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# An Exact Solution Procedure for Multi-Item Two-Echelon Spare Parts Inventory Control Problem with Batch Ordering in the Central Warehouse

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## Abstract

We consider a multi-item two-echelon inventory system in which the central warehouse operates under a  $(Q, R)$  policy, and the local warehouses implement basestock policy. An exact solution procedure is proposed to find the inventory control policy parameters that minimize the system-wide inventory holding and fixed ordering cost subject to an aggregate mean response time constraint at each facility.

## Key words:

Inventory, Branch and Price, Two-Echelon, Multi-Item, Batch Ordering, Spare Parts

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## 1. Introduction and Related Literature

In this study, a multi-item two-echelon spare parts inventory system is considered. The system consists of a central warehouse and a number of local warehouses, each of which (including the central warehouse) can respond to external customer demand. The central warehouse also responds to the replenishment orders from local warehouses, implying that it has both internal and external demands to satisfy. The stocks at the central warehouse are replenished from an outside supplier. We assume that the outside supplier has ample stock. Unsatisfied demand is backordered at all locations for each of which an aggregate service level target is set.

In this system, parts are often slow moving at local warehouses, and accordingly, the usual practice is to follow a basestock policy. However, at the central warehouse, parts move faster due to the accumulation of internal and external demands. Batching decisions at the central warehouse are important, especially when there exist some sort of fixed procurement or transportation costs, or production smoothing requirements of the supplier. Under these conditions it is more reasonable for the central warehouse to operate under  $(Q, R)$  policy. whereas for local warehouses, operating under basestock policies still makes sense.

In this paper, our objective is to find inventory control policy parameters that will minimize the sum of inventory holding and fixed ordering costs subject to constraints on the aggregate mean response time, which is demand weighted average of response times. We propose an exact solution procedure based on a branch-and-price algorithm. The procedure corresponds to solving the Lagrangean dual problem by using a column generation method, and then using this solution as a lower bound in a branch and bound algorithm. Although the procedure is proposed for the problem with fixed ordering costs and aggregate mean response time constraints, it can be extended to problems with target order frequency constraints and/or backorder costs. In addition, although our main concern is spare parts inventory systems, the procedure is valid for any distribution system with a similar structure. Finding the optimal policy parameters for a multi-item two-echelon inventory system is generally difficult. Even a moderate-size problem involves thousands of stock keeping units and many warehouses, which makes the problem very complicated. Therefore, decomposing the problem by facilities and/or items, then applying an iterative procedure to combine the resulting subproblems is common [5], [9], [1], [4], [12]. In addition, evaluating the objective function and the constraints requires solving nested convolutions, which is a typical burden even for the single-item case. Therefore, many approximations are proposed to overcome these difficulties. The METRIC [11] and the negative binomial approximation [8] are the most well-known approaches of this kind, in which lead time demand distribution at the lower echelon

facility is approximated. Although these approximations are proposed for single item case, they are also used to solve multi-item problems [9], [1], [4], [12]. The main drawback of these approximations is that they do not guarantee feasibility [12].

To the best of our knowledge, there is no exact solution algorithm for finding optimal policy parameters of the multi-item two-echelon inventory problem that we pose, although there exists exact procedures for single item problems [2]. Even though approximations and heuristics are prevalent approaches, an exact solution procedure is still desirable because of two reasons:

- Cost reductions: Even a small percentage improvement on total costs corresponds to very significant reductions in absolute terms in the spare part business.
- Benchmarking purposes: Due to lack of an exact solution algorithm, heuristics in the literature are usually compared to each other, or lower bounds, or simulation optimization results. However, not all of these benchmark solutions can guarantee high performance. Their performances may differ depending on the problem parameters leading to difficulties in assessing the performance of a proposed heuristic [1], [4]. In addition, it is possible to use an exact solution algorithm to test the performance of a lower bound for small size problems, before using it as a benchmark for larger problems.

The outline of this paper is as follows. In Section 2, we first specify the problem environment and then formulate the problem. In Section 3, the branch-and-price algorithm and the basic procedures used in the algorithm are presented. Finally, in Section 4, we provide some computational results, followed by a discussion on possible further applications.

## 2. The model

We consider a two-echelon distribution network in which the lower echelon comprises a set,  $N$ , of local warehouses, each is denoted by  $n = 1, 2, \dots |N|$ , while the upper echelon corresponds to a central warehouse, which is denoted by  $n = 0$ . There is a set,  $I$ , of parts, each is denoted by  $i = 1, 2, \dots |I|$ . In this system, we assume that the external customer demand for part  $i$  at warehouse  $n \in N \cup \{0\}$  occurs according to a Poisson process with rate  $\lambda_{in}$ . The external demand is independent across parts and warehouses. In addition to external demands, the central warehouse also faces internal demands from local warehouses. Internal and external demands are not differentiated and are satisfied according to the FCFS rule. We assume part-specific holding costs for all facilities and part-specific fixed ordering costs for the central warehouse. There is no incentive for joint ordering of different part types. The demand that can not be satisfied from stock is backordered. Warehouses have no capacity restrictions.

As for the control policies, for each part  $i \in I$ , local warehouse  $n \in N$  operates under a basestock level  $S_{in}$ , whereas the central warehouse operates under a batch ordering policy with reorder point  $R_i$  and order quantity  $Q_i$ . The system operates as follows: Whenever a demand for any part  $i$  arrives at warehouse  $n \in N \cup \{0\}$ , it is immediately satisfied from stocks if there is an available part; otherwise, the demand is backordered. In both cases, if the external demand is served by a local warehouse, an order of size one is placed at the central warehouse. This internal request is satisfied within a constant transportation lead time of  $T_{in}$ , if the part is available in the central warehouse. Otherwise, the internal demand is backordered as well. In any case, if the inventory position of the central warehouse drops to reorder level  $R_i$ , an order of size  $Q_i$  is placed at the outside supplier. It is assumed that the supplier has ample stock and can always satisfy requests for part  $i$  in a constant lead time of  $T_{i0}$ .

Based on this system definition, our problem can be stated as to find policy parameters minimizing the sum of the inventory holding and fixed ordering costs subject to constraints on the demand weighted average of individual part response times over all parts at each warehouse, which we refer to as aggregate mean response time. Our notation is given in Table 1.

**Insert Table 1 here.**

For sake of brevity, we omit the parameters that the variables depend on (unless there is ambiguity) e.g.,  $I_{in}(t, Q_i, R_i, S_{in})$  is simply denoted as  $I_{in}(t)$ . Also, when our focus is on limiting behavior of a stochastic variable, we omit the time component, e.g.,  $I_{in} = \lim_{t \rightarrow \infty} I_{in}(t)$ . Similarly, demands during the respective lead times at the central warehouse and the local warehouse  $n \in N$  are shortly denoted by  $Y_{i0}$  and  $Y_{in}$ , respectively.

Let  $\Lambda_n = \sum_{i \in I} \lambda_{in}$  denote the total demand rate for warehouse  $n \in N \cup \{0\}$ . By using Little's law, the aggregate mean response time at warehouse  $n \in N$  can be expressed as

$$W_n(\vec{Q}, \vec{R}, \vec{S}) = \sum_{i \in I} \frac{\lambda_{in}}{\Lambda_n} E[W_{in}(Q_i, R_i, S_{in})] = \sum_{i \in I} \frac{E[B_{in}(Q_i, R_i, S_{in})]}{\Lambda_n}.$$

Similarly, we obtain  $W_0(\vec{Q}, \vec{R}) = \sum_{i \in I} \frac{E[B_{i0}(Q_i, R_i)]}{\Lambda_0}$ . Accordingly, the problem (P) is formulated as follows.

$$\text{Min } Z = \sum_{i \in I} \left[ c_i h \left( E[I_{i0}(Q_i, R_i)] + \sum_{n \in N} E[I_{in}(Q_i, R_i, S_{in})] \right) + \frac{\lambda_{i0} K_i}{Q_i} \right] \quad (1)$$

$$\text{s.t.} \quad \sum_{i \in I} \frac{E[B_{i0}(Q_i, R_i)]}{\Lambda_0} \leq W_0^{\max}, \quad (2)$$

$$\sum_{i \in I} \frac{E[B_{in}(Q_i, R_i, S_{in})]}{\Lambda_n} \leq W_n^{\max}, \quad \text{for } \forall n \in N, \quad (3)$$

$$Q_i \geq 1, R_i \geq -1, S_{in} \geq 0, \quad \text{and } Q_i, R_i, S_{in} \in \mathbb{Z}, \quad \text{for } \forall i \in I, \forall n \in N.$$

In problem P, the objective function (1) minimizes the systemwide inventory holding and fixed ordering costs. Note that, since we assume full backordering, variable ordering costs are not included in the objective function. The following constraints are for aggregate mean response times at central and local warehouses: Constraint (2) and (3) guarantee that aggregate mean response times do not exceed  $W_0^{\max}$  and  $W_n^{\max}$ , respectively. The rest of this section is devoted to deriving the expected inventory levels in (1) and backorder levels in (2) and (3).

Since the local warehouses operate under basestock policies, any demand arrival, concurrently triggers an order at the central warehouse, the demand at the central warehouse is the sum of Poisson random variables and its distribution is also Poisson due to superpositioning. Thus, the inventory balance equation for part  $i$  at the central warehouse at time  $t + T_{i0}$  is given by

$$I_{i0}(t + T_{i0}) - B_{i0}(t + T_{i0}) = IP_{i0}(t) - Y_{i0}(t, t + T_{i0}). \quad (4)$$

Consequently, since  $IP_{i0}$  is uniformly distributed between  $R_i + 1$  and  $R_i + Q_i$  [2], the steady state distributions of  $I_{i0}$  and  $B_{i0}$  are as follows:

$$P\{I_{i0}(Q_i, R_i) = x\} = \begin{cases} \frac{1}{Q_i} \sum_{k=\max(R_i+1, x)}^{R_i+Q_i} P\{Y_{i0} = k - x\}, & \text{for } 1 \leq x \leq R_i + Q_i, \\ 1 - \sum_{x=1}^{R_i+Q_i} P\{I_{i0}(Q_i, R_i) = x\}, & \text{for } x = 0, \end{cases} \quad (5)$$

$$P\{B_{i0}(Q_i, R_i) = x\} = \begin{cases} \frac{1}{Q_i} \sum_{k=1}^{Q_i} P\{Y_{i0} = R_i + k + x\}, & \text{for } x \geq 1, \\ 1 - \sum_{x=1}^{\infty} P\{B_{i0}(Q_i, R_i) = x\}, & \text{for } x = 0, \end{cases}$$

where  $Y_{i0}$  has a Poisson distribution with parameter  $\lambda_{i0}T_{i0}$ . Further, we evaluate the steady state distributions of the inventory levels at local warehouses. The relevant inventory balance equation for each part  $i$  at local warehouse  $n$  is given by

$$IP_{in}(t) = S_{in} = I_{in}(t) - B_{in}(t) + X_{in}(t), \quad (6)$$

where,  $X_{in}(t)$  can be established as

$$X_{in}(t) = B_{i0}^{(n)}(t - T_{in}) + Y_{in}(t - T_{in}, t). \quad (7)$$

Note that  $B_{i0}^{(n)}(t, Q_i, R_i)$  can be obtained by conditioning on  $B_{i0}(Q_i, R_i)$  as

$$P\{B_{i0}^{(n)}(Q_i, R_i) = x\} = \sum_{y=x}^{\infty} P\{B_{i0}^{(n)}(Q_i, R_i) = x | B_{i0}(Q_i, R_i) = y\} \cdot P\{B_{i0}(Q_i, R_i) = y\}, \quad \text{for } x \geq 0,$$

where  $B_{i0}^{(n)}|B_{i0}$  is binomially distributed with parameters  $B_{i0}$  and  $\frac{\lambda_{in}}{\lambda_{i0}}$  [8]. Similarly, by using (7), the steady state distribution of  $X_{in}(Q_i, R_i)$  can be expressed in terms of the distribution of  $B_{i0}^{(n)}(Q_i, R_i)$  as

$$P\{X_{in}(Q_i, R_i) = x\} = \sum_{y=0}^x P\{Y_{in} = y\} \cdot P\{B_{i0}^{(n)}(Q_i, R_i) = x - y\}, \quad \text{for } x \geq 0,$$

where  $Y_{in}$  has a Poisson distribution with parameter  $\lambda_{in}T_{in}$ . As a result, from (6), the steady state distribution of  $I_{in}(Q_i, R_i, S_{in})$  is

$$P\{I_{in}(Q_i, R_i, S_{in}) = x\} = \begin{cases} P\{X_{in}(Q_i, R_i) = S_{in} - x\}, & \text{for } 1 \leq x \leq S_{in}, \\ 1 - \sum_{x=1}^{S_{in}} P\{I_{in}(Q_i, R_i, S_{in}) = x\}, & \text{for } x = 0. \end{cases} \quad (8)$$

Using the distributions of inventory levels in (5) and (8), the expected inventory costs in the objective function (1) can be derived. Finally, the expected backorder expressions in constraints (2) and (3) are

$$E[B_{i0}] = E[Y_{i0}] - R_i - \frac{(Q_i + 1)}{2} + E[I_{i0}], \quad (9)$$

$$E[B_{in}] = E[X_{in}] - S_{in} + E[I_{in}], \quad (10)$$

which avoids solving nested set of convolutions.

### 3. An Exact Solution Procedure

In this section, an exact solution procedure based on a branch-and-price algorithm is proposed. Branch-and-price is a generalization of branch and bound algorithm with LP-relaxation. A column generation method is used to obtain a lower bound for each subproblem (node) of the branch and bound tree. In Section 3.1, a high-level description of the algorithm is provided, then in Sections 3.2-3.4, the basic procedures that are used as building blocks of the algorithm are explained.

#### 3.1. Branch-and-Price

In the branch-and-price algorithm, at each node of the branch and bound tree, first, by iterating a column generation algorithm (Section 3.2) we obtain the Lagrangean dual solution of the corresponding node, then, by applying a greedy heuristic (Section 3.4) on the corresponding Lagrangean dual solution, we find a feasible solution. The former solution is used as a lower bound for the corresponding node, and the latter is used as a candidate for the global upper bound to tighten the bounding scheme and expedite the procedure. Depending on these bounds, a node is either fathomed, or further explored by branching. The procedure is repeated until all nodes are fathomed.

As a lower bound, we consider the Lagrangean dual solution because of three reasons. (i) The Lagrangean relaxation makes it possible to decompose the multi-item problem into multiple single-item problems. (ii) The Lagrangean dual of our problem has no integrality property [10]. As a direct consequence of that, the Lagrangean dual bound is guaranteed to be a better lower bound than the one that LP-relaxation gives. (iii) It is known to be a tight lower bound for multi-item two-echelon inventory problems with basestock control policies [12].

At each iteration of the algorithm, we select the node that provides the lowest average of lower and upper bounds to explore first, because of the superior performance observed in the experiments. As for the branching decision, we consider variable dichotomy, which corresponds to imposing branching constraints on the original variables. That is, any fractional  $Q_i$  or  $R_i$  or  $S_{in}$  whose remainder is closer to 1/2 is selected for branching.

In Section 3.2, we introduce the column generation method. As discussed in Section 3.2, the problem  $P$  decomposes by part after implementing the method. In Section 3.3, a subroutine is proposed to solve each of these single-item two-echelon subproblems. Finally, in Section 3.4, we obtain an upper bound for  $P$  is introduced.

### 3.2. Column Generation Method

In this section, first, we introduce the column generation method, based on its implementation on the root node. The additional requirements to implement the algorithm to non-root nodes will be discussed later.

The method requires defining an alternative formulation of problem  $P$ , known as extensive formulation [10]. Let  $L$  denote the set of control policy alternatives  $(\vec{S}_i, Q_i, R_i)$ , that is columns, for each part  $i$ , and  $x_{il}$  indicate whether policy (column)  $l \in L$  for part  $i$  is selected or not, and let  $C_{il} = c_i h E[I_{i0}(Q_i^l, R_i^l)] + c_i h \sum_{n \in N} E[I_{in}(S_{in}^l, Q_i^l, R_i^l)] + \frac{\lambda_{i0} K_i}{Q_i}$ ,  $A_{i0} = \frac{E[B_{i0}(Q_i^l, R_i^l)]}{\Lambda_0}$ ,  $A_{in} = \frac{E[B_{in}(S_{in}^l, Q_i^l, R_i^l)]}{\Lambda_n}$ , and for  $i \in I$ ,  $n \in N$ , and  $l \in L$ , where  $(S_{in}^l, Q_i^l, R_i^l)$  denotes policy  $l \in L$ , then the alternative formulation for the problem is given as follows.

$$\begin{aligned} \text{Min } Z &= \sum_{i \in I} \sum_{l \in L} C_{il} x_{il} \\ \text{s.t.} \\ \sum_{i \in I} \sum_{l \in L} A_{in} x_{il} &\leq W_n^{\max}, \quad \text{for } \forall n \in N, \quad (\alpha_n) \\ \sum_{l \in L} x_{il} &= 1, \quad \text{for } \forall i \in I, \quad (\beta_i) \\ x_{il} &= 0/1, \quad \text{for } \forall n \in I, \forall l \in L. \end{aligned}$$

The problem is known as the Master Problem ( $MP$ ). It is a tighter formulation than problem  $P$ , corresponds to a special case of the set packing problem, which is known to be  $\mathcal{NP}$ -hard [7]. However, it is sufficient to solve the linear relaxation of  $MP$  to obtain the Lagrangean dual solution, because the linear relaxation of this formulation corresponds to the dual of the Lagrangean dual problem. In order to solve this problem, one can start with a small subset of solutions instead of working with the complete solution space, and then generate only the solutions that improve the objective function value at each iteration. This restricted version of the problem is known as the Restricted Master Problem ( $RMP$ ). This pricing procedure is known as the column generation. In the light of these, letting  $C_i(\vec{S}_i, Q_i, R_i) = c_i h E[I_{i0}(Q_i, R_i)] + c_i h \sum_{n \in N} E[I_{in}(S_{in}, Q_i, R_i)] + \frac{\lambda_{i0} K_i}{Q_i}$ ,  $A_{i0} = \frac{E[B_{i0}(Q_i, R_i)]}{\Lambda_0}$ , and  $A_{in} = \frac{E[B_{in}(S_{in}, Q_i, R_i)]}{\Lambda_n}$  for  $i \in I$  and  $n \in N$ , we introduce the column generation (pricing) problem ( $CG$ ) as

$$\begin{aligned} \text{Min } \sum_{i \in I} \left( C_i(\vec{S}_i, Q_i, R_i) - \sum_{n \in N \cup \{0\}} \alpha_n A_{in} - \beta_i \right) \\ \text{s.t.} \\ Q_i \geq 1, R_i \geq -1, \vec{S}_i \geq \vec{0}, \quad \text{and } Q_i, R_i, S_{in} \in Z, \quad \text{for } \forall i \in I, \forall n \in N, \end{aligned}$$

where  $\alpha_n$  for each  $n \in N$  and  $\beta_i$  for each  $i \in I$  are the dual variables (or equivalently Lagrangean multipliers) of problem  $MP$ , which can be obtained from the solution of  $RMP$ . In this sense,  $CG$  is equivalent to the Lagrangean relaxation of problem  $P$  [3]. The problem  $CG$  is decomposable by parts. Hence, we decompose this pricing problem into  $|I|$  subproblems, each of which is denoted by  $SP_i(\vec{\theta})$ , where  $\vec{\theta} = [\theta_1, \theta_2, \dots, \theta_{|M|}]$  and  $\theta_n = \frac{-\alpha_n}{\Lambda_n}$ .

$$\begin{aligned} \text{Min } Z_i &= c_i h \left( E[I_{i0}(Q_i, R_i)] + \sum_{n \in N} E[I_{in}(S_{in}, Q_i, R_i)] \right) + \frac{\lambda_{i0} K_i}{Q_i} - \theta_0 E[B_{i0}(Q_i, R_i)] \\ &\quad - \sum_{n \in N} \theta_n E[B_{in}(S_{in}, Q_i, R_i)] \\ \text{s.t.} \\ Q_i \geq 1, R_i \geq -1, \vec{S}_i \geq \vec{0}, \quad \text{and } Q_i, R_i, S_{in} \in Z, \quad \text{for } \forall n \in N. \end{aligned} \tag{11}$$

After finding the optimal objective function value of each  $SP_i(\vec{\theta})$ , i.e.,  $Z_i(\vec{\theta})$ , then the optimal objective function value of  $CG$ , i.e.,  $Z(\vec{\theta})$ , can be obtained using  $Z(\vec{\theta}) = \sum_{i \in I} (Z_i(\vec{\theta}) - \beta_i)$ . Finally, if the pricing problem  $CG$  yields a negative optimal objective function value, i.e.,  $Z(\vec{\theta}) < 0$ , then the combination of solutions of subproblems are added to set  $L$  as a new promising column (solution). Otherwise, the optimality is achieved and we conclude that the optimal solution of the  $LP$  relaxation of  $RMP$  becomes optimal for the  $LP$  relaxation of  $MP$  as well.

The method requires an initial feasible solution for the  $LP$  relaxation of  $MP$ . For this purpose, first, the order quantities are determined by using the EOQ model. Then, using these order quantities, assuming that the target

aggregate mean response time at each warehouse  $n \in N$ , i.e.,  $W_n^{\max}$ , should be reached by each part individually, the initial values for the remaining control parameters, i.e.,  $R_i^0$  and  $S_{in}^0$  for  $i \in I$  and  $n \in N$ , are obtained. This corresponds to obtaining  $Q_i^0$ ,  $R_i^0$ , and  $S_{in}^0$  by using the following nested formulas in the given order.

$$\begin{aligned} Q_i^0 &= \lfloor EOQ_i \rfloor, \\ R_i^0 &= \text{Min} \left\{ R_i \in \{-1, 0, 1, \dots\} : \frac{E[B_{i0}(Q_i^0, R_i)]}{\lambda_{i0}} \leq W_0^{\max} \right\}, \\ S_{in}^0 &= \text{Min} \left\{ S_{in} \in \{0, 1, 2, \dots\} : \frac{E[B_{in}(S_{in}, Q_i^0, R_i^0)]}{\lambda_{in}} \leq W_n^{\max} \right\}. \end{aligned}$$

Related with the non-root nodes, we have some additional considerations.

- Any column generated by a parent node is introduced also to a child node as long as that column satisfies the branching constraint dedicated to the corresponding child node.
- Although, pricing problem  $CG$  is an unconstrained problem at the root node, it will involve branching constraints at non-root nodes.

### 3.3. Solution Procedure for Subproblems: Single-item Two-echelon Batch Ordering Problem

To the best of our knowledge, there is no exact solution algorithm proposed for a single-item two-echelon batch ordering problem,  $SP_i(\vec{\theta})$ . Therefore, we develop an algorithm based on the result that when  $Q_i$  and  $R_i$  are fixed it is easy to find the optimal  $S_{in}$ , i.e.,  $S_{in}^*(Q_i, R_i)$ . For this purpose, two nested loops are required; the outer loop searches for the optimal  $Q_i$ , the inner loop searches for the optimal  $R_i$  for a fixed  $Q_i$ , whereas an innermost subroutine optimizes  $S_{in}$  for given values of  $Q_i$  and  $R_i$ . In Section 3.3.1, we derive the optimality conditions for the problem that is solved by the innermost subroutine of this algorithm. In order to reduce the search space, upper bounds  $Q_i^{UB}$ ,  $R_i^{UB}$ , and lower bounds  $Q_i^{LB}$  and  $R_i^{LB}$ , are proposed for the optimal values for  $Q_i$  and  $R_i$ , respectively. For a given value of  $Q_i$ ,  $R_i^{UB}$  ( $R_i^{LB}$ ) is obtained by optimizing  $R_i$  for  $S_{in} = 0$  ( $S_{in} \rightarrow \infty$ ) for all  $n \in N$ . Similarly, we obtain  $Q_i^{LB}$ , by optimizing  $Q_i$  for  $R_i \rightarrow \infty$  and  $S_{in} \rightarrow \infty$  for all  $n \in N$ . Finally, by modifying the upper bound proposed by Gallego [6] for single-echelon problems,  $Q_i^{UB} = \sqrt{\frac{2K_i\lambda_{i0} + (c_i h + p)\lambda_{i0}T_{i0}}{H_i}}$  is obtained, where  $H_i = \frac{c_i h p}{c_i h + p}$  and  $p = \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}}$ . The details of how we obtain these bounds and the corresponding proofs are given in the Appendix. Finally, we note that the algorithm can be extended to problems where there is an explicit cost of backordering.

#### 3.3.1. Finding Optimal Solution for Subproblems for Given Values of Reorder Level and Order Quantity

For a given part  $i \in I$ , and given values of  $Q_i$  and  $R_i$ ,  $SP_i(\vec{\theta})$  reduces to  $|N|$  independent subproblems, each of which is denoted by  $SP_{in}(\theta_n, Q_i, R_i)$ .

$$\begin{aligned} &\text{Min } c_i h E[I_{in}(S_{in}, Q_i, R_i)] + \theta_n E[B_{in}(S_{in}, Q_i, R_i)] \\ &\text{s.t.} \\ &S_{in} \geq 0, \text{ and } \in \mathbb{Z}. \end{aligned}$$

By using equation (10), the objective function in  $SP_{in}(\theta_n, Q_i, R_i)$  can be restated as

$$G(S_{in}) = (c_i h + \theta_n) E[I_{in}(S_{in}, Q_i, R_i)] + \theta_n E[X_{in}(Q_i, R_i)] - \theta_n S_{in}.$$

**Proposition 1.**  $G(S_{in})$  is unimodal.

*Proof.* Let  $\Delta_x$  and  $\Delta_x^2$  be the first and second order difference equations with respect to variable  $x$ , respectively. Then,  $\Delta_{S_{in}}^2 G(S_{in}) \geq 0$ , is a sufficient condition for  $G(S_{in})$  to be unimodal. First, from (8) we have

$$E[I_{in}(S_{in}, Q_i, R_i)] = \sum_{x=1}^{S_{in}-1} (S_{in} - x) \cdot P\{X_{in}(Q_i, R_i) = x\}.$$

Using this result, next, we have

$$\begin{aligned}\Delta_{S_{in}} G(S_{in}) &= (c_i h + \theta_n) \sum_{x=1}^{S_{in}} P\{X_{in}(Q_i, R_i) = x\} - \theta_n, \\ \Delta_{S_{in}}^2 G(S_{in}) &= (c_i h + \theta_n) P\{X_{in}(Q_i, R_i) = S_{in} + 1\},\end{aligned}$$

which satisfies  $\Delta_{S_{in}}^2 G(S_{in}) \geq 0$ . □

**Proposition 2.** *The optimal solution of  $SP_{in}(\theta_n, Q_i, R_i)$  is*

$$\text{Min}_{S_{in} \in \{0, 1, 2, \dots\}} \left\{ S_{in} : \sum_{x=0}^{S_{in}} P(X_{in}(Q_i, R_i) = x) \geq \frac{\theta_n}{c_i h + \theta_n} \right\}. \quad (12)$$

*Proof.* As a direct consequence of Proposition 1, the optimal  $S_{in}$  is the smallest integer that satisfies the first order condition, i.e.,  $\Delta_{S_{in}} G(S_{in}) > 0$ , which gives (12). □

Thus, it follows from Proposition 2 that the optimality condition for problem  $SP_{in}(\theta_n, Q_i, R_i)$  leads to the Newsboy result, i.e., no-stockout probability is at least equal to  $\frac{\theta_n}{c_i h + \theta_n}$ .

### 3.4. Generating Upper Bounds

We obtain an upper bound for each node of the branch and bound tree using a greedy heuristic. The greedy heuristic is a fast, simple local search method that can generate a feasible solution from a dual solution. It is easy to implement and also known to perform well for multi-item two-echelon inventory control problems [5], [12]. The main idea of the heuristic is as follows: Starting with a promising dual solution, which is infeasible for the primal problem, the algorithm iterates to a new solution that is as close to the feasible region as possible, while incurring as low additional cost as possible. This procedure is repeated until a feasible solution is obtained.

Recall that  $\vec{Q}$ ,  $\vec{R}$ , and  $\vec{S}$  are vectors of order quantities, reorder levels, and basestock levels, respectively. Then, one can define the maximum constraint violation, for given values of  $\vec{Q}$ ,  $\vec{R}$ , and  $\vec{S}$ ,  $\omega(\vec{Q}, \vec{R}, \vec{S})$ , as

$$\omega(\vec{Q}, \vec{R}, \vec{S}) = \max_{n \in N \cup \{0\}} \left\{ \left( W_n(\vec{Q}, \vec{R}, \vec{S}) - W_n^{\max} \right)^+ \right\}.$$

Also, let  $Z(\vec{Q}, \vec{R}, \vec{S})$  be the objective function value for given values of (1)  $\vec{Q}$ ,  $\vec{R}$ , and  $\vec{S}$ ,  $E(\vec{Q}, \vec{R}, \vec{S})$  be the neighborhood of  $(\vec{Q}, \vec{R}, \vec{S})$ . We define  $E(\vec{Q}, \vec{R}, \vec{S})$  such that it includes all vectors  $[\vec{Q}, \vec{R}, \vec{S}] + \varepsilon$ , where  $\varepsilon$  is a vector in which exactly one of the entries is one, and the rest is zero. Then, a greedy move can be formally described as determining the solution  $(\vec{Q}', \vec{R}', \vec{S}') \in E(\vec{Q}, \vec{R}, \vec{S})$  that yields the maximum  $r(\vec{Q}', \vec{R}', \vec{S}')$  ratio, where

$$r(\vec{Q}', \vec{R}', \vec{S}') = \frac{\omega(\vec{Q}', \vec{R}', \vec{S}') - \omega(\vec{Q}, \vec{R}, \vec{S})}{Z(\vec{Q}', \vec{R}', \vec{S}') - Z(\vec{Q}, \vec{R}, \vec{S})}.$$

## 4. Computational Results and Possible Further Applications

To have an insight on how large problems can be solved by the by the exact algorithm, we provide some computational results. To this purpose, we consider problems with 5-13 parts and 2-4 local warehouses. For each pair of these parameters, 10 random instances are generated, where the demand rate, the fixed ordering cost, the unit variable cost and the lead time are randomly generated, while other parameters are not varied as shown in Table 2. The demand rates for each part  $i$  is generated from a uniform distribution  $U[0.01, 0.05]$ . Further, by multiplying this random number with another uniform random number generated from  $U[0.5, 1.5]$ , we obtain part-specific location-dependent demand rates, i.e.,  $\lambda_{in}$ , for each part  $i$  and warehouse  $n \in N \cup \{0\}$ . The algorithm is coded in MATLAB 7.0 and the experiment is run on Intel Centrino 2.53 GHZ processor with 4 GB RAM. The time limit is set as 1 hour. The results of the experiment are presented in Table 3.



**Insert Table 2 here.**

**Insert Table 3 here.**

As it can be seen in Table table3, for a few local warehouse and a limited number of critical parts, the exact solution procedure that we propose can be used. Nevertheless, for larger problems one needs to use heuristic approaches. For this purpose, the column generation method and the greedy heuristic that we introduce can be made use of. However, devising such heuristics falls out of the scope of this paper.

The branch-and-price algorithm that we introduce can be extended to multi-item two-echelon inventory control problems with alternative service levels constraints, e.g., aggregate probability of no stockout, aggregate ready rate. In this case, the unit shortage costs in the pricing problem are replaced by relevant shortage costs definitions. This will bring changes in the subproblems and the single-item two-echelon subroutine that solves these subproblems. One can still utilize our branch-and-price algorithm after adopting bounds, optimality conditions for the innermost subroutine, accordingly. The algorithm can also be extended to problems with the ordering frequency limits at the central warehouse. Although this will increase the number of both constraints in the original problem and the master problem of the corresponding system by one, this will not further effect any of the solution procedures in the algorithm. Since the pricing problem corresponds to the Lagrangean relaxation problem, applying the column generation method corresponds to introducing Lagrangean multipliers for the corresponding constraint, which is equivalent to introducing fixed ordering costs for the central warehouse.

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## Appendix<sup>1</sup> Bounds on the Reorder Levels and Order Quantities

In this part, we develop lower and upper bounds on the optimal  $R_i$  (for a given value of  $Q_i$ ) and the optimal  $Q_i$  for subproblem  $SP_i(\vec{\theta})$ . First, for a given value of  $Q_i$ , an upper (lower) bound on the optimal  $R_i$  is obtained by optimizing  $R_i$  for  $S_{in} = 0$  ( $S_{in} \rightarrow \infty$ ) for all  $n \in N$ , through Theorem 1, Corollary 1, and Proposition 3 (4). Similarly, we obtain a lower bound on the optimal  $Q_i$ , by optimizing  $Q_i$  for  $R_i \rightarrow \infty$  and  $S_{in} \rightarrow \infty$  for all  $n \in N$ , through Theorem 1, Corollary 2, and Proposition 5. Finally, an upper bound on the optimal  $Q_i$  is obtained through Proposition 6 by modifying the upper bound proposed by Gallego [6] for single-echelon problems.

First, to simplify the derivations, we obtain expressions that make it possible to express relevant inventory terms in terms of  $E[I_{i0}(Q_i, R_i)]$ . Since  $Y_{i0}$  is distributed according to a Poisson distribution with mean  $\lambda_{i0}T_{i0}$  and due to (9),  $E[B_{i0}(Q_i, R_i)]$  can be expressed in terms of  $E[I_{i0}(Q_i, R_i)]$  as

$$E[B_{i0}(Q_i, R_i)] = \lambda_{i0}T_{i0} - R_i - \frac{Q_i + 1}{2} + E[I_{i0}(Q_i, R_i)]. \quad (13)$$

<sup>1</sup>This part can be made available on line upon request.

Also, by using (7), (13), the distribution of  $Y_{i0}$  and that  $B_{i0}^{(n)}|B_{i0}$  is binomially distributed with parameters  $B_{i0}$  and  $\frac{\lambda_{in}}{\lambda_{i0}}$ ,  $E[X_{in}(Q_i, R_i)]$  can be represented in terms of  $E[I_{i0}(Q_i, R_i)]$  as

$$E[X_{in}(Q_i, R_i)] = \frac{\lambda_{in}}{\lambda_{i0}} \left( \lambda_{i0} T_{i0} - R_i - \frac{Q_i + 1}{2} + E[I_{i0}(Q_i, R_i)] \right) + \lambda_{in} T_{in}. \quad (14)$$

Further, by using (10) and (14), we obtain

$$\begin{aligned} E[B_{in}(S_{in}, Q_i, R_i)] &= \left( \frac{\lambda_{in}}{\lambda_{i0}} \left( \lambda_{i0} T_{i0} - R_i - \frac{Q_i + 1}{2} + E[I_{i0}(Q_i, R_i)] \right) + \lambda_{in} T_{in} \right) - S_{in} \\ &+ E[I_{in}(S_{in}, Q_i, R_i)]. \end{aligned} \quad (15)$$

**Theorem 1.** For a given value of  $Q_i$ , the objective function of problem  $SP_i(\vec{\theta})$  is unimodal in each of the following cases

- $S_{in}$  approaches infinity, for all  $n \in N$ ,
- $S_{in} = 0$ , for all  $n \in N$ .

*Proof.* For given values of  $Q_i$ , and  $S_{in}$  for each  $n \in N$ , by substituting equations (13) and (15), the objective function of problem  $SP_i(\vec{\theta})$  can be simplified as follows.

$$\begin{aligned} G(R_i) &= \left( c_i h + \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) E[I_{i0}(Q_i, R_i)] + \sum_{n \in N} \left( c_i h + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) E[I_{in}(S_{in}, Q_i, R_i)] \\ &+ \frac{\lambda_{i0} K_i}{Q_i} + \left( \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \left( \lambda_{i0} T_{i0} - R_i - \frac{Q_i + 1}{2} \right) + \sum_{n \in N} \theta_n (\lambda_{in} T_{in} - S_{in}), \end{aligned}$$

$$\begin{aligned} \Delta_{R_i} G(R_i) &= G(R_i + 1) - G(R_i) = \left( c_i h + \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \Delta_{R_i} (E[I_{i0}(Q_i, R_i)]) \\ &+ \sum_{n \in N} \left( c_i h + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \left( \Delta_{R_i} E[I_{in}(S_{in}, Q_i, R_i)] \right) - \left( \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right), \end{aligned}$$

$$\begin{aligned} \Delta_{R_i}^2 G(R_i) &= \Delta G(R_i + 1) - \Delta G(R_i) = \left( c_i h + \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \Delta_{R_i}^2 (E[I_{i0}(Q_i, R_i)]) \\ &+ \sum_{n \in N} \left( c_i h + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \left( \Delta_{R_i}^2 E[I_{in}(S_{in}, Q_i, R_i)] \right). \end{aligned}$$

Since  $\Delta_{R_i}^2 G(R_i) \geq 0$  is a sufficient condition for  $G(R_i)$  to be unimodal, the proof will be complete if we show that

$$\Delta_{R_i}^2 E[I_{i0}(Q_i, R_i)] + \sum_{n \in N} \Delta_{R_i}^2 E[I_{in}(S_{in}, Q_i, R_i)] \geq 0. \quad (16)$$

First, using (5), we have

$$E[I_{i0}(Q_i, R_i)] = \frac{1}{Q_i} \sum_{k=R_i+1}^{R_i+Q_i} \sum_{x=0}^{k-1} P(Y_{i0} \leq x). \quad (17)$$

Then, we can obtain

$$\Delta_{R_i} E[I_{i0}(Q_i, R_i)] = E[I_{i0}(Q_i, R_i + 1)] - E[I_{i0}(Q_i, R_i)] = \frac{1}{Q_i} \sum_{x=R_i+1}^{R_i+Q_i} P(Y_{i0} \leq x), \quad (18)$$

$$\Delta_{R_i}^2 (E [I_{i0} (Q_i, R_i)]) = P(Y_{i0} \leq R_i + Q_i + 1) - P(Y_{i0} \leq R_i + 1) > 0. \quad (19)$$

Similarly, due to (14), the difference equations for  $E [X_{in} (Q_i, R_i)]$  can be expressed as

$$\begin{aligned} \Delta_{R_i} E [X_{in} (Q_i, R_i)] &= \frac{\lambda_{in}}{\lambda_{i0}} \left( -1 + \Delta_{R_i} E [I_{i0} (Q_i, R_i)] \right), \\ \Delta_{R_i}^2 E [X_{in} (Q_i, R_i)] &= \frac{\lambda_{in}}{\lambda_{i0}} \Delta_{R_i}^2 E [I_{i0} (Q_i, R_i)]. \end{aligned} \quad (20)$$

Note that from (19), (20) and  $\sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} < 1$ , we have

$$\Delta_{R_i}^2 E [I_{i0} (Q_i, R_i)] > \sum_{n \in N} \Delta_{R_i}^2 E [X_{in} (Q_i, R_i)] > 0. \quad (21)$$

By definition,

$$I_{in} (S_{in}, Q_i, R_i) = \begin{cases} S_{in} - X_{in} (Q_i, R_i) & \text{for } S_{in} - X_{in} (Q_i, R_i) > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

Then, it follows from (22) that for any  $n \in N$ ,

- $\Delta_{R_i} E [I_{in} (S_{in}, Q_i, R_i)] = \Delta_{R_i}^2 E [I_{in} (S_{in}, Q_i, R_i)] = 0$ , when  $S_{in} = 0$ ,
- $\Delta_{R_i} E [I_{in} (S_{in}, Q_i, R_i)] = -\Delta_{R_i} E [X_{in} (Q_i, R_i)]$ , and  $\Delta_{R_i}^2 E [I_{in} (S_{in}, Q_i, R_i)] = -\Delta_{R_i}^2 E [X_{in} (Q_i, R_i)]$ , when  $S_{in}$  approaches infinity.

Finally, using these results and (21), one can establish (16) for each cases.  $\square$

**Corollary 1.** For a given value of  $Q_i$ , as  $S_{in}$  approaches infinity for each  $n \in N$ , the optimal  $R_i$  for  $SP_i(\vec{\theta})$ , i.e.,  $R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$ , is given by

$$\text{Min}_{R_i \in \{-1, 0, 1, \dots\}} \left\{ R_i : \frac{1}{Q_i} \sum_{x=R_i+1}^{R_i+Q_i} P(Y_{i0} \leq x) \geq \frac{\theta_0 - c_i h \cdot \sum_{n=1}^{|N|} \frac{\lambda_{in}}{\lambda_{i0}}}{c_i h (1 - \sum_{n=1}^{|N|} \frac{\lambda_{in}}{\lambda_{i0}}) + \theta_0} \right\}. \quad (23)$$

Similarly, for a given value of  $Q_i$ , when  $S_{in}$  is set to zero for all  $n \in N$ , the optimal  $R_i$  for  $SP_i(\vec{\theta})$ , i.e.,  $R_i^*(\vec{S}_i = \vec{0}, Q_i)$ , is given by

$$\text{Min}_{R_i \in \{-1, 0, 1, \dots\}} \left\{ R_i : \frac{1}{Q_i} \sum_{x=R_i+1}^{R_i+Q_i} P(Y_{i0} \leq x) \geq \frac{\theta_0 + \sum_{n=1}^{|N|} \theta_n \frac{\lambda_{in}}{\lambda_{i0}}}{c_i h + \theta_0 + \sum_{n=1}^{|N|} \theta_n \frac{\lambda_{in}}{\lambda_{i0}}} \right\}. \quad (24)$$

*Proof.* For a given value of  $Q_i$ , and when  $S_{in}$  approaches infinity for each  $n \in N$ , by using (13), (15) and  $\lim_{S_{in} \rightarrow \infty} E [B_{in} (S_{in}, Q_i, R_i)] = 0$ , the objective function of problem  $SP_i(\vec{\theta})$  can be simplified as follows.

$$\begin{aligned} G(R_i) &= \left( c_i h \left( 1 - \sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} \right) + \theta_0 \right) E [I_{i0} (Q_i, R_i)] + \frac{\lambda_{i0} K_i}{Q_i} + c_i h \sum_{n \in N} \lim_{S_{in} \rightarrow \infty} S_{in} \\ &+ \left( \theta_0 - c_i h \sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} \right) \left( \lambda_{i0} T_{i0} - R_i - \frac{Q_i + 1}{2} \right). \end{aligned}$$

Then, by using (18), we have

$$\Delta_{R_i} G(R_i) = \left( c_i h \left( 1 - \sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} \right) + \theta_0 \right) \left( \frac{1}{Q_i} \sum_{k=R_i+1}^{R_i+Q_i} P(Y_{i0} \leq x) \right) + \left( c_i h \sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} - \theta_0 \right).$$

Finally, it follows from Theorem 1 that the smallest  $R_i$  that satisfies  $\Delta G(R_i) > 0$  is optimal, and applying this result yields  $R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$ .

Similarly, for a given value of  $Q_i$ , when  $\vec{S}_i = \vec{0}$ , by using equations (13) to (15) and  $E[I_{in}(S_{in} = 0, Q_i, R_i)] = 0$  in  $SP_i(\vec{\theta})$ , the relevant objective function can be stated as

$$\begin{aligned} G(R_i) &= \left( c_i h + \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) E[I_{i0}(Q_i, R_i)] + \sum_{n \in N} \theta_n \lambda_{in} T_{in} \\ &+ \frac{\lambda_{i0} K_i}{Q_i} + \left( \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \left( \lambda_{i0} T_{i0} - R_i - \frac{Q_i + 1}{2} \right). \end{aligned}$$

In a similar manner, by substituting (18) in  $\Delta G(R_i)$ , the smallest  $R_i$  that satisfies  $\Delta G(R_i) > 0$  gives  $R_i^*(\vec{S}_i = \vec{0}, Q_i)$ .  $\square$

**Proposition 3.**  $R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$  is a lower bound on the optimal  $R_i$  for a given value of  $Q_i$ .

*Proof.* Using equations (13) to (15), the objective function of problem  $SP_i(\vec{\theta})$  can be simplified as follows.

$$\begin{aligned} G(\vec{S}_i, R_i) &= \left( c_i h \left( 1 - \sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} \right) + \theta_0 \right) E[I_{i0}(Q_i, R_i)] + c_i h \sum_{n \in N} (S_{in} - \lambda_{in} T_{in}) + \frac{\lambda_{i0} K_i}{Q_i} \\ &+ \left( \theta_0 - c_i h \sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} \right) \left( \lambda_{i0} T_{i0} - R_i - \frac{Q_i + 1}{2} \right) + \sum_{n \in N} (c_i h + \theta_n) E[B_{in}(S_{in}, Q_i, R_i)]. \end{aligned}$$

Based on this result, in order to show that  $R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$  is a lower bound on the optimal  $R_i$  for a given value of  $Q_i$ , it is sufficient to show that for any  $R_i < R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$  and any  $\vec{S}_i$ ,  $\Delta G(\vec{S}_i, R_i) < 0$  is satisfied, i.e., any  $R_i$  such that  $R_i < R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$  cannot be optimal. Thus, first we use (18) to obtain

$$\begin{aligned} \Delta G_{R_i}(\vec{S}_i, R_i) &= \left( c_i h \left( 1 - \sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} \right) + \theta_0 \right) \frac{1}{Q_i} \sum_{k=R_i+1}^{R_i+Q_i} P(Y_{i0} \leq x) - \left( \theta_0 - c_i h \sum_{n \in N} \frac{\lambda_{in}}{\lambda_{i0}} \right) \\ &+ \sum_{n \in N} (c_i h + \theta_n) (E[B_{in}(S_{in}, Q_i, R_i + 1)] - E[B_{in}(S_{in}, Q_i, R_i)]). \end{aligned} \quad (25)$$

Next, it follows from Corollary 1 that  $R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$  is the smallest integer that satisfies the condition (23). Then, for any  $R_i < R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$  the condition (23) is not satisfied. Using this result and  $E[B_{in}(S_{in}, Q_i, R_i + 1)] \leq E[B_{in}(S_{in}, Q_i, R_i)]$  in (25), we obtain that for any  $Q_i$ ,  $R_i < R_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, Q_i)$  and  $S_{in}$ ,  $\Delta G(\vec{S}_i, R_i) < 0$ .  $\square$

**Proposition 4.**  $R_i^*(\vec{S}_i = \vec{0}, Q_i)$  is an upper bound on the optimal  $R_i$  for a given value of  $Q_i$ .

*Proof.* Similar to the proof of Proposition 3, in order to show that  $R_i^*(\vec{S}_i = \vec{0}, Q_i)$  is an upper bound on the optimal  $R_i$  for a given value of  $Q_i$ , it is sufficient to show that for any  $R_i > R_i^*(\vec{S}_i = \vec{0}, Q_i)$  and any  $S_{in}$ ,  $G(\vec{S}_i, R_i - 1) < 0$  is satisfied, i.e., any  $R_i$  such that  $R_i > R_i^*(\vec{S}_i = \vec{0}, Q_i)$  cannot be optimal. To do so, first, by using equations (13) to (15), the objective function of problem  $SP_i(\vec{\theta})$  can be rewritten as

$$\begin{aligned} G(\vec{S}_i, R_i) &= \left( c_i h + \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) E[I_{i0}(Q_i, R_i)] + \sum_{n \in N} (c_i h + \theta_n) E[I_{in}(S_{in}, Q_i, R_i)] \\ &+ \frac{\lambda_{i0} K_i}{Q_i} + \left( \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \left( \lambda_{i0} T_{i0} - R_i - \frac{Q_i + 1}{2} \right) + \sum_{n \in N} \theta_n (\lambda_{in} T_{in} - S_{in}). \end{aligned}$$

Then, by using (18), we have

$$\begin{aligned} \Delta G_{R_i}(\vec{S}_i, R_i - 1) &= \left( c_i h + \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \frac{1}{Q_i} \sum_{k=R_i}^{R_i+Q_i-1} P(Y_{i0} \leq x) - \left( \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) \\ &+ \sum_{n \in N} (c_i h + \theta_n) (E[I_{in}(S_{in}, Q_i, R_i)] - E[I_{in}(S_{in}, Q_i, R_i - 1)]). \end{aligned} \quad (26)$$

Note that from Corollary 1,  $R_i = R_i^*(\vec{S}_i = \vec{0}, Q_i)$  is known to be the smallest integer that satisfies the condition (24). Then, for any  $R_i > R_i^*(\vec{S}_i = \vec{0}, Q_i)$ , the condition (24) is satisfied even for  $R_i - 1$ . Using this result and  $E[I_{in}(S_{in}, Q_i, R_i)] \geq E[I_{in}(S_{in}, Q_i, R_i - 1)]$  in (26), we obtain that for any  $Q_i, R_i > R_i^*(\vec{S}_i = \vec{0}, Q_i)$  and  $S_{in}$ ,  $\Delta G_{R_i}(\vec{S}_i, R_i - 1) < 0$ .  $\square$

**Corollary 2.** As  $R_i$  and  $S_{in}$  approach infinity for each  $n \in N$ , the optimal  $Q_i$  for  $SP_i(\vec{\theta})$ , i.e.,  $Q_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, \lim_{R_i \rightarrow \infty} R_i)$ , is given by

$$\text{Min}_{Q_i \in \{1, 2, 3, \dots\}} \left\{ Q_i : (Q_i + 1) Q_i \geq \frac{2K_i \lambda_{i0}}{c_i h} \right\}. \quad (27)$$

*Proof.* The proof is similar to that of Corollary 1.  $\square$

**Proposition 5.**  $Q_i^*(\lim_{\vec{S}_i \rightarrow \vec{\infty}} \vec{S}_i, \lim_{R_i \rightarrow \infty} R_i)$  is a lower bound on the optimal  $Q_i$ .

*Proof.* The proof is similar to that of Proposition 3.  $\square$

**Proposition 6.**  $\sqrt{\frac{2K_i \lambda_{i0} + (c_i h + p) \lambda_{i0} T_{i0}}{H_i}}$ , where  $H_i = \frac{c_i h p}{c_i h + p}$ , and  $p = \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}}$ , is an upper bound on the optimal  $Q_i$ .

*Proof.* From (6) and (7) we have

$$E[B_{in}(S_{in}, Q_i, R_i)] = E[I_{in}(S_{in}, Q_i, R_i)] + \frac{\lambda_{in}}{\lambda_{i0}} E[B_{i0}(Q_i, R_i)] + \lambda_{in} T_{in} - S_{in}.$$

Using this result, the objective function of  $SP_i(\vec{\theta})$  can be rewritten as

$$\begin{aligned} G(\vec{S}_i, Q_i, R_i) &= c_i h E[I_{i0}(Q_i, R_i)] + \frac{\lambda_{i0} K_i}{Q_i} + \left( \theta_0 + \sum_{n \in N} \theta_n \frac{\lambda_{in}}{\lambda_{i0}} \right) E[B_{i0}(Q_i, R_i)] \\ &+ \sum_{n \in N} (c_i h + \theta_n) E[I_{in}(S_{in}, Q_i, R_i)] + \sum_{n \in N} \theta_n (\lambda_{in} T_{in} - S_{in}). \end{aligned} \quad (28)$$

In order to show that  $\sqrt{\frac{2K_i \lambda_{i0} + (c_i h + p) \lambda_{i0} T_{i0}}{H_i}}$  is an upper bound on the optimal  $Q_i$ , similar to the proof of Proposition 4, it is sufficient to show that  $G(\vec{S}_i, Q_i, R_i) > G(\vec{S}_i, Q_i - 1, R_i)$  for any  $Q_i > \sqrt{\frac{2K_i \lambda_{i0} + (c_i h + p) \lambda_{i0} T_{i0}}{H_i}}$ ,  $\vec{S}_i$  and  $R_i$ . Note that  $\sqrt{\frac{2K_i \lambda_{i0} + (c_i h + p) \lambda_{i0} T_{i0}}{H_i}}$  is an upper bound on the optimal  $Q_i$  for the objective function that consists of the first three terms in (28). Thus, we know that  $G(\vec{S}_i, Q_i, R_i) > G(\vec{S}_i, Q_i - 1, R_i)$  is guaranteed for the first three terms in (28). Using  $E[I_{in}(S_{in}, Q_i, R_i)] \geq E[I_{in}(S_{in}, Q_i - 1, R_i)]$ , and that  $\sum_{n \in N} \theta_n (\lambda_{in} T_{in} - S_{in})$  is constant with respect to  $Q_i$ , we have  $G(\vec{S}_i, Q_i, R_i) > G(\vec{S}_i, Q_i - 1, R_i)$ .  $\square$

Table 1: General Notation

$i$	Part index, $i \in I$
$n$	Warehouse index $n \in N \cup \{0\}$
$c_i$	Unit variable cost of part $i$
$h$	Inventory carrying charge
$K_i$	Fixed ordering cost of part $i$ at the central warehouse
$\lambda_{in}$	Annual demand rate for part $i$ at local warehouse $n \in N$
$\lambda_{i0}$	Annual demand rate (sum of internal and external) for part $i$ at the central warehouse
$\Lambda_n$	Total annual demand rate for warehouse $n \in N$
$T_{i0}$	Lead time for part $i$ at the central warehouse from the outside supplier
$T_{in}$	Transportation lead time from the central warehouse to local warehouse $n \in N$ for part $i$
$W_n^{\max}$	Target aggregate mean response time at warehouse $n \in N \cup \{0\}$
$R_i$	Reorder level for part $i$ at the central warehouse
$Q_i$	Order quantity for part $i$ at the central warehouse
$S_{in}$	Basestock level for part $i$ at local warehouse $n \in N$
$\vec{S}_i$	$[S_{i1}, S_{i2}, \dots, S_{i N }] =$ Vector of basestock levels for part $i$
$\vec{S}$	$[\vec{S}_1, \vec{S}_2, \dots, \vec{S}_{ I }] =$ Vector of basestock levels
$\vec{Q}$	$[Q_1, Q_2, \dots, Q_{ I }] =$ Vector of order quantities
$\vec{R}$	$[R_1, R_2, \dots, R_{ I }] =$ Vector of reorder levels
$I_{in}(t, Q_i, R_i, S_{in})$	On-hand inventory level for part $i$ at warehouse $n \in N$ at time $t$
$I_{i0}(t, Q_i, R_i)$	On-hand inventory level for part $i$ at the central warehouse at time $t$
$IP_{i0}(t, Q_i, R_i)$	Inventory position for part $i$ at the central warehouse at time $t$
$X_{in}(t, Q_i, R_i, S_{in})$	Number of outstanding orders for part $i$ at warehouse $n \in N$ at time $t$
$Y_{in}(t, t + \tau)$	Demand accumulated for part $i$ at warehouse $n \in N \cup \{0\}$ in time interval $(t, t + \tau)$
$B_{in}(t, Q_i, R_i, S_{in})$	Backorder level for part $i$ at warehouse $n \in N$ at time $t$
$B_{i0}(t, Q_i, R_i)$	Backorder level for part $i$ at the central warehouse at time $t$
$B_{i0}^{(n)}(t, Q_i, R_i)$	Backorder level of local warehouse $n$ for part $i$ at the central warehouse at time $t$
$W_{in}(t, Q_i, R_i, S_{in})$	Response time for part $i$ at warehouse $n \in N$ at time $t$
$W_{i0}(t, Q_i, R_i)$	Response time for part $i$ at the central warehouse at time $t$
$W_n(\vec{Q}, \vec{R}, \vec{S})$	Aggregate mean response time at warehouse $n \in N$ at time $t$
$W_0(\vec{Q}, \vec{R})$	Aggregate mean response time at the central warehouse at time $t$

Table 2: Parameter values for the testbed.

Factors	$\lambda_{in}$ (units/days)	$c_i$ (\$/unit)	$K_i$ (\$/order)	$h$ (per year)	$T_{i0}$ (days)	$W_0^{\max}$ (days)	$W_n^{\max}$ (days)	$W_n^{\max}$ (days)
Values	U[0.01, 0.05] × U[0.5, 1.5]	U[1000, 5000]	U[50, 150]	U[0.5, 1.5]	0.25	0.5	1	0.1

Table 3: The results of the experiment.

Parameters		Average CPU time	Average number of columns generated	Average number of branches generated	Number of instances solved within 1 hr (out of 10 instances)
Number of parts	Number of local warehouses				
5	2	3.84	441.70	105.60	10
6	2	8.36	770.70	176.20	10
7	2	34.87	2410.10	530.80	10
8	2	63.04	3907.80	833.40	10
9	2	99.48	6128.60	1318.60	10
10	2	169.19	8754.30	1865.40	10
11	2	210.40	10537.00	2221.20	10
12	2	255.89	11444.90	2430.00	10
13	2	611.36	22842.60	4914.60	9
5	3	11.86	923.10	197.80	10
6	3	26.15	1664.90	339.60	10
7	3	105.93	5688.70	1118.80	10
8	3	240.90	11325.30	2168.80	10
9	3	389.15	16250.00	3096.20	10
10	3	1212.15	46283.20	8832.60	10
11	3	1429.37	47138.50	9072.80	9
5	4	22.70	1213.70	238.20	10
6	4	82.10	3512.90	664.20	10
7	4	379.64	13837.80	2503.20	10
8	4	1190.03	38189.80	6775.40	9