theta5) theta132dot theta131dot ...
  sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta5) sin(theta131) sin(theta132) theta132ddot ...
+ cos(theta132) sin(theta5) sin(theta131) theta132ddot ...
+ cos(theta5) 2 cos(theta6) sin(theta6) theta131dot 2  ...
  cos(theta5) 2 cos(theta6) sin(theta6) theta132dot 2 ...
  2 cos(theta6) cos(theta131) 2 sin(theta6) theta132dot 2 ...
+ cos(theta6) cos(theta132) 2 sin(theta6) theta131dot 2 ...
  cos(theta6) cos(theta132) 2 sin(theta6) theta132dot 2 ...
  2 cos(theta5) cos(theta6) 2 sin(theta132) theta5dot theta131dot ...
+ 2 cos(theta6) 2 cos(theta132) sin(theta5) theta131dot theta5dot ...
+ cos(theta5) 2 cos(theta6) cos(theta131) 2 sin(theta6) theta132dot 2 ...
  2 cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta6) theta131dot 2 ...
+ 2 cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta6) theta132dot 2 ...
+ cos(theta6) cos(theta131) 2 cos(theta132) 2 sin(theta6) theta132dot 2 ...
+ cos(theta5) cos(theta131) cos(theta132) sin(theta131) theta132dot 2
```

```
...
+ 2 cos(theta6) cos(theta131) sin(theta6) theta5dot theta132dot ...
  2 cos(theta5) cos(theta131) sin(theta132) theta131dot theta132dot ...
+ 2 cos(theta131) cos(theta132) sin(theta5) theta131dot theta132dot ...
+ cos(theta131) sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
  2 cos(theta5) 2 cos(theta6) cos(theta131) 2 cos(theta132) 2 sin(theta6) theta132dot 2 ...
  2 cos(theta5) cos(theta6) 2 cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
+ 2 cos(theta5) cos(theta6) 2 cos(theta132) sin(theta131) theta132dot theta5dot ...
+ 2 cos(theta5) cos(theta6) 2 cos(theta131) sin(theta132) theta131dot theta132dot ...
  2 cos(theta6) 2 cos(theta131) cos(theta132) sin(theta5) theta131dot theta132dot ...
  2 cos(theta6) 2 cos(theta131) sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
+ 2 cos(theta6) 2 sin(theta5) sin(theta131) sin(theta132) theta5dot theta132dot ...
  2 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta6) sin(theta132) theta131dot 2 ...
+ 2 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta6) sin(theta132) theta132dot 2 ...
  2 cos(theta5) cos(theta6) sin(theta5) sin(theta6) sin(theta131) theta131dot theta132dot ...
+ 2 cos(theta6) cos(theta132) sin(theta6) sin(theta131) sin(theta132) theta131dot theta132dot ...
  2 cos(theta5) cos(theta6) cos(theta131) 2 cos(theta132) sin(theta5) sin(theta6) sin(theta132) theta132dot 2 ...
+ 4 cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) sin(theta6) sin(theta131) theta131dot theta132dot ...
  4 cos(theta5) 2 cos(theta6) cos(theta132) sin(theta6) sin(theta131) sin(theta132) theta131dot theta132dot) m7 r71 ...
...
+ (cos(theta131) sin(theta6) theta132ddot ...
 cos(theta5) sin(theta5) sin(theta6) theta131dot 2 ...
+ cos(theta5) sin(theta5) sin(theta6) theta132dot 2 ...
+ cos(theta132) sin(theta6) sin(theta132) theta131dot 2 ...
 cos(theta132) sin(theta6) sin(theta132) theta132dot 2 ...
 cos(theta5) cos(theta6) sin(theta132) theta131ddot ...
+ cos(theta6) cos(theta132) sin(theta5) theta131ddot ...
 cos(theta5) cos(theta131) 2 sin(theta5) sin(theta6) theta132dot 2 ...
+ 2 cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) theta131dot 2 ...
  2 cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) theta132dot 2 ...
  2 cos(theta5) 2 cos(theta132) sin(theta6) sin(theta132) theta131dot 2 ...
+ 2 cos(theta5) 2 cos(theta132) sin(theta6) sin(theta132) theta132dot 2 ...
+ cos(theta131) 2 cos(theta132) sin(theta6) sin(theta132) theta132dot 2 ...
  2 cos(theta5) 2 sin(theta6) sin(theta131) theta132dot theta131dot ...
  2 cos(theta132) 2 sin(theta6) sin(theta131) theta131dot theta132dot ...
```

```
+ 2 cos(theta5) cos(theta6) cos(theta132) theta131dot theta5dot ...
  cos(theta5) cos(theta6) cos(theta132) theta131dot theta132dot ...
  cos(theta5) cos(theta6) sin(theta131) sin(theta132) theta132dot 2 ...
+ cos(theta6) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
+ cos(theta5) cos(theta6) cos(theta132) sin(theta131) theta132ddot ...
+ 2 cos(theta6) sin(theta5) sin(theta132) theta131dot theta5dot ...
  cos(theta6) sin(theta5) sin(theta132) theta132dot theta131dot ...
+ cos(theta6) sin(theta5) sin(theta131) sin(theta132) theta132ddot ...
  cos(theta5) cos(theta6) cos(theta131) sin(theta131) sin(theta132) theta132dot 2 ...
+ cos(theta6) cos(theta131) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
+ 2 cos(theta5) cos(theta6) sin(theta131) sin(theta132) theta132dot theta5dot ...
  2 cos(theta6) cos(theta132) sin(theta5) sin(theta131) theta5dot theta132dot ...
+ 2 cos(theta5) cos(theta131) 2 cos(theta132) 2 sin(theta5) sin(theta6) theta132dot 2 ...
  2 cos(theta5) 2 cos(theta131) 2 cos(theta132) sin(theta6) sin(theta132) theta132dot 2 ...
+ 4 cos(theta5) 2 cos(theta132) 2 sin(theta6) sin(theta131) theta132dot theta131dot ...
+ 4 cos(theta5) cos(theta132) sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot theta131dot) m7 r73 ...
 ...
+ (cos(theta6) sin(theta132) theta131dot theta132dot ...
  cos(theta6) cos(theta132) theta131ddot ...
  cos(theta6) cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta5) sin(theta6) theta132dot 2 ...
  cos(theta131) sin(theta5) sin(theta6) theta132ddot ...
  cos(theta6) sin(theta131) sin(theta132) theta132ddot ...
  cos(theta5) cos(theta132) 2 sin(theta6) theta131dot 2 ...
+ cos(theta5) cos(theta132) 2 sin(theta6) theta132dot 2 ...
  cos(theta5) cos(theta131) 2 cos(theta132) 2 sin(theta6) theta132dot 2 ...
  cos(theta6) cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta132) sin(theta5) sin(theta6) sin(theta132) theta131dot 2 ...
+ cos(theta132) sin(theta5) sin(theta6) sin(theta132) theta132dot 2 ...
  cos(theta131) 2 cos(theta132) sin(theta5) sin(theta6) sin(theta132) theta132dot 2 ...
+ 2 cos(theta132) 2 sin(theta5) sin(theta6) sin(theta131) theta132dot theta131dot ...
  2 cos(theta5) cos(theta132) sin(theta6) sin(theta131) sin(theta132) theta131dot theta132dot) m7 rGO1 ...
 ...
+ (g cos(theta6) sin(theta131) ...
+ g cos(theta5) cos(theta131) cos(theta132) sin(theta6) ...
+ g cos(theta131) sin(theta5) sin(theta6) sin(theta132)) m7 ;

B(15) = (cos(theta6) cos(theta131) theta132ddot ...
+ 2 sin(theta6) theta5dot theta6dot ...
+ cos(theta5) cos(theta6) sin(theta5) theta131dot 2 ...
  cos(theta5) cos(theta6) sin(theta5) theta132dot 2 ...
```

```
cos(theta6) cos(theta132) sin(theta132) theta131dot 2 ...
+ cos(theta6) cos(theta132) sin(theta132) theta132dot 2 ...
 2 cos(theta131) sin(theta6) theta6dot theta132dot ...
 2 cos(theta6) sin(theta131) theta132dot theta131dot ...
+ cos(theta5) sin(theta6) sin(theta132) theta131ddot ...
cos(theta132) sin(theta5) sin(theta6) theta131ddot ...
sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132ddot
  ...
+ cos(theta5) cos(theta6) cos(theta131) 2 sin(theta5) theta132dot 2
  ...
 2 cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) theta131dot 2
    ...
+ 2 cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) theta132dot 2
   ...
+ 2 cos(theta5) 2 cos(theta6) cos(theta132) sin(theta132) theta131dot
   2 ...
 2 cos(theta5) 2 cos(theta6) cos(theta132) sin(theta132) theta132dot
   2 ...
cos(theta6) cos(theta131) 2 cos(theta132) sin(theta132) theta132dot
   2 ...
+ 2 cos(theta5) 2 cos(theta6) sin(theta131) theta131dot theta132dot
  ...
+ 2 cos(theta6) cos(theta132) 2 sin(theta131) theta132dot
  theta131dot ...
+ 2 cos(theta5) cos(theta6) sin(theta132) theta6dot theta131dot ...
 2 cos(theta6) cos(theta132) sin(theta5) theta131dot theta6dot ...
+ cos(theta5) cos(theta132) sin(theta6) theta132dot theta131dot ...
+ cos(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot 2
  ...
cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot 2
  ...
cos(theta5) cos(theta132) sin(theta6) sin(theta131) theta132ddot
  ...
+ sin(theta5) sin(theta6) sin(theta132) theta132dot theta131dot ...
 2 cos(theta5) cos(theta6) cos(theta132) sin(theta131) theta132dot
  theta6dot ...
 2 cos(theta5) cos(theta131) cos(theta132) sin(theta6) theta132dot
  theta131dot ...
cos(theta5) cos(theta131) sin(theta6) sin(theta131) sin(theta132)
  theta132dot 2 ...
+ cos(theta131) cos(theta132) sin(theta5) sin(theta6) sin(theta131)
  theta132dot 2 ...
 2 cos(theta6) sin(theta5) sin(theta131) sin(theta132) theta6dot
  theta132dot ...
 2 cos(theta131) sin(theta5) sin(theta6) sin(theta132) theta131dot
  theta132dot ...
 2 cos(theta5) cos(theta6) cos(theta131) 2 cos(theta132) 2 sin(theta5
  ) theta132dot 2 ...
+ 2 cos(theta5) 2 cos(theta6) cos(theta131) 2 cos(theta132) sin(
  theta132) theta132dot 2 ...
 4 cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta131)
  theta131dot theta132dot ...
 4 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta131)
  sin(theta132) theta132dot theta131dot) m7 r71 ...
 ...
+ (theta5dot 2 ...
+ theta132dot 2 ...
+ cos(theta5) 2 theta131dot 2 ...
cos(theta5) 2 theta132dot 2 ...
+ cos(theta132) 2 theta131dot 2 ...
```

```
  cos(theta132) 2 theta132dot 2 ...
  2 cos(theta131) theta5dot theta132dot ...
+ cos(theta5) 2 cos(theta131) 2 theta132dot 2 ...
  2 cos(theta5) 2 cos(theta132) 2 theta131dot 2 ...
+ 2 cos(theta5) 2 cos(theta132) 2 theta132dot 2 ...
+ cos(theta131) 2 cos(theta132) 2 theta132dot 2 ...
  2 cos(theta5) 2 cos(theta131) 2 cos(theta132) 2 theta132dot 2 ...
  2 cos(theta5) cos(theta132) sin(theta5) sin(theta132) theta131dot 2 ...
+ 2 cos(theta5) cos(theta132) sin(theta5) sin(theta132) theta132dot 2 ...
  2 cos(theta5) sin(theta5) sin(theta131) theta131dot theta132dot ...
+ 2 cos(theta132) sin(theta131) sin(theta132) theta131dot theta132dot ...
  2 cos(theta5) cos(theta131) 2 cos(theta132) sin(theta5) sin(theta132) theta132dot 2 ...
+ 4 cos(theta5) cos(theta132) 2 sin(theta5) sin(theta131) theta131dot theta132dot ...
  4 cos(theta5) 2 cos(theta132) sin(theta131) sin(theta132) theta132dot theta131dot) m7 r73 ...
 ...
+ (cos(theta5) cos(theta131) theta132ddot ...
  sin(theta5) theta132dot 2 ...
  cos(theta132) 2 sin(theta5) theta131dot 2 ...
+ cos(theta132) 2 sin(theta5) theta132dot 2 ...
  cos(theta131) 2 cos(theta132) 2 sin(theta5) theta132dot 2 ...
+ cos(theta5) cos(theta132) sin(theta132) theta131dot 2 ...
  cos(theta5) cos(theta132) sin(theta132) theta132dot 2 ...
+ cos(theta5) cos(theta131) 2 cos(theta132) sin(theta132) theta132dot 2 ...
  2 cos(theta5) cos(theta132) 2 sin(theta131) theta131dot theta132dot ...
  2 cos(theta132) sin(theta5) sin(theta131) sin(theta132) theta131dot theta132dot) m7 rGO1 ...
 ...
+ (g cos(theta131) cos(theta132) sin(theta5) ...
  g cos(theta5) cos(theta131) sin(theta132)) m7 ;

B(16) = (cos(theta6) theta6dot theta5dot ...
  cos(theta131) sin(theta6) theta132ddot ...
  cos(theta6) cos(theta131) theta6dot theta132dot ...
+ cos(theta5) cos(theta6) sin(theta132) theta131ddot ...
  cos(theta6) cos(theta132) sin(theta5) theta131ddot ...
+ sin(theta6) sin(theta131) theta131dot theta132dot ...
  cos(theta5) cos(theta6) cos(theta132) theta131dot theta5dot ...
+ cos(theta5) cos(theta6) cos(theta132) theta131dot theta132dot ...
+ cos(theta5) cos(theta6) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta6) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
  cos(theta5) cos(theta6) cos(theta132) sin(theta131) theta132ddot ...
  cos(theta6) sin(theta5) sin(theta132) theta5dot theta131dot ...
  cos(theta5) sin(theta6) sin(theta132) theta131dot theta6dot ...
+ cos(theta132) sin(theta5) sin(theta6) theta6dot theta131dot ...
+ cos(theta6) sin(theta5) sin(theta132) theta131dot theta132dot ...
  cos(theta6) sin(theta5) sin(theta131) sin(theta132) theta132ddot ...
  cos(theta5) cos(theta6) cos(theta131) cos(theta132) theta132dot
```

```
theta131dot ...
cos(theta5) cos(theta6) sin(theta131) sin(theta132) theta132dot theta5dot ...
+ cos(theta6) cos(theta132) sin(theta5) sin(theta131) theta132dot theta5dot ...
+ cos(theta5) cos(theta132) sin(theta6) sin(theta131) theta132dot theta6dot ...
cos(theta6) cos(theta131) sin(theta5) sin(theta132) theta132dot theta131dot ...
+ sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot theta6dot) J711 ...
...
+ ( cos(theta6) cos(theta131) theta132ddot ...
2 sin(theta6) theta5dot theta6dot ...
+ 2 cos(theta131) sin(theta6) theta6dot theta132dot ...
+ cos(theta6) sin(theta131) theta131dot theta132dot ...
cos(theta5) sin(theta6) sin(theta132) theta131ddot ...
+ cos(theta132) sin(theta5) sin(theta6) theta131ddot ...
+ sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132ddot ...
+ cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) theta131dot 2 ...
cos(theta5) 2 cos(theta6) cos(theta132) sin(theta132) theta131dot 2 ...
cos(theta5) cos(theta6) sin(theta5) sin(theta132) 2 theta131dot 2 ...
+ cos(theta6) cos(theta132) sin(theta5) 2 sin(theta132) theta131dot 2 ...
2 cos(theta5) cos(theta6) sin(theta132) theta131dot theta6dot ...
+ 2 cos(theta6) cos(theta132) sin(theta5) theta6dot theta131dot ...
cos(theta5) cos(theta132) sin(theta6) theta131dot theta132dot ...
cos(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot 2 ...
+ cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot 2 ...
+ cos(theta5) cos(theta132) sin(theta6) sin(theta131) theta132ddot ...
sin(theta5) sin(theta6) sin(theta132) theta131dot theta132dot ...
+ 2 cos(theta5) cos(theta6) cos(theta132) sin(theta131) theta6dot theta132dot ...
+ 2 cos(theta5) cos(theta131) cos(theta132) sin(theta6) theta132dot theta131dot ...
+ cos(theta5) cos(theta131) sin(theta6) sin(theta131) sin(theta132) theta132dot 2 ...
cos(theta131) cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot 2 ...
+ 2 cos(theta6) sin(theta5) sin(theta131) sin(theta132) theta6dot theta132dot ...
+ 2 cos(theta131) sin(theta5) sin(theta6) sin(theta132) theta131dot theta132dot ...
cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) sin(theta131) 2 theta132dot 2 ...
+ cos(theta5) 2 cos(theta6) cos(theta132) sin(theta131) 2 sin(theta132) theta132dot 2 ...
+ cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta131) theta131dot theta132dot ...
+ cos(theta5) cos(theta6) sin(theta5) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
cos(theta6) cos(theta132) sin(theta5) 2 sin(theta131) 2 sin(theta132) theta132dot 2 ...
```

```
cos(theta6) cos(theta132) 2 sin(theta5) 2 sin(theta131) theta131dot theta132dot ...
cos(theta5) 2 cos(theta6) sin(theta131) sin(theta132) 2 theta131dot theta132dot ...
+ cos(theta6) sin(theta5) 2 sin(theta131) sin(theta132) 2 theta131dot theta132dot ...
+ 4 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta131) sin(theta132) theta132dot theta131dot) J712 ...
...
+ (cos(theta6) cos(theta131) theta132dot theta6dot ...
cos(theta6) theta6dot theta5dot ...
cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) theta131dot 2 ...
+ cos(theta5) 2 cos(theta132) sin(theta6) sin(theta132) theta131dot 2 ...
+ cos(theta5) sin(theta5) sin(theta6) sin(theta132) 2 theta131dot 2 ...
cos(theta132) sin(theta5) 2 sin(theta6) sin(theta132) theta131dot 2 ...
cos(theta5) cos(theta6) cos(theta132) theta5dot theta131dot ...
cos(theta6) sin(theta5) sin(theta132) theta131dot theta5dot ...
+ cos(theta5) sin(theta6) sin(theta132) theta131dot theta6dot ...
cos(theta132) sin(theta5) sin(theta6) theta131dot theta6dot ...
sin(theta5) 2 sin(theta6) sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
+ cos(theta5) cos(theta6) cos(theta131) cos(theta132) theta131dot theta132dot ...
+ cos(theta5) cos(theta6) cos(theta131) sin(theta131) sin(theta132) theta132dot 2 ...
cos(theta6) cos(theta131) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
cos(theta5) cos(theta6) sin(theta131) sin(theta132) theta5dot theta132dot ...
+ cos(theta6) cos(theta132) sin(theta5) sin(theta131) theta132dot theta5dot ...
cos(theta5) cos(theta132) sin(theta6) sin(theta131) theta132dot theta6dot ...
+ cos(theta6) cos(theta131) sin(theta5) sin(theta132) theta132dot theta131dot ...
sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot theta6dot ...
+ cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) sin(theta131) 2 theta132dot 2 ...
cos(theta5) 2 cos(theta132) sin(theta6) sin(theta131) 2 sin(theta132) theta132dot 2 ...
cos(theta5) 2 cos(theta132) 2 sin(theta6) sin(theta131) theta132dot theta131dot ...
cos(theta5) sin(theta5) sin(theta6) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
+ cos(theta132) sin(theta5) 2 sin(theta6) sin(theta131) 2 sin(theta132) theta132dot 2 ...
+ cos(theta5) 2 sin(theta6) sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
+ cos(theta132) 2 sin(theta5) 2 sin(theta6) sin(theta131) theta132dot theta131dot ...
4 cos(theta5) cos(theta132) sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot theta131dot) J722 ...
...
+ (cos(theta6) theta6dot theta5dot ...
cos(theta6) cos(theta131) theta132dot theta6dot ...
```

```
+ cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) theta131dot 2 ...
  cos(theta5) 2 cos(theta132) sin(theta6) sin(theta132) theta131dot 2 ...
  cos(theta5) sin(theta5) sin(theta6) sin(theta132) 2 theta131dot 2 ...
+ cos(theta132) sin(theta5) 2 sin(theta6) sin(theta132) theta131dot 2 ...
+ cos(theta5) cos(theta6) cos(theta132) theta5dot theta131dot ...
+ cos(theta6) sin(theta5) sin(theta132) theta131dot theta5dot ...
  cos(theta5) sin(theta6) sin(theta132) theta131dot theta6dot ...
+ cos(theta132) sin(theta5) sin(theta6) theta131dot theta6dot ...
+ sin(theta5) 2 sin(theta6) sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
  cos(theta5) cos(theta6) cos(theta131) cos(theta132) theta131dot theta132dot ...
  cos(theta5) cos(theta6) cos(theta131) sin(theta131) sin(theta132) theta132dot 2 ...
+ cos(theta6) cos(theta131) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
+ cos(theta5) cos(theta6) sin(theta131) sin(theta132) theta5dot theta132dot ...
  cos(theta6) cos(theta132) sin(theta5) sin(theta131) theta132dot theta5dot ...
+ cos(theta5) cos(theta132) sin(theta6) sin(theta131) theta132dot theta6dot ...
  cos(theta6) cos(theta131) sin(theta5) sin(theta132) theta132dot theta131dot ...
+ sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot theta6dot ...
  cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) sin(theta131) 2 theta132dot 2 ...
+ cos(theta5) 2 cos(theta132) sin(theta6) sin(theta131) 2 sin(theta132 ) theta132dot 2 ...
+ cos(theta5) 2 cos(theta132) 2 sin(theta6) sin(theta131) theta132dot theta131dot ...
+ cos(theta5) sin(theta5) sin(theta6) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
  cos(theta132) sin(theta5) 2 sin(theta6) sin(theta131) 2 sin(theta132 ) theta132dot 2 ...
  cos(theta5) 2 sin(theta6) sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
  cos(theta132) 2 sin(theta5) 2 sin(theta6) sin(theta131) theta132dot theta131dot ...
+ 4 cos(theta5) cos(theta132) sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot theta131dot) J733;

B(17) = (sin(theta6) theta6dot theta5dot ...
  cos(theta131) sin(theta6) theta132dot theta6dot ...
  cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) theta131dot 2 ...
+ cos(theta5) 2 cos(theta6) cos(theta132) sin(theta132) theta131dot 2 ...
+ cos(theta5) cos(theta6) sin(theta5) sin(theta132) 2 theta131dot 2 ...
  cos(theta6) cos(theta132) sin(theta5) 2 sin(theta132) theta131dot 2 ...
+ cos(theta5) cos(theta132) sin(theta6) theta5dot theta131dot ...
+ cos(theta5) cos(theta6) sin(theta132) theta131dot theta6dot ...
  cos(theta6) cos(theta132) sin(theta5) theta131dot theta6dot ...
```

```
+ sin(theta5) sin(theta6) sin(theta132) theta131dot theta5dot ...
  cos(theta5) cos(theta6) cos(theta132) sin(theta131) theta6dot theta132dot ...
  cos(theta5) cos(theta131) cos(theta132) sin(theta6) theta131dot theta132dot ...
  cos(theta5) cos(theta131) sin(theta6) sin(theta131) sin(theta132) theta132dot 2 ...
+ cos(theta131) cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot 2 ...
+ cos(theta5) sin(theta6) sin(theta131) sin(theta132) theta5dot theta132dot ...
  cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot theta5dot ...
  cos(theta6) sin(theta5) sin(theta131) sin(theta132) theta132dot theta6dot ...
  cos(theta131) sin(theta5) sin(theta6) sin(theta132) theta132dot theta131dot ...
+ cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) sin(theta131) 2 theta132dot 2 ...
  cos(theta5) 2 cos(theta6) cos(theta132) sin(theta131) 2 sin(theta132 ) theta132dot 2 ...
  cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta131) theta132dot theta131dot ...
  cos(theta5) cos(theta6) sin(theta5) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
+ cos(theta6) cos(theta132) sin(theta5) 2 sin(theta131) 2 sin(theta132 ) theta132dot 2 ...
+ cos(theta6) cos(theta132) 2 sin(theta5) 2 sin(theta131) theta132dot theta131dot ...
+ cos(theta5) 2 cos(theta6) sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
  cos(theta6) sin(theta5) 2 sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
  4 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta131) sin(theta132) theta132dot theta131dot) J711 ...
 ...
+ (+ 2 cos(theta6) theta6dot theta5dot ...
  cos(theta131) sin(theta6) theta132ddot ...
  2 cos(theta6) cos(theta131) theta6dot theta132dot ...
+ cos(theta5) cos(theta6) sin(theta132) theta131ddot ...
  cos(theta6) cos(theta132) sin(theta5) theta131ddot ...
+ sin(theta6) sin(theta131) theta132dot theta131dot ...
+ cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) theta131dot 2 ...
  cos(theta5) 2 cos(theta132) sin(theta6) sin(theta132) theta131dot 2 ...
  cos(theta5) sin(theta5) sin(theta6) sin(theta132) 2 theta131dot 2 ...
+ cos(theta132) sin(theta5) 2 sin(theta6) sin(theta132) theta131dot 2 ...
+ cos(theta5) cos(theta6) cos(theta132) theta131dot theta132dot ...
+ cos(theta5) cos(theta6) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta6) cos(theta132) sin(theta5) sin(theta131) theta132dot 2 ...
  cos(theta5) cos(theta6) cos(theta132) sin(theta131) theta132ddot ...
  2 cos(theta5) sin(theta6) sin(theta132) theta6dot theta131dot ...
+ 2 cos(theta132) sin(theta5) sin(theta6) theta6dot theta131dot ...
+ cos(theta6) sin(theta5) sin(theta132) theta132dot theta131dot ...
```

```
  cos(theta6) sin(theta5) sin(theta131) sin(theta132) theta132ddot
    ...
+ sin(theta5) 2 sin(theta6) sin(theta131) sin(theta132) 2 theta132dot
     theta131dot ...
   2 cos(theta5) cos(theta6) cos(theta131) cos(theta132) theta132dot
    theta131dot ...
  cos(theta5) cos(theta6) cos(theta131) sin(theta131) sin(theta132)
    theta132dot 2 ...
+ cos(theta6) cos(theta131) cos(theta132) sin(theta5) sin(theta131)
    theta132dot 2 ...
+  2 cos(theta5) cos(theta132) sin(theta6) sin(theta131) theta6dot
    theta132dot ...
   2 cos(theta6) cos(theta131) sin(theta5) sin(theta132) theta131dot
    theta132dot ...
+  2 sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot
    theta6dot ...
  cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) sin(theta131) 2
    theta132dot 2 ...
+ cos(theta5) 2 cos(theta132) sin(theta6) sin(theta131) 2 sin(theta132
    ) theta132dot 2 ...
+ cos(theta5) 2 cos(theta132) 2 sin(theta6) sin(theta131) theta131dot
     theta132dot ...
+ cos(theta5) sin(theta5) sin(theta6) sin(theta131) 2 sin(theta132) 2
    theta132dot 2 ...
  cos(theta132) sin(theta5) 2 sin(theta6) sin(theta131) 2 sin(theta132
    ) theta132dot 2 ...
  cos(theta5) 2 sin(theta6) sin(theta131) sin(theta132) 2 theta131dot
     theta132dot ...
  cos(theta132) 2 sin(theta5) 2 sin(theta6) sin(theta131) theta131dot
     theta132dot ...
+  4 cos(theta5) cos(theta132) sin(theta5) sin(theta6) sin(theta131)
    sin(theta132) theta132dot theta131dot) J712 ...
...
+ (  cos(theta6) cos(theta131) theta132ddot ...
  sin(theta6) theta5dot theta6dot ...
+ cos(theta131) sin(theta6) theta6dot theta132dot ...
+ cos(theta6) sin(theta131) theta131dot theta132dot ...
  cos(theta5) sin(theta6) sin(theta132) theta131ddot ...
+ cos(theta132) sin(theta5) sin(theta6) theta131ddot ...
+ sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132ddot
    ...
+ cos(theta5) cos(theta132) sin(theta6) theta131dot theta5dot ...
  cos(theta5) cos(theta6) sin(theta132) theta131dot theta6dot ...
+ cos(theta6) cos(theta132) sin(theta5) theta131dot theta6dot ...
  cos(theta5) cos(theta132) sin(theta6) theta131dot theta132dot ...
  cos(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot 2
    ...
+ cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot 2
    ...
+ cos(theta5) cos(theta132) sin(theta6) sin(theta131) theta132ddot
    ...
+ sin(theta5) sin(theta6) sin(theta132) theta5dot theta131dot ...
  sin(theta5) sin(theta6) sin(theta132) theta131dot theta132dot ...
+ cos(theta5) cos(theta6) cos(theta132) sin(theta131) theta6dot
    theta132dot ...
+ cos(theta5) cos(theta131) cos(theta132) sin(theta6) theta131dot
    theta132dot ...
+ cos(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot
    theta5dot ...
  cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta5dot
```

```
theta132dot ...
+ cos(theta6) sin(theta5) sin(theta131) sin(theta132) theta132dot theta6dot ...
+ cos(theta131) sin(theta5) sin(theta6) sin(theta132) theta132dot theta131dot) J722 ...
 ...
+ (cos(theta131) sin(theta6) theta132dot theta6dot ...
  sin(theta6) theta6dot theta5dot ...
+ cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) theta131dot 2 ...
  cos(theta5) 2 cos(theta6) cos(theta132) sin(theta132) theta131dot 2 ...
  cos(theta5) cos(theta6) sin(theta5) sin(theta132) 2 theta131dot 2 ...
+ cos(theta6) cos(theta132) sin(theta5) 2 sin(theta132) theta131dot 2 ...
  cos(theta5) cos(theta132) sin(theta6) theta5dot theta131dot ...
  cos(theta5) cos(theta6) sin(theta132) theta131dot theta6dot ...
+ cos(theta6) cos(theta132) sin(theta5) theta131dot theta6dot ...
  sin(theta5) sin(theta6) sin(theta132) theta131dot theta5dot ...
+ cos(theta5) cos(theta6) cos(theta132) sin(theta131) theta6dot theta132dot ...
+ cos(theta5) cos(theta131) cos(theta132) sin(theta6) theta131dot theta132dot ...
+ cos(theta5) cos(theta131) sin(theta6) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta131) cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot 2 ...
  cos(theta5) sin(theta6) sin(theta131) sin(theta132) theta5dot theta132dot ...
+ cos(theta132) sin(theta5) sin(theta6) sin(theta131) theta132dot theta5dot ...
+ cos(theta6) sin(theta5) sin(theta131) sin(theta132) theta132dot theta6dot ...
+ cos(theta131) sin(theta5) sin(theta6) sin(theta132) theta132dot theta131dot ...
  cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) sin(theta131) 2 theta132dot 2 ...
+ cos(theta5) 2 cos(theta6) cos(theta132) sin(theta131) 2 sin(theta132) theta132dot 2 ...
+ cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta131) theta132dot theta131dot ...
+ cos(theta5) cos(theta6) sin(theta5) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
  cos(theta6) cos(theta132) sin(theta5) 2 sin(theta131) 2 sin(theta132) theta132dot 2 ...
  cos(theta6) cos(theta132) 2 sin(theta5) 2 sin(theta131) theta132dot theta131dot ...
  cos(theta5) 2 cos(theta6) sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
+ cos(theta6) sin(theta5) 2 sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
+ 4 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta131) sin(theta132) theta132dot theta131dot) J733 ;

B(18) = (cos(theta5) cos(theta6) 2 cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta5) 2 cos(theta6) sin(theta6) sin(theta132) 2 theta131dot 2 ...
  cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta6) sin(theta131)
```

```
2 theta132dot 2 ...
  cos(theta5) cos(theta6) 2 cos(theta131) sin(theta132) theta131dot
  theta132dot ...
+ cos(theta5) cos(theta6) 2 sin(theta132) theta5dot theta131dot ...
  2 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta6) sin(
  theta131) 2 sin(theta132) theta132dot 2 ...
+ 2 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta6) sin(
  theta132) theta131dot 2 ...
  cos(theta5) cos(theta131) cos(theta132) sin(theta6) 2 sin(theta131)
  theta132dot 2 ...
+ cos(theta5) cos(theta131) sin(theta6) 2 sin(theta132) theta131dot
  theta132dot ...
+ cos(theta5) cos(theta132) sin(theta6) 2 sin(theta131) theta5dot
  theta132dot ...
  cos(theta5) sin(theta6) 2 sin(theta132) theta5dot theta131dot ...
+ cos(theta6) 2 cos(theta131) cos(theta132) sin(theta5) theta131dot
  theta132dot ...
+ cos(theta6) 2 cos(theta131) sin(theta5) sin(theta131) sin(theta132)
  theta132dot 2 ...
  cos(theta6) 2 cos(theta132) sin(theta5) theta5dot theta131dot ...
+ cos(theta6) cos(theta131) 2 sin(theta6) theta132dot 2 ...
  cos(theta6) cos(theta132) 2 sin(theta5) 2 sin(theta6) theta131dot 2
  ...
  2 cos(theta6) cos(theta132) sin(theta5) 2 sin(theta6) sin(theta131)
  sin(theta132) theta131dot theta132dot ...
  cos(theta6) sin(theta5) 2 sin(theta6) sin(theta131) 2 sin(theta132)
  2 theta132dot 2 ...
+ cos(theta6) sin(theta6) theta5dot 2   cos(theta131) cos(theta132)
  sin(theta5) sin(theta6) 2 theta131dot theta132dot ...
  cos(theta131) sin(theta5) sin(theta6) 2 sin(theta131) sin(theta132)
  theta132dot 2 ...
+ cos(theta132) sin(theta5) sin(theta6) 2 theta5dot theta131dot ...
+ 2 cos(theta5) 2 cos(theta6) cos(theta132) sin(theta6) sin(theta131)
  sin(theta132) theta132dot theta131dot ...
  cos(theta5) cos(theta6) 2 cos(theta132) sin(theta131) theta132dot
  theta5dot ...
  2 cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) sin(theta6)
  sin(theta131) theta132dot theta131dot ...
+ 2 cos(theta5) cos(theta6) sin(theta5) sin(theta6) sin(theta131) sin(
  theta132) 2 theta132dot theta131dot ...
  cos(theta6) 2 sin(theta5) sin(theta131) sin(theta132) theta132dot
  theta5dot ...
  2 cos(theta6) cos(theta131) sin(theta6) theta132dot theta5dot ...
+ sin(theta5) sin(theta6) 2 sin(theta131) sin(theta132) theta132dot
  theta5dot) J711 ...
...
+ (2 cos(theta5) 2 cos(theta6) 2 cos(theta132) sin(theta131) sin(
  theta132) theta131dot theta132dot ...
  cos(theta5) 2 cos(theta6) 2 cos(theta132) 2 sin(theta131) 2
  theta132dot 2 ...
  cos(theta5) 2 cos(theta6) 2 sin(theta132) 2 theta131dot 2 ...
+ cos(theta5) 2 cos(theta132) 2 sin(theta6) 2 sin(theta131) 2
  theta132dot 2 ...
  2 cos(theta5) 2 cos(theta132) sin(theta6) 2 sin(theta131) sin(
  theta132) theta131dot theta132dot ...
+ cos(theta5) 2 sin(theta6) 2 sin(theta132) 2 theta131dot 2 ...
  2 cos(theta5) cos(theta6) 2 cos(theta132) sin(theta5) sin(theta131)
  2 sin(theta132) theta132dot 2 ...
+ 2 cos(theta5) cos(theta6) 2 cos(theta132) sin(theta5) sin(theta132)
  theta131dot 2 ...
```

```
+ 2 cos(theta5) cos(theta6) 2 sin(theta5) sin(theta131) sin(theta132) 2 theta131dot theta132dot ...
  4 cos(theta5) cos(theta6) cos(theta131) cos(theta132) sin(theta6) sin(theta131) theta132dot 2 ...
+ 4 cos(theta5) cos(theta6) cos(theta131) sin(theta6) sin(theta132) theta131dot theta132dot ...
+ 4 cos(theta5) cos(theta6) cos(theta132) sin(theta6) sin(theta131) theta5dot theta132dot ...
+ 2 cos(theta5) cos(theta132) 2 sin(theta5) sin(theta6) 2 sin(theta131) theta131dot theta132dot ...
+ 2 cos(theta5) cos(theta132) sin(theta5) sin(theta6) 2 sin(theta131) 2 sin(theta132) theta132dot 2 ...
  2 cos(theta5) cos(theta132) sin(theta5) sin(theta6) 2 sin(theta132) theta131dot 2 ...
+ cos(theta6) 2 cos(theta131) 2 theta132dot 2 ...
  2 cos(theta6) 2 cos(theta131) theta5dot theta132dot ...
  cos(theta6) 2 cos(theta132) 2 sin(theta5) 2 theta131dot 2 ...
  cos(theta6) 2 sin(theta5) 2 sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
+ cos(theta6) 2 theta5dot 2 ...
  4 cos(theta6) cos(theta131) cos(theta132) sin(theta5) sin(theta6) theta131dot theta132dot ...
  4 cos(theta6) cos(theta131) sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta132dot 2 ...
+ 4 cos(theta6) sin(theta5) sin(theta6) sin(theta131) sin(theta132) theta5dot theta132dot ...
  cos(theta131) 2 sin(theta6) 2 theta132dot 2 + 2 cos(theta131) sin(theta6) 2 theta5dot theta132dot ...
+ cos(theta132) 2 sin(theta5) 2 sin(theta6) 2 theta131dot 2 ...
+ sin(theta5) 2 sin(theta6) 2 sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
  sin(theta6) 2 theta5dot 2 ...
  2 cos(theta5) cos(theta6) 2 cos(theta132) 2 sin(theta5) sin(theta131) theta132dot theta131dot ...
  4 cos(theta5) cos(theta6) sin(theta6) sin(theta132) theta131dot theta5dot ...
  2 cos(theta5) sin(theta5) sin(theta6) 2 sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
  2 cos(theta6) 2 cos(theta132) sin(theta5) 2 sin(theta131) sin(theta132) theta132dot theta131dot ...
+ 4 cos(theta6) cos(theta132) sin(theta5) sin(theta6) theta131dot theta5dot ...
+ 2 cos(theta132) sin(theta5) 2 sin(theta6) 2 sin(theta131) sin(theta132) theta132dot theta131dot) J712 ...
...
+ (cos(theta5) 2 cos(theta6) cos(theta132) 2 sin(theta6) sin(theta131) 2 theta132dot 2 ...
+ cos(theta5) 2 cos(theta6) sin(theta6) sin(theta132) 2 theta131dot 2 ...
  cos(theta5) cos(theta6) 2 cos(theta131) cos(theta132) sin(theta131) theta132dot 2 ...
+ cos(theta5) cos(theta6) 2 cos(theta131) sin(theta132) theta131dot theta132dot ...
  cos(theta5) cos(theta6) 2 sin(theta132) theta5dot theta131dot ...
+ 2 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta6) sin(theta131) 2 sin(theta132) theta132dot 2 ...
  2 cos(theta5) cos(theta6) cos(theta132) sin(theta5) sin(theta6) sin(theta132) theta131dot 2 ...
+ cos(theta5) cos(theta131) cos(theta132) sin(theta6) 2 sin(theta131) theta132dot 2 ...
```

```
  cos(theta5) cos(theta131) sin(theta6) 2 sin(theta132) theta131dot theta132dot ...
  cos(theta5) cos(theta132) sin(theta6) 2 sin(theta131) theta5dot theta132dot ...
+ cos(theta5) sin(theta6) 2 sin(theta132) theta5dot theta131dot ...
  cos(theta6) 2 cos(theta131) cos(theta132) sin(theta5) theta131dot theta132dot ...
  cos(theta6) 2 cos(theta131) sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
+ cos(theta6) 2 cos(theta132) sin(theta5) theta5dot theta131dot ...
  cos(theta6) cos(theta131) 2 sin(theta6) theta132dot 2 ...
+ cos(theta6) cos(theta132) 2 sin(theta5) 2 sin(theta6) theta131dot 2 ...
+ 2 cos(theta6) cos(theta132) sin(theta5) 2 sin(theta6) sin(theta131) sin(theta132) theta131dot theta132dot ...
+ cos(theta6) sin(theta5) 2 sin(theta6) sin(theta131) 2 sin(theta132) 2 theta132dot 2 ...
  cos(theta6) sin(theta6) theta5dot 2 ...
+ cos(theta131) cos(theta132) sin(theta5) sin(theta6) 2 theta131dot theta132dot ...
+ cos(theta131) sin(theta5) sin(theta6) 2 sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta132) sin(theta5) sin(theta6) 2 theta5dot theta131dot ...
  2 cos(theta5) 2 cos(theta6) cos(theta132) sin(theta6) sin(theta131) sin(theta132) theta132dot theta131dot ...
+ cos(theta5) cos(theta6) 2 cos(theta132) sin(theta131) theta132dot theta5dot ...
+ 2 cos(theta5) cos(theta6) cos(theta132) 2 sin(theta5) sin(theta6) sin(theta131) theta132dot theta131dot ...
  2 cos(theta5) cos(theta6) sin(theta5) sin(theta6) sin(theta131) sin(theta132) 2 theta132dot theta131dot ...
+ cos(theta6) 2 sin(theta5) sin(theta131) sin(theta132) theta132dot theta5dot ...
+ 2 cos(theta6) cos(theta131) sin(theta6) theta132dot theta5dot ...
  sin(theta5) sin(theta6) 2 sin(theta131) sin(theta132) theta132dot theta5dot) J722 ...
 ...
+ (cos(theta132) sin(theta5) theta131dot theta5dot ...
  sin(theta5) sin(theta132) theta131ddot ...
  cos(theta5) cos(theta132) sin(theta131) theta132dot 2 ...
  cos(theta5) sin(theta132) theta131dot theta5dot ...
  cos(theta5) cos(theta132) theta131ddot ...
+ cos(theta5) sin(theta132) theta132dot theta131dot ...
  cos(theta132) sin(theta5) theta131dot theta132dot ...
  sin(theta5) sin(theta131) sin(theta132) theta132dot 2 ...
  cos(theta5) sin(theta131) sin(theta132) theta132ddot ...
+ cos(theta132) sin(theta5) sin(theta131) theta132ddot ...
+ cos(theta5) cos(theta132) sin(theta131) theta132dot theta5dot ...
  cos(theta5) cos(theta131) sin(theta132) theta132dot theta131dot ...
+ cos(theta131) cos(theta132) sin(theta5) theta132dot theta131dot ...
+ sin(theta5) sin(theta131) sin(theta132) theta132dot theta5dot) J733 ...
 ...
+ M673dp ;

 X = A B

C = inv(A);
```

```
C = simplifyFraction(C);

D = collect(simplify(C(1:2,:)),[m6, m7, J611, J613, J622, J633, J711, J712, J722, J733]);

  Dsubs = subs(D,...
  [m6, m7, J611, J613, J622, J633, J711, J712, J722, J733, r51, r52, r53, r61, r62, r63,...
   r71, r72, r73, r561, r562, r563, r651, r652, r653, r671, r672, r673, rGO1, rGO2, rGO3, r761, r762, r763],...
   ...
  [1e34530.59, 1e32557.47, 18898872.13e9, 1485177.89e9, 110603524.64e9, 93783786.61e9,...
   11824640.51e9, 1679850.64e9, 24252622.46e9, 15203370.83e9, 1e3  98.45, 0, 1e3  (60.66), 0, 0, 1e3  (168.02),...
   1e3  39.19, 0, 1e3  (16.28), 1e3  167.65, 0, 1e3  235.56, 0, 1e3  (173.83), 1e3  168.02,...
   0, 0, 1e3  303.02, 1e3  266.1, 0, 0, 1e3  (39.19), 0, 1e3  151.28]);

  D = simplify(eval(D));



E = simplify(D B);

E = simplifyFraction(E);

GlobalVariables

  E = simplify(eval(E));

Esubs = subs(E,...
[m6, m7, J611, J613, J622, J633, J711, J712, J722, J733, r51, r52, r53, r61, r62, r63,...
r71, r72, r73, r561, r562, r563, r651, r652, r653, r671, r672, r673, rGO1, rGO2, rGO3, r761, r762, r763],...
...
[1e34530.59, 1e32557.47, 18898872.13e9, 1485177.89e9, 110603524.64e9, 93783786.61e9,...
11824640.51e9, 1679850.64e9, 24252622.46e9, 15203370.83e9, 1e3  98.45, 0, 1e3  (60.66), 0, 0, 1e3  (168.02),...
1e3  39.19, 0, 1e3  (16.28), 1e3  167.65, 0, 1e3  235.56, 0, 1e3  (173.83), 1e3  168.02,...
0, 0, 1e3  303.02, 1e3  266.1, 0, 0, 1e3  (39.19), 0, 1e3  151.28]);

Esubs = simplify(Esubs);

Esubs = collect(Esubs, [M562dp, M673dp, theta5, theta6, theta131, theta132]);
```

Listing D.14: Newton-Euler equations for the stabilizer

```
clear
clc

syms J511 J512 J513 J521 J522 J523 J531 J532 J533 ...
     J611 J612 J613 J621 J622 J623 J631 J632 J633 ...
     J711 J712 J713 J721 J722 J723 J731 J732 J733 real ...
```

```
syms F561dp F562dp F563dp M561dp M562dp M563dp ...
     F671dp F672dp F673dp M671dp M672dp M673dp ...

syms Fg1 Fg2 Fg3 Mg1 Mg2 Mg3 ...

syms theta5(t) theta6(t) theta131(t) theta132(t)

syms m5 m6 m7 g

syms r51 r52 r53 r61 r62 r63 r71 r72 r73

syms r561 r562 r563 r651 r652 r653 r671 r672 r673 ...
     rGO1 rGO2 rGO3 r761 r762 r763

 Basic Column Matrices

u1 = [1; 0; 0];
u2 = [0; 1; 0];
u3 = [0; 0; 1];

 Relations with Derivatives

theta131dot = diff(theta131);

theta131doubledot = diff(theta131dot);

theta132dot = diff(theta132);

theta132doubledot = diff(theta132dot);

theta5dot = diff(theta5);

theta5doubledot = diff(theta5dot);

theta6dot = diff(theta6);

theta6doubledot = diff(theta6dot);

 Inertia matrices

J5 = [J511 J512 J513; J521 J522 J523; J531 J532 J533];  Ineria dyadic resolved in Reference Frame 5

J6 = [J611 J612 J613; J621 J622 J623; J631 J632 J633];  Ineria dyadic resolved in Reference Frame 6

J7 = [J711 J712 J713; J721 J722 J723; J731 J732 J733];  Ineria dyadic resolved in Reference Frame 7

 Transformation Matrices

T5612 = simplify(BasicRotationMatrix(3,theta5) BasicRotationMatrix(1,sym(pi)/2));  C(5,6)

T6712 = simplify(BasicRotationMatrix(3,theta6));  C(6,7)

T5712 = T5612  T6712 ;  C(5,7)

T2513 = simplify(BasicRotationMatrix(2,theta131) BasicRotationMatrix
```

```
    (1,theta132));   C(0,5)

 Force Definitions

F 56 dp = [F 561 dp; F 562 dp; F 563 dp];

F 67 dp = [F 671 dp; F 672 dp; F 673 dp];

F g = [F g 1; F g 2; F g 3];



 Force Tranformations

F 56 p = simplify(T 56 12  F 56 dp);

F 67 p = simplify(T 67 12  F 67 dp);

 Force Components

F s body = formula(F 56 p); F 561 p = F s body (1); F 562 p = F s body (2);
    F 563 p = F s body (3);

F s body = formula(F 67 p); F 671 p = F s body (1); F 672 p = F s body (2);
    F 673 p = F s body (3);

 Moment Definitions

M 56 dp = [M 561 dp; M 562 dp; M 563 dp];

M 67 dp = [M 671 dp; M 672 dp; M 673 dp];

M g = [M g 1; M g 2; M g 3];

 Moment Tranformations

M 56 p = simplify(T 56 12  M 56 dp);

M 67 p = simplify(T 67 12  M 67 dp);

 Moment Components

M s body = formula(M 56 p); M 561 p = M s body (1); M 562 p = M s body (2);
    M 563 p = M s body (3);

M s body = formula(M 67 p); M 671 p = M s body (1); M 672 p = M s body (2);
    M 673 p = M s body (3);



 Newton  Euler Equations

 Body 5

W 5 = simplify( m 5  g ( T 25 13  u 3));
    Weight vector resolved in Reference Frame 5

 Definition of motion vectors

w 5 = simplify(T 25 13  (theta132dot  u 1 + theta131dot  u 2));
```

```
    Angular velocity vector resolved in Reference Frame 5

w5tilda = SkewSymmetricMatrix(w5);                                        Skew
    Symmetric Matrix of the angular velocity matrix

alfa5 = simplify(diff(w5,t));
                                                  Angular acceleration
    vector resolved in Reference Frame 5
                                            Definition of r j

r5 = r51 u1 + r52 u2 + r53 u3;
    Position vector of C5 resolved in Reference Frame 5

V5 = simplify(diff(r5,t) + w5tilda r5);
    Velocity vector of C5 resolved in Reference Frame 5

a5 = simplify(diff(V5) + w5tilda V5);
    Acceleration vector of C5 resolved in Reference Frame 5

 Force equations

FE5 = Fg   F56p + W5   m5 a5;

FE5body = formula(FE5);

FE51 = FE5body(1); FE52 = FE5body(2); FE53 = FE5body(3);

 Definition of Moment Arms

r5G =   r5;

r5Gtilda = SkewSymmetricMatrix(r5G);

r56 = [r561 r562 r563];

r56tilda = SkewSymmetricMatrix(r56);

 Moment equations

ME5 = Mg   M56p + r5Gtilda Fg   r56tilda F56p   (
    DyadicDotProduct(J5,alfa5) + w5tilda DyadicDotProduct(J5,w5))
    ;

ME5body = formula(ME5);

ME51 = ME5body(1); ME52 = ME5body(2); ME53 = ME5body(3);

 Body 6

W6 = simplify( m6 g (T5612 (T2513 u3)));
    Weight vector resolved in Reference Frame 6

 Definition of motion vectors

w6 = simplify(T5612 (w5 + u3 theta5dot));
                                      Angular velocity vector resolved in
    Reference Frame 6

w6tilda = SkewSymmetricMatrix(w6);                                        Skew
    Symmetric Matrix of the angular velocity matrix
```

```
alfa6 = simplify(diff(w6,t));   Angular acceleration vector resolved in Reference Frame 6

rGO = [rGO1; rGO2; rGO3];   Offset distance between Point O   Ground

r6 = simplify(T5612  rGO + r61  u1 + r62  u2 + r63  u3);   Position vector of C6 resolved in Reference Frame 6

V6 = simplify(diff(r6,t) + w6tilda  r6);   Velocity vector of C6 resolved in Reference Frame 6

a6 = simplify(diff(V6) + w6tilda  V6);   Acceleration vector of C6 resolved in Reference Frame 6

 Force equations

FE6 = F56dp   F67p + W6   m6  a6;

FE6body = formula(FE6);

FE61 = FE6body(1); FE62 = FE6body(2); FE63 = FE6body(3);

 Definition of Moment Arms

r65 = [r651; r652; r653];

r65tilda = SkewSymmetricMatrix(r65);

r67 = [r671; r672; r673];

r67tilda = SkewSymmetricMatrix(r67);

 Moment equations

ME6 = M56dp   M67p + r65tilda  F56dp   r67tilda  F67p   (
    DyadicDotProduct(J6,alfa6) + w6tilda  DyadicDotProduct(J6,w6))
    ;

ME6body = formula(ME6);

ME61 = ME6body(1); ME62 = ME6body(2); ME63 = ME6body(3);

 Body 7

W7 = simplify( m7  g  (T5712  (T2513  u3)));   Weight vector resolved in Reference Frame 7

 Definition of motion vectors

w7 = simplify(T6712  (w6 + u3  theta6dot));   Angular velocity vector resolved in Reference Frame 7

w7tilda = SkewSymmetricMatrix(w7);   Skew Symmetric Matrix of the angular velocity matrix

alfa7 = simplify(diff(w7,t));
```

```
                                                    Angular acceleration vector resolved in Reference Frame 7

r7 = simplify(T5712 rGO + r71 u1 + r72 u2 + r73 u3);                    Position vector of C7 resolved in Reference Frame 7

V7 = simplify(diff(r7,t) + w7tilda r7);    Velocity vector of C7 resolved in Reference Frame 7

a7 = simplify(diff(V7) + w7tilda V7);    Acceleration vector of C7 resolved in Reference Frame 7

 Force equations

FE7 = F67dp + W7   m7 a7;

FE7body = formula(FE7);

FE71 = FE7body(1); FE72 = FE7body(2); FE73 = FE7body(3);

 Definition of Moment Arms

r76 = [r761; r762; r763];

r76tilda = SkewSymmetricMatrix(r76);

 Moment equations

ME7 = M67dp + r76tilda F67dp   (DyadicDotProduct(J7,alfa7) + w7tilda DyadicDotProduct(J7,w7));

ME7body = formula(ME7);

ME71 = ME7body(1); ME72 = ME7body(2); ME73 = ME7body(3);
```

Listing D.15: Newton-Euler equations for the stabilizer

```
clear
clc

syms Js011 Js012 Js013 Js021 Js022 Js023 Js031 Js032 Js033 ...
     Js111 Js112 Js113 Js121 Js122 Js123 Js131 Js132 Js133 ...
     Js211 Js212 Js213 Js221 Js222 Js223 Js231 Js232 Js233 real ...

syms Fs011dp Fs012dp Fs013dp Ms011dp Ms012dp Ms013dp ...
     Fs121dp Fs122dp Fs123dp Ms121dp Ms122dp Ms123dp ...

syms Fg1 Fg2 Fg3 Mg1 Mg2 Mg3 ...

syms theta1(t) theta2(t) theta131(t) theta132(t)

syms ms0 ms1 ms2

syms r11 r12 r13 r21 r22 r23 r31 r32 r33 r41 r42 r43

syms r101 r102 r103 r121 r122 r123 r211 r212 r213 r251 r252 r253...
```

```
     r301 r302 r303 r341 r342 r343 r431 r432 r433 r451 r452
        r453...
     r521 r522 r523 r541 r542 r543 r501 r502 r503 rp1 rp2 rp3

syms da1 da2 da3

  Basic Column Matrices

u1 = [1; 0; 0];
u2 = [0; 1; 0];
u3 = [0; 0; 1];

  Relations with Derivatives

theta1dot = diff(theta1);

theta1doubledot = diff(theta1dot);

theta2dot = diff(theta2);

theta2doubledot = diff(theta2dot);

theta131dot = diff(theta131);

theta131doubledot = diff(theta131dot);

theta132dot = diff(theta132);

theta132doubledot = diff(theta132dot);

  Inertia matrices

Js3 = [Js011 Js012 Js013; Js021 Js022 Js023; Js031 Js032
    Js033];   Ineria dyadic resolved in Reference Frame 0

Js1 = [Js111 Js112 Js113; Js121 Js122 Js123; Js131 Js132
    Js133];   Ineria dyadic resolved in Reference Frame 1

Js2 = [Js211 Js212 Js213; Js221 Js222 Js223; Js231 Js232
    Js233];   Ineria dyadic resolved in Reference Frame 2

  Transformation Matrices

Ts01 = simplify(BasicRotationMatrix(3,theta1) BasicRotationMatrix
    (1, sym(pi)/2));   Cs(0,1)

Ts12 = simplify(BasicRotationMatrix(3,theta2));   Cs(1,2)

Ts02 = Ts01   Ts12 ;   Cs(0,2)

T2513 = simplify(BasicRotationMatrix(2,theta131) BasicRotationMatrix
    (1,theta132));   C(0,5)

  Force Definitions

Fs01dp = [Fs011dp; Fs012dp; Fs013dp];

Fs12dp = [Fs121dp; Fs122dp; Fs123dp];

Fg = [Fg1; Fg2; Fg3];
```

```
 Force Tranformations

F s 0 1 p = simplify(T s 0 1  F s 0 1 dp);

F s 1 2 p = simplify(T s 1 2  F s 1 2 dp);

 Force Components

F s body = formula(F s 0 1 p); F s 0 1 1 p = F s body(1); F s 0 1 2 p = F s body (2); F s 0 1 3 p = F s body(3);

F s body = formula(F s 1 2 p); F s 0 3 1 p = F s body(1); F s 0 3 2 p = F s body (2); F s 0 3 3 p = F s body(3);

 Moment Definitions

M s 0 1 dp = [M s 0 1 1 dp; M s 0 1 2 dp; M s 0 1 3 dp];

M s 1 2 dp = [M s 1 2 1 dp; M s 1 2 2 dp; M s 1 2 3 dp];

M g = [M g 1; M g 2; M g 3];

 Moment Tranformations

M s 0 1 p = simplify(T s 0 1  M s 0 1 dp);

M s 1 2 p = simplify(T s 1 2  M s 1 2 dp);

 Moment Components

M s body = formula(M s 0 1 p); M s 0 1 1 p = M s body(1); M s 0 1 2 p = M s body (2); M s 0 1 3 p = M s body(3);

M s body = formula(M s 1 2 p); M s 0 3 1 p = M s body(1); M s 0 3 2 p = M s body (2); M s 0 3 3 p = M s body(3);



 Newton  Euler Equations

 Body S 0

W S 0 = simplify( m 0 g u 3);         Weight vector resolved in Reference Frame 0

 Definition of motion vectors

w 5 = simplify(T 2 5 1 3 (theta 1 3 2 dot u 1 + theta 1 3 1 dot u 2)); Angular velocity vector resolved in Reference Frame 5

w 5 tilda = SkewSymmetricMatrix(w 5);                               Skew Symmetric Matrix of the angular velocity matrix

alfa 5 = diff(w 5 ,t); Angular acceleration vector resolved in Reference Frame 5

r j = simplify(d a 2 (T 2 5 1 3 u 3));
```

```
    Definition of r j

r 5 = r j   r p1  u 1   r p2  u 2   r p3  u 3;
    Position vector of C 5  resolved in Reference Frame 5

V 5 = simplify(diff(r 5 ,t) + w 5tilda  r 5);
    Velocity vector of C 5  resolved in Reference Frame 5

a 5 = simplify(diff(V 5) + w 5tilda  V 5);
    Acceleration vector of C 5  resolved in Reference Frame 5
```

Listing D.16: Matlab fuction to calculate skew-symmetric matrix

```
function X = SkewSymmetricMatrix(x)

Xbody = formula(x);

X = [0  Xbody(3) Xbody(2); Xbody(3) 0  Xbody(1);  Xbody(2) Xbody(1)
    0];

end
```

Listing D.17: Calculation of LQR parameters for the stabilizer

```
clear; clc;

Ts = 5e 4;

Q(1,1) = 3.2828e+05;
Q(2,2) = 0.2;
Q(3,3) = 3.2828e+05;
Q(4,4) = 0.2;

p = 1;

R = p [1  0; 0 1];

theta131 = pi/180 (16:0.1:16);
theta132 = pi/180 (16:0.1:16);
K = zeros(2,4,length(theta131),length(theta132));

for i = 1:length(theta131)

    for j = 1:length(theta132)

        [A,B] = LinearizedSystemModel(theta131(i),0, 0, theta132(j),0,
            0);

        [K(:,:,i,j),  ,  ] = lqrd(A,B,Q,R,Ts);

    end

end
```

Listing D.18: Calculation of PID parameters for the stabilizer

```
clear;
clc;
```

```
GlobalVariablesv1;

   Parameters for theta131 = 16   theta132 = 0;

  P T 5 = 19.8940285880982; I T 5 = 18.6266907488659; D T 5 =
   5.2173870978178; N T 5 = 109.079405084324;

   Parameters for theta131 = 16   theta132 = 0;

  P T 6 = 4.51825275804428; I T 6 = 7.10337841040921; D T 6 =
   0.415377321109234; N T 6 = 2285.65595208321;

systemName =  LagrangeStabilizerModelPIDv02 ;

opensystem(systemName);

Theta5star = [ systemName,  /Integrator1 ];
Theta6star = [ systemName,  /Integrator3 ];
PID5 = [ systemName,  /PID5 ];
PID6 = [ systemName,  /PID6 ];
Theta131 = [ systemName,  /theta131deg ];
Theta132 = [ systemName,  /theta132deg ];

t131 = 16:2:16;
t132 = 16:2:16;

for i = 1:length(t131)
    for j = 1:length(t132)

        setparam(Theta131, Value ,num2str(t131(i)))
        setparam(Theta132, Value ,num2str(t132(j)))

          io5(1) = linio(PID5,1, input );
          io5(2) = linio(Theta5star,1, openoutput );
          linsys5 = linearize(systemName,io5);

          io6(1) = linio(PID6,1, input );
          io6(2) = linio(Theta6star,1, openoutput );
          linsys6 = linearize(systemName,io6);

          pidTuner(linsys5)

        disp( tune );

          setparam(systemName, SimulationCommand , Start );

          while strcmp(getparam(systemName, SimulationStatus ) ,  running
    )
              pause(0.01)
          end

        if strcmp(getparam(systemName, SimulationStatus ) ,  stopped )
            P5(i,j) = str2double(getparam(PID5,  P ));
            I5(i,j) = str2double(getparam(PID5,  I ));
            D5(i,j) = str2double(getparam(PID5,  D ));
            N5(i,j) = str2double(getparam(PID5,  N ));
            P6(i,j) = str2double(getparam(PID6,  P ));
            I6(i,j) = str2double(getparam(PID6,  I ));
            D6(i,j) = str2double(getparam(PID6,  D ));
```

```
            N 6 (i,j) = str2double(getparam(PID6,  N ));
        end
    end
end
```

Listing D.19: Solution of Ricatti equation

```
function dXdt  = mRiccati(X, A, B,R,Q)
  r1=randi([1 1000]);
  r2=randi([1 1000]);
  r3=randi([1 1000]);
  r4=randi([1 1000]);
  r5=randi([1 1000]);
  r6=randi([1 1000]);
  Q = diag([r1 r2 r3 r4 r5 r6] );
  R = 1 diag([1 1 1 1]);
X = reshape(X, size(A));  Convert from "n 2" by 1 to "n" by "n"
dXdt = A .  X + X A   (X B) inv(R) (B .  X) + Q;  Determine derivative
  dXdtS= dXdt(:);  Convert from "n" by "n" to "n 2" by 1, algebraic Riccati
    equation (ARE)
```

D.4 Additional Real Time Target Codes

Listing D.20: Filter design for noisy signals

```
t = theta6star.time;                Time vector
figure
plot(t, theta6starvalues);
ylabel( Ampltude );
xlabel( Time(secs) );
axis tight;
title( Noisy Input Signal );

Fs = 1/Ts;                          Sampling frequency
L = length(theta6starvalues);       Length of signal
y = theta6starvalues;

  figure
  plot(Fs t(1:50),y(1:50))
  xlabel( time (milliseconds) )

NFFT = 2 nextpow2(L);   Next power of 2 from length of y
Y = fft(y,NFFT)/L;
f = Fs/2 linspace(0,1,NFFT/2+1);

  Plot single sided amplitude spectrum.
figure
plot(f,2 abs(Y(1:NFFT/2+1)))
title( Single Sided Amplitude Spectrum of y(t) )
xlabel( Frequency (Hz) )
ylabel(  Y(f)  )
```

Listing D.21: Parameters with Roketsan profiles loaded before the experiments started

```
clear; clc; close all;
```

```
  load( C: MATLAB Work Zafer03 01 2019 Disturber Roketsan Data roketsanprofil1.mat )
load( C: Users AsYA Desktop26052019 usb yedek Zafer03 01 2019 Disturber Roketsan Data roketsanprofil1.mat )
theta131roketsan = timeseries(5 theta1311,t);
theta132roketsan = timeseries(5 theta1321,t);

load( WorkspaceKayitlari PIDParametersv01.mat )

  Global variables

Ts = 5e 4;   Ts is imported from MAT file. Use this to overwrite it.

eff = 0.8;

pulse2degree = 360/250000;   Number of encoder pulses for a one revolution of GB shaft.

statictorque = 3.75;   N.m
zerovelocitytolerance = 0.02;   deg/sec

g = 9.81;

g 0 = [0; 0;  g];

  Masses

m 6 = 1e 3 4530.59;       kg

m 7 = 1e 3 2557.47;       kg

  Inertia Terms

J Motor GB = 8786000e 9;     kg.m 2  Inertia of rotating parts at the output of gearbox

J 611 = 19787090.89e 9;        J 612 = 330917.07e 9;        J 613 = 2341197.11e 9;        kg.m 2
J 621 = 330917.07e 9;        J 622 = 112319139.40e 9 + J Motor GB;        J 623 = 6005.85e 9;        kg.m 2
J 631 = 2341197.11e 9;        J 632 = 6005.85e 9;        J 633 = 94615995.73e 9;        kg.m 2

J 711 = 18797121.45e 9;        J 712 = 5730.37e 9;        J 713 = 1902546.14e 9;        kg.m 2
J 721 = 5730.37e 9;        J 722 = 13470946.42e 9;        J 723 = 9452.80e 9;        kg.m 2
J 731 = 1902546.14e 9;        J 732 = 9452.80e 9;        J 733 = 8094810.07e 9 + J Motor GB;   kg.m 2

J 6 = [J 611 J 612 J 613; J 621 J 622 J 623; J 631 J 632 J 633];   Ineria dyadic resolved in Reference Frame 6   kg.m 2

J 7 = [J 711 J 712 J 713; J 721 J 722 J 723; J 731 J 732 J 733];   Ineria dyadic resolved in Reference Frame 7   kg.m 2

  Center of Mass

r 61 = 0.43;          r 62 = 3.09;    r 63 = 1e 3  (166.93);
```

```
    Defined in Frame 6

r71 = 1e 3  38.33;     r72 = 0;        r73 = 1e 3  (19.40);
    Defined in Frame 7

rGO1 = 1e 3  266.1;    rGO2 = 0;       rGO3 = 0;
    Defined in Frame 5

  PID Controller Parameters

InitialConditionForFilter = 0;

InitialConditionForIntegrator = 0;

FilterOutMax = 48;

FilterOutMin =  48;

IntegratorOutMax = 48;

IntegratorOutMin =  48;

LowersaturationLimit =  48;   N.m

UppersaturationLimit = 48;   N.m

  Disturber Parameters

a1 = 783.4e3;

a3 = 193e3;

b1 = 193e3;

s410 = 783.4e3;

s420 = 783.4e3;

dpiston = 40e3;

reqstroke = 300e3;

Apiston = pi dpiston2/4;

convltmin2m3s = 1/1000/60;

  Qmax1 = 33 convltmin2m3s;   m3/s

  Qmax2 = 24 convltmin2m3s;   m3/s

  Intermediate Calculations

nmotor = 1500;    1/s

Fullstrokevolume = 376991.12e9;    m3

tfast = 21.4;    s

tslow = 22.1;    s

```

```
Qpump = nmotor   9.8e 3;    m 3 / s

QQfast = Fullstrokevolume/t fast;    m 3 / s

Q max 1 lt = QQfast   1000   60   t fast/t slow;    liter/min

Q max 2 lt = QQfast   1000   60;    liter/min

Q max 1 = Q max 1 lt   conv lt min 2 m3 s;    m 3 / s

Q max 2 = Q max 2 lt   conv lt min 2 m3 s;    m 3 / s

  Intermediate Calculations

s 41 max = Q max 1 /Apiston;

s 42 max = Q max 2 /Apiston;

t 1 min = 2 reqstroke/ s 41 max;      Min coefficient must be 1.6 while
    working with sine (Now coef=3)

t 2 min = 2 reqstroke/ s 42 max;      Min coefficient must be 1.6 while
    working with sine

theta131amplitute = pi/30;

theta132amplitute = pi/30;
```

Listing D.22: Plotting and logging experiment results

```
close all;

date = 20190526 ;
counter = 21 ;
  success = stepresponseLQR5deg ;
  success = stepresponsealteredPID5deg ;
success = successroketsanLQR ;
  success = sensorreading ;
  success = vibrationslidingpiston ;

Valve1values = reshape(Valve1.signals.values,max(size(Valve1.signals.
    values)),1);
Valve2values = reshape(Valve2.signals.values,max(size(Valve2.signals.
    values)),1);

e41values = reshape(e41.signals.values,max(size(e41.signals.values)),1)
    ;
e42values = reshape(e42.signals.values,max(size(e42.signals.values)),1)
    ;

s41values = reshape(s41.signals.values,max(size(s41.signals.values)),1)
    ;
s42values = reshape(s42.signals.values,max(size(s42.signals.values)),1)
    ;
s41refvalues = reshape(s41ref.signals.values,max(size(s41ref.signals
    .values)),1);
s42refvalues = reshape(s42ref.signals.values,max(size(s42ref.signals
    .values)),1);

theta5starinclinevalues = reshape(theta5starincline.signals.values,
```

```
    max(size(theta5starincline.signals.values)),1);
theta6starinclinevalues = reshape(theta6starincline.signals.values,
    max(size(theta6starincline.signals.values)),1);

theta5starvalues = reshape(theta5starcalculated.signals.values,max(
    size(theta5starcalculated.signals.values)),1);
theta6starvalues = reshape(theta6starcalculated.signals.values,max(
    size(theta6starcalculated.signals.values)),1);
  theta5starfilteredvalues = reshape(theta5starlowpass.signals.
    values,max(size(theta5starlowpass.signals.values)),1);
  theta6starfilteredvalues = reshape(theta6starlowpass.signals.
    values,max(size(theta6starlowpass.signals.values)),1);

  theta5stardotvalues = reshape(theta5stardot.signals.values,max(
    size(theta5stardot.signals.values)),1);
  theta6stardotvalues = reshape(theta6stardot.signals.values,max(
    size(theta6stardot.signals.values)),1);

encoder5values = reshape(encoder5degree.signals.values,max(size(
    encoder5degree.signals.values)),1);
encoder6values = reshape(encoder6degree.signals.values,max(size(
    encoder6degree.signals.values)),1);

T5correctedvalues = reshape(T5corrected.signals.values,max(size(
    T5corrected.signals.values)),1);
T6correctedvalues = reshape(T6corrected.signals.values,max(size(
    T6corrected.signals.values)),1);
T5voltagevalues = reshape(T5voltage.signals.values,max(size(
    T5voltage.signals.values)),1);
T6voltagevalues = reshape(T6voltage.signals.values,max(size(
    T6voltage.signals.values)),1);

mkdir([ C: Users Host Fuze Yuku Desktop Grafikler stabilizer  ,date],[
    counter,   ,success])

foldername = [ C: Users Host Fuze Yuku Desktop Grafikler stabilizer  ,date
    ,   ,counter,   ,success];

save([foldername,  DATA.mat ])

figure
plot(s41.time, s41values)
hold on
plot(s41ref.time, s41refvalues)
xlabel( Time [s] , interpreter , latex )
ylabel(  s  41   [mm] , interpreter , latex )
legend(  Actual , Reference  , interpreter , latex )
savefig(gcf,[foldername,  s41.fig ])
saveas(gcf,[foldername,  s41 ] , epsc )

figure
plot(s42.time, s42values)
hold on
plot(s42ref.time, s42refvalues)
xlabel( Time [s] , interpreter , latex )
ylabel(  s  42   [mm] , interpreter , latex )
legend(  Actual , Reference  , interpreter , latex )
savefig(gcf,[foldername,  s42.fig ])
saveas(gcf,[foldername,  s42 ] , epsc )
```

```
figure
plot(e41.time, e41values)
xlabel( Time [s] , interpreter , latex )
ylabel(  e  41   [mm] , interpreter , latex )
savefig(gcf,[foldername,  e41.fig ])
saveas(gcf,[foldername,  e41 ] , epsc )

figure
plot(e42.time, e42values)
xlabel( Time [s] , interpreter , latex )
ylabel(  e  42   [mm] , interpreter , latex )
savefig(gcf,[foldername,  e42.fig ])
saveas(gcf,[foldername,  e42 ] , epsc )

figure
plot(Valve1.time, Valve1values)
xlabel( Time [s] , interpreter , latex )
ylabel(  x 1  [mm] , interpreter , latex )
savefig(gcf,[foldername,  x1.fig ])
saveas(gcf,[foldername,  x1 ] , epsc )

figure
plot(Valve2.time, Valve2values)
xlabel( Time [s] , interpreter , latex )
ylabel(  x 2  [mm] , interpreter , latex )
savefig(gcf,[foldername,  x2.fig ])
saveas(gcf,[foldername,  x2 ] , epsc )

figure
plot(theta5starcalculated.time, theta5starvalues)
  hold on
  plot(theta5starlowpass.time, theta5starfilteredvalues)
xlabel( Time [s] , interpreter , latex )
ylabel(   theta5    [    circ   ] , interpreter , latex )
  legend(  Calculated  ,  Filtered  , interpreter , latex )
savefig(gcf,[foldername,  theta5star.fig ])
saveas(gcf,[foldername,  theta5star ] , epsc )

figure
plot(theta6starcalculated.time, theta6starvalues)
  hold on
  plot(theta6starlowpass.time, theta6starfilteredvalues)
xlabel( Time [s] , interpreter , latex )
ylabel(   theta6    [    circ   ] , interpreter , latex )
  legend( Calculated , Filtered , interpreter , latex )
savefig(gcf,[foldername,  theta6star.fig ])
saveas(gcf,[foldername,  theta6star ] , epsc )

figure
plot(theta5starincline.time, theta5starinclinevalues
    theta5starinclinevalues(1))
hold on
plot(encoder5degree.time, encoder5values)
xlabel( Time [s] , interpreter , latex )
ylabel(   theta5    [    circ   ] , interpreter , latex )
legend(  Inclinemeter , Encoder  , interpreter , latex )
savefig(gcf,[foldername,  theta5inclineencoder.fig ])
saveas(gcf,[foldername,  theta5inclineencoder ] , epsc )

figure
```

```
plot(theta6starincline.time, theta6starinclinevalues theta6starinclinevalues(1))
hold on
plot(encoder6degree.time, encoder6values)
xlabel( Time [s] , interpreter , latex )
ylabel(  theta6   [   circ  ] , interpreter , latex )
legend( Inclinemeter , Encoder  , interpreter , latex )
savefig(gcf,[foldername,  theta6inclineencoder.fig ])
saveas(gcf,[foldername,  theta6inclineencoder ] , epsc )

  figure
  plot(theta5stardot.time, theta5stardotvalues)
  xlabel( Time [s] , interpreter , latex )
  ylabel(  dot  theta5    [   circ  /s] , interpreter , latex )
  savefig(gcf,[foldername,  theta5stardot.fig ])
  saveas(gcf,[foldername,  theta5stardot ] , epsc )

  figure
  plot(theta6stardot.time, theta6stardotvalues)
  xlabel( Time [s] , interpreter , latex )
  ylabel(  dot  theta6    [   circ  /s] , interpreter , latex )
  savefig(gcf,[foldername,  theta6stardot.fig ])
  saveas(gcf,[foldername,  theta6stardot ] , epsc )

figure
plot(T5corrected.time, T5correctedvalues)
hold on
plot(T5voltage.time, T5voltagevalues 48/9)
xlabel( Time [s] , interpreter , latex )
ylabel(  T 5  [N.m] , interpreter , latex )
legend( Calculated , saturated  , interpreter , latex )
savefig(gcf,[foldername,  Torque5.fig ])
saveas(gcf,[foldername,  Torque5 ] , epsc )

figure
plot(T6corrected.time, T6correctedvalues)
hold on
plot(T6voltage.time, T6voltagevalues 48/9)
xlabel( Time [s] , interpreter , latex )
ylabel(  T 6  [N.m] , interpreter , latex )
legend( Calculated , saturated  , interpreter , latex )
savefig(gcf,[foldername,  Torque6.fig ])
saveas(gcf,[foldername,  Torque6 ] , epsc )

  Logging experiment parameters into a TXT file

fileID = fopen([foldername,  parameters.txt ] , w+ );

fprintf(fileID, Ts =  g n n r ,Ts);
fprintf(fileID, Thetaxstars are calculated using linear encoders and motor encoders. n n r );
fprintf(fileID, Not filtered. n n r );
  fprintf(fileID, Low pass filter w/ Passband Freq: 60Hz, stopbandFreq: 90 Hz n n r );
  fprintf(fileID, Moving average is not applied. n n r );
fprintf(fileID, Moving average is applied on 30 samples. n n r );
fprintf(fileID, Coefficient c in the transfer function approximation s/(c s + 1) used for linearization: inf n n r );
  fprintf(fileID, Counter Torque reduced 1 N.m. n n r );
```

```
fclose(fileID);
```