VAN KAMPEN THEOREM FOR PERSISTENT FUNDAMENTAL GROUP

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ABSTRACT

VAN KAMPEN THEOREM FOR PERSISTENT FUNDAMENTAL GROUP

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Persistent homotopy is one of the newest algebraic topology methods in order to understand and capture topological features of discrete objects or point data clouds (the set of points with metric defined on it). On the other hand, in algebraic topology, the Van Kampen Theorem is a great tool to determine fundamental group of complicated spaces in terms of simpler subspaces whose fundamental groups are already known. In this thesis, we show that Van Kampen Theorem is still valid for the persistent fundamental group. Finally, we show that interleavings, a way to compare persistences, among subspaces imply interleavings among total spaces.

Keywords: Persistent Homotopy, Van Kampen Theorem, Fundamental Group, Interleaving.

ÖZ

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Kararlı homotopi ayrık nesnelerin veya noktasal veri bulutlarının (üzerinde bir metrik tanımlanmış noktalar kümesi) topolojik özelliklerini yakalamak ve anlamak için kullanılan en yeni cebirsel topoloji metotlarından biridir. Öte yandan, cebirsel topolojide, Van Kampen Teoremi bir uzayın temel grubunu temel grubu bilinen görece daha basit altuzaylarının yardımıyla belirlenmesini sağlayan iyi bir yöntemdir. Bu tezde, Van Kampen Teoremi'nin kararlı temel grubu içinde hala geçerli olduğunu ispatlayacağız. Son olarak, alt uzaylar arasındaki yakın değişimlerin, kararlılık dizilerinin karşılaştırmak için kullanılan bir yöntem, tüm uzaylar arasındaki yakın değişimlere neden olduğunu göstereceğiz.

Anahtar Kelimeler: Kararlı Homotopi, Van Kampen Teoremi, Temel Grup, Değişim.

To my family and friends

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1D	1 Dimensional
2D	2 Dimensional
3D	3 Dimensional

CHAPTER 1

INTRODUCTION

Persistent homology is one of the algebraic topology methods for capturing topological features of spaces such as components, holes and graph structure of a given data. By data we mean set of discrete points with a metric. We construct topological spaces out of set of data points by giving structure on it, for instance; cell complex, simplicial complex and $\check{C}$ech complex.

Persistent homology is used extensively in several fields such as statistics [3], protein structures [22], breast cancer [24] and structures of amorphous solids [18]. Morever, interest on persistent homology is increasing day by day in a varied applications of independent domains such as social network and sensor network analysis [8], classification of image processing [21] in addition to chemistry, biology, medicine and astrophysics which are mentioned above. Therefore, lots of researchers from different branches of science are now using persistent homology for their studies.

The persistence concept developed firstly in the study of Patrizio Frosini and his colleaques in 1990. They acquainted size theory, a formalism which is equivalent to 0-dimensional persistent homology [13]. In 2000, Letscher, Edelsbrunner and Zomorodian introduced persistent homology with other related concepts, for example; persistence diagram and algorithm [11]. In 2009, G. Carlsson has carried persistent homology to topological data analysis [5].

Furthermore, Letscher [20], introduced persistent homotopy groups and applied this concept for detecting if a complex is knotted and if the knotting can be undone in a larger complex. This was the first successful computation in homotopy theory by using persistent homotopy. Letscher defined persistent homotopy in a manner very similar to how persistent homology is a generalization of the usual homology.

Note that, homotopy groups and homology groups capture different information. For instance, one of the invariants we can attach to a knot is the fundamental group of its complement. If two knots are equivalent, then the corresponding fundamental groups are isomorphic. On the other hand, for any knot K in S^3, the 0-th and 1-st homology group of complement of K in S^3 is $\mathbb{Z}$ and the homology of complement K in S^3 is trivial for all higher dimensions. Consequently, homology group cannot distinguish knots and that is why it is not a good invariant for knot theory.

Knot theory is the study of closed curves in three dimensions, and their possible deformations without self intersections. The first question in knot theory is whether such a curve is truly knotted or one can deform it in space into a standard unknotted curve like a circle. The second question is whether any two given curves represent different knots or are really the same knot in the sense that one can be continuously deformed into the other.

However, homotopy groups are more difficult to work with compared to homology groups. Fortunately, The Van Kampen Theorem provides an easier way for computing fundamental groups.

Interleaving is a way of measuring the distance between persistence modules [6]. By studying at algebraic level directly, the authors in [6] have provided a mean of comparing the persistence diagrams of functions defined over different spaces by using the interleavings. For many persistence modules, this distance equals the bottleneck distance between the corresponding persistence diagrams. Interleavings and the resulting interleaving distance have been extensively studied both for the persistence modules [6] for Reeb graphs [9], for zig-zag persistence modules [2], for multiparameter persistence modules [19]. In the last section, we introduce interleaving notion for fundamental group and obtain similar stability results as in the general setting of persistence modules.

We use stability results for interleavings between fundamental groups and distances defined in terms of interleavings called interleaving distances. As mentioned above, interleavings are tools for quantifying the similarity between multidimensional filtrations and persistence modules. Interleaving distances are generalizations of the bottleneck distance to pseudometrics on multidimensional filtrations and on multidimensional persistence modules. The most basic type of interleavings, called δ-interleavings, were introduced for 1D persistence filtrations and persistence modules by Chazal [6]. This thesis uses a generalization of δ-interleavings given in [6] to be able to talk about δ-interleavings on fundamental groups. We also make use of interleaving distances in terms of these generalized δ-interleavings.

Note that in this thesis we always use integers as coefficient rings for computing homology groups, unless otherwise stated.

To sum up, in chapter 2, we will give some basic definitions from algebraic topology to be used in the following sections. In chapter 3, we will introduce persistent homotopy groups specifically persistent fundamental group and we will show that the Van Kampen Theorem can be applied to persistent fundamental group as well. In chapter 4, we will show that interleavings among subspaces imply interleavings among total spaces.

CHAPTER 2

BACKGROUND

In this chapter, we first give some basic definitions from algebraic topology. For details one can see [17] and also, [14]. Then, we roughly introduce persistent homology in Section 2.2. Finally, in the last section we briefly mention persistence module and persistence diagram. For much more details one can see [12, 10, 16, 5].

2.1 Some Basic Definitions in Algebraic Topology

In this section, we give some basic definitions such as simplicial complexes and we define basic concepts such as boundary, cycle and chain of a simplicial complex in order to define homology and then persistent homology or use them in the next sections when necessary.

Definition 1 *An* ***n-simplex****, denoted by* Δ^n*, is an* n*-dimensional polytope with convex hull determined by a collection of* $(n+1)$ *points in a Euclidean space* $\mathbb{R}^k$*. Intuitively, it is analogous to a triangle in each dimension n with* $(n+1)$ *vertices e.g. a triangle for* $n = 2$ *and a tetrahedron for* $n = 3$*. Formally, a standard* n*-simplex in* $\mathbb{R}^k$ *is defined as*

$$\{\sum_{i=0}^{n} t_i u_i : 0 \leq t_i \leq 1 \, \textit{for} \; i = 0, 1, \ldots, n \;\; \textit{and} \;\; \sum_{i=0}^{n} t_i = 1\}$$

where $u_0, u_1, \ldots, u_n$ *are geometrically independent points of* $\mathbb{R}^k$*, that is,* $\langle u_1 - u_0 \rangle$*,* $\langle u_2 - u_0 \rangle$*,. . .,* $\langle u_n - u_0 \rangle$ *are linearly independent vectors.*

Definition 2 *We say that the set of simplices* S *forms a* ***simplicial complex*** *if the following conditions hold:*

(i) if $\sigma \in S$ and τ is a face of σ, denoted by $\tau \leq \sigma$, then $\tau \in S$ and

(ii) if $\sigma_1, \sigma_2 \in S$ then $\sigma_1 \cap \sigma_2 = \emptyset$ or a face of both.

Definition 3 *A linear combination of n-simplices in a simplicial complex K is called an* ***n-chain****, which is shown by a finite formal sum $c = \sum_{i=0}^{k} a_i\sigma_i$, where each σ_i is an n-simplex and $a_i \in \mathbb{Z}$. Moreover, the n-chains form a free abelian group under addition $'+'$ where the identity is 0-chain as $0 = \sum_{i=0}^{k} 0\sigma_i$, and the inverse of a chain c is $-c = \sum_{i=0}^{k}(-a_i)\sigma_i$ since $c + (-c) = 0$ and we are working with $\mathbb{Z}$ coefficients. This group is called the n-th chain group which is denoted by $C_n := C_n(K)$. Also, this operation is associative and commutative since addition in $\mathbb{Z}$ is associative and commutative.*

Definition 4 *The sum of $(n-1)$-dimensional faces of an n-simplex is called its* ***boundary****. In other words, let $\sigma = [u_0,u_1,\ldots,u_n]$ be an n-simplex with the listed vertices and then* ***the boundary*** *of σ is given by*

$$\partial_n\sigma = \sum_{j=0}^{n}(-1)^j[u_0,\ldots,\hat{u}_j,\ldots,u_n].$$

Here, $\hat{u}_j$ component is omitted. If we have an n-chain $c = \sum_{i=0}^{k} a_i\sigma_i$, then boundary of c is defined as $\partial_n c = \sum_{i=0}^{k} a_i\partial_n\sigma_i$. Therefore, the boundary map can be written as $\partial_n : C_n \to C_{n-1}$.

Moreover, since the boundary map satisfies $\partial_n(c+c^{'}) = \partial_n c + \partial_n c^{'}$, we can conclude that boundary operator is a group homomorphism between the chain groups. We will therefore refer to ∂_n as the boundary homomorphism. The chain complex is the sequence of chain groups connected by boundary homomorphisms.

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots .$$

Definition 5 *If an n-chain $c = \sum_{i=0}^{k} a_i\sigma_i$ of a simplicial complex K has an empty boundary, i.e., $\partial c = 0$, then it is called an* ***n-cycle****. Note that n-cycle forms a subgroup of C_n which can be defined to be the kernel of the n-th boundary homomorphism denoted as $Z_n := Z_n(K) = \ker\partial_n$.*

Definition 6 *If an n-chain $c = \sum_{i=0}^{k} a_i\sigma_i$ of a simplicial complex K is a boundary of an $(n+1)$-chain $d \in C_{n+1}$, i.e., $c = \partial d$, then it is called an **n-boundary**. Note that n-boundaries form a subgroup of C_n which can be defined as the image of the $(n+1)$-st boundary homomorphism which is denoted by the following $B_n := B_n(K) = \operatorname{Im} \partial_{n+1}$.*

Now, we are ready to give the definition of the homology of a topological space on which we put a simplicial complex structure.

Definition 7 ***The n-th homology group** of a simplicial comlex K is defined to be the quotient group $H_n(K) = Z_n(K)/B_n(K)$. The rank of this group is called **the n-th Betti number** and it is denoted by $\beta_n = rank(H_n)$.*

We have mentioned about homology so far. Now it is time to give the definiton of homotopy between two continuous functions and homotopy equivalence between topological spaces.

Definition 8 *A **homotopy** H between two continuous functions $f, g : X \to Y$ is a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. In this case, we say f and g are homotopic and denoted by $f \simeq g$.*

Definition 9 *A map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. In this case, the spaces X and Y are said to be **homotopy equivalent** or they have **the same homotopy type** and it is denoted by $X \simeq Y$.*

Definition 10 *A **path** in a space X is a continuous function $f : I \to X$ where I is the unit interval $[0,1]$. If endpoints are the same i.e. $f(0) = x_0 = x_1 = f(1)$, then we say a **loop** instead of a path. In this case x_0 is called a basepoint for the loop. A homotopy of paths in X is a family of paths $f_t : I \to X$, $0 \le t \le 1$, such that the endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t and the associated function $F : I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous. The set of all homotopic loops $[f]$ in X at the basepoint x_0 is denoted by $\pi_1(X, x_0)$.*

*Moreover, one can see that $\pi_1(X, x_0)$ becomes a group under the product defined as the concatenation of loops. This group is called **fundamental group** or **first homotopy group** of X at the basepoint x_0. Furthermore, the fundamental group is independent of the basepoint if the space is path connected. In other words, by changing the basepoint of a path connected space, we get a new group which is isomorphic to the first one. Therefore, we will denote the fundamental group of X by $\pi_1(X)$ for path connected spaces, shortly.*

More generally, fundamental group of X at the basepoint x_0 can be considered as follows,

$$\pi_1(X, x_0) = \{\gamma : S^1 \to X \mid \gamma(s_0) = x_0\}/\simeq$$

where $s_0 \in S^1$. Similarly, higher homotopy groups of X at the basepoint x_0 can be considered as follows,

$$\pi_k(X, x_0) = \{\gamma : S^k \to X \mid \gamma(s_0) = x_0\}/\simeq$$

where $s_0 \in S^k$.

2.2 Persistent Homology

In this section, after the given definitions we will be ready to give the definition of persistent homology.

Definition 11 *[15] Any unordered sequence of points in an Euclidean n-dimensional space $\mathbb{R}^n$ can be considered as a **data set**.*

One type of data set for which global features are present and significant is the so-called point cloud data coming from physical objects in 2D or 3D.

Definition 12 *[15] Given a collection of points S in Euclidean space $\mathbb{R}^n$, **the Čech complex** C_ϵ, is a simplicial complex whose k-simplices are determined by unordered $(k+1)$-tuples of points $\{(x_0, x_1, x_2, ..., x_k) \in S : x_i \in \mathbb{R}\ for\ all\ i = 0, 1, 2, ..., k\}$ whose closed $\epsilon/2$-ball neighborhoods have a point of common intersection.*

Definition 13 *[15] Given a collection of points S in Euclidean space $\mathbb{R}^n$,* ***the Vietoris-Rips complex*** *R_ϵ, is a simplicial complex whose k-simplices are determined by unordered $(k+1)$-tuples of points $\{(x_0, x_1, x_2, ..., x_k) \in S : x_i \in \mathbb{R}\ for\ all\ i = 0, 1, 2, ..., k\}$ which are pairwise within distance ϵ.*

If we have X, a subset of $\mathbb{R}^n$, we can construct the Vietoris-Rips complex as follows: Let d denote the chosen metric and for any $\epsilon > 0$,

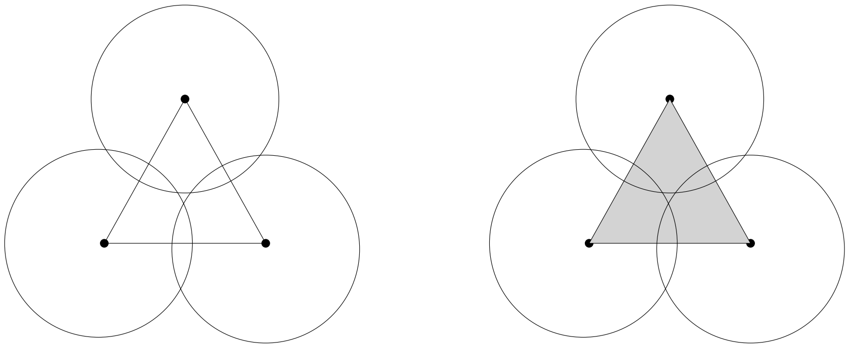
1. add a 0-simplex for each point in X,
2. for $x_1, x_2 \in X$, add a 1-simplex between x_1, x_2 if $d(x_1, x_2) \leq \epsilon$,
3. for $x_1, x_2, x_3 \in X$, add a 2-simplex with vertices x_1, x_2, x_3 if $d(x_1, x_2), d(x_1, x_3), d(x_2, x_3) \leq \epsilon$,

$$\vdots$$

n. for $x_1, x_2, \ldots, x_n \in X$, add an $(n-1)$-simplex with vertices $x_1, x_2, \ldots, x_n$ if $d(x_i, x_j) \leq \epsilon$ for $1 \leq i, j \leq n$; that is, if all the points are within a distance of ϵ from each other.

There is a difference between $\check{C}$ech and Vietoris-Rips complexes. For example, in Figure 2.1 on the same set of three vertices with distance of 1 unit between each other, the figure on the left is the $\check{C}$ech complex and the figure on the right is the Vietoris-Rips complex with $\epsilon = 1.1$. Note that, while the $\check{C}$ech complex is just a triangle (the triple intersection of the discs is empty), the Vietoris-Rips complex contains also a 2-simplex and hence, it is homotopy equivalent to a 2-disc.

Observe that, since X is sitting in some Euclidean space, a metric space in particular, the union of the balls with radius epsilon forms some topological space $X(\epsilon)$ that is in a sense an approximation of X. The Nerve Theorem [16] states that the $X(\epsilon)$ and C_ϵ have same homotopy type. Therefore, if we want to get information about point cloud, we can choose an ϵ and study the topology of the corresponding $\check{C}$ech complex C_ϵ. Despite this correspondence, it is really hard to compute $\check{C}$ech complexes. The reason is that, to determine the existence of an n-simplex one has to check all the subsets of



(a) $\check{C}$ech complex (b) Vietoris-Rips complex

Figure 2.1: The difference between $\check{C}$ech and Vietoris Rips complexes.

size n. If the dimension of the simplices increase then the required time will increase exponentially. Hence computing the entire space is exponentially proportional to the size of the space. To overcome this weakness of $\check{C}$ech complexes, one uses Vietoris-Rips complexes which is less time consuming.

Whether we use Vietoris-Rips or $\check{C}$ech complexes to construct global object from the point cloud data sets we need to make a choice of the parameter ϵ. If we choose ϵ too small then our complex will be just a discrete set of points, if we choose ϵ too large then we will have a single high dimensional simplex. It is the reason why we do not fix the value of ϵ. To capture the topology of data set we start with small ϵ values and increase the values of ϵ in small amount over the time. Note that, by the process above, we get a nested sequence of simplicial complexes.

Now, let us define the filtration of a simplicial complex to be used in persistent homology.

Definition 14 *Let K be a simplicial complex. A **filtration** is a sequence of sets*

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{n-1} \subseteq K_n = K$$

such that each K_i is a simplicial subcomplex of K and obtained by

$$K_i := f^{-1}((-\infty, i]) = \{x \in K : f(x) \leq i\}$$

*where $f\colon K \to \mathbb{R}$ is a filtering function. A simplicial complex K with a filtering function f is called **filtered space**. For any real number i, the preimage $f^{-1}(i)$ will* *give a* ***level set*** *that consists of all the elements of K having the height i. Then* ***the sublevel set*** *can be defined to be all the points of K having height less than or equal to i.*

Definition 15 *Let K be a filtered space. By applying the homology functor throughout the filtration, for $0 \leq u < v \leq n$,* ***the k-th persistent homology group*** *$H_k^{u,v}(K)$ is defined as the image of the homomorphism $i_k^{u,v}\colon H_k(K_u) \to H_k(K_v)$ induced by the inclusion map $i\colon K_u \hookrightarrow K_v$,*

$$H_k^{u,v}(K) := \operatorname{Im}(i_k^{u,v}).$$

Equivalently, the persistent homology groups consist of the homology classes of K_u that are still alive at K_v. Also, one can see another definition of persistent homology mentioned in [20].

Definition 16 *[20] For any k,* ***the k-th persistent homology group*** *can also be defined as*

$$H_k^p(K_u) := \frac{Z_k(K_u)}{B_k(K_{u+p}) \cap Z_k(K_u)}$$

where K_u is the u-th element of the filtration, and B_k and Z_k denote, k-boundary and k-cycles, respectively.

Equivalently, the persistent homology groups consist of the homology classes of K_u that are still alive at K_{u+p}. Moreover, note that $B_k(K_u) \subset B_k(K_{u+p})$ and the more p increases the more we take bigger quotients of cycles. This follows that persistent homology groups become smaller as p increases.

Persistence diagram is one of the basic invariants for understanding the topology of spaces or point cloud. For instance, the following spaces (see Figure 2.2) do not have the same persistence diagram although they have the same homotopy type and hence their all homology groups and homotopy groups are isomorphic. Their persistence diagrams, in terms of first homology group classes (1-cyles), are given in Figure 2.3.

Note that we have distinguished the spaces X and Y by their first persistence diagrams using the same filtering function and filtration levels. By using two different

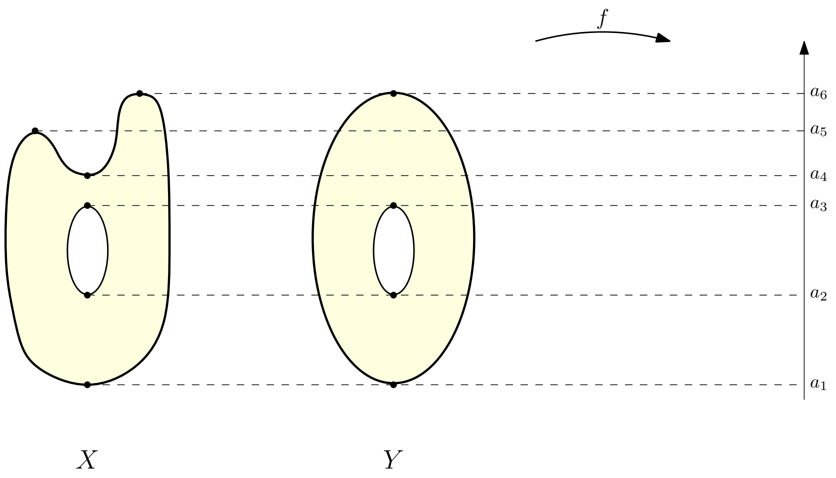


Figure 2.2: Filtration for topological spaces.

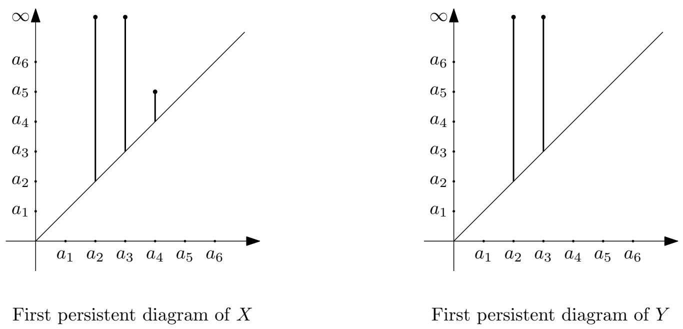


Figure 2.3: Persistence diagram of topological spaces.

filtering functions $f\colon K \to \mathbb{R}$ and $g\colon K \to \mathbb{R}$ on the same space K, one may have different persistence diagrams denoted by $Dgm_p(f)$ and $Dgm_p(g)$, where p stands for the dimesion of the diagram, respectively.

Now, one may ask if we use two filtering functions on the same space K where one of them is just a perturbation of the other then are their corresponding diagrams similar? The answer is yes under some conditions, but before giving the statement of stability theory for persistence digrams we will give some definitions.

Definition 17 *A real valued function $f\colon K \to \mathbb{R}$ is said to be **tame**, where f is a filtering function, if the homology groups of every sublevel set have finite ranks and there are only finitely many filtration levels i across which the homology groups are not isomorphic.*

Definition 18 *Let K be a topological space with two tame functions $f, g\colon K \to \mathbb{R}$. Given a bijection ι between two diagrams, we take the supremum L_∞-distance between matched points and define **the bottleneck distance** between persistence diagrams, denoted by $d_B(Dgm_p(f), Dgm_p(g))$ by taking the infimum over all supremums,*

$$d_B(Dgm_p(f), Dgm_p(g)) = \inf_{\iota} \sup_{u \in K} \{\, ||u - \iota(u)||_\infty \}.$$

Recall that L_∞-distance between two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is the following $||u-v||_\infty = max\{|u_1 - v_1|,\ |u_2 - v_2|\}$. Note that in the persistence diagrams we may have point at infinity which means the class never dies, so keep in mind that we are working on extended plane. Furthermore, in the bijection we mentioned above we matched the points at infinity and we assume that the L_∞-distance between these points is zero. Now, we are ready to state in what sense persistence is stable.

Theorem 2.2.1 *[10] Let K be a topological space with two tame functions $f\colon K \to \mathbb{R}$ and $g\colon K \to \mathbb{R}$. Then for each dimension p the bottleneck distance between the dimension p persistence diagrams is bounded from above by the difference between the functions,*

$$d_B(Dgm_p(f), Dgm_p(g)) \le ||f(k) - g(k)||_\infty.$$

Therefore, measuring the distance between functions by taking the supremum of the absolute difference between corresponding values,

$$||f - g||_\infty = sup\{|f(k) - g(k)| : k \in K\}$$

the stability theorem above states that if the filtering functions are close with respect to sup metric, then corresponding persistence diagrams are also closed (similar) to each other.

2.3 Persistence Module

In algebraic topology, we use abelian groups as a coefficient to define homology. Similarly, we can use abelian groups as a coefficient for persistent homology. Furthermore, we can define persistent homology with coefficients modules or rings because of the fact that persistent homology is defined by using the homomorphisms between the homology groups.

Zomorodian and Carlsson [29] used homology groups to construct a module structure over the polynomial ring $\mathbb{F}[x]$ when $\mathbb{F}$ is a field. Note that in this case $\mathbb{F}[x]$ is a principal ideal domain. Since we are working on principal ideal domain the classification of finitely generated modules over principle ideal domain is very simple that is if M is a finitely generated module then M can be represented as the direct sum of finitely generated free modules and torsion modules (see Theorem 2.3.1).

Now, let K be a filtered space such that $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{n-1} \subseteq K_n = K$.

Definition 19 *A **persistence complex** $\mathscr{C}$ is a family of chain complexes $\{C_*(K_i) : 0 \leq i \leq n\}$ over a commutative ring with unity R, together with chain maps $f^i : C_*(K_i) \to C_*(K_{i+1})$, so that we have the following diagram:*

$$C_*(K_0) \xrightarrow{f^0} C_*(K_1) \xrightarrow{f^1} C_*(K_2) \xrightarrow{f^2} \cdots \xrightarrow{f^{n-1}} C_*(K_n).$$

We expand a portion of the persistence complex below:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \big\downarrow \partial^{i}_{k+1} & & \big\downarrow \partial^{i+1}_{k+1} & & \big\downarrow \partial^{i+2}_{k+1} & & \\
\cdots & \xrightarrow{f^{i-1}} & C_k(K_i) & \xrightarrow{f^{i}} & C_k(K_{i+1}) & \xrightarrow{f^{i+1}} & C_k(K_{i+2}) & \xrightarrow{f^{i+2}} & \cdots \\
 & & \big\downarrow \partial^{i}_{k} & & \big\downarrow \partial^{i+1}_{k} & & \big\downarrow \partial^{i+2}_{k} & & \\
\cdots & \xrightarrow{f^{i-1}} & C_{k-1}(K_i) & \xrightarrow{f^{i}} & C_{k-1}(K_{i+1}) & \xrightarrow{f^{i+1}} & C_{k-1}(K_{i+2}) & \xrightarrow{f^{i+2}} & \cdots \\
 & & \big\downarrow \partial^{i}_{k-1} & & \big\downarrow \partial^{i+1}_{k-1} & & \big\downarrow \partial^{i+2}_{k-1} & & \\
\cdots & \xrightarrow{f^{i-1}} & C_{k-2}(K_i) & \xrightarrow{f^{i}} & C_{k-2}(K_{i+1}) & \xrightarrow{f^{i+1}} & C_{k-2}(K_{i+2}) & \xrightarrow{f^{i+2}} & \cdots \\
 & & \big\downarrow \partial^{i}_{k-2} & & \big\downarrow \partial^{i+1}_{k-2} & & \big\downarrow \partial^{i+2}_{k-2} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

In the above diagram, each column is a chain complex of K_i where i is the filtration level of K, and the chain maps f^i connect chain complexes of successively larger simplicial complexes in the filtration together so that diagram can be commutative.

Recall that a graded ring is a ring $< R, +, . >$ equipped with a direct sum decomposition of abelian groups R_i where $i \in \mathbb{Z}$, i.e. $R = \bigoplus_i R_i$. A graded module M over a graded ring R is a module equipped with a direct sum decomposition, $M = \bigoplus_i M_i$, where $i \in \mathbb{Z}$.

Definition 20 *[28]* ***The k-th persistence module*** *of K, $\mathcal{H}_k(K)$, is the family of k-th homology modules $H_k(K_i)$ with module homomorphisms $\mu^i_k : H_k(K_i) \to H_k(K_{i+1})$. A persistence module is said to be of finite type if each component module is finitely generated and there exists some integer m such that the maps μ^i_k are isomorphisms for all $i \geq m$.*

The k-th persistence module can be given the structure of a graded module over the polynomial ring $R[x]$:

$$\mathcal{H}_k(K) = \bigoplus_{i=0}^{n} H_k(K_i)$$

where the action of x is given by $x.(\sum_{i=0}^{n} m_i) = \sum_{i=0}^{n} \mu^i_k(m_i)$ for any $m_i \in H_k(K_i)$ [28].

In other words, the action of x shifts the grading upward by 1. By this action, the persistent module makes a connection between homologies of different complexes in the filtration. Hence, it contains all of the information needed for the p-persistent k-th homology modules $H_k^p(K_i)$ from the i-th filtration level.

Example 1 *The following is an example of action mentioned above.*

$$x^3.(m_0, m_1, m_2, ...) = (0, 0, 0, \mu_k^0(m_0), \mu_k^1(m_1), ...)$$

where $m_i \in H_k(K_i)$.

Likewise in algebra if we work on principal ideal domain we can also use structure theorem of modules for persistence modules.

Theorem 2.3.1 *[28] Suppose* $\mathcal{H}_k(K)$ *is over the polynomial ring* $\mathbb{F}[x]$, *where* $\mathbb{F}$ *is a field. Then*

$$\mathcal{H}_k(K) = (\bigoplus_i (x^{a_i})) \oplus (\bigoplus_j (x^{b_j}/x^{c_j}))$$

where the sums range over $1 \leq i \leq M$ *and* $1 \leq j \leq N$ *for non-negative integers* M, N *and* a_i, b_j, c_j *are non-negative integer powers of* x.

Indeed, the numbers a_i and b_j represent index in the filtration where k-dimensional homology generators are born at K_{a_i} and K_{b_j} and the number c_j represents the death of a k-dimensional homology generator which merges with an older k-dimensional homology generator at subcomplex K_{b_j}. While the free part of the above sum carries the information about essential classes which born at K_{a_i}, the torsion part gives the classes which born and die through the filtration.

CHAPTER 3

VAN KAMPEN THEOREM FOR PERSISTENT FUNDAMENTAL GROUP

In this chapter, in Section 3.1 we state the Van Kampen Theroem. For more details and the proof one can see Hatcher's Algebraic Topology Book [17, p. 43-46]. Futhermore, in Section 3.2, we define persistent fundamental group which is first defined by Letscher [20]. Finally, in Section 3.3 we state and prove Van Kampen Theorem in persistent fundamental group set up.

3.1 Van Kampen Theorem

Let $\varphi : (X, x_0) \to (Y, y_0)$ be a basepoint preserving map. Then φ induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, defined by composing loops $f : I \to X$ based at x_0 with φ, that is, $\varphi_*[f] = [\varphi \circ f]$. One can show that π_1 is a functor from the category of topological spaces to the category of groups. Moreover, if φ is a homotopy equivalence between X and Y, then the induced map φ_* is an isomorphism. In other words, if X and Y have the same homotopy type, then their fundemantal groups are isomorphic.

We use these facts to introduce Van Kampen Theorem which is used for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundemantal groups are already known.

Theorem 3.1.1 *[17] If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the induced homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective.*

If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is

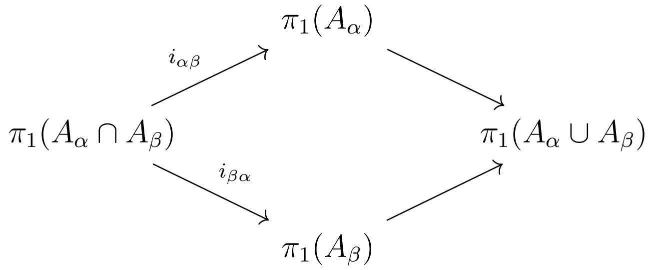


Figure 3.1: Commutative diagram for $A_\alpha \cup A_\beta$.

the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ *where* $w \in \pi_1(A_\alpha \cap A_\beta)$*, and so* Φ *induces an isomorphism, that is,* $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$*.*

3.2 Persistent Homotopy

Persistent homotopy generalizes homotopy groups in a similar way persistent homology generalizes homology. This concept was first introduced by Letscher [20].

Let K be a filtered space with basepoint $k_0 \in K$

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{n-1} \subseteq K_n = K.$$

Definition 21 *By applying the first homotopy functor throughout the filtration, for* $0 \leq u \leq v \leq n$, ***the persistent fundamental group*** *of* K, $\pi_1^{u,v}(K)$ *is defined as the image of the homomorphism* $i^{u,v}\colon \pi_1(K_u) \to \pi_1(K_v)$ *which is induced by the inclusion map* $i\colon K_u \hookrightarrow K_v$*,*

$$\pi_1^{u,v}(K) := \mathrm{Im}(i^{u,v}).$$

Equivalently, the persistent fundamental group consists of fundamental group elements of K_u that are still alive at K_v. Like persistent homology, persistent homotopy uses the generators in one space K_u along with the larger equivalence relation coming from K_{u+p}. Also, one can see another definition of persistent fundamental group mentioned in Letscher's article is the following.

Definition 22 *[20] **The persistent fundamental group** is defined for connected K_u as*

$$\pi_1^p(K_u) = \{\gamma : [0,1] \to K_u \mid \gamma(0) = \gamma(1) = k_0\}/\simeq$$

where $\gamma_0 \simeq \gamma_1$ if there exists a homotopy between the two curves in K_{u+p} that fixes $k_0 \in K_u$.

If K_i is not connected then one can calculate its persistent homotopy groups on each component separately. Similarly, we can define higher persistent homotopy groups.

Definition 23 *[20] **The k-th persistent homotopy group** is defined as*

$$\pi_k^p(K_i) = \{\gamma : S^k \to K_i \mid \gamma(s_0) = k_0\}/\simeq$$

where s_0 and k_0 are fixed point in S^k and K, respectively, and two maps are equivalent if there is a homotopy between them in K_{i+p} that fixes the image of s_0.

Theorem 3.2.1 *[20] Given a filtration K_i with each K_i connected, $\pi_1^p(K_i)$ satisfies the following properties*

1. *$\pi_1^p(K_i)$ is a group where multiplication is concatenation of curves.*
2. *For $k > 1$, $\pi_k^p(K_i)$ is an abelian group where multiplication is the usual multiplication in higher homotopy groups.*
3. *The map $\pi_k(K_i) \to \pi_k^p(K_i)$ that sends equivalence classes of maps from the sphere to K_i to their larger equivalence classes in K_{i+p} is a group homomorphism.*
4. *$\iota_* : \pi_k(K_i) \to \pi_k(K_{i+p})$ is the map induced by inclusion, then $\pi_k^p(K_i)$ is isomorphic to the image of $\iota_\star$.*
5. *If $\pi_k^p(K_i) = 0$ for $k \leq n-1$ ($n \geq 2$), then $H_k^p(K_i) = 0$ for $k \leq n-1$ and $H_n^p(K_i) \cong h(\pi_n^p(K_i))$ where $h : \pi_n(K_{i+p}) \to H_n(K_{i+p})$ is the Hurewicz map [23].*

The final statement implies that persistent homotopy carries all the information of the persistent homology group and uses the Hurewicz map. When $k = 1$, the map h abelianizes the fundamental group to obtain the first homology group.

Like homotopy groups, persistent homotopy groups can detect features that persistent homology cannot. In particular, homology is not capable of detecting knotting. However, if all of the K_i and their complements are assumed to be connected, then it is conjectured that persistent homotopy groups and the maps between them completely determine the filtration $K_1 \subset K_2 \subset ... \subset K_n$ up to isotopy. This is similar to the result in classical knot theory that any knot is characterized by the fundamental group of its complement up to mirror image.

Unfortunately, calculating the persistent homotopy groups can be very difficult in general. In particular, it can be difficult to write down a presentation for the group. Furthermore, even if we could find a nice presentation, the word problem and group triviality problems are undecidable for arbitrary groups. However, if we are working with submanifold of S^3, these problems are algorithmically decidable [25] but impractical.

3.3 Van Kampen Theorem For Persistent Fundamental Group

Let X be a based topological space that is decomposed as the union of path-connected open subsets A and B each of which contains the basepoint $x_0 \in X$. Let $\phi\colon X \to \mathbb{R}$ be a continuous function filtering X. For $u \in \mathbb{R}$, a sublevel set of X obtained by ϕ is defined as

$$X_u = \{x \in X \mid \phi(x) < u\}.$$

Since the restriction of ϕ also filters A and B, these sublevel sets can also be defined for A and B, and $X_u = A_u \cup B_u$. Note also that these sublevel sets provide a filtration of the corresponding spaces. If in addition for an arbitrary filtration level u, each intersection $A_u \cap B_u$ is path connected, then by Van Kampen Theorem (see for example [17]), the map

$$\phi_u\colon \pi_1(A_u) * \pi_1(B_u) \to \pi_1(X_u)$$

defined as the restriction of the filtering function to the u-th level of the filtration is surjective. Let i_{uA} and i_{uB} denote the maps induced by the inclusions from $A_u \cap B_u$ to A and B, respectively. The kernel of ϕ_u is the normal subgroup N_u generated by all elements of the form $i_{uA}(x)i_{uB}(x)^{-1}$ for $x \in \pi_1(A_u \cap B_u)$ and hence ϕ_u induces an isomorphism

$$\pi_1(X_u) \cong (\pi_1(A_u) * \pi_1(B_u))/N_u.$$

Consider the diagram in Figure 3.2 connecting the u- and v-th levels ($u < v$) of the filtration where the maps f_{uv}, g_{uv}, h_{uv} and k_{uv} are the homomorphisms induced by the inclusions.

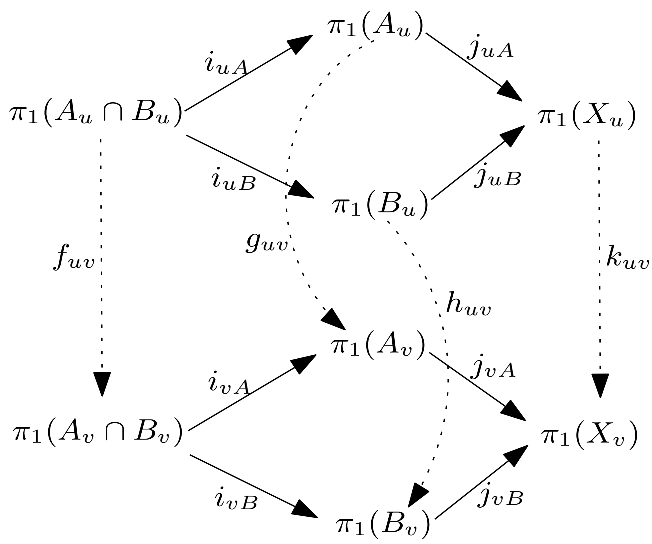


Figure 3.2: The homomorphisms f_{uv}, g_{uv}, h_{uv}, k_{uv} between fundamental groups which are induced by inclusion maps.

Remark 1 *Note that by Definition 21 we have, for every $u < v \in \mathbb{R}$,*

1. $\operatorname{Im} f_{uv} = \pi_1^{u,v}(A \cap B)$,
2. $\operatorname{Im} g_{uv} = \pi_1^{u,v}(A)$,
3. $\operatorname{Im} h_{uv} = \pi_1^{u,v}(B)$,
4. $\operatorname{Im} k_{uv} = \pi_1^{u,v}(X)$.

Now, consider the diagram of persistent fundamental groups in Figure 3.3, where $\alpha = i_{vA}|_{\operatorname{Im} f_{uv}}$, $\beta = i_{vB}|_{\operatorname{Im} f_{uv}}$, $\gamma = j_{vA}|_{\operatorname{Im} g_{uv}}$, and $\delta = j_{vB}|_{\operatorname{Im} h_{uv}}$.

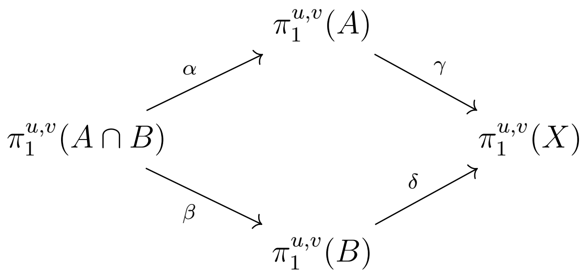


Figure 3.3: Commutative diagram for persistent fundamental group.

Let $\phi^{u,v} \colon \pi_1^{u,v}(A) * \pi_1^{u,v}(B) \to \pi_1^{u,v}(X)$ be the map such that

$$\phi^{u,v}(g_1 g_2) = \gamma(g_1)\delta(g_2)$$

with $g_1 \in \pi_1^{u,v}(A)$ and $g_2 \in \pi_1^{u,v}(B)$. That is, $\phi^{u,v}$ is the extension of the homomorphisms γ and δ.

Theorem 3.3.1 *The homomorphism $\phi^{u,v} \colon \pi_1^{u,v}(A) * \pi_1^{u,v}(B) \to \pi_1^{u,v}(X)$ is surjective.*

Proof 1 *Let $x \in \pi_1^{u,v}(X)$. Then, by definition $x \in \pi_1(X_v)$ and there exists $x_1 \in \pi_1(X_u)$ such that $x = k_{uv}(x_1)$. Since ϕ_u is surjective, by Van Kampen Theorem at level u, there exists $a_u \in \pi_1(A_u)$ and $b_u \in \pi_1(B_u)$ such that*

$$\phi_u(a_u b_u) = x_1 = j_{uA}(a_u) j_{uB}(b_u).$$

By Van Kampen Theorem and naturality of the diagram in Figure 3.2 is commutative, hence we have

$$k_{uv} \circ j_{uA} = j_{vA} \circ g_{uv}$$

$$k_{uv} \circ j_{uB} = j_{vB} \circ h_{uv}.$$

So, we have

$$\begin{aligned} x = k_{uv}(x_1) &= k_{uv}(j_{uA}(a_u) j_{uB}(b_u)) \\ &= k_{uv}(j_{uA}(a_u)) k_{uv}(j_{uB}(b_u)) \\ &= j_{vA}(g_{uv}(a_u)) j_{vB}(h_{uv}(b_u)) \\ &= j_{vA}(x_v) j_{vB}(y_v) \end{aligned}$$

where $x_v = g_{uv}(a_u) \in \mathrm{Im}(g_{uv})$ *and* $y_v = h_{uv}(b_u) \in \mathrm{Im}(h_{uv})$. *By definition of* γ *and* δ, $x = \gamma(x_v)\delta(y_v) = \phi^{u,v}(x_v y_v)$. *Hence,* $\phi^{u,v}$ *is surjective.*

Let $N_v = \{i_{vA}(w)i_{vB}(w)^{-1} |$ for all $w \in \pi_1(A_v \cap B_v)\}$. Let $p \in \pi_1(X_v)$, and $s_1 s_2$, $r_1 r_2$ be two factorizations of p, which are mapped to p by ϕ_v. Surjectivity of ϕ_v is equivalent to saying that every $p \in \pi_1(X_v)$ has a factorization. By the proof of Van Kampen Theorem for $X_v = A_v \cup B_v$, it is known that any two factorizations of p are equivalent. So any two factorizations of any element in $\pi_1(X_v)$ are equivalent.

Let

$$\begin{aligned} N_{uv} &= \{\alpha(w)\beta(w)^{-1} |\, for\ all\ w \in \pi_1^{u,v}(A \cap B)\} \\ &= \{\alpha(w)\beta(w)^{-1} |\, for\ all\ w \in \pi_1^{u,v}(A \cap B) = \mathrm{Im}(f_{uv}) \leq \pi_1(A_v \cap B_v)\} \\ &= \{i_{vA}(w)i_{vB}(w)^{-1} |\, for\ all\ w \in \mathrm{Im}(f_{uv}) = \pi_1^{u,v}(A \cap B)\} \end{aligned}$$

and

$$Q_{uv} :\ = (\pi_1^{u,v}(A) * \pi_1^{u,v}(B)) \diagup N_{uv},$$

where $\varphi_{uv} : Q_{uv} \to \pi_1^{u,v}(X)$ is defined by $\varphi_{uv}((s_1 s_2)N_{uv}) = \phi^{u,v}(s_1 s_2)$.
Let $p \in \pi_1^{u,v}(X) = \mathrm{Im}(k_{uv}) \leq \pi_1(X_v)$. If we can show that any two factorizations of p are equivalent, this will say that the map φ_{uv} induced by $\phi^{u,v}$ is injective, hence $\ker \phi^{u,v} = N_{uv}$.

Theorem 3.3.2 *The map* φ_{uv} *induced by* $\phi^{u,v}$ *is injective, that is,* $\ker \phi^{u,v} = N_{uv}$.

Proof 2 *Assume that there exists an element* $p \in \pi_1^{u,v}(X)$ *such that* p *has two factorizations* $s_1 s_2$ *and* $r_1 r_2$ *which are not equivalent. Since* $p \in \pi_1(X_v)$ *and* s_1, $r_1 \in \pi_1(A_v)$, s_2, $r_2 \in \pi_1(B_v)$, *then* $s_1 s_2 \in \pi_1(A_v) * \pi_1(B_v)$, $r_1 r_2 \in \pi_1(A_v) * \pi_1(B_v)$ *are two non-equivalent factorizations of* p *in* $\pi_1(X_v)$. *But this is not true by Van Kampen Theorem for* $\pi_1(X_v)$. *Therefore, our assumption is false and any two factorizations of* $p \in \pi_1^{u,v}(X)$ *are equivalent, and* φ_{uv} *is an injective map.*

Then we have $\ker \phi^{u,v} = N_{uv}$. *Since* $\phi^{u,v}$ *is surjective map and* $\ker \phi^{u,v} = N_{uv}$ *by first isomorphism theorem, we get*

$$(\pi_1^{u,v}(X) \cong \pi_1^{u,v}(A) * \pi_1^{u,v}(B)) \diagup N_{uv}.$$

Remark 2 *Note that in the previous Theorem 3.3.1 we have covered space X by two open path-connected spaces A and B that is why it is not necessary to check that the triple intersection of covering elements is path-connected. For those who use more than two elements for covering X must check that each triple intersection of covering elements is path-connected, otherwise the Van Kampen Theorem for persistent fundamental group may fail.*

CHAPTER 4

INTERLEAVING FOR FUNDAMENTAL GROUPS

The notion of interleaving gives a way of measuring the distance between persistence modules. The stability of persistence diagrams as in Theorem 2.2.1 requires that the space X is triangulable, f and g are continuous, and f and g are tame (they have finitely many critical values). Despite these restrictions, the stability result has many applications such as inferring local homology from sampled stratified spaces [1], quantifying homology classes [7] and persistence-sensitive simplification functions on 2-manifolds [12]. However these additional conditions, although reasonable for practical applications are not always satisfied in theory. Moreover, the fact that the functions f and g defined over the same space is a strong restriction. There are cases requiring to compare persistence diagrams defined over different spaces. For example in [26, 27] the persistence diagrams for compact geodesic spaces and their finite sample points are compared. Because of these reasons new stability results that do not suffer from the above limitations are introduced.

4.1 Interleaving

By working at the algebraic level directly, the authors in [6] have provided a way of comparing the persistence diagrams of functions defined over different spaces by using interleavings. Interleavings and the resulting interleaving distance have been extensively studied both for the persistence modules [6] for Reeb graphs [9], for zigzag persistence modules [2] and for multiparameter persistence modules [19].

Let X be a filtered space

$$\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_u \subseteq \ldots \subseteq X_n = X.$$

By applying either the homology functor or the homotopy functor we obtain an induced filtration of X consisting of either abelian groups (in the homology group case) with homomorphisms between homology groups or just groups (in the fundamental group case) with homomorphisms between fundamental groups. Throughout this section we will refer to this induced filtration as **persistence** which will be denoted by $\{H_k(X_u)\}$ and $\{\pi_k(X_u)\}$ for persistent homology groups and persistent homotopy groups, respectively [26]. Since our essential study is about fundamental groups we will use $\{\pi_1(X_u)\}$ notation from now on.

Due to computational difficulties of persistent homotopy groups (mentioned in introduction), the stability theory of persistent homotopy groups is not easy when we compared to the stability of persistent homology. One of the main goals of this thesis is to show that being δ-interleaved between fundamental group of subspaces of two different topological spaces implies being δ-interleaved between fundamental groups of total spaces. Hence, thanks to the Van Kampen Theorem 3.1.1 one can understand the similarity between two complicated topological spaces by using similarity of subspaces (easy to compute fundamental groups) throughout the filtration levels.

Let X be a based topological space that is decomposed as a union of path-connected open subsets A and B each of which contains the basepoint $x_0 \in X$. Let

$$\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_u \subseteq \ldots \subseteq X_n = X$$

be a filtration of X with $X_u = A_u \cup B_u$ for all $u \geq 0$.

Let X' be another topological space decomposed as a union of path-connected open subsets A' and B' with the filtration

$$\emptyset = X'_0 \subseteq X'_1 \subseteq \ldots \subseteq X'_u \subseteq \ldots \subseteq X'_n = X'$$

with $X'_u = A'_u \cup B'_u$ for all $u \geq 0$.

In this section, we prove that if the persistence $\{\pi_1(A_u)\}$ (in the sense of [26, 27]) is δ-interleaved with the persistence $\{\pi_1(A'_u)\}$ and if the persistence $\{\pi_1(B_u)\}$ is δ-interleaved with the persistence $\{\pi_1(B'_u)\}$ where $\delta > 0$, then the persistence $\{\pi_1(X_u)\}$ is δ-interleaved with the persistence $\{\pi_1(X'_u)\}$. This may allow one to compare persistence diagrams on different spaces through their subspaces by applying Van Kampen Theorem for each level of filtration.

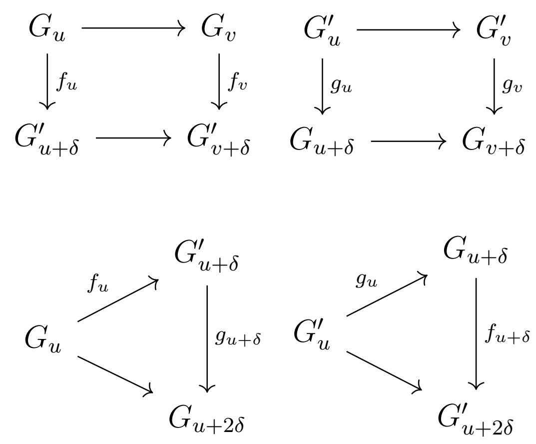


Figure 4.1: Commutative diagrams of δ-interleaved persistences $\{G_u\}$ and $\{G'_u\}$.

Now, let us begin with the general definition of δ-interleaving of any persistence.

Definition 24 *[27] A* ***δ-interleaving*** *between two different persistences $\{G_u\}$ and $\{G'_u\}$ is a collection of homomorphisms $f_u : G_u \to G'_{u+\delta}$ and $g_u : G'_u \to G_{u+\delta}$ for all u such that the diagrams in Figure 4.1 commute for all $u \leq v$ where the horizontal maps are induced by the inclusions.*

Also, the homomorphisms f_u and g_u in the definition are called **bonding maps**. Note that Definiton 24 is a general definition for any persistence mentioned above we will also use this definition in the case of fundamental groups.

Remark 3 *[19] If $\{G_u\}$ and $\{G'_u\}$ are δ-interleaved, then they are also ϵ-interleaved for any $\epsilon > \delta$.*

For the proof, it is enough to show that the corresponding four diagrams in Figure 4.1 commute for any $\epsilon > \delta$.

As a result one can define the interleaving distance between two persistences as follows.

Definition 25 *[4] Let $\{G_u\}$ and $\{G'_u\}$ be two persistences. Then* ***the interleaving distance*** *$d_I(\{G_u\}, \{G'_u\})$ between $\{G_u\}$ and $\{G'_u\}$ is defined as*

$$d_I(\{G_u\}, \{G'_u\}) = \inf\{\delta \in [0, \infty] : \{G_u\} \ and \ \{G'_u\} \ are \ \delta - interleaved\}.$$

Note that if no such δ exists, we accept that $d_I(\{G_u\}, \{G'_u\}) = \infty$ as a convention.

An isomorphism h between persistences $\{G_u\}$ and $\{G'_u\}$ is a collection of isomorphisms $h_u : G_u \to G'_u$ which commutes with the corresponding bonding maps. The isomorphism is denoted by $\{G_u\} \cong \{G'_u\}$

Remark 4 *[19] Two persistences are 0-interleaved if and only if they are isomorphic. Moreover, if persistences $\{G_u\}$ and $\{G'_u\}$ are ϵ-interleaved and $\{G'_u\}$ and $\{G''_u\}$ are δ-interleaved, then $\{G_u\}$ and $\{G''_u\}$ are $(\epsilon + \delta)$-interleaved.*

4.2 Interleaving For Fundamental Groups

Since all homology groups and homotopy groups (except the fundamental group) are abelian, one can consider them as module. Moreover, if we work with field coefficients we can use the structure theorem (see Theorem 2.3.1). In this case one can put two filtrations on this space and can sketch their persistence diagrams. The distance between these persistence diagrams is so called the bottleneck distance. We know that this distance is less than the distance between filtering functions of the space (sup metric). For details see Section 2.2 and Theorem 2.2.1. Also, the distance between persistence diagrams (bottleneck distance) can bounded by interleaving distance between persistences defined above rather than the distance between the two filtering functions.

Now, let $\{G_u\}$ and $\{G'_u\}$ be two persistences where G_u and G'_u are vector spaces for all u. If they are δ-interleaved (see Definition 24), then

$$d_B(\{G_u\}, \{G'_u\}) \leq \delta. \text{ [27]}$$

However, since fundamental group of a space is not always an abelian group the stability we mentioned in Section 2.2 is not applicable. Fortunately, thanks to Van Kampen Theorem, we will be able to understand the similarity of two complicated spaces by using their subspaces which are much easier to compute compared to the space itself. Therefore, our aim is to show that interleaving among subspaces implies interleaving among total spaces by using Van Kampen Theorem for each level.

Now, since we have the definition of δ-interleaving of any persistences we are ready to establish our results about the interleaving of fundamental groups of two different topological spaces X and X'.

Let X be a based topological space that is decomposed as a union of path-connected open subsets A and B each of which contains the basepoint $x_0 \in X$. Let

$$\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_u \subseteq \ldots \subseteq X_n = X$$

be a filtration of X with $X_u = A_u \cup B_u$ for all $u \geq 0$.

Let X' be another topological space decomposed as a union of path-connected open subsets A' and B' with the filtration

$$\emptyset = X'_0 \subseteq X'_1 \subseteq \ldots \subseteq X'_u \subseteq \ldots \subseteq X'_n = X'$$

with $X'_u = A'_u \cup B'_u$ for all $u \geq 0$.

Theorem 4.2.1 *If $\{\pi_1(A_u)\}$ is δ-interleaved with $\{\pi_1(A'_u)\}$ and $\{\pi_1(B_u)\}$ is δ - interleaved with $\{\pi_1(B'_u)\}$ for $\delta > 0$, then $\{(\pi_1(A_u) * \pi_1(B_u))/N_u\}$ is δ-interleaved with $\{(\pi_1(A'_u) * \pi_1(B'_u))/N'_u\}$.*

Remark 5 *Note that for the simplicity in Proof 3, we abuse the notation such that the word $a_1 b_1 \ldots a_n b_n$ may be notated as $a_u b_u$.*

Proof 3 *Assume $\{\pi_1(A_u)\}$ is δ-interleaved with $\{\pi_1(A'_u)\}$, then there exists a collection of homomorphisms $m_u : \pi_1(A_u) \to \pi_1(A'_{u+\delta})$ and $n_u : \pi_1(A'_u) \to \pi_1(A_{u+\delta})$ for all u such that the diagrams in Figure 4.2 commute,*

where g_{uv}, $g_{uv+\delta}$, $g_{uv+2\delta}$, g'_{uv}, $g'_{uv+\delta}$ and $g'_{uv+2\delta}$ are the homomorphisms induced by the inclusion maps. Thus, we have

$$m_v \circ g_{uv} = (g'_{uv+\delta}) \circ m_u, \tag{41}$$

$$g_{uv+2\delta} = (n_{u+\delta}) \circ m_u, \tag{42}$$

$$n_v \circ g'_{uv} = (g_{uv+\delta}) \circ n_u, \tag{43}$$

$$g'_{uv+2\delta} = (m_{u+\delta}) \circ n_u. \tag{44}$$

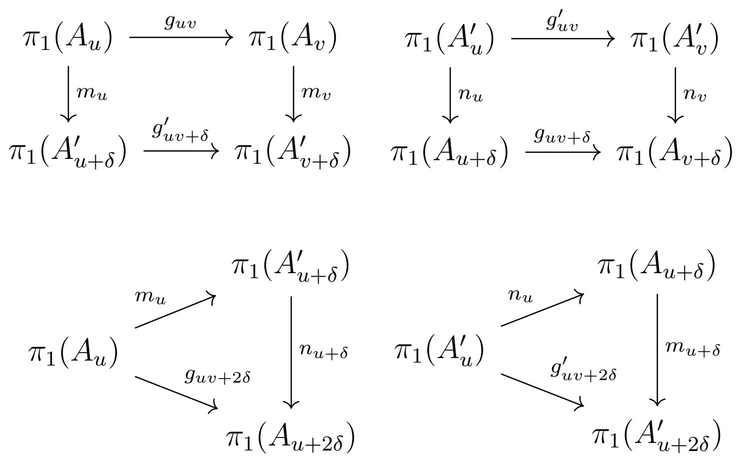


Figure 4.2: Commutative diagrams of δ-interleaved persistences $\{\pi_1(A_u)\}$ and $\{\pi_1(A'_u)\}$.

Similarly, there is a collection of homomorphisms for all u*,* $s_u : \pi_1(B_u) \to \pi_1(B'_{u+\delta})$ *and* $k_u : \pi_1(B'_u) \to \pi_1(B_{u+\delta})$ *such that the diagrams in Figure 4.3 commute,*

where h_{uv}*,* $h_{uv+\delta}$*,* $h_{uv+2\delta}$*,* h'_{uv}*,* $h'_{uv+\delta}$ *and* $h'_{uv+2\delta}$ *are the homomorphisms induced by the inclusions. So, we have*

$$s_v \circ h_{uv} = (h'_{uv+\delta}) \circ s_u, \tag{45}$$

$$h_{uv+2\delta} = (k_{u+\delta}) \circ s_u, \tag{46}$$

$$k_v \circ h'_{uv} = (h_{uv+\delta}) \circ k_u, \tag{47}$$

$$h'_{uv+2\delta} = (s_{u+\delta}) \circ k_u. \tag{48}$$

Now, let

$$t_u : \pi_1(A_u) * \pi_1(B_u) \to \pi_1(A_v) * \pi_1(B_v),$$

$$y_u : (\pi_1(A_u) * \pi_1(B_u))/N_u \to \pi_1(A_u) * \pi_1(B_u)$$

and

$$z_v : \pi_1(A_v) * \pi_1(B_v) \to (\pi_1(A_v) * \pi_1(B_v))/N_v$$

be the maps defined as

$$t_u(a_1 b_1 \dots a_n b_n) = g_{uv}(a_1) h_{uv}(b_1) \dots g_{uv}(a_n) h_{uv}(b_n)$$

$$
\begin{array}{ccc}
\pi_1(B_u) & \xrightarrow{h_{uv}} & \pi_1(B_v) \\
\downarrow{\scriptstyle s_u} & & \downarrow{\scriptstyle s_v} \\
\pi_1(B'_{u+\delta}) & \xrightarrow{h'_{uv+\delta}} & \pi_1(B'_{v+\delta})
\end{array}
\qquad
\begin{array}{ccc}
\pi_1(B'_u) & \xrightarrow{h'_{uv}} & \pi_1(B'_v) \\
\downarrow{\scriptstyle k_u} & & \downarrow{\scriptstyle k_v} \\
\pi_1(B_{u+\delta}) & \xrightarrow{h_{uv+\delta}} & \pi_1(B_{v+\delta})
\end{array}
$$

$$
\begin{array}{ccc}
 & & \pi_1(B'_{u+\delta}) \\
 & \nearrow^{s_u} & \downarrow{\scriptstyle k_{u+\delta}} \\
\pi_1(B_u) & \searrow_{h_{uv+2\delta}} & \\
 & & \pi_1(B_{u+2\delta})
\end{array}
\qquad
\begin{array}{ccc}
 & & \pi_1(B_{u+\delta}) \\
 & \nearrow^{k_u} & \downarrow{\scriptstyle s_{u+\delta}} \\
\pi_1(B'_u) & \searrow_{h'_{uv+2\delta}} & \\
 & & \pi_1(B'_{u+2\delta})
\end{array}
$$

Figure 4.3: Commutative diagrams of δ-interleaved persistences $\{\pi_1(B_u)\}$ and $\{\pi_1(B'_u)\}$.

and

$$y_u((ab)N_u) = ab, z_v(ab) = (ab)N_v,$$

respectively, for all u, v. *These maps are homomorphisms by their definitions.*

Moreover, the map

$$\sigma_u = z_v \circ t_u \circ y_u : (\pi_1(A_u) * \pi_1(B_u))/N_u \to (\pi_1(A_v) * \pi_1(B_v))/N_v$$

defined as

$$\sigma_u((ab)N_u) = (z_v \circ t_u \circ y_u)((ab)N_u) = (g_{uv}(a)h_{uv}(b))N_v$$

is a homomorphism.

Similarly, let

$$t'_u : \pi_1(A'_u) * \pi_1(B'_u) \to \pi_1(A'_v) * \pi_1(B'_v),$$

$$y'_u : (\pi_1(A'_u) * \pi_1(B'_u))/N'_u \to \pi_1(A'_u) * \pi_1(B'_u)$$

and

$$z'_u : \pi_1(A'_v) * \pi_1(B'_v) \to (\pi_1(A'_v) * \pi_1(B'_v))/N'_v$$

be maps defined as

$$t'_u(a'_1 b'_1 \dots a'_n b'_n) = g'_{uv}(a'_1)h'_{uv}(b'_1) \dots g'_{uv}(a'_n)h'_{uv}(b'_n)$$

and

$$y'_u((a'b')N'_u) = a'b', z'_v(a'b') = (a'b')N'_v,$$

respectively for all u, v. *By definition, these maps are homomorphisms and thus the map* $\alpha_u = z'_v \circ t'_u \circ y'_u : (\pi_1(A'_u) * \pi_1(B'_u))/N'_u \to (\pi_1(A'_v) * \pi_1(B'_v))/N'_v$ *defined as*

$$\alpha_u((a'b')N'_u) = (z'_v \circ t'_u \circ y'_u)((a'b')N'_u) = (g'_{uv}(a')h'_{uv}(b'))N'_v$$

is a homomorphism.

The map

$$p_u : (\pi_1(A_u) * \pi_1(B_u))/N_u \to (\pi_1(A'_{u+\delta}) * \pi_1(B'_{u+\delta}))/N'_{u+\delta}$$

defined as

$$p_u((a_1b_1)N_u \ldots (a_nb_n)N_u) = (m_u(a_1)s_u(b_1) \ldots m_u(a_n)s_u(b_n))N'_{u+\delta}$$

is a homomorphism for all u. *Similarly, the map*

$$q_u : (\pi_1(A'_u) * \pi_1(B'_u))/N'_u \to (\pi_1(A_{u+\delta}) * \pi_1(B_{u+\delta}))/N_{u+\delta}$$

defined as

$$q_u((a'_1b'_1)N'_u \ldots (a'_nb'_n)N'_u) = (n_u(a'_1)k_u(b'_1) \ldots n_u(a'_n)k_u(b'_n))N_{u+\delta}$$

is also a homomorphism for all u.

Now, we show that the topmost diagram in Figure 4.4 commutes:

Let $(a_ub_u)N_u \in (\pi_1(A_u) * \pi_1(B_u))/N_u$. *Then*

$$\begin{aligned}
(p_v \circ \sigma_u)((a_ub_u)N_u) &= p_v(\sigma_u((a_ub_u)N_u)) \\
&= p_v((g_{uv}(a_u)h_{uv}(b_u))N_v) \\
&= p_v((g_{uv}(a_u)N_v)(h_{uv}(b_u)N_v)) \\
&= (m_v(g_{uv}(a_u))s_v(h_{uv}(b_u)))N'_{v+\delta} \\
&= ((m_v \circ g_{uv})(a_u)(s_v \circ h_{uv})(b_u))N'_{v+\delta}.
\end{aligned}$$

$$
\begin{array}{ccc}
(\pi_1(A_u) * \pi_1(B_u))/N_u & \xrightarrow{\sigma_u} & (\pi_1(A_v) * \pi_1(B_v))/N_v \\
\downarrow{\scriptstyle p_u} & & \downarrow{\scriptstyle p_v} \\
(\pi_1(A'_{u+\delta}) * \pi_1(B'_{u+\delta}))/N'_{u+\delta} & \xrightarrow{\alpha_{u+\delta}} & (\pi_1(A'_{v+\delta}) * \pi_1(B'_{v+\delta}))/N'_{v+\delta}
\end{array}
$$

$$
\begin{array}{ccc}
(\pi_1(A'_u) * \pi_1(B'_u))/N'_u & \xrightarrow{\alpha_u} & (\pi_1(A'_v) * \pi_1(B'_v))/N'_v \\
\downarrow{\scriptstyle q_u} & & \downarrow{\scriptstyle q_v} \\
(\pi_1(A_{u+\delta}) * \pi_1(B_{u+\delta}))/N_{u+\delta} & \xrightarrow{\sigma_{u+\delta}} & (\pi_1(A_{v+\delta}) * \pi_1(B_{v+\delta}))/N_{v+\delta}
\end{array}
$$

$$
\begin{array}{ccc}
 & & (\pi_1(A'_{u+\delta}) * \pi_1(B'_{u+\delta}))/N'_{u+\delta} \\
 & \overset{p_u}{\nearrow} & \\
(\pi_1(A_u) * \pi_1(B_u))/N_u & & \downarrow{\scriptstyle q_{u+\delta}} \\
 & \underset{\sigma_u}{\searrow} & \\
 & & (\pi_1(A_{u+2\delta}) * \pi_1(B_{u+2\delta}))/N_{u+2\delta}
\end{array}
$$

$$
\begin{array}{ccc}
 & & (\pi_1(A_{u+\delta}) * \pi_1(B_{u+\delta}))/N_{u+\delta} \\
 & \overset{q_u}{\nearrow} & \\
(\pi_1(A'_u) * \pi_1(B'_u))/N'_u & & \downarrow{\scriptstyle p_{u+\delta}} \\
 & \underset{\alpha_u}{\searrow} & \\
 & & (\pi_1(A'_{u+2\delta}) * \pi_1(B'_{u+2\delta}))/N'_{u+2\delta}
\end{array}
$$

Figure 4.4: Commutative diagrams of δ-interleaved persistences $\{(\pi_1(A_u) * \pi_1(B_u))/N_u\}$ and $\{(\pi_1(A'_u) * \pi_1(B'_u))/N'_u\}$.

and

$$
\begin{aligned}
(\alpha_{u+\delta} \circ p_u)((a_u b_u)N_u) &= \alpha_{u+\delta}(p_u((a_u b_u)N_u)) \\
&= \alpha_{u+\delta}((m_u(a_u)s_u(b_u))N'_{u+\delta}) \\
&= (g'_{uv+\delta}(m_u(a_u))h'_{uv+\delta}(s_u(b_u))N'_{v+\delta} \\
&= ((g'_{uv+\delta} \circ m_u)(a_u)(h'_{uv+\delta} \circ s_u)(b_u))N'_{v+\delta}.
\end{aligned}
$$

Since $m_v \circ g_{uv} = (g'_{uv+\delta}) \circ m_u$ *by equality* 41 *and* $s_v \circ h_{uv} = (h'_{uv+\delta}) \circ s_u$ *by equality* 45*, we get* $p_v \circ \sigma_u = \alpha_{u+\delta} \circ p_u$*. Thus the topmost diagram in Figure 4.4 is commutative.*

Similarly, one can easily show that the other diagrams in Figure 4.4 are commutative.

Therefore, we have shown that if $\{\pi_1(A_u)\}$ *is* δ*-interleaved with* $\{\pi_1(A'_u)\}$ *and if* $\{\pi_1(B_u)\}$ *is* δ*-interleaved with* $\{\pi_1(B'_u)\}$*, then* $\{(\pi_1(A_u) * \pi_1(B_u))/N_u\}$ *is* δ*- interleaved with* $\{(\pi_1(A'_u) * \pi_1(B'_u))/N'_u\}$*.*

Note that $(\pi_1(A_u) * \pi_1(B_u))/N_u$ is isomorphic to $\pi_1(X_u)$ for each u, by Van Kampen Theorem.

Remark 6 *Moreover, if persistences* $\{\pi_1(A_u)\}$ *and* $\{\pi_1(A'_u)\}$ *are* ϵ*-interleaved and* $\{\pi_1(B_u)\}$ *and* $\{\pi_1(B'_u)\}$ *are* δ*-interleaved, then* $\{\pi_1(X_u)\}$ *and* $\{\pi_1(X'_u)\}$ *are* τ*-interleaved where* $\tau = max\{\epsilon, \delta\}$*.*

Remark 6 is an immediate consequence of Remark 3. Since without loss of generality if $\epsilon > \delta$ then by Remark 3 $\{\pi_1(B_u)\}_{u>0}$ and $\{\pi_1(B'_u)\}$ are also ϵ-interleaved so by Theorem 4.2.1, $\{\pi_1(X_u)\}$ and $\{\pi_1(X'_u)\}$ are ϵ-interleaved. Therefore, one can generalize the statement of Theorem 4.2.1 to different interleaving values.

Theorem 4.2.2 *The persistence* $\{\pi_1(X_u)\}$ *(*$\{\pi_1(X'_u)\}$*) is isomorphic (*0*-interleaved) to the persistence* $\{(\pi_1(A_u) * \pi_1(B_u))/N_u\}$ *(*$\{(\pi_1(A'_u) * \pi_1(B'_u))/N'_u\}$*).*

Proof 4 *It is enough to show that the persistence* $\{\pi_1(X_u)\}$ *is isomorphic (0- interleaved) to the persistence* $\{(\pi_1(A_u) * \pi_1(B_u))/N_u\}$*. The isomorphism between* $\{\pi_1(X'_u)\}$ *and* $\{(\pi_1(A'_u) * \pi_1(B'_u))/N'_u\}$ *can be seen similarly. We only show that*

$$
\begin{array}{ccc}
\pi_1(X_u) & \xrightarrow{k_{uv}} & \pi_1(X_v) \\
\downarrow{\scriptstyle r_u} & & \downarrow{\scriptstyle r_v} \\
(\pi_1(A_u) * \pi_1(B_u))/N_u & \xrightarrow{\sigma_u} & (\pi_1(A_v) * \pi_1(B_v))/N_v
\end{array}
$$

$$
\begin{array}{ccc}
(\pi_1(A_u) * \pi_1(B_u))/N_u & \xrightarrow{\sigma_u} & (\pi_1(A_v) * \pi_1(B_v))/N_v \\
\downarrow{\scriptstyle s_u} & & \downarrow{\scriptstyle s_v} \\
\pi_1(X_u) & \xrightarrow{k_{uv}} & \pi_1(X_v)
\end{array}
$$

$$
\begin{array}{ccc}
 & & (\pi_1(A_u) * \pi_1(B_u))/N_u \\
\pi_1(X_u) & \xrightarrow{r_u} & \downarrow{\scriptstyle s_u} \\
 & \xrightarrow{id} & \pi_1(X_u)
\end{array}
$$

$$
\begin{array}{ccc}
 & & \pi_1(X_u) \\
(\pi_1(A_u) * \pi_1(B_u))/N_u & \xrightarrow{s_u} & \downarrow{\scriptstyle r_u} \\
 & \xrightarrow{id} & (\pi_1(A_u) * \pi_1(B_u))/N_u
\end{array}
$$

Figure 4.5: Commutative diagrams of isomorphic persistences.

topmost diagram in Figure 4.5 commutes since second diagram can be done similarly and commutativity of the last two diagrams are trivial because r_u and s_u are isomorphisms by Van Kampen Theorem at each level u and we define them as inverses of each other,

By the proof of Van Kampen Theorem for persistent fundamental groups in Figure 3.2 we have the commutative diagrams in Figure 4.6. Now, by the diagram in Figure 3.2, we have

$$k_{uv} \circ j_{uA} = j_{vA} \circ g_{uv}$$

$$k_{uv} \circ j_{uB} = j_{vB} \circ h_{uv}.$$

$$\begin{array}{ccccccc}
\pi_1(A_u) & \xrightarrow{j_{uA}} & \pi_1(X_u) & \quad & \pi_1(B_u) & \xrightarrow{j_{uB}} & \pi_1(X_u) \\
\downarrow{\scriptstyle g_{uv}} & & \downarrow{\scriptstyle k_{uv}} & & \downarrow{\scriptstyle h_{uv}} & & \downarrow{\scriptstyle k_{uv}} \\
\pi_1(A_v) & \xrightarrow{j_{vA}} & \pi_1(X_v) & & \pi_1(B_v) & \xrightarrow{j_{vB}} & \pi_1(X_v)
\end{array}$$

Figure 4.6: Commutative diagrams from Van Kampen Theorem.

Now, let $z_u \in \pi_1(X_u)$, then by Van Kampen Theorem there exists $a_u \in \pi_1(A_u)$ and $b_u \in \pi_1(B_u)$ such that $z_u = j_{uA}(a_u)j_{uB}(b_u)$.

Let $r_u : \pi_1(X_u) \to (\pi_1(A_u) * \pi_1(B_u))/N_u$ defined as $r_u(z_u) = (a_u b_u)N_u$ for each u.

Now,

$$\begin{aligned}
r_v(k_{uv}(z_u)) &= r_v(k_{uv}(j_{uA}(a_u)j_{uB}(b_u))) \\
&= r_v(k_{uv}(j_{uA}(a_u))k_{uv}(j_{uB}(b_u))) \\
&= r_v(j_{vA}(g_{uv}(a_u))j_{vB}(h_{uv}(b_u))) \\
&= (g_{uv}(a_u)h_{uv}(b_u))N_v
\end{aligned}$$

For other direction,

$$\begin{aligned}
\sigma_u(r_u(z_u)) &= \sigma_u(r_u(j_{uA}(a_u)j_{uB}(b_u))) \\
&= \sigma_u((a_u b_u)N_u) \\
&= (g_{uv}(a_u)h_{uv}(b_u))N_v
\end{aligned}$$

Hence, we have shown that the topmost diagram commutes. Similarly, other diagrams in Figure 4.5 can be shown that they commute.

We know that by Remark 4 if the persistences $\{\pi_1(A_u)\}$ and $\{\pi_1(B_u)\}$ are δ- interleaved, and if the persistences $\{\pi_1(B_u)\}$ and $\{\pi_1(C_u)\}$ are ϵ-interleaved, then the persistences $\{\pi_1(A_u)\}$ and $\{\pi_1(C_u)\}$ are $(\delta+\epsilon)$-interleaved. Hence, as a consequence of Theorem 4.2.1 and 4.2.2, we get the following result.

Corollary 4.2.2.1 *If $\{\pi_1(A_u)\}$ is δ-interleaved with $\{\pi_1(A'_u)\}$ and $\{\pi_1(B_u)\}$ is δ-interleaved with $\{\pi_1(B'_u)\}$ where $\delta > 0$, then the persistence $\{\pi_1(X_u)\}$ is δ- interleaved with the persistence $\{\pi_1(X'_u)\}$ for $\delta > 0$.*

Proof 5 *By Theorem 4.2.2 we have shown that the persistence* $\{\pi_1(X_u)\}$ *is isomorphic (*0*-interleaved) to the persistence* $\{(\pi_1(A_u) * \pi_1(B_u))/N_u\}$*. Also, by Theorem 4.2.1 we have* $\{(\pi_1(A_u)*\pi_1(B_u))/N_u\}$ *is* δ*-interleaved with* $\{(\pi_1(A'_u)*\pi_1(B'_u))/N'_u\}$*. Again by Theorem 4.2.2 we have the persistence* $\{(\pi_1(A'_u) * \pi_1(B'_u))/N'_u\}$ *is isomorphic (*0*-interleaved) to* $\{\pi_1(X'_u)\}$*. Hence, by Remark 4 we get the following result, the persistence* $\{\pi_1(X_u)\}$ *is* δ*-interleaved with the persistence* $\{\pi_1(X'_u)\}$ *for* $\delta > 0$*.*

Hence, we have shown that interlaving among subspaces imply interleaving among total spaces.

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