UTILIZATION OF OUTLIER-ADJUSTED LEE-CARTER MODEL IN MORTALITY ESTIMATION ON WHOLE LIFE ANNUITIES

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ABSTRACT

UTILIZATION OF OUTLIER-ADJUSTED LEE-CARTER MODEL IN MORTALITY ESTIMATION ON WHOLE LIFE ANNUITIES

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Annuity and its pricing are very critical to the insurance companies for their financial liabilities. Companies aim to adjust the prices of annuity by choosing the forecasting model that fits best to their historical data. While doing it, there may be outliers in the historical data influencing the model. These outliers can be arisen from environmental conditions and extraordinary events such as weak health system, outbreak of war, occurrence of a contagious disease. These conditions and events impact mortality of populations and influence the life expectancy. So, using future mortality estimates that are not generated by the model that includes all of these factors, can influence on the financial strength of the life insurance industry. Therefore, these outliers should be taken into account as well while forecasting mortality rates and calculating annuity prices.

Although there are many discrete and stochastic models that can be used to forecast mortality rates, the most widely known and used of these is Lee-Carter model [18]. Fundamentally, Lee-Carter model uses some time-varying parameters and age-specific components. The parameter, which is inspired and used by many other researchers, is the mortality index κ_t, that Lee and Carter take as the basis in their model. Once, mortality index is forecasted correctly, then death probabilities of individuals and the prices of annuity can be estimated.

In case when there exist extremes in the mortality rates, outlier-adjusted model developed by Chan [7] can be used. This approach implements some iteration integrated in original Lee-Carter model to find better model that fits to historical data. In this thesis, we aim to find out whether there is a difference between models that consider mortality jumps and models that do not take into account jumps effects in terms of annuity pricing. Finally, we test the annuity price fluctuations among different countries and come to conclusion on the effects of different models on country characteristics.

For this comparison, Canada as a developed country with high longevity risk and Russia as an emerging country with jumps in its mortality history are considered. In addition to Canada and Russia, data of UK, Japan and Bulgaria are analyzed to provide ease of interpretation in terms of country characteristics. The results of this thesis support the usages of outlier-adjusted models for specific countries in term of annuity pricing.

Keywords: Mortality Rates, Annuity Pricing, Outliers, Outlier-Adjusted Lee-Carter Model

ÖZ

UÇ DEĞER İÇİN DÜZELTİLMİŞ LEE-CARTER MODELİNİN TAM HAYAT ANÜİTE HESAPLAMALARINDAKİ ÖLÜM TAHMİNİNDE KULLANIMI

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Anüite ve anüite fiyatlarının hesaplanması hayat sigorta şirketleri için finansal sorumlulukları açısından kritik bir değere sahiptir. Şirketler, kendi geçmiş datalarına en iyi uyan tahmin modelini kullanarak anüite fiyatlarını hesaplamayı hedeflerler. Bunu yaparken, geçmiş datalarında kullandıkları modeli etkileyen uç değerler olabilir. Bu uç değerler; zayıf sağlık sistemi, savaşın patlak vermesi veya salgın hastalık gibi sıradışı olaylardan ve çevresel etkenlerden kaynaklanabilirler. Bu durumlar ve olaylar ölümlülük oranlarını ve yaşam beklentisini olumsuz yönde etkilerler. Bu yüzden, tüm bu faktörleri dahil etmeyen bir modelden üretilen ölümlülük oranları, hayat sigorta endüstrisinin finansal gücü üzerinde etkiye neden olur. Bunlardan ötürü, ölümlülük oranları tahmin edilirken ve anüite fiyatları hesaplanırken bu uç değerler de hesaba katılmalıdır.

Ölümlülük oranlarını tahmin etmek için kullanılan birden çok kesikli ve stokastik modeller olmasına rağmen, bunlardan en çok kullanılanı Lee-Carter modelidir [18]. Temel olarak, Lee-Carter modeli zamana bağlı değişen parametre ve yaşa bağlı bileşen kullanır. Lee ve Carter'ın kendi modelinde temel aldığı ölümlülük indeksi κ_t, diğer birçok araştırmacılar tarafından kullanılan bir parametre olmuştur. Ölümlülük indeksi tahmin edildiği zaman, bireylerin ölüm olasılıkları ve anüite fiyatları hesaplanabilir.

Ölümlülük oranlarında uç değerler olduğu durumda, Chan [7] tarafından geliştirilen uç değer için düzeltilmiş model kullanılabilir. Bu yaklaşım, geçmiş dataya daha iyi uyan bir model bulabilmek için orijinal Lee-Carter modeline entegre edilmiş bir döngü kullanır. Bu tezde, uç değerleri göz önünde bulunduran modellerle bulundurmayan modeller arasında anüite fiyatlama bakımından fark olup olmadığı incelenecektir. Ek olarak, birden çok ülke için bu iki modelin anüite fiyatlarının üzerindeki etkisi test edilecek ve modellerin ülke özellikleri ile ilişkisi incelenecektir.

Bu karşılaştırma için, uzun ömürlülük riskine sahip gelişmiş bir ülke olan Kanada datası ile ölümlülük datasında uç değerleri barındıran gelişmekte bir ülke olan Rusya datası alınacaktır. Kanada ve Rusya'ya ek olarak, ülke özellikleri bakımından kolay bir çıkarım sağlaması açısından Birleşik Krallık, Japonya ve Bulgaristan dataları da incelenecektir. Bu tezin sonundaki çıkarımlar, anüite fiyatlama üzerine belirli ülkeler için uç değer için düzeltilmiş model kullanımını desteklemektedir.

Anahtar Kelimeler: Ölümlülük Oranları, Anüite Fiyatlama, Uç Değerler, Uç Değer için Düzeltilmiş Lee-Carter Modeli

To My Family

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CHAPTER 1

INTRODUCTION

Life expectancy has increased significantly for some countries during the 20th century and it continues to increase in the 21st century as well however, this increasing trend is not shown for some countries. The Human Mortality Database [15] demonstrates that Canadian, Japanese and Briton life expectancy at birth from 1960 to 2010 rose from 70.98 to 81.38 years, from 67.70 to 82.93 years and from 71.02 to 80.40 years for total population respectively. Meanwhile, the life expectancy at birth of Bulgarian population did not go up as rapidly as above-mentioned countries with the rise from 69.17 to 73.72 years for the same period. This situation is even worse for Russian population as the life expectancy at birth remained almost the same as 68.70 and 68.92 for the same period [15].

These differences between countries can be explained by lots of reasons which some of them are environmental conditions, economic crisis, lower incomes, the weak health care system and some extraordinary events that are outbreak of war, occurrence of a contagious disease and important changes in economic or political policies [8, 25]. These conditions and events impact mortality of populations influencing the life expectancy. At this point, in addition to aiming to increase life expectancy, countries should consider the mortality rates individually. This is because the fact that future estimates of mortality rates are used in calculations like pricing annuities, life insurances and retirement payments, by insurance industry and government agencies. By calculating these, they make critical policy decisions on the retirement and insurance systems. Among these decisions, the most important ones are pension policies. The insurance premiums and annuities, which are one of the most significant income items of the financial sector, are also calculated by using the estimated future mortality rates. Therefore, using future mortality estimates that are generated by the model that does not include all of these factors, can have an impact on the financial strength of the life insurance industry and the stability of the pension system of a country.

Although human mortality trends show little fluctuation in most of the time, they may have outliers at some points for the reasons mentioned above. These outliers in the human mortality rates are usually referred to as extremes or jumps in the literature. Mortality jumps are rare, but their presence could alter the long-term mortality trends by triggering a large number of unexpected deaths thereby may also affect future estimates. As Stracke and Heinen [23] estimated, additional claims received from unexpected pandemic would cost nearly €5 billion (50% of the market's total annual gross profit) not even in the worst scenario. With another example, the earthquake and tsunami occurred in southern Asia in 2004 made nearly 130,000 people missing and killed 180,000 as mentioned in Guy Carpenter report [14]. The report also indicates that if the event would occur in a more economically developed area, the life insurance industry would not have enough capacity to cover all claims.

In the light of all these significant indicators, companies should model the dynamics of mortality over time and to do this, many models have been introduced. These models are divided into two different methods. The first of these methods is method that is paying attention to force of mortality by using continuous time processes while the other one is concentrating mortality rates directly by discrete time processes. While some of the researchers who use continuous time models in their studies are Biffis [3] and Cairns, Blake and Dowd [6], Lee and Carter [18] developed one of the most popular and earliest discrete time models that is still being used by lots of researchers.

Regardless of the methods, the presence of mortality jumps could lead to influences both in the sample and partial autocorrelation functions, which could cause erroneous in the model [7]. For this reason, the errors arising from models that are not coordinated with mortality jumps should be analyzed cautiously such as problems caused by false assumption of the mortality and longevity risks in the model that many researchers emphasize on.

The first of these problems is, under optimistic assumptions, companies may overstate the forecasted mortality rates, therefore understate the life expectancy of the population thus their deficit as pension payments and annuities would increase. On the contrary, under pessimistic assumptions, in the environmental conditions and exogenous events that abovementioned are encountered, the financial institutions and the insurance industry will be in an insolvency position as they will enter into a fast payment process for life insurance policies [11].

In both assumptions, the use of the inadequate model affects pricing and reserve allocation for life insurances and annuities. This situation poses big threats to the solvency and price competitiveness of life insurance companies. To overcome this risk, building forecasted tables including effects of mortality jumps is essential. Hence, appropriate mortality forecasting models should be used to predict this situation [5].

After the model is constructed, it is applied to historical mortality rates to obtain future estimates. At this point, the efficiencies of different models should be examined to choose between each model. The easiest method to compare models is Akaike Information Criterion (AIC) introduced by Akaike [20] that takes the number of observations, parameters in the model and standard deviation of residuals as its components. In addition to this, the variation of prices of annuities and life insurances can be analyzed to check how durability of insurance industry changes with different models.

The simplicity and being the most widely used mortality forecasting model in the world [17] have led us to take Lee-Carter model as our base model to forecast long run mortality trends. Another reason for choosing this model is that the additions make on this model can be applied more easily to other models as future studies. Afterwards, we use some approaches that allow us to detect and adjust the effects of mortality jumps within base model. Finally, we test the annuity price fluctuations between models among different countries and come to conclusion on the effects of different models on annuity pricing in the basis of country characteristics.

1.1 Aim of the Thesis

The securitization of insurance industry is a critical subject as the strength of it reflects the stability of country economy. The Global Economy [24] indicates that the average of insurance company assets including annuities and pension plans for 2016 in the world as perfect of Gross Domestic Product (GDP) was 16.48%. It can be said that a decrease of this number would have impact on country's economy on its own. We approach this subject from the perspective of mortality jumps that influencing the model thus, changing the prices of annuities that creates more liabilities for companies.

In the literature ,there are studies that include the approaches to determine the effect of models that incorporate with mortality jumps on securitization [10, 21] and it is seen that finding model's effect on the prices of whole life annuities and how the liability of the companies changes are not considered deeply. In this thesis, we aim to find out whether there is a difference between models that consider mortality jumps and models that do not take into account jumps effects in terms of annuity pricing. We take some approaches and practices improved by Chen and Liu [9] and Chan [7] as a method to consider mortality jumps in time-series data. The reasons that we apply these approaches in this thesis are that these studies are relatively new and there is not many studies done on them in the literature.

While using these approaches, we consider calculating annuity prices for different age groups as 0, 30 and 70 in 2060. In addition to this, we compare the data of two countries to comprehend how the model works according to the characteristics of the countries. In order to differentiate the effects of the mortality jumps on countries, we compare a developed and a developing country in terms of their financial strength. The population of Canada as a developed country that we do not expect many jumps in its mortality rates and Russia as a developing country which is similar to the Canada as its demographical features but experienced many wars in its past and has problems with the stability of economy influencing its mortality rates throughout history, are chosen for comparison. Moreover, although not included in the comparison, the populations of Japan, Bulgaria and United Kingdom are also analyzed in order to provide ease of interpretation in terms of country characteristics. MATLAB is used in the calculations and all the steps during the implementation of the applied model.

Consequently, we aim to reach a conclusion that mortality jumps should be considered during the process of estimating future mortality rates, especially for the countries whose population is exposed to jumps related to migration due to economy, individual incomes, health care system and/or underwent many wars and diseases in its past. The outcomes of this thesis encourage the utilizations of outlier-adjusted models for specific countries and support the remarks on the use of applied models in the literature.

The thesis is organized as follows. Chapter 2 includes literature review on original and applied models while Chapter 3 shows the methodology of the models and the performance criteria. The iteration cycle that needs to be done during the process is also explained in the Chapter 3. Chapter 4 contains the implementation and results of the applied model. Chapter 4 gives also the results of the difference of annuity prices between selected countries. Chapter 5 concludes results with a brief discussion and contains some comments for further research.

CHAPTER 2

LITERATURE REVIEW

A number of stochastic mortality models are introduced to model the dynamics of mortality over time. Among these models, Lee-Carter [18] is the most widely used one for forecasting mortality. Lee-Carter is originated by a new method for extrapolation of age patterns and trends in mortality. In despite of its simplicity and quickly applicability, it has some weakness that some historical patterns may not be seen in the future so, present developments may be missed. Nevertheless, Census Bureau of US uses the Lee-Carter model as a benchmark for their forecast of US life expectancy [16].

Lee-Carter model has been extended by Renshaw and Haberman [22] and Brouhns et al. [5] where Renshaw and Haberman describe a method for modelling reduction factors using regression methods within the generalized linear modelling (GLM) framework, Brouhns et al. use Poisson regression model to forecast age-sex-specific mortality rates. However, none of these approaches include possible outliers that the mortality data has.

Significance of outlier detection-adjustment is taking place in the literature over the years. Some researchers develop models that integrate outlier effects which some of them are Lin and Cox [20], Cox et al. [12], Chen and Cox [10]. These researchers improve their models to allocate outlier locations by using a discrete-time Markov chain, Poisson distribution and independent Bernoulli distribution respectively. To calculate outlier severity, they employ normal distribution and double exponential jumps theorem.

The study of Li and Chan [19] use the approach to take into account possible outliers based in the original Lee-Carter model. They create outlier-adjusted time-series that is used with Lee-Carter model by using the iteration process developed by Chen and Liu [9] to determine the locations of outliers and adjust their effects. However, the iteration process of Chen and Liu is based on the study of Chang et al. [8]. Chang et al. develop the iteration process to incorporate outlier effects thereafter, Chen and Liu improve their process for the joint estimation of model parameters and outlier effects.

In this thesis, a part of the iteration process developed by Li and Chan is used. While Li and Chan eliminate some of possible outliers for finding them insignificant in their iteration process thus, narrowing range of possible outliers, we take all the possible outliers without elimination and generate outlier-adjusted time-series for being used to forecast mortality rates.

CHAPTER 3

METHODOLOGY

In this chapter, we present all the steps of applied model, starting with the Lee-Carter model which is our base model in implementation of this thesis. After that, we give notations and explanations on the process of outlier-adjusted model that built on the base model with the help of the iteration cycle. Then, how the performance criterion that should be performed to compare models, is explained immediately before the definition of the projection of whole life annuities.

3.1 The Lee-Carter Model

Lee-Carter model [18] is a method that is used for long-term mortality forecasts based on a combination of standard time series methods and an approach to handling with the age distribution of mortality. Model basically defines the logarithm of age-specific central death rates ($m_{x,t}$) as the sum of an independent of time age-specific component (a_x) and another element that is the product of a time-varying parameter (κ_t) which is also called as the mortality index and an age-specific component (b_x) that represents how mortality rate at each age changes when the mortality index varies. Extrapolation of mortality index under standard linear time-series methods forms the basis of Lee-Carter model.

Mathematically, the Lee-Carter model can be symbolized as follows,

$$ln(m_{x,t}) = a_x + b_x \kappa_t + \epsilon_{x,t} \tag{3.1}$$

where x shows the age and t represents the time. $m_{x,t}$ is the age-specific central death rate for age x at time t, a_x stands for the age pattern of death rates, b_x is the age-specific reactions to the time-varying factor, κ_t indicates for the mortality index in the year t while $\epsilon_{x,t}$ is the error term that captures the age-specific influences not reflected in the model for age x and time t.

Since the variables on the right-hand side of the model are unobservable, ordinary least square method cannot be used to fit the model. Moreover, it is a widely known fact that model is overparameterized. To obtain a unique solution, some restrictions should be implemented on parameters which are,

$$\sum_x b_x = 1 \quad \text{and} \quad \sum_t \kappa_t = 0 \tag{3.2}$$

When these restrictions applied to the model then, the age pattern of death rates becomes the average value of $ln(m_{x,t})$ over time and that is,

$$a_x = \frac{1}{T}\sum_{t=1}^{T} ln(m_{x,t}) \tag{3.3}$$

where T is a length of the series of mortality data.

Lee and Carter [18] also suggest a two-stage estimation procedure to overcome the problem mentioned above. In the first stage, singular value decomposition (SVD) method is applied to $\{ln(m_{x,t}) - a_t\}$ matrix to get estimates of b_x and κ_t. In the second stage, estimates of κ_t is re-estimated by using iteration with the values of b_x and a_t acquired in the first step.

$$D_t = \sum_x (N_{x,t} exp(a_x + b_x \kappa_t)) \tag{3.4}$$

where D_t shows the total number of deaths in year t and $N_{x,t}$ is the exposure to risk of age x in time t.

By implying second estimation, we ensure that number of deaths equals to the actual number of deaths thus mortality index fits correctly to the historical data. After that, the orthodox Box and Jenkin's approach [4] is employed to generate an autoregressive integrated moving average (ARIMA) model for the mortality index, κ_t. This approach can be done easily with the functions of $estimate$ and $arima$ in MATLAB. Once the ARIMA model and its parameters are obtained, the outlier detection and adjustment procedure begins.

3.2 Outlier Analysis

The outlier analysis consists of two issues which the first one is the determination of the location of the outlier values that may exist in the mortality index and the second one is finding and adjusting the effects of these outliers if any exists. For the first issue, Chang et al. [8] mentions that the value of standardized statistics of outlier effects should be found in order to detect outliers. For the second issue, more complex approaches and processes should be applied to standardized statistics of outlier effects as Chen and Liu state [9].

Furthermore, there are two types of problems that can be encountered in the outlier detection and adjustment procedure [9]. The first of these is that having outlier in a mortality data may cause an error in model selection while the second one is that even the model is selected correctly, the effect of outliers can significantly affect the estimation of model parameters. The approach of Chen and Liu partly solves the second problem while the first one stays the same. Since this approach is the newest method that can be found in the literature regardless of abovementioned shortcomings, we use this method with a little change that is described in the following sections.

3.3 Outlier Models

Let Z_t an outlier-free time series follows an ARIMA(p,d,q) model and is written as follows,

$$\phi(B)(1-B)^d Z_t = \theta(B)\alpha_t \tag{3.5}$$

$$\phi(B) = 1 - \phi_1 B - ... - \phi_p B^p$$

$$\theta(B) = 1 - \theta_1 B - ... - \theta_q B^q$$

where B is the backshift operator such that $B^s Z_t = Z_{t-s}$ and a_t represents white noise random variables with mean 0 and constant variance σ^2.

It is known that outliers are non-repetitive interferences of exogenous events and time series with outliers form as outlier-free time series plus the effects of emergent outliers, denoted as $\Delta_t(T, w)$, where T and w are the location and the size of outlier respectively. Then we can have,

$$Y_t = Z_t + \Delta_t(T, w) \tag{3.6}$$

where Y_t is the time-series with outliers and t indicates the years.

In the literature, generally, four types of outliers are considered. These are Innovational Outlier (IO), Additive Outlier (AO), Temporary Change (TC) and Level Shift (LS) [26] [9] [19]. While an AO influences only single observation that is its location, an IO affects all observations with some decreasing pattern after T years become until the effect vanishes. This situation is slightly different for Temporary Change and Level Shift types. The effect of outlier remains the same influencing all observations after T years become for LS type and it decreases until to reach zero point by almost linearly for TC type.

It is believed that large portion of the outliers comprises of Additive Outlier (AO) and Innovational Outlier (IO) [8]. Since we focus on short time effects of mortality jumps that arise from extraordinary events that affecting mortality rates for a short time, we give our effort on these two intervention models. The effects of these two types can be illustrated as follows,

$$AO : \Delta_t(T, w) = w D_t^T$$

$$IO : \Delta_t(T, w) = \frac{\theta(B)}{\phi(B)(1-B)^d} w D_t^T$$

where D_t^T is a variable that becomes 1 in presence of outliers otherwise 0 in absence of outlier at time T.

More than one outlier can be found in a time series thus, following model should be used when m outliers exist.

$$Y_t = Z_t + \sum_{i=1}^{m} \Delta_t(T_i, w_i) \tag{3.7}$$

Equation (3.7) illustrates that time-series with outliers are outlier-free time-series plus sum of the effects of outliers that are observed.

3.4 Outlier Detection

The outlier detection method of Chang et al. [8] is grounded on the effects of outliers on the residuals of the model. As they state, the values of standardized statistics of outlier effects should be calculated to detect possible outliers. To achive this, once Z_t is constructed as shown in equation (3.5) like ARIMA(p,d,q) model, polynomial of $\pi(B)$ should be defined as follow,

$$\Pi(B) = \frac{\phi(B)(1-B)^d}{\theta(B)} = 1 - \Pi_1 B_1 - \Pi_2 B_2 - ... \tag{3.8}$$

where π_j weights of outliers that are found at location T, influencing the years after T so, $j \geq T$. While the distance between j and T increases, π_j becomes zero as the effect of outlier at location T doesn't impact on distance mortality values.

Then, equation (3.5) can be written as,

$$\Pi(B)Z_t = \alpha_t \tag{3.9}$$

and equation (3.6) can be expressed as,

$$\hat{e}_t = \Pi(B)Y_t \quad \text{for} \quad t = 1, 2, 3, ... \tag{3.10}$$

where $\hat{e}_t$ defines the residuals that are obtained from time-series with outliers. After necessary calculations are done, we can have following equations.

$$AO : \hat{e}_t = wD_t^T + \alpha_t$$

$$IO : \hat{e}_t = w\Pi(B)D_t^T + \alpha_t$$

Above equations can be symbolized as a general time-series structure as follow,

$$\hat{e}_t = wd(j,t) + \alpha_t \tag{3.11}$$

where $j = (AO, IO)$; $d(j,t) = 0$ for both types with $t < T$; $d(j,T) = 1$ for both types; when $k \geq 1$ the following equations can be written.

$$d(AO, T+k) = 0$$

$$d(IO, T+k) = -\Pi_k$$

It is clear to reach the conclusion that the effect of an AO is contained only at particular point T, whereas the effect of IO is dispersed through the time after at time point T.

Consequently, from least squares theory, the effect of an outlier at $t = t_1$ can be formed for both types as follows,

$$\hat{w}_{AO}(t_i) = \hat{e}_{t_i} \tag{3.12}$$

$$\hat{w}_{IO}(t_i) = \frac{\sum_{t=t_1}^{t_n} \hat{e}_t d_{IO,t}}{\sum_{t=t_1}^{t_n} d_{IO,t}^2} \tag{3.13}$$

where t_n illustrates the last year that the data has.

As Chen and Liu [9] state, because of the effects of AO and IO in the last observation equal to $\hat{e}_n$, it is not possible to distinguish the type of an outlier at the end of the series. For locating possible outliers, analyzing maximum value of the standardized statistics should be implemented as proposed by Chang et al. [8]. To do this, the standardize effects of outliers should be divided as follow,

$$\hat{\tau}_{AO}(t_i) = \frac{\hat{w}_{AO}(t_i)}{\hat{\sigma}_\alpha} \tag{3.14}$$

$$\hat{\tau}_{IO}(t_i) = \frac{\hat{w}_{IO}(t_i)}{\hat{\sigma}_\alpha} \left(\sum_{t=t_1}^{n} d_{IO,t}^2 \right)^{1/2} \tag{3.15}$$

where $\hat{\sigma}_a$ is the estimation of residual standard deviation.

Hence, possible location of an outlier can be determined by looking the standardized values when they are greater than value of C that is chosen as positive constant. In order to decide whether the outlier is a form of AO or IO when both of their effects are greater than C, we follow a simple rule described by Fox [13] which is choosing the type of outlier whose effect is greater than the other type of outlier. In order to achieve a high degree of sensitivity in locating the outliers, we take the value C equals to 3.0 as Chang et al. [8] recommend.

Standard deviation of residuals should be calculated to reach numerical value for maximum value of standardized statistics as seen from the equations (3.14) and (3.15).

3.4.1 Estimation of Standard Deviation of Residuals, $\hat{\sigma}_a$

To calculate residual standard deviation, there are more than one method in the literature. The first three of them are the a% trimmed method, the omit-one method and the median absolute deviation (MAD) method.

Since all of these three methods come up with close results to each other as Chen and Liu [9] state in their study, MAD is used in calculations of this thesis because its fast computability. The MAD estimation is defined as follow,

$$\hat{\sigma}_\alpha = 1.483 \times \text{median}\{|\hat{e}_t - \tilde{e}|\}$$

where $\tilde{e}$ is the median of the estimated residuals [2].

3.5 Outlier Adjustment and Iteration Process

After the locations of possible outliers are found, the effects of these outliers should be adjusted in order to estimate new model parameters and outlier effects again. To accomplish this, an iteration cycle that is repeated until no more outliers are found, is needed. When the iteration stops, ultimate ARIMA model and its parameters are going to be identified for being used to forecast mortality index, κ_t. The iteration process is outlined as follow, *Step 1*: Use Box and Jenkin's approach [4] to identify the order of the underlying ARIMA(p,d,q) model.

Step 2: Compute the residuals of mortality index that is found from original Lee-Carter model.

Step 3: Calculate coefficient of $\pi(B)$ and then, outlier effects of AO and IO accordingly.

Step 4: Evaluate the standard deviation of residuals obtained in *Step 2*.

Step 5: Compute standardized statistics for AO and IO for all time points to decide whether there is an outlier in the series. Then, determine the type of outliers by comparing values with pre-determined C value.

Step 6: If there is no outlier found in *Step 5*, then stop, the series is outlier-free or outlier-adjusted. Otherwise, remove the effects of outliers by defining new residuals $\hat{e}_t = \hat{e}_t - \hat{w}_{A0} = 0$ at T for AO and $\hat{e}_t = \hat{e}_t - \hat{w}_{IO}\Pi(B)$ for $t > T$ for IO.

Step 7: Re-calculate standard deviation of residuals with adjusted residuals and go to *Step 5*. Repeat this cycle until no further outlier can be identified.

Step 8: After the locations of all possible outliers are found, remove effects of outliers from mortality index at determinated locations by the same method defined in *Step 6* as reaching to have new mortality index.

Step 9: All the cycle from *Step 1* starts again using new data of mortality index until no further outlier is found after mortality index is changed.

Step 10: The final ARIMA model and its parameters are used to create ultimate forecasting model for mortality index.

The difference of this iteration with the iteration that is mentioned in the study of Chen and Liu [9] is that Chen and Liu eliminate some of the possible outliers by comparing their standardized statistics that are calculated with another formula, with the pre-defined C value again in the *step 8*. However, this iteration does not eliminate any possible outliers and removes all their effects from mortality index in order to get fully outlier-adjusted time-series.

3.6 The Akaike Information Criterion (AIC)

If there is more than one model, some comparison should be performed to compare the performance of models. During the iteration process, the number of parameters and the variance of the model may change thus, they can have impact on the accuracy of models. One of the performance criterion that can be computed between models, is Akaike Information Criterion introduced by Akaike [1] which is defined as,

$$AIC = nln(\hat{\sigma}_a^2) + 2M$$

where n is the number of observations in time-series, $\hat{\sigma}_a$ is the standard deviation of residuals and M denotes the number of estimated parameters in the model. Smaller AIC value is preferable for choosing the better model as it represents better fitting of the model as Akaike [1] mentions. This criterion is used to compare the performances of original Lee-Carter model and outlier-adjusted Lee-Carter model.

3.7 Forecasting Mortality Index and Death Probabilities of Individuals

Once the iteration is completed and the ultimate ARIMA model and its parameters are found, estimation of mortality index can be forecasted. Later on, once forecasted mortality index is set, death probabilities of individuals can be computed easily with the following formula,

$$q_x = \frac{m_x}{1 + (1 - c_x)m_x} \tag{3.16}$$

where m_x is the forecasted central death rate for age x, c_x is the average number of years lived within the age interval x and $x + 1$. As in the protocol of Human Mortality Database [15] states, the number of 0.5 is taken as c_x for all ages except 0. For the beginning age, the last observed numbers in the data for all ages are used.

Then, the probability of surviving from age x to x+1, p_x, can be illustrated as follow,

$$p_x = 1 - q_x \tag{3.17}$$

3.8 Projection of Whole Life Annuity

Whole life annuity is a financial product sold by insurance companies that pays annually or at different intervals payments to a person for the time one lives, beginning at a stated age. Annuities are generally purchased by investors who want to provide a fixed income during their retirement. There are two perspectives in terms of annuities. One of them is the perspective of buyers that they make payments to the insurance company in the period called accumulation while the other one is that companies make payments to buyers. However, the total amount of the annuity or the price of annuity can be paid to companies by buyers as well.

Once death probabilities of individuals thus, surviving probabilities are obtained, the price of whole life annuities can be calculated. There are two types of whole life annuities which are called “due” and “immediate”. The all payments start at the beginning of stated period in “due” type while in “immediate” type, payments are made at the end of stated period. The formulas of two types can be written as follow,

$$Due \rightarrow \ddot{a}_x = 1 + p_x\vartheta + p_x p_{x+1}\vartheta^2 + \ldots + p_x p_{x+1} ... p_{w-1}\vartheta^w$$

$$Immediate \rightarrow a_x = p_x\vartheta + p_x p_{x+1}\vartheta^2 + \ldots + p_x p_{x+1} ... p_{w-1}\vartheta^w$$

where $\ddot{a}_x$ and a_x are the present values of whole life annuities with 1 unit payments for the age x, ϑ is the discount rate that is calculated as $1/(1 + i)$, i shows interest rate for a given period and w is the last age that can be attended.

CHAPTER 4

IMPLEMENTATION

4.1 Countries Selected

While observing the variability of the annuity prices between models can be an important indicator, it is not enough in order to understand how the models change with different data. At this point, it may be useful to compare different countries to better understanding of the impacts of models. The comparison between developed and developing countries is more meaningful on the basis of achieving explicit outcomes. For all these reasons, we compare the populations of Canada as developed country and Russia as developing country.

The reasons for being selected of the populations of Canada and Russia for comparison is that there are different troubles that they try to cope with. While Canada has longevity risk, Russia has issues with weak health system, economy and other environmental considerations.

In addition to Canada and Russia, without the comparison, the populations of Japan and UK as developed countries and Bulgaria as developing country according to Human Development Index Report in 2016 [27] are analyzed. These countries are chosen for being in different regions. While comparing two models under different country characteristics, we consider calculating the prices of annuities that start at 0, 30 and 70 ages.

4.2 The Data

The central death rates, exposure-to-risk and the number of total deaths in a year are required to implement the Lee-Carter model and complete outlier-adjusted process. We obtain the required data from The Human Mortality Database [15].

The data is taken by each age for total population and contain up to 110 years for all selected countries. The ranges of cover period for each country: Canada (1921-2016), Russia (1959-2014), Japan (1947-2017), UK (1922-2016) and Bulgaria (1947-2010). Mortality rates, death and surviving probabilities of individuals are forecasted by using MATLAB till 2060 year for all countries.

4.3 Empirical Analysis

In this section, firstly, the plots of death probabilities of individuals for different ages by time and then, the parameters of original Lee-Carter model and the results of iteration process that is made on original LC model are presented for the populations of selected countries. The performance differences between models and the plots of forecasted mortality index till 2060 year for selected countries are also illustrated.

Finally, the annuity prices and their differences between models for 0, 30 and 70 ages in 2060 year are given not only for the populations of Canada and Russia, but also for Japan, UK and Bulgaria. In addition to results, the comments and inferences are also made.

4.3.1 Death Probabilities of Individuals, $q_{x,t}$

In F igure (4.1) and (4.2), we present the death probabilities of individuals at all ages for selected countries taken from historical mortality data to be able to observe possible outliers in mortality data. While all countries have some fluctuations in their death probabilities among the years, for Russia and Bulgaria, sharper ups and downs can be seen thus, we expect more outliers in their mortality index as well.

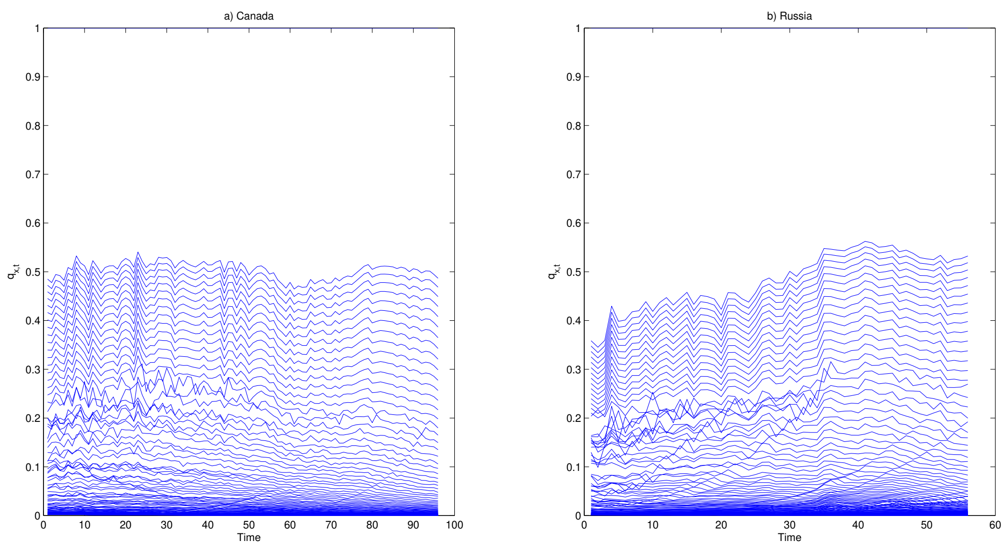

Figure 4.1: Death Probabilities: a)Canada (1921-2016) b)Russia (1959-2014)

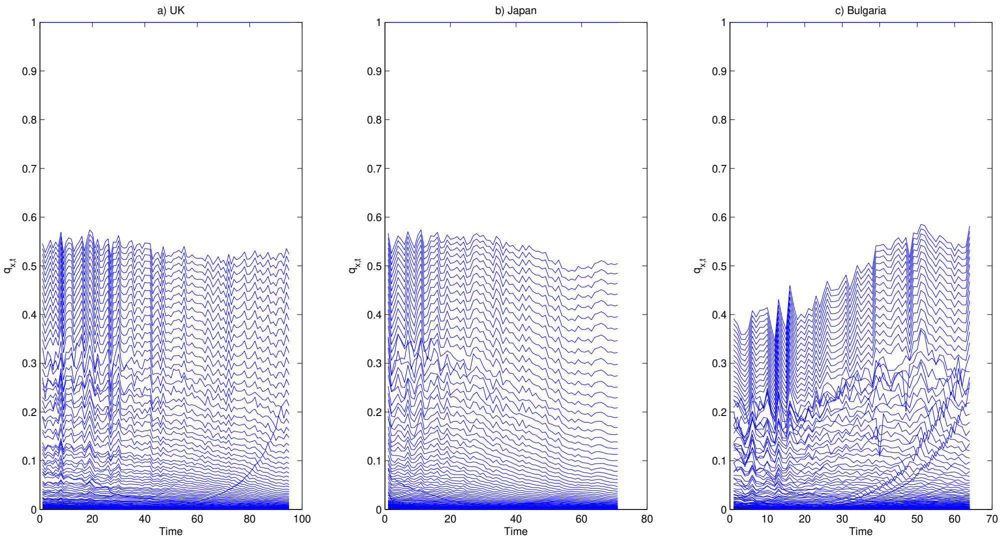

Figure 4.2: Death Probabilities: a)UK (1922-2016) b)Japan (1947-2017) c)Bulgaria (1947-2010)

4.3.2 Parameters of Original Lee-Carter Model

The plots of estimated values of a_x, b_x and κ_t that obtained in the second stage of original Lee-Carter model for the population of Russia and Canada are given in figure (4.3) and for UK, Japan and Bulgaria in figure (4.4).

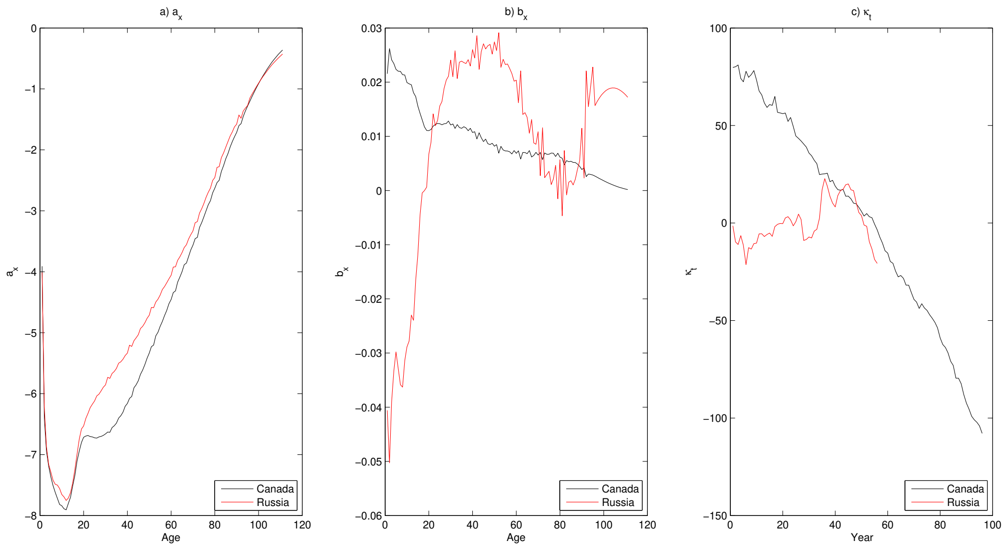

Figure 4.3: Original Lee-Carter Model Parameters for Canada and Russia: a)a_x, b)b_x, c)κ_t

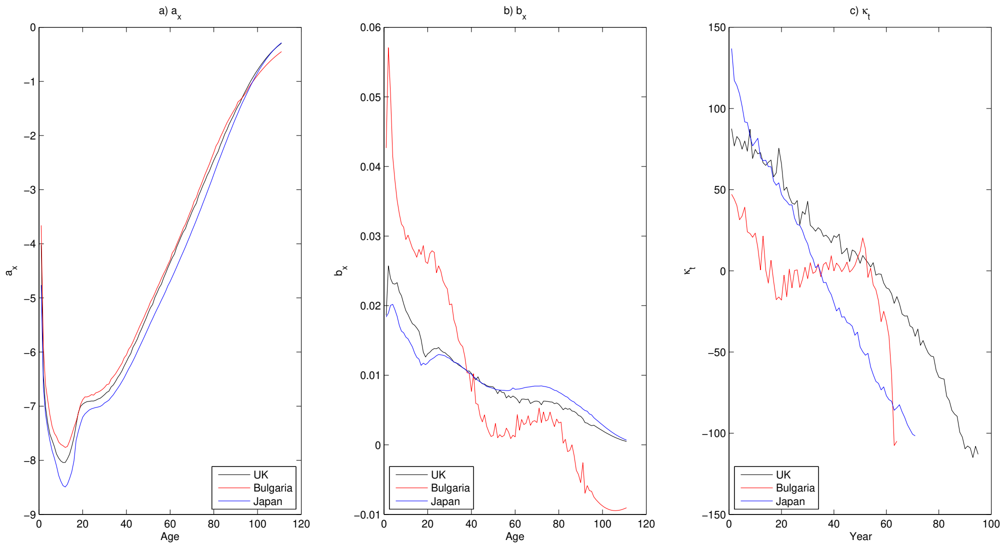

Figure 4.4: Original Lee-Carter Model Parameters for UK, Japan and Bulgaria: a)a_x, b)b_x, c)κ_t

Since it is shown and explained in equation (3.1), a_x is the age pattern of death rates, b_x is the age-specific reaction to time factor and κ_t is the mortality index. Although, there is not any significant difference between countries on basis of age pattern of death rates, great differences are seen for b_x and κ_t.

The plot of a_x shows that the average values of $ln(m_{x,t})$ over years are quite similar between countries. However, every age reacts to mortality improvements very differently as can be seen in the plots of b_x. It means that while mortality improvements benefit younger generation in Canada, they provide more benefit to adult people in Russia even still the level of benefit is lower in most of the times than Canada. This pattern can be seen for Bulgarian data as well. Bulgarian population reacts unequally to mortality improvements as the mortality index of younger age groups benefit much more than older age groups. Not surprisingly, the plot of κ_t clarifies that mortality rates decrease significantly in Canada, Japan and UK thus, create problems on longevity risk, while Russia and Bulgaria have fluctuating mortality trends creating unstable mortality rates over years. The exact values of a_x, b_x and κ_t can be found in the Appendix A.

4.3.3 Iteration Cycle

The locations, types and effects of outliers found in the iteration cycle are analyzed and given in Table (4.1) for Russia and Table (4.2) for Canada.

The Bayesian Information Criterion (BIC) is used to select the degrees p and q of an ARIMA model. To identify best lags for ARIMA model, the loglikelihood objective function value and number of coefficients for each fitted model are stored. Then, these values are used for *aicbic* function to calculate the BIC measure of models. The smallest value is chosen for our best fitted model.

Table 4.1: Summary of the Proposed Iteration Process for Russian Data

ARIMA Model	Number of Iteration	Parameters of ARIMA model							Outliers			
		c_0	θ_1	θ_2	θ_3	ϕ_1	ϕ_2	$\hat{\sigma}^2$	Time Point	Type	w	τ
(3,1,2)	1	-0.93	-0.01	-0.75	0.28	0.33	1.00	13.04	6	IO	-5.34	-3.10
									28	AO	-9.32	-3.31
									35	AO	11.52	4.09
	2	-0.09	-0.10	0.70	0.01	0.18	-0.63	18.83	6	IO	-5.95	-4.28
									7	AO	14.27	6.27
									53	AO	-6.84	-3.01
	3	-0.04	1.54	-0.94	0.22	-1.31	0.56	16.42	7	AO	7.04	3.00
	4	-0.52	0.14	-0.67	-0.05	-0.03	0.84	18.01	-	-	-	-
(0,1,0)	5	-0.33	-	-	-	-	-	20.50	36	AO	17.65	4.23
	6	-0.33	-	-	-	-	-	18.26	-	-	-	-
(0,1,0)	7	-0.33	-	-	-	-	-	18.26	37	AO	14.10	3.55
	8	-0.33	-	-	-	-	-	15.79	7	IO	-8.43	-3.09
	9	-0.33	-	-	-	-	-	14.76	7	IO	-8.38	-3.21
									8	AO	14.97	4.07
	10	-0.33	-	-	-	-	-	20.09	-	-	-	-
(0,1,0)	11	-0.33	-	-	-	-	-	20.09	7	IO	11.75	4.20
									8	IO	-12.40	-4.43
	12	-0.33	-	-	-	-	-	23.16	8	IO	-18.53	-6.55
									9	AO	20.88	5.20
	13	-0.33	-	-	-	-	-	18.65	8	IO	-8.27	-3.07
									9	AO	18.65	4.88
	14	-0.33	-	-	-	-	-	29.45	-	-	-	-
(2,1,2)	15	-0.25	0.38	-0.56	-	-0.86	1.00	20.78	8	AO	17.73	4.27
	16	-0.06	0.24	0.47	-	-0.07	-0.47	13.57	-	-	-	-
(0,1,0)	17	-0.33	-	-	-	-	-	14.49	38	AO	9.20	3.03
									54	AO	-10.33	-3.40
	18	-0.33	-	-	-	-	-	15.31	-	-	-	-
(0,1,0)	19	-0.33	-	-	-	-	-	15.31	54	IO	7.68	3.77
									55	AO	-15.30	-5.38
	20	-0.33	-	-	-	-	-	13.87	-	-	-	-
(0,1,0)	21	-0.33	-	-	-	-	-	13.87	56	AO/IO	-9.86	-3.39
	22	-0.17	-	-	-	-	-	12.10	-	-	-	-
(0,1,0)	23	-0.17	-	-	-	-	-	12.10	-	-	-	-

Table 4.2: Summary of the Proposed Iteration Process for Canadian Data

ARIMA Model	Number of Iteration	Parameters of ARIMA Model			Outliers			
		c_0	θ_1	$\hat{\sigma}^2$	Time Point	Type	w	τ
(0,1,0)	1	-1.90	-	5.38	6	AO	7.32	3.13
					17	IO	6.10	3.73
	2	-1.90	-	5.34	17	IO	6.27	4.09
					18	AO	-12.48	-5.66
	3	-1.90	-	6.47	-	-	-	-
(0,1,0)	4	-1.90	-	6.47	17	IO	-9.07	-5.87
					18	AO	12.37	5.58
	5	-1.90	-	4.49	18	IO	6.92	4.35
	6	-1.90	-	5.77	18	IO	-6.46	4.30
					19	AO	-11.70	-4.95
	7	-1.90	-	7.81	19	AO	-7.26	-3.09
	8	-1.90	-	9.80	-	-	-	-
(1,1,0)	9	-2.49	-0.35	8.60	18	IO	-13.58	-7.57
					19	AO	12.30	5.40
	10	-1.79	0.06	4.71	19	AO	9.27	4.26
					20	AO	7.47	3.43
	11	-1.91	-0.01	5.98	9	IO	5.02	3.07
	12	-1.97	-0.04	5.75	11	AO	-6.62	-3.20
	13	-2.00	-0.06	6.13	-	-	-	-
(0,1,0)	14	-1.90	-	6.15	19	IO	-7.62	-4.99
					20	AO	6.69	3.04
	15	-1.90	-	4.76	11	IO	4.50	3.05
					19	IO	-4.55	-3.08
					20	AO	7.83	3.69
	16	-1.90	-	5.56	11	IO	4.64	3.25
					12	AO	-9.58	-4.66
	17	-1.90	-	5.67	-	-	-	-
(0,1,0)	18	-0.32	-	4.13	-	-	-	-

The parameters of ARIMA model show the values that obtained in iteration process. While c_o and $\hat{\sigma}^2$ represent the constant value within the model and the estimation of the variation of a_t that is shown in the equation (3.5) respectively. The other parameters indicate AR and MA parameters.

w stands for the effect of outlier while t is standardized statistic of it. Time point of outlier reflects outlier that found exactly at that specific point of time. During the iteration process, three different ARIMA models are found for Russian data and two different for Canadian data.

Iteration process continues until no outliers are found two times in a row. It is seen that the effects of outliers that are type of IO, are more powerful than the effects of AO because they influence not only one time point as AO but also the times after that specific time point. Since more outliers are found as the number of iteration number increases, relationship between number of iteration number and outliers can be constructed.

Ultimately, both data become ARIMA(0,1,0) once iteration process is done then, they become outlier-adjusted time-series which means that no more outliers can be identified. Moreover, it can be inferred that time-series obtained from ARIMA model that contains a_t with higher variance, has more fluctuations in its mortality index as seen from comparison between the results of Canada and Russia. Moreover, reductions in variance of a_t are seen for both Canada and Russia which indicates that the mortality index that are used to establish outlier-adjusted models shows less variation thus, the outlier-adjusted models are superior to original models. It is also interesting to note that as Chen and Liu [9] specify, the type of an outlier that is identified in the last observation of time-series cannot be distinguished between AO and IO as seen in 56$^{\text{th}}$ time point of Russian data.

The results of iteration process for the populations of UK, Japan and Bulgaria can be found in the Table (4.3) (4.4) (4.5) respectively.

Table 4.3: Summary of the Proposed Iteration Process for Briton Data

ARIMA Model	**Number of Iteration**	**Parameters of ARIMA model**			**Outliers**			
		c_0	ϕ_1	$\hat{\sigma}^2$	**Time Point**	**Type**	w	t
(0,1,1)	1	-2.09	-0.51	25.20	8	IO	13.28	3.16
					19	IO	17.14	4.08
	2	-2.09	-0.48	18.91	9	AO	-14.34	-3.01
					21	AO	-18.14	-3.81
	3	-2.09	-0.44	17.50	-	-	-	-
(0,1,1)	4	-2.09	-0.44	17.50	-	-	-	-

Table 4.4: Summary of the Proposed Iteration Process for Japanese Data

ARIMA Model	**Number of Iteration**	**Parameters of ARIMA model**							**Outliers**			
		c_0	θ_1	θ_2	ϕ_1	ϕ_2	ϕ_3	$\hat{\sigma}^2$	**Time Point**	**Type**	w	t
(1,1,3)	1	-0.47	0.84	-	-1.24	0.16	0.08	9.67	2	AO	-11.61	-3.41
	2	-2.44	0.30	-	-0.47	-0.02	0.29	11.39	-	-	-	-
(2,1,2)	3	-1.46	1.20	-0.62	-1.53	1.00	-	8.46	-	-	-	-

Table 4.5: Summary of the Proposed Iteration Process for Bulgarian Data

ARIMA Model	Number of Iteration	Parameters of ARIMA model								Outliers			
		c_0	θ_1	θ_2	θ_3	ϕ_1	ϕ_2	ϕ_3	$\hat{\sigma}^2$	Time Point	Type	w	t
(0,1,0)	1	-2.27	-	-	-	-	-	-	108.61	63	AO	-41.04	-4.10
	2	-2.27	-	-	-	-	-	-	102.30	-	-	-	-
(1,1,2)	3	-1.29	0.58	-	-	-0.94	0.61	-	80.06	64	AO/IO	-26.97	-3.19
	4	-0.21	0.95	-	-	-1.37	0.55	-	64.44	53	AO	-24.11	-3.00
	5	-1.52	0.33	-	-	-0.79	0.68	-	66.49	21	AO	22.47	3.01
										62	AO	-23.68	-3.18
	6	-1.71	0.27	-	-	-0.65	0.69	-	66.13	-	-	-	-
(3,1,0)	7	-1.49	-0.19	0.04	0.51	-	-	-	59.24	23	AO	21.43	3.02
	8	-1.51	-0.17	0.05	0.48	-	-	-	59.11	-	-	-	
(0,1,3)	9	-2.08	-	-	-	-0.14	0.21	0.54	58.31	-	-	-	-

As we expect, mortality index of Bulgaria has more variance in itself consequently, it presents unstable or nonlinear mortality rates. The variance goes down to 58.31 from 108.61 between original and outlier-adjusted model. The data of Japan performs the best among all countries since just one outlier is found. Furthermore, while the models for UK and Canada do not change during iteration process, they change for Russia, Japan and Bulgaria which means that the models generated from original Lee-Carter method do not reflect the historical data truly.

Table 4.6: Years with Outliers for the Selected Countries

Canada	Russia		UK	Japan	Bulgaria
1926	1964		1929	1948	1967
1929	1965		1930		1969
1931	1966		1940		1999
1932	1967		1942		2008
1937	1986				2009
1938	1993				2010
1939	1994				
1940	1995				
	1996				
	2011				
	2012				
	2013				
	2014				

Table (4.6) illustrates the corresponding years with outliers that found in iteration process for the selected countries. As we expect, more outliers are observed for Russian population.

When the corresponding years are associated with real historical events, it can be said that wars and economic crises play a major role in changing mortality rates thus, having outliers in mortality data. The Second World War affects UK in 1940 and 1942 as it has detected mortality outliers at that years.

In addition to wars, some important accidents such as Chernobyl Accident that happened in Ukraine which was in USSR (Union of Soviet Socialist Republics) in 1986 influence more than one country increasing their mortality rates. After the dissolution of the USSR in 1990 causing separation into 15 different countries, makes fluctuations in mortality rates between 1993 and 1996 for Russia.

Moreover, the effects of economic crises can be seen in Canada between 1929-1939 years identified as Great Depression, in Russia between 2011-2014 years as having a sharp decrease in individual's income and purchase power and in Bulgaria at 1999.

It is also interesting to note that the data of Canada does not contain any outliers after 1940 year whereas Russia has outliers that are identified in 21st century too. This situation explains the unstable mortality trends that Russia has even in the present. Moreover, Bulgaria also shows the same pattern as Russia. It has outliers in the very end of series which cannot be distinguished among the types of outlier.

Table (4.7) presents the performances between models for selected countries. It is seen that ultimate models for all populations are superior to the original models as they have lower AIC values. Although there is more decline in AIC values for Canadian data, outlier-adjusted models are more preferable than original models for all countries on basis of AIC criterion.

Table 4.7: Model Performance Indicators for the Selected Countries

	Values of AIC	
	Original Lee-Carter Model	**Outlier-Adjusted Lee-Carter Model**
Canada	653.97	477.68
Russia	130.05	120.02
UK	306.44	285.55
Japan	186.12	163.64
Bulgaria	298.76	280.17

4.3.4 Forecasted Mortality Index, κ_t

Mortality index is forecasted until 2060 year for all populations by using ultimate ARIMA model and its parameters are obtained in the end of the iteration. The forecasted values are drawn with 95% confidence interval. Since a_x of ARIMA model for Russian and Bulgarian data have more variance and they have fluctuations in their mortality rates, the confidence interval of them cover more space in forecasted area.

The exact values of mortality index that obtained after end of the iteration and forecasted values are given in the Appendix B. In addition to mortality index data, the surviving probabilities of individuals for all ages in 2060 year that calculated with formula (3.16) using forecasted mortality index, can be found in the Appendix C.

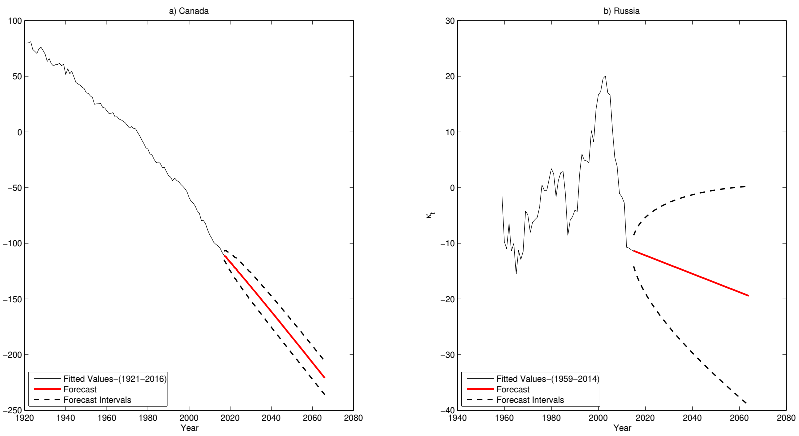

Figure 4.5: Forecasted Mortality Index for the Selected Countries: a)Canada, b)Russia

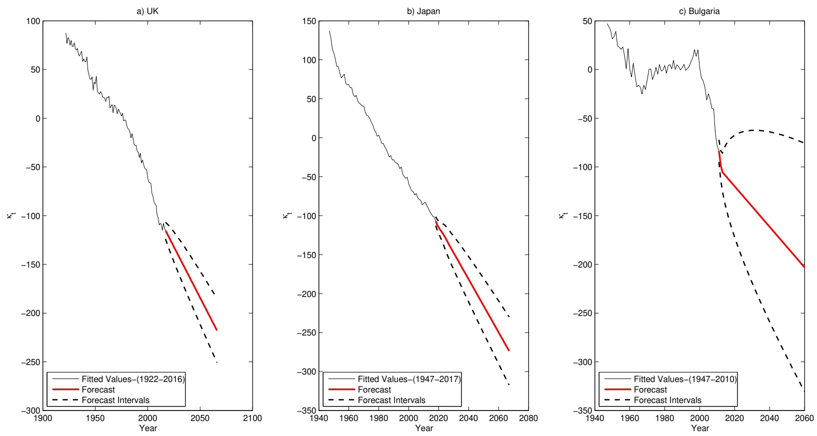

Figure 4.6: Forecasted Mortality Index for the Selected Countries: a)UK, b)Japan, c)Bulgaria

4.3.5 Whole Life Annuity Pricing Under Outlier-Adjusted Lee-Carter Model

The prices of whole life annuities (due) for ages 0, 30 and 70 in 2060 year for both original and outlier-adjusted Lee-Carter models for Canadian and Russian data are presented in Table (4.8). The interest rate i, is chosen an arbitrary representative value as 0.03 in all calculations.

We find that there are significant differences between the variation of the prices of annuity of both countries. This result is more obvious for the comparison of annuities start at age 0. There is an increase of 11% in the price of annuity between models in Russian data while there is only 0.22% for Canadian data. The numerical results suggest that more amount of premium should be collected by life insurance companies for whole life annuities start at age 0 in Russia but this conclusion cannot be reached for the Canada population. Surprisingly, there are decreases for the annuities start at ages 30 and 70 for Russian data which means that while the mortality improvements work after 30 years and reducing death rates, they are not enough in general as the prices of annuities start at age 0 increase.

Table 4.8: The Prices of Whole Life Annuities (Due) in 2060, Canada and Russia

	Age	Prices of Whole Life Annuities		
		Original Lee-Carter Model	**Outlier-Adjusted Lee-Carter Model**	**Difference**
Canada	0	31.59	31.66	0.22%
	30	27.57	27.73	0.56%
	70	14.48	14.77	2.01%
Russia	0	25.53	28.43	11.36%
	30	25.36	24.13	-4.82%
	70	10.08	9.59	-4.87%

In table (4.9), no visible differences as in Russian data, can be seen. Even though Japan has some fluctuations in the prices of annuities that start at 30 and 70 ages by around 2-5%, in general there are not any differences between prices like Canadian data. However, for Bulgarian data, sharper variations could been expected as being a developing country, but the differences do not exceed 0.71%. Even though Bulgaria has unstable mortality index as Russia, the prices of annuities do not change with the same ratio like in Russian data. Thus, we can conclude that the outliers in Russian data have impact on annuity prices much more than Bulgarian data. Moreover, we can come to a conclusion that the criteria of being developed or developing country is not enough alone for expecting more differences between the prices of annuities that are estimated with original and outlier-adjusted models.

Table 4.9: The Prices of Whole Life Annuities (Due) in 2060: UK, Japan and Bulgaria

	Age	Prices of Whole Life Annuities		
		Original Lee-Carter Model	**Outlier-Adjusted Lee-Carter Model**	**Difference**
UK	0	31.53	31.52	0.00%
	30	27.38	27.38	-0.01%
	70	13.95	13.95	-0.03%
Japan	0	31.97	32.04	0.22%
	30	28.46	29.89	5.02%
	70	16.36	16.73	2.25%
Bulgaria	0	30.84	30.81	-0.10%
	30	25.60	25.53	-0.28%
	70	10.44	10.37	-0.71%

4.3.6 Pricing the Annuity Portfolio

It is useful to show the annuity pricing on the portfolio for a more comprehensive assessment of the achieved result. In order to create this portfolio, 10.000 people are randomly generated from uniform distribution, aged between 15 and 75 years. Then, the prices of whole life annuities on the portfolio are calculated for both original and outlier-adjusted Lee Carter model. As can be seen from the Table (4.10), while the difference between models for Russian data is around 5%, for other countries this number does not exceed 1%. In larger and more diverse portfolios, this difference can be more pronounced.

Table 4.10: Prices for the Portfolio of Whole Life Annuities (Due) in 2060

	Prices of Whole Life Annuities for Portfolio				
	Canada	**Russia**	**UK**	**Japan**	**Bulgaria**
Original Lee-Carter Model	235,410	205,380	232,350	248,130	208,430
Outlier-Adjusted Lee-Carter Model	237,500	194,990	232,320	250,480	207,610
Difference	0.89%	-5.06%	-0.01%	0.95%	-0.39%

Moreover, in addition to whole life annuities, the prices of term life annuities are also examined. Although there are no sharp differences in prices of term life insurance like in whole life insurance, there are more differences for Russia than other countries, indicating that Russian data is more sensitive to mortality jumps. The prices of term insurance on portfolio for all selected countries can be seen in the Table (4.11).

Table 4.11: Prices for the Portfolio of Term Life Annuities (Due) in 2060

Insurance Type		**Prices of Term Life Annuities for Portfolio**				
		Canada	**Russia**	**UK**	**Japan**	**Bulgaria**
5 Year Term Insurance	**Original Lee-Carter Model**	55,461	54,750	54,190	55,641	50,500
	Outlier-Adjusted Lee-Carter Model	55,495	54,366	54,190	55,665	50,015
	Difference	0.06%	-0.70%	0.00%	0.04%	-0.96%
10 Year Term Insurance	**Original Lee-Carter Model**	93,980	91,298	93,808	94,668	92,397
	Outlier-Adjusted Lee-Carter Model	94,111	90,061	93,806	94,762	92,269
	Difference	0.14%	-1.35%	0.00%	0.10%	-0.14%
30 Year Term Insurance	**Original Lee-Carter Model**	190,730	174,350	189,280	196,450	177,210
	Outlier-Adjusted Lee-Carter Model	191,730	168,310	189,260	197,410	176,690
	Difference	0.52%	-3.46%	-0.01%	0.49%	-0.29%

CHAPTER 5

CONCLUSIONS AND COMMENTS

The prices of annuities may be considered as a cost for life insurance companies as they create liability for them in the future. The annuity prices that are calculated from the model which does not include all the significant factors that may influence mortality data, can have serious impacts on the financial strength of life insurance companies. Therefore, factors influencing the forecasting model should be taken into account. One of these factors is the outliers that are likely to be in the mortality data.

In this thesis, models with and without considering outliers in the historical mortality data, have been examined in different scenarios to measure the effect of any variation in the mortality rates thus, surviving probabilities on the price of whole life annuities. We have applied a different form of the outlier-adjusted Lee-Carter model developed by Chen and Liu [9]. While Chen and Liu use the iteration that eliminates some possible outliers during the outlier detection process, we take all possible outliers and generate outlier-adjusted model using all of them.

Our aim has been to determine the effects of original and outlier-adjusted model on the price of annuities not just for one country but among countries. While the populations of Canada and Russia are chosen for comparison, the data of UK, Japan and Bulgaria are also analyzed for better explication of the differences between countries. Moreover, calculating whole life annuity prices has been made at ages 0, 30 and 70.

Implementation results show that using outlier-adjusted model in calculation of forecasting mortality rates is critical on the annuity prices for the countries that have many outliers in their mortality rates. This inference cannot be used to distinguish between developed and developing countries because significant variations in the annuity prices between models have been found for Russian data but not for Bulgarian data as both countries are developing countries. In conclusion, life insurance companies or other related institutions should truly consider outliers in their forecasting model, especially the companies that working for population of countries with severe fluctuations in their mortality rates.

For the future studies, it would be useful to investigate outlier-adjusted scheme with Cairns-Blake-Dowd stochastic mortality model. This approach would also be used to model and value catastrophic mortality bonds. In addition to the prices of whole life annuities, the costs of life insurances would be calculated.

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APPENDIX A

Table A.1: Parameter Estimates of Original Lee-Carter Model for Canada and Russia

Age	Canada		Russia		Age	Canada		Russia	
	a_x	b_x	a_x	b_x		a_x	b_x	a_x	b_x
0	-3.9088	0.0216	-4.0106	-0.0406	56	-4.7189	0.0071	-4.2420	0.0225
1	-6.4007	0.0262	-6.2147	-0.0503	57	-4.6369	0.0068	-4.1825	0.0216
2	-6.9422	0.0241	-6.8681	-0.0390	58	-4.5252	0.0074	-4.1171	0.0202
3	-7.1910	0.0234	-7.1597	-0.0333	59	-4.4546	0.0069	-4.0572	0.0204
4	-7.3621	0.0224	-7.2845	-0.0298	60	-4.3414	0.0073	-3.9220	0.0162
5	-7.5076	0.0220	-7.4190	-0.0328	61	-4.3146	0.0058	-3.9175	0.0221
6	-7.6210	0.0220	-7.4864	-0.0358	62	-4.1706	0.0070	-3.8130	0.0140
7	-7.7180	0.0214	-7.4998	-0.0363	63	-4.0869	0.0070	-3.7551	0.0144
8	-7.8185	0.0213	-7.5562	-0.0314	64	-4.0019	0.0068	-3.6877	0.0135
9	-7.8389	0.0199	-7.6537	-0.0289	65	-3.8849	0.0074	-3.6025	0.0106
10	-7.8878	0.0197	-7.6936	-0.0278	66	-3.8564	0.0062	-3.5556	0.0131
11	-7.9053	0.0195	-7.7521	-0.0230	67	-3.7515	0.0065	-3.4631	0.0088
12	-7.8027	0.0181	-7.7130	-0.0240	68	-3.6415	0.0070	-3.3907	0.0085
13	-7.6980	0.0173	-7.6238	-0.0165	69	-3.5653	0.0066	-3.3309	0.0109
14	-7.5165	0.0156	-7.4655	-0.0119	70	-3.4575	0.0070	-3.1956	0.0027
15	-7.3408	0.0142	-7.2430	-0.0045	71	-3.4355	0.0057	-3.1765	0.0116
16	-7.1067	0.0130	-6.9930	-0.0005	72	-3.2725	0.0069	-3.0385	0.0024
17	-6.9421	0.0118	-6.7458	0.0000	73	-3.1925	0.0067	-2.9652	0.0030
18	-6.8003	0.0111	-6.5791	0.0006	74	-3.0991	0.0067	-2.8799	0.0035
19	-6.7159	0.0110	-6.5300	0.0067	75	-2.9989	0.0069	-2.7785	0.0011
20	-6.6947	0.0113	-6.4059	0.0089	76	-2.9018	0.0069	-2.6983	0.0021
21	-6.6869	0.0119	-6.3245	0.0141	77	-2.8333	0.0063	-2.6274	0.0046
22	-6.7047	0.0122	-6.2299	0.0120	78	-2.7051	0.0069	-2.4998	-0.0015
23	-6.7107	0.0124	-6.1705	0.0130	79	-2.6290	0.0062	-2.4556	0.0057
24	-6.7250	0.0123	-6.1201	0.0156	80	-2.5369	0.0060	-2.2877	-0.0047
25	-6.7284	0.0121	-6.0363	0.0164	81	-2.4994	0.0047	-2.2754	0.0074
26	-6.7102	0.0123	-6.0088	0.0189	82	-2.3531	0.0055	-2.1262	-0.0008
27	-6.7014	0.0124	-5.9531	0.0204	83	-2.2560	0.0053	-2.0497	0.0015
28	-6.6850	0.0128	-5.8957	0.0211	84	-2.1464	0.0054	-1.9576	0.0018
29	-6.6597	0.0121	-5.8559	0.0241	85	-2.0617	0.0052	-1.8633	0.0010
30	-6.6292	0.0123	-5.7313	0.0210	86	-1.9552	0.0052	-1.7751	0.0020
31	-6.6337	0.0115	-5.7490	0.0258	87	-1.8643	0.0049	-1.7044	0.0039
32	-6.5565	0.0122	-5.6693	0.0207	88	-1.7861	0.0045	-1.6233	0.0055
33	-6.5273	0.0116	-5.6361	0.0237	89	-1.7114	0.0039	-1.5774	0.0115
34	-6.4832	0.0115	-5.5862	0.0239	90	-1.6020	0.0042	-1.4267	0.0023
35	-6.4009	0.0118	-5.5002	0.0236	91	-1.5699	0.0026	-1.4817	0.0221
36	-6.3619	0.0116	-5.4775	0.0234	92	-1.4503	0.0030	-1.3613	0.0155
37	-6.3029	0.0110	-5.4341	0.0241	93	-1.3617	0.0030	-1.3151	0.0191
38	-6.2080	0.0115	-5.3757	0.0230	94	-1.2797	0.0029	-1.2705	0.0228
39	-6.1615	0.0107	-5.3350	0.0260	95	-1.2017	0.0027	-1.1508	0.0157
40	-6.0782	0.0109	-5.2044	0.0245	96	-1.1260	0.0024	-1.0864	0.0164
41	-6.0449	0.0095	-5.2253	0.0286	97	-1.0527	0.0022	-1.0242	0.0171
42	-5.9033	0.0107	-5.1317	0.0224	98	-0.9820	0.0020	-0.9643	0.0176
43	-5.8565	0.0097	-5.0854	0.0256	99	-0.9140	0.0018	-0.9066	0.0181
44	-5.7866	0.0090	-5.0326	0.0271	100	-0.8489	0.0016	-0.8512	0.0184
45	-5.6830	0.0095	-4.9314	0.0261	101	-0.7866	0.0014	-0.7982	0.0187
46	-5.6117	0.0086	-4.8953	0.0267	102	-0.7273	0.0012	-0.7476	0.0189
47	-5.5214	0.0085	-4.8373	0.0270	103	-0.6710	0.0011	-0.6994	0.0189
48	-5.4200	0.0087	-4.7653	0.0252	104	-0.6177	0.0009	-0.6536	0.0189
49	-5.3347	0.0082	-4.7119	0.0274	105	-0.5675	0.0007	-0.6102	0.0188
50	-5.2300	0.0085	-4.5849	0.0258	106	-0.5203	0.0006	-0.5692	0.0186
51	-5.2027	0.0069	-4.5905	0.0291	107	-0.4761	0.0005	-0.5305	0.0184
52	-5.0526	0.0081	-4.4979	0.0227	108	-0.4349	0.0004	-0.4940	0.0180
53	-4.9925	0.0075	-4.4495	0.0242	109	-0.3965	0.0003	-0.4598	0.0177
54	-4.8941	0.0073	-4.3814	0.0233	110	-0.3609	0.0002	-0.4276	0.0172
55	-4.8166	0.0073	-4.2914	0.0234					

Year	κ_t		Year	κ_t	
	Canada	Russia		Canada	Russia
1	79.7811	-1.4494	49	8.0633	5.4551
2	80.1212	-9.7273	50	5.9371	3.8115
3	81.0636	-10.9708	51	3.5756	-1.0646
4	74.0866	-6.4536	52	4.8545	-1.6716
5	72.3963	-11.3872	53	3.2283	-9.5614
6	77.8235	-21.3352	54	2.7106	-13.3764
7	74.7038	-12.6227	55	-0.5289	-18.6744
8	76.0919	-13.3691	56	-3.5001	-20.6903
9	78.1630	-10.5875	57	-7.2131	
10	73.3407	-10.3022	58	-10.3265	
11	67.7214	-5.5493	59	-14.2169	
12	65.6737	-5.5026	60	-15.4131	
13	61.5633	-6.9143	61	-19.7168	
14	59.2722	-5.8978	62	-20.6601	
15	60.7143	-5.1532	63	-24.6255	
16	60.3287	-6.8486	64	-27.5831	
17	64.8844	-1.8508	65	-26.8964	
18	56.7600	-0.6671	66	-28.3661	
19	56.4316	-0.2061	67	-31.8567	
20	56.0823	-0.3113	68	-31.8858	
21	56.4105	2.5859	69	-35.5835	
22	52.1156	3.2292	70	-39.1356	
23	54.1145	1.6071	71	-40.5022	
24	49.5076	-1.4459	72	-43.8000	
25	44.6419	0.6931	73	-41.3919	
26	43.3877	4.5466	74	-43.5928	
27	42.0536	1.8218	75	-44.7805	
28	40.4942	-8.9330	76	-47.0776	
29	38.9280	-8.4381	77	-48.8583	
30	35.8908	-7.1642	78	-50.9519	
31	34.5141	-7.6813	79	-53.7561	
32	32.1931	-4.1317	80	-58.9477	
33	30.4270	-3.2183	81	-62.2854	
34	24.8042	2.5138	82	-63.8280	
35	25.1027	16.9632	83	-66.5778	
36	25.2303	22.7689	84	-70.8851	
37	25.5613	18.8931	85	-73.0324	
38	21.1987	13.6665	86	-79.5788	
39	21.8945	10.2332	87	-79.5678	
40	18.9012	8.2521	88	-82.4048	
41	17.1354	14.0576	89	-87.9356	
42	16.7940	16.6560	90	-92.3373	
43	17.3058	17.2671	91	-95.4172	
44	13.7558	19.6060	92	-99.0878	
45	13.7777	20.0536	93	-100.8770	
46	12.3799	17.0304	94	-102.2092	
47	10.1981	16.6059	95	-104.0307	
48	9.8383	10.4174	96	-107.8780	

Table A.2: Parameter Estimates of Original Lee-Carter Model for UK, Japan and Bulgaria

Age	UK		Japan		Bulgaria	
	a_x	b_x	a_x	b_x	a_x	b_x
0	-4.0224	0.0186	-4.7632	0.0184	-3.6595	0.0427
1	-6.4233	0.0257	-6.6488	0.0189	-5.9047	0.0571
2	-7.0086	0.0239	-7.1485	0.0200	-6.6020	0.0505
3	-7.3073	0.0232	-7.4247	0.0202	-6.8908	0.0414
4	-7.5167	0.0231	-7.6362	0.0194	-7.1600	0.0381
5	-7.6288	0.0233	-7.8208	0.0185	-7.3729	0.0352
6	-7.7439	0.0220	-7.9567	0.0172	-7.4970	0.0332
7	-7.8814	0.0214	-8.1012	0.0163	-7.5625	0.0317
8	-7.9587	0.0203	-8.2790	0.0161	-7.6626	0.0313
9	-8.0113	0.0193	-8.3867	0.0155	-7.7067	0.0295
10	-8.0418	0.0187	-8.4679	0.0152	-7.7305	0.0301
11	-8.0394	0.0180	-8.4932	0.0146	-7.7616	0.0292
12	-7.9736	0.0174	-8.4440	0.0141	-7.7435	0.0283
13	-7.8857	0.0172	-8.3359	0.0132	-7.6462	0.0277
14	-7.7481	0.0168	-8.2144	0.0126	-7.5269	0.0270
15	-7.5768	0.0163	-8.0110	0.0123	-7.3926	0.0281
16	-7.3476	0.0151	-7.6346	0.0114	-7.2824	0.0273
17	-7.1188	0.0133	-7.4673	0.0117	-7.1349	0.0286
18	-7.0054	0.0126	-7.3220	0.0115	-6.9685	0.0263
19	-6.9620	0.0131	-7.2035	0.0118	-6.8822	0.0261
20	-6.9342	0.0133	-7.1453	0.0121	-6.8292	0.0275
21	-6.9156	0.0136	-7.1065	0.0123	-6.8289	0.0279
22	-6.9115	0.0138	-7.0710	0.0125	-6.8152	0.0277
23	-6.9048	0.0138	-7.0499	0.0128	-6.7873	0.0247
24	-6.9045	0.0140	-7.0360	0.0130	-6.7961	0.0257
25	-6.8902	0.0136	-7.0246	0.0129	-6.7537	0.0250
26	-6.8687	0.0133	-7.0163	0.0129	-6.7406	0.0236
27	-6.8537	0.0132	-7.0055	0.0128	-6.7196	0.0229
28	-6.8229	0.0129	-6.9796	0.0126	-6.6911	0.0224
29	-6.7864	0.0127	-6.9408	0.0124	-6.6534	0.0201
30	-6.7453	0.0123	-6.9168	0.0121	-6.6030	0.0202
31	-6.7228	0.0119	-6.8885	0.0119	-6.5905	0.0179
32	-6.6455	0.0118	-6.8364	0.0117	-6.4998	0.0170
33	-6.6094	0.0115	-6.7917	0.0117	-6.4556	0.0151
34	-6.5445	0.0113	-6.7292	0.0114	-6.4037	0.0145
35	-6.4709	0.0111	-6.6706	0.0111	-6.3365	0.0141
36	-6.4147	0.0109	-6.6111	0.0109	-6.2709	0.0127
37	-6.3452	0.0106	-6.5426	0.0106	-6.2046	0.0103
38	-6.2570	0.0106	-6.4748	0.0103	-6.1066	0.0103
39	-6.1780	0.0103	-6.3924	0.0102	-6.0538	0.0077
40	-6.0967	0.0099	-6.3149	0.0097	-5.9515	0.0102
41	-6.0295	0.0093	-6.2424	0.0096	-5.8943	0.0060
42	-5.9098	0.0098	-6.1620	0.0092	-5.8101	0.0058
43	-5.8411	0.0091	-6.0817	0.0090	-5.7170	0.0041
44	-5.7577	0.0088	-5.9909	0.0088	-5.6268	0.0033
45	-5.6542	0.0087	-5.9073	0.0086	-5.5325	0.0043
46	-5.5678	0.0085	-5.8214	0.0084	-5.4471	0.0027
47	-5.4688	0.0084	-5.7331	0.0083	-5.3600	0.0018
48	-5.3786	0.0084	-5.6471	0.0082	-5.2672	0.0012
49	-5.2738	0.0083	-5.5559	0.0081	-5.1807	0.0013
50	-5.1822	0.0080	-5.4655	0.0080	-5.0763	0.0031
51	-5.1197	0.0074	-5.3827	0.0079	-5.0026	0.0010
52	-4.9944	0.0078	-5.2951	0.0079	-4.9175	0.0015
53	-4.9069	0.0075	-5.2100	0.0078	-4.8205	0.0011
54	-4.8127	0.0075	-5.1281	0.0078	-4.7373	0.0014
55	-4.7477	0.0067	-5.0443	0.0078	-4.6412	0.0024

Age	UK		Japan		Bulgaria	
	a_x	b_x	a_x	b_x	a_x	b_x
56	-4.6332	0.0071	-4.9597	0.0078	-4.5538	0.0018
57	-4.5435	0.0068	-4.8737	0.0079	-4.4791	0.0009
58	-4.4386	0.0070	-4.7735	0.0082	-4.3846	0.0014
59	-4.3465	0.0068	-4.6977	0.0080	-4.2993	0.0011
60	-4.2582	0.0065	-4.6101	0.0080	-4.1677	0.0044
61	-4.1866	0.0060	-4.5196	0.0080	-4.1083	0.0021
62	-4.0626	0.0066	-4.4286	0.0081	-4.0129	0.0037
63	-3.9684	0.0066	-4.3348	0.0082	-3.9147	0.0029
64	-3.8717	0.0065	-4.2416	0.0083	-3.8209	0.0031
65	-3.7816	0.0066	-4.1503	0.0083	-3.7146	0.0042
66	-3.7190	0.0059	-4.0558	0.0083	-3.6397	0.0028
67	-3.6068	0.0061	-3.9599	0.0084	-3.5383	0.0034
68	-3.5111	0.0063	-3.8619	0.0084	-3.4346	0.0035
69	-3.4120	0.0063	-3.7628	0.0084	-3.3423	0.0032
70	-3.3219	0.0062	-3.6633	0.0084	-3.2136	0.0053
71	-3.2494	0.0058	-3.5623	0.0085	-3.1432	0.0032
72	-3.1167	0.0063	-3.4600	0.0084	-3.0211	0.0047
73	-3.0223	0.0063	-3.3578	0.0084	-2.9260	0.0034
74	-2.9232	0.0062	-3.2531	0.0083	-2.8248	0.0036
75	-2.8352	0.0062	-3.1474	0.0082	-2.7164	0.0047
76	-2.7348	0.0061	-3.0423	0.0080	-2.6217	0.0036
77	-2.6559	0.0059	-2.9338	0.0078	-2.5251	0.0026
78	-2.5505	0.0060	-2.8254	0.0078	-2.4146	0.0037
79	-2.4567	0.0058	-2.7164	0.0076	-2.3131	0.0031
80	-2.3688	0.0055	-2.6033	0.0074	-2.1938	0.0033
81	-2.2943	0.0050	-2.4957	0.0072	-2.1275	0.0002
82	-2.1749	0.0053	-2.3915	0.0070	-2.0276	0.0008
83	-2.0803	0.0051	-2.2845	0.0068	-1.9420	-0.0010
84	-1.9741	0.0052	-2.1830	0.0065	-1.8530	-0.0008
85	-1.8921	0.0049	-2.0791	0.0063	-1.7656	0.0000
86	-1.7923	0.0048	-1.9757	0.0062	-1.6931	-0.0015
87	-1.7104	0.0046	-1.8787	0.0058	-1.6212	-0.0037
88	-1.6332	0.0042	-1.7782	0.0057	-1.5521	-0.0043
89	-1.5390	0.0041	-1.6800	0.0055	-1.4839	-0.0054
90	-1.4578	0.0037	-1.5864	0.0051	-1.3816	-0.0026
91	-1.3960	0.0032	-1.4912	0.0049	-1.3461	-0.0069
92	-1.2976	0.0031	-1.3987	0.0048	-1.3012	-0.0058
93	-1.2168	0.0029	-1.3186	0.0044	-1.2490	-0.0066
94	-1.1395	0.0028	-1.2310	0.0043	-1.1873	-0.0067
95	-1.0581	0.0028	-1.1467	0.0039	-1.1094	-0.0075
96	-0.9868	0.0026	-1.0656	0.0036	-1.0509	-0.0079
97	-0.9181	0.0024	-0.9874	0.0034	-0.9945	-0.0082
98	-0.8523	0.0022	-0.9123	0.0031	-0.9401	-0.0085
99	-0.7894	0.0020	-0.8405	0.0029	-0.8878	-0.0088
100	-0.7294	0.0018	-0.7720	0.0026	-0.8376	-0.0090
101	-0.6725	0.0017	-0.7071	0.0024	-0.7895	-0.0092
102	-0.6186	0.0015	-0.6457	0.0021	-0.7435	-0.0093
103	-0.5677	0.0013	-0.5880	0.0019	-0.6997	-0.0094
104	-0.5200	0.0012	-0.5340	0.0017	-0.6579	-0.0094
105	-0.4752	0.0010	-0.4836	0.0015	-0.6182	-0.0095
106	-0.4334	0.0009	-0.4368	0.0013	-0.5805	-0.0094
107	-0.3946	0.0008	-0.3936	0.0011	-0.5447	-0.0094
108	-0.3586	0.0007	-0.3539	0.0010	-0.5109	-0.0093
109	-0.3253	0.0006	-0.3174	0.0008	-0.4790	-0.0092
110	-0.2946	0.0005	-0.2842	0.0007	-0.4488	-0.0090

Year	κ_t		
	UK	**Japan**	**Bulgaria**
1	87.5672	136.9620	47.1508
2	76.8844	117.2758	44.1142
3	82.7157	114.1561	40.2393
4	80.6115	109.0003	31.4904
5	74.9646	101.3409	33.7065
6	79.9808	91.7051	39.2496
7	73.6858	91.3370	24.0485
8	87.1864	83.1329	22.9755
9	69.2045	76.9657	20.7022
10	74.8137	79.2714	23.2652
11	72.0382	81.6208	14.2429
12	72.8111	69.7652	0.5904
13	66.5722	67.8254	21.4032
14	64.9430	68.1315	0.4986
15	66.8447	64.1993	-7.5488
16	68.2516	64.1068	6.4619
17	57.8670	55.3068	-6.0704
18	60.3477	52.7273	-17.7425
19	75.5515	54.2213	-16.0962
20	65.8205	47.0931	-18.0253
21	49.5898	44.3345	-2.6924
22	51.5360	42.7805	-15.9467
23	45.4721	40.6449	1.1154
24	41.7823	40.6362	-10.4823
25	41.0076	32.5923	0.2194
26	43.3016	28.5804	0.6259
27	28.3944	28.0228	-10.2491
28	36.4934	24.7248	-5.2318
29	34.8504	19.9668	2.1962
30	42.8199	16.5923	-4.8021
31	28.0630	10.5314	5.0400
32	26.4766	7.3659	-1.5276
33	24.4351	1.9465	-0.5535
34	26.6265	3.4106	4.2529
35	25.1244	-1.2048	-3.6838
36	21.2314	-7.2387	3.8630
37	21.9492	-7.1802	5.1775
38	20.2562	-11.8415	0.5808
39	17.3138	-14.6747	9.2990
40	21.6229	-19.7146	0.0088
41	20.5818	-24.6940	4.7871
42	22.6323	-22.7837	2.8114
43	10.4666	-28.4861	-0.4822
44	11.9864	-28.2840	1.9644
45	13.9401	-31.7545	5.3773
46	5.6910	-32.5793	-0.7793
47	12.8804	-34.1347	0.8451
48	11.9299	-39.6300	3.4033

Year	κ_t		
	UK	**Japan**	**Bulgaria**
49	9.1722	-37.7711	7.9279
50	4.6485	-46.8545	12.0153
51	9.4726	-49.7860	20.3307
52	7.0362	-51.9793	13.5634
53	5.0417	-50.7927	-4.0230
54	2.3344	-59.1853	1.6600
55	5.0364	-63.8347	-8.6567
56	-2.6130	-68.0627	-11.6275
57	-1.8759	-69.3249	-18.3780
58	-2.0879	-73.3179	-31.3263
59	-7.5116	-71.4705	-24.9274
60	-10.4420	-76.8990	-31.1069
61	-11.2423	-79.3030	-40.1524
62	-14.2060	-80.3001	-64.2510
63	-20.0271	-85.8772	-107.5654
64	-15.8684	-84.3025	-104.9127
65	-20.2177	-82.4362	
66	-26.5950	-86.8338	
67	-27.7931	-90.2136	
68	-27.7363	-94.6601	
69	-33.9173	-97.6105	
70	-34.9872	-100.6149	
71	-40.3239	-101.6441	
72	-35.8014		
73	-45.8523		
74	-42.9030		
75	-47.1363		
76	-50.6448		
77	-52.4286		
78	-52.8864		
79	-61.3232		
80	-65.4341		
81	-66.2767		
82	-66.6204		
83	-77.2017		
84	-79.8700		
85	-85.9355		
86	-88.1989		
87	-89.5878		
88	-99.3559		
89	-102.4328		
90	-109.5734		
91	-107.8769		
92	-109.1019		
93	-114.9955		
94	-108.0175		
95	-112.9689		
96			

APPENDIX B

Table B.1: Outlier-Adjusted and Forecasts Values of κ_t for Canada and Russia

Year	Outlier-Adjusted κ_t		Year	Outlier-Adjusted κ_t		Year	Forecasted κ_t	
	Canada	Russia		Canada	Russia		Canada	Russia
1	79.7811	-1.4494	49	8.1596	5.4551	2015	-	-11.3496
2	80.1212	-9.7273	50	5.9371	3.8115	2016	-	-11.5146
3	81.0636	-10.9708	51	3.5756	-1.0646	2017	-110.8756	-11.6796
4	74.0866	-6.4536	52	4.8545	-1.6716	2018	-112.3108	-11.8446
5	72.3963	-11.3872	53	3.2283	-2.7234	2019	-114.8884	-12.0096
6	70.5008	-10.0465	54	2.7106	-10.7278	2020	-116.9554	-12.1746
7	74.7038	-15.5200	55	-0.5289	-10.8524	2021	-118.8964	-12.3396
8	76.0919	-11.2631	56	-3.5001	-11.1846	2022	-121.5292	-12.5046
9	73.1394	-12.8950	57	-7.2131		2023	-123.1330	-12.6696
10	69.9104	-11.4409	58	-10.3265		2024	-125.8814	-12.8346
11	63.3708	-4.2073	59	-14.2169		2025	-127.6371	-12.9996
12	65.8444	-4.8969	60	-15.4131		2026	-130.0725	-13.1646
13	61.3791	-8.0421	61	-19.7168		2027	-132.2649	-13.3296
14	59.3093	-6.2232	62	-20.6601		2028	-134.2843	-13.4946
15	60.4064	-5.7525	63	-24.6255		2029	-136.8308	-13.6596
16	60.5360	-5.3295	64	-27.5831		2030	-138.6615	-13.8246
17	61.5813	-3.4596	65	-26.8964		2031	-141.2493	-13.9896
18	59.4651	0.4777	66	-28.3661		2032	-143.2140	-14.1546
19	60.9412	-0.4847	67	-31.8567		2033	-145.5739	-14.3196
20	51.6547	-0.5911	68	-31.8858		2034	-147.8369	-14.4846
21	56.7571	1.3891	69	-35.5835		2035	-149.9282	-14.6496
22	52.2940	3.3948	70	-39.1356		2036	-152.4101	-14.8146
23	54.4052	2.4741	71	-40.5022		2037	-154.4006	-14.9796
24	49.4196	-1.6221	72	-43.8000		2038	-156.8866	-15.1446
25	44.4793	1.2972	73	-41.3919		2039	-158.9869	-15.3096
26	43.1364	2.6802	74	-43.5928		2040	-161.3104	-15.4746
27	42.0576	2.8896	75	-44.7805		2041	-163.6124	-15.6396
28	40.2303	-1.1758	76	-47.0776		2042	-165.7644	-15.8046
29	38.9014	-8.5656	77	-48.8583		2043	-168.1997	-15.9696
30	35.2096	-5.8552	78	-50.9519		2044	-170.3016	-16.1346
31	34.5600	-5.2158	79	-53.7561		2045	-172.7247	-16.2996
32	32.3186	-4.0328	80	-58.9477		2046	-174.9125	-16.4646
33	30.9225	-4.2818	81	-62.2854		2047	-177.2215	-16.6296
34	24.8429	2.3884	82	-63.8280		2048	-179.5445	-16.7946
35	25.2414	6.0254	83	-66.5778		2049	-181.7454	-16.9597
36	25.3609	4.9260	84	-70.8851		2050	-184.1483	-17.1247
37	25.5524	4.7912	85	-73.0324		2051	-186.3275	-17.2897
38	22.1293	4.4650	86	-79.5788		2052	-188.7125	-17.4547
39	21.5425	10.2332	87	-79.5678		2053	-190.9568	-17.6197
40	19.0494	8.2521	88	-82.4048		2054	-193.2630	-17.7847
41	16.6713	14.0576	89	-87.9356		2055	-195.5968	-17.9497
42	16.7417	16.6560	90	-92.3373		2056	-197.8360	-18.1147
43	17.3548	17.2671	91	-95.4172		2057	-200.2168	-18.2797
44	13.4728	19.6060	92	-99.0878		2058	-202.4492	-18.4447
45	13.6749	20.0536	93	-100.8770		2059	-204.8121	-18.6097
46	11.5375	17.0304	94	-102.2092		2060	-207.0927	-18.7747
47	10.7827	16.6059	95	-104.0307				
48	9.6473	10.4174	96	-107.8780				

Table B.2: Outlier-Adjusted and Forecasts Values of κ_t for UK, Japan and Bulgaria

Year	Outlier-Adjusted κ_t			Year	Outlier-Adjusted κ_t			Year	Forecasted κ_t		
									UK	Japan	Bulgaria
								2011	-	-	-83.5418
	UK	Japan	Bulgaria		UK	Japan	Bulgaria	2012	-	-	-98.3297
1	87.5672	136.9620	47.1508	49	9.7359	-37.7711	7.9279	2013	-	-	-105.3223
2	76.8844	128.8857	44.1142	50	4.6485	-46.8545	12.0153	2014	-	-	-107.4030
3	82.7157	114.1561	40.2393	51	9.4726	-49.7860	20.3307	2015	-	-	-109.4837
4	80.6115	109.0003	31.4904	52	7.0362	-51.9793	13.5634	2016	-	-	-111.5644
5	74.9646	101.3409	33.7065	53	5.0417	-50.7927	20.0902	2017	-115.3644	-	-113.6451
6	79.9808	91.7051	39.2496	54	2.3344	-59.1853	1.6600	2018	-117.4556	-106.9908	-115.7258
7	73.6858	91.3370	24.0485	55	5.0364	-63.8347	-8.6567	2019	-119.5468	-111.5656	-117.8065
8	73.9083	83.1329	22.9755	56	-2.6130	-68.0627	-11.6275	2020	-121.6380	-115.1568	-119.8872
9	77.5150	76.9657	20.7022	57	-1.8759	-69.3249	-18.3780	2021	-123.7292	-118.0533	-121.9679
10	71.3538	79.2714	23.2652	58	-2.0879	-73.3179	-31.3263	2022	-125.8204	-120.7325	-124.0486
11	69.9907	81.6208	14.2429	59	-7.5116	-71.4705	-24.9274	2023	-127.9116	-123.5857	-126.1293
12	71.6845	69.7652	0.5904	60	-10.4420	-76.8990	-31.1069	2024	-130.0028	-126.7826	-128.2100
13	65.0469	67.8254	21.4032	61	-11.2423	-79.3030	-40.1524	2025	-132.0940	-130.2822	-130.2907
14	64.0190	68.1315	0.4986	62	-14.2060	-80.3001	-40.5751	2026	-134.1853	-133.9292	-132.3714
15	66.6393	64.1993	-7.5488	63	-20.0271	-85.8772	-66.5206	2027	-136.2765	-137.5637	-134.4521
16	68.6705	64.1068	6.4619	64	-15.8684	-84.3025	-77.9403	2028	-138.3677	-141.0912	-136.5328
17	58.4217	55.3068	-6.0704	65	-20.2177	-82.4362		2029	-140.4589	-144.4984	-138.6135
18	60.9055	52.7273	-17.7425	66	-26.5950	-86.8338		2030	-142.5501	-147.8285	-140.6942
19	58.3447	54.2213	-16.0962	67	-27.7931	-90.2136		2031	-144.6413	-151.1416	-142.7749
20	57.6583	47.0931	-18.0253	68	-27.7363	-94.6601		2032	-146.7325	-154.4823	-144.8556
21	62.9389	44.3345	-25.1638	69	-33.9173	-97.6105		2033	-148.8237	-157.8668	-146.9363
22	48.9774	42.7805	-15.9467	70	-34.9872	-100.6149		2034	-150.9149	-161.2863	-149.0170
23	44.5275	40.6449	-20.3130	71	-40.3239	-101.6441		2035	-153.0061	-164.7206	-151.0977
24	40.1698	40.6362	-10.4823	72	-35.8014			2036	-155.0973	-168.1504	-153.1784
25	40.0228	32.5923	0.2194	73	-45.8523			2037	-157.1885	-171.5659	-155.2591
26	42.4508	28.5804	0.6259	74	-42.9030			2038	-159.2798	-174.9669	-157.3398
27	28.7448	28.0228	-10.2491	75	-47.1363			2039	-161.3710	-178.3595	-159.4205
28	37.1146	24.7248	-5.2318	76	-50.6448			2040	-163.4622	-181.7512	-161.5012
29	35.3286	19.9668	2.1962	77	-52.4286			2041	-165.5534	-185.1470	-163.5819
30	43.0994	16.5923	-4.8021	78	-52.8864			2042	-167.6446	-188.5483	-165.6626
31	27.6116	10.5314	5.0400	79	-61.3232			2043	-169.7358	-191.9536	-167.7433
32	26.2023	7.3659	-1.5276	80	-65.4341			2044	-171.8270	-195.3602	-169.8240
33	24.8451	1.9465	-0.5535	81	-66.2767			2045	-173.9182	-198.7660	-171.9047
34	27.3157	3.4106	4.2529	82	-66.6204			2046	-176.0094	-202.1698	-173.9854
35	25.6180	-1.2048	-3.6838	83	-77.2017			2047	-178.1006	-205.5720	-176.0661
36	21.8808	-7.2387	3.8630	84	-79.8700			2048	-180.1918	-208.9733	-178.1468
37	21.7140	-7.1802	5.1775	85	-85.9355			2049	-182.2830	-212.3746	-180.2275
38	20.4469	-11.8415	0.5808	86	-88.1989			2050	-184.3743	-215.7765	-182.3082
39	17.1915	-14.6747	9.2990	87	-89.5878			2051	-186.4655	-219.1791	-184.3889
40	21.3107	-19.7146	0.0088	88	-99.3559			2052	-188.5567	-222.5821	-186.4696
41	21.0525	-24.6940	4.7871	89	-102.4328			2053	-190.6479	-225.9852	-188.5503
42	22.6844	-22.7837	2.8114	90	-109.5734			2054	-192.7391	-229.3882	-190.6310
43	10.6451	-28.4861	-0.4822	91	-107.8769			2055	-194.8303	-232.7909	-192.7117
44	12.3748	-28.2840	1.9644	92	-109.1019			2056	-196.9215	-236.1935	-194.7924
45	13.9802	-31.7545	5.3773	93	-114.9955			2057	-199.0127	-239.5959	-196.8731
46	5.7340	-32.5793	-0.7793	94	-108.0175			2058	-201.1039	-242.9984	-198.9538
47	13.3721	-34.1347	0.8451	95	-112.9689			2059	-203.1951	-246.4009	-201.0345
48	12.6021	-39.6300	3.4033	96				2060	-205.2863	-249.8035	-203.1152

APPENDIX C

Table C.1: Surviving Probabilities by Age in 2060 for Canada and Russia

Age	p_x		Age	p_x		Age	p_x		Age	p_x	
	Canada	**Russia**		**Canada**	**Russia**		**Canada**	**Russia**		**Canada**	**Russia**
0	0.999769	0.962444	31	0.999877	0.998038	62	0.996400	0.983175	93	0.870564	0.828411
1	0.999993	0.994873	32	0.999886	0.997662	63	0.996073	0.982291	94	0.857264	0.832313
2	0.999993	0.997837	33	0.999868	0.997716	64	0.995531	0.980764	95	0.840488	0.789124
3	0.999994	0.998548	34	0.999858	0.997608	65	0.995539	0.977900	96	0.821734	0.779445
4	0.999994	0.998801	35	0.999855	0.997381	66	0.994160	0.977915	97	0.801588	0.769454
5	0.999994	0.998890	36	0.999842	0.997312	67	0.993872	0.973779	98	0.780135	0.759155
6	0.999995	0.998902	37	0.999814	0.997229	68	0.993930	0.971709	99	0.757508	0.748558
7	0.999995	0.998907	38	0.999813	0.996998	69	0.992752	0.971254	100	0.733884	0.737672
8	0.999995	0.999058	39	0.999771	0.997047	70	0.992648	0.961853	101	0.709471	0.726512
9	0.999994	0.999184	40	0.999758	0.996536	71	0.990162	0.966986	102	0.684521	0.715099
10	0.999994	0.999233	41	0.999669	0.996859	72	0.991008	0.955209	103	0.659294	0.703454
11	0.999994	0.999338	42	0.999699	0.996126	73	0.989776	0.952406	104	0.634077	0.691603
12	0.999990	0.999299	43	0.999620	0.996184	74	0.988808	0.948817	105	0.609143	0.679580
13	0.999987	0.999334	44	0.999527	0.996085	75	0.988148	0.940957	106	0.584762	0.667415
14	0.999978	0.999285	45	0.999523	0.995591	76	0.986879	0.937354	107	0.561175	0.655146
15	0.999966	0.999222	46	0.999389	0.995481	77	0.984109	0.935862	108	0.538589	0.642814
16	0.999945	0.999074	47	0.999310	0.995234	78	0.984032	0.918929	109	0.517178	0.630458
17	0.999915	0.998825	48	0.999273	0.994704	79	0.980037	0.925750	110	0.000000	0.000000
18	0.999888	0.998628	49	0.999114	0.994644	80	0.977224	0.895028			
19	0.999877	0.998714	50	0.999076	0.993734	81	0.969668	0.914353			
20	0.999880	0.998604	51	0.998683	0.994143	82	0.970118	0.885767			
21	0.999894	0.998627	52	0.998818	0.992762	83	0.965814	0.882283			
22	0.999902	0.998428	53	0.998560	0.992612	84	0.962492	0.872246			
23	0.999906	0.998362	54	0.998353	0.991950	85	0.957489	0.858573			
24	0.999907	0.998361	55	0.998210	0.991213	86	0.952638	0.849222			
25	0.999903	0.998246	56	0.997966	0.990620	87	0.945161	0.844175			
26	0.999905	0.998277	57	0.997609	0.989880	88	0.935727	0.836514			
27	0.999905	0.998231	58	0.997650	0.988911	89	0.922157	0.846347			
28	0.999912	0.998150	59	0.997200	0.988269	90	0.918785	0.793729			
29	0.999896	0.998180	60	0.997149	0.985498	91	0.884940	0.860358			
30	0.999896	0.997816	61	0.995990	0.986943	92	0.882354	0.825120			

Table C.2: Surviving Probabilities by Age in 2060 for UK, Japan and Bulgaria

Age	p_x			Age	p_x		
	UK	Japan	Bulgaria		UK	Japan	Bulgaria
0	0.999606	0.999913	0.999996	**56**	0.997719	0.998999	0.992781
1	0.999992	0.999989	1.000000	**57**	0.997354	0.998934	0.99058
2	0.999993	0.999995	1.000000	**58**	0.997205	0.998904	0.99063
3	0.999994	0.999996	1.000000	**59**	0.996799	0.99876	0.989295
4	0.999995	0.999996	1.000000	**60**	0.996289	0.998656	0.993626
5	0.999996	0.999996	1.000000	**61**	0.995573	0.998538	0.989346
6	0.999995	0.999995	0.999999	**62**	0.995534	0.998425	0.991432
7	0.999995	0.999995	0.999999	**63**	0.995107	0.998304	0.98902
8	0.999995	0.999995	0.999999	**64**	0.994547	0.998177	0.988431
9	0.999994	0.999995	0.999999	**65**	0.99409	0.998019	0.989633
10	0.999993	0.999995	0.999999	**66**	0.992851	0.997843	0.985278
11	0.999992	0.999995	0.999999	**67**	0.992299	0.997664	0.985421
12	0.999990	0.999994	0.999999	**68**	0.991781	0.99745	0.984187
13	0.999989	0.999991	0.999998	**69**	0.990959	0.99719	0.981703
14	0.999986	0.999988	0.999998	**70**	0.989896	0.996892	0.986356
15	0.999982	0.999985	0.999998	**71**	0.988167	0.996581	0.977653
16	0.999971	0.999972	0.999997	**72**	0.987891	0.996153	0.981487
17	0.999947	0.999970	0.999998	**73**	0.986657	0.995692	0.973603
18	0.999932	0.999963	0.999996	**74**	0.985138	0.995123	0.97186
19	0.999935	0.999961	0.999995	**75**	0.983601	0.994451	0.974947
20	0.999937	0.999962	0.999996	**76**	0.981785	0.993591	0.965328
21	0.999940	0.999962	0.999996	**77**	0.979329	0.992535	0.953991
22	0.999942	0.999963	0.999996	**78**	0.977534	0.991507	0.958752
23	0.999941	0.999965	0.999993	**79**	0.974417	0.990054	0.949038
24	0.999943	0.999966	0.999994	**80**	0.969962	0.988393	0.944204
25	0.999937	0.999965	0.999993	**81**	0.964828	0.986421	0.892568
26	0.999933	0.999964	0.999990	**82**	0.962333	0.984121	0.893732
27	0.999930	0.999963	0.999988	**83**	0.957212	0.981676	0.839163
28	0.999923	0.999960	0.999987	**84**	0.953333	0.978099	0.830552
29	0.999917	0.999957	0.999978	**85**	0.946012	0.97441	0.84214
30	0.999906	0.999952	0.999978	**86**	0.939736	0.970683	0.778047
31	0.999895	0.999948	0.999964	**87**	0.93177	0.965024	0.656377
32	0.999885	0.999943	0.999952	**88**	0.921241	0.959662	0.598404
33	0.999873	0.999939	0.999927	**89**	0.911534	0.953431	0.490634
34	0.999859	0.999930	0.999912	**90**	0.897699	0.944918	0.650404
35	0.999841	0.999920	0.999899	**91**	0.879051	0.93635	0.303905
36	0.999825	0.999912	0.999856	**92**	0.865586	0.928632	0.38271
37	0.999801	0.999899	0.999753	**93**	0.848882	0.915637	0.295005
38	0.999783	0.999882	0.999726	**94**	0.833626	0.904995	0.255203
39	0.999747	0.999868	0.999506	**95**	0.822573	0.887175	0.1404
40	0.999703	0.999840	0.999673	**96**	0.803457	0.870368	0.072596
41	0.999645	0.999822	0.999178	**97**	0.783138	0.851671	0.009424
42	0.999635	0.999790	0.999080	**98**	0.761732	0.831083	0
43	0.999549	0.999762	0.998584	**99**	0.739398	0.808666	0
44	0.999479	0.999725	0.998155	**100**	0.716321	0.784552	0
45	0.999417	0.999686	0.998351	**101**	0.692725	0.758949	0
46	0.999334	0.999635	0.997492	**102**	0.668841	0.732134	0
47	0.999248	0.999593	0.996755	**103**	0.644923	0.704444	0
48	0.999173	0.999546	0.995963	**104**	0.62122	0.676263	0
49	0.999072	0.999490	0.995732	**105**	0.597968	0.647987	0
50	0.998904	0.999423	0.996665	**106**	0.575392	0.620018	0
51	0.998679	0.999363	0.994543	**107**	0.553679	0.59273	0
52	0.998643	0.999298	0.994572	**108**	0.532991	0.56646	0
53	0.998427	0.999229	0.993633	**109**	0.513452	0.541486	0
54	0.998253	0.999162	0.993378	**110**	0	0	0
55	0.997823	0.999079	0.994098				