KNOTTED SOLUTIONS OF MAXWELL EQUATIONS

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ABSTRACT

KNOTTED SOLUTIONS OF MAXWELL EQUATIONS

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We review Rañada's knotted solutions of Maxwell equations in this thesis. We explain the method behind these constructions and its relation to topology. The method starts with the null field assumption. We obtain the initial conditions of the electromagnetic field using the Hopf map. Then we find the time-dependent fields using Fourier transformation techniques. Finally, we analyze the properties of general torus knotted solutions and show that these also include the non-null fields.

Keywords: electromagnetism, knotted solutions, Hopf map

ÖZ

MAXWELL DENKLEMLERİNİN DÜĞÜMLÜ ÇÖZÜMLERİ

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Bu tezde Rañada'nın bulduğu, Maxwell denklemlerinin düğümlü çözümleri incelenmiştir. Bu çözümlere ulaşmak için kullanılan yöntem ve bunun topoloji ile ilişkisi açıklanmıştır. Çözümlere boş alan (İng: null field) varsayımı ile başlanmıştır. Hopf gönderimi (İng: Hopf map) kullanılarak elektromanyetik alanın başlangıç koşulları elde edilmiştir. Daha sonra Fourier dönüşüm teknikleri kullanılarak zamana bağlı alanlar elde edilmiştir. Son olarak genel simit düğümlü (İng: torus knotted) çözümlerin özellikleri irdelenmiş, bunların boş olmayan (İng: non-null) alanları da içerdiği gösterilmiştir.

Anahtar Kelimeler: elektromanyetizma, düğümlü çözümler, Hopf gönderimi

to my family with love

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NOTATION

In this work I will use the following notations, units and assumptions:

Vectors will be shown with boldface (e.g. $\mathbf{B}$ for magnetic field).

Minkowski metric is taken as $\eta_{\mu\nu} = diag(1, -1, -1, -1)$.

Speed of light $c = 1$.

Reduced Planck constant $\hbar = h/2\pi = 1$.

Vacuum permittivity $\varepsilon_0 = 1$.

Vacuum permeability $\mu_0 = 1$.

Greek indices such as μ,ν take values 0,1,2,3, whereas the Latin indices such as *i,j,k* take values 1,2,3.

Levi-Civita symbol $\epsilon_{ijk} = \epsilon^{0ijk}$ and $\epsilon^{\mu\nu\alpha\beta}$ is a totally antisymmetric tensor with $\epsilon^{0123} = 1$. Minkowski metric $\eta_{\mu\nu}$ is used to raise and lower the indices.

Einstein summation convention is used: Whenever an expression contains an index as a superscript and a subscript, a summation is implied over all the allowed values of that index.

CHAPTER 1

INTRODUCTION

There has been a long standing relationship between Physics and Geometry. One of the best examples of this relationship can be seen in the Theory of General Relativity, in which the gravitational force is interpreted as the curvature of space-time. Topological concepts play an important role in Quantum Physics too. As Atiyah [2] says, this is not surprising, since "both quantum theory and topology lead from continuous to the discrete". The reflections of this relationship can also be seen in the theory of electromagnetism. Many scientists studied the magnetic and electric fields in the nineteenth century. The common point of all these studies was the usage of lines of force for describing these fields. Faraday thought the field lines as real physical objects and during the nineteenth century there were many attempts to explain the field lines in terms of the streamlines and vorticity of ether. It is thought that the electromagnetic phenomena is a result of the motion of ether particles. In his paper "On vortex atoms" [3], Kelvin suggested that the atoms are knots of the vortex lines of ether. But these ideas have not been supported for a long period. With the advances in the other theories such as relativity and quantum mechanics, which explained the electromagnetic phenomena satisfactorily, the idea of lines of force remained in the background. Although the topological studies has not been in the main stage at every field of physics, there were many studies using topological concepts. One of them was the study of Gauss [4]. Considering two linked circuits, Gauss established the relation between the magnetic field induced by the currents along the circuits and the linking number, which is a topological invariant. As we will see in this work, Dirac [5] also used topological arguments for his studies and proposed a magnetic monopole field (analog of an electric charge field) which results in the quantization of electric charge. In 1959 Aharonov and Bohm [6] showed the relationship between electromagnetism and topology in their work, which has been known as the *Aharonov-Bohm effect*. In 1977 Trautman [7] showed that the magnetic field corresponding to a magnetic monopole can be constructed using Hopf fibration. In 1989 Rañada [8] used Hopf map to construct linked electromagnetic configurations. The possibility of generating these linked field solutions experimentally is a topic of great interest. Since the topology is very important in plasma physics, it is one of the promising research areas to generate these solutions . As reviewed by Arrayás, Bouwmeester and Trueba in "Knots in electromagnetism" [1], there are plasma configurations in which the magnetic field lines approximates plasma torus knots, leading to the prediction of topological solitons in plasma and it is an active research topic.

This study can be considered as a review of works done by Rañada, Arrayás and Trueba [8], [9], [10], [1]. The main point of this work is to explain the idea and the topological background for the construction of Rañada fields.

In Chapter 2, we begin with the Maxwell equations, and explain the gauge transformations under which these equations remain invariant. Then we will show how the topological arguments affect the physics for a monopole field as Dirac proposed. Using the results of the monopole field we will explain the effects of these results classically and quantum mechanically. At the end of Chapter 2, we will give a brief information about the fiber bundles, the Hopf map and we will mention how Trautman used topology for constructing the magnetic monopole field.

In Chapter 3, we will start by writing Maxwell equations in the language of differential forms. Then we will explain the required conditions in order to write the electromagnetic fields in a more compact form. We will interpret this compact form in terms of level curves. After that we will show how to construct electromagnetic fields using complex maps. We will analyze the topological properties of these maps and the relationship between these maps and the helicity of the fields. Then we will show the duality of helicity, which explains the relation between the helicity of the electromagnetic wave and the particle interpretation of helicity. As a last step, we will show how to construct the knotted solutions of Maxwell equations. To construct these solutions, we will start with defining the initial conditions of the electromagnetic fields. Then using Fourier transform techniques we will obtain the time dependent field equations.

Lastly, we will analyze some properties of these knotted solutions.

CHAPTER 2

MAXWELL THEORY AND THE TOPOLOGICAL PROPERTIES OF A MONOPOLE FIELD

In this chapter we will start by giving a brief information about Maxwell equations and the gauge transformations under which the electromagnetic fields are left invariant. Then we will study a magnetic monopole field which is proposed by Dirac. The importance of this field is due to its relation to the quantization of the electric charge. Then we will write Maxwell equations in the language of differential forms which gives a better understanding of the electromagnetic fields in terms of the vector potentials.

2.1 Maxwell Equations

The basic laws of the electromagnetism in microscopic form are given by Maxwell equations as follows

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned} \tag{2.1}$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi\rho \\ \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= 4\pi\mathbf{j} \end{aligned} \tag{2.2}$$

where ρ and $\mathbf{j}$ represent the charge and current densities, respectively. The first line in (2.1) tells us that there is no such thing as a magnetic charge, whereas the second one shows that a changing magnetic field produces an electric field. The first line of (2.2) tells us that the total charge inside a closed volume may be obtained by integrating the electric field $\mathbf{E}$ over the surface of the volume. The second equation of (2.2) tells that the current density and the changing electric field produces a magnetic field. We see that the first and the second pair of equations are not symmetric due to the charge and current densities. When there are no sources; that is, $\rho = 0$ and $\mathbf{j} = 0$, Maxwell equations reduce to

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= 0, \\ \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= 0. \end{aligned} \tag{2.4}$$

These are almost symmetric equations and they are called source-free Maxwell equations. Using the first equation of (2.3), we can write

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A},$$

where $\mathbf{A}$ is called the vector potential. We will see in section 2.2 that it is not always possible to define a global vector potential but we can define it locally. Inserting the vector potential into the second line of (2.3), we obtain

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t},$$

where V represents the scalar potential. It is easy to see that the vector and scalar potentials that define the magnetic and electric fields are not unique. Since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \mathcal{X}) = 0$, we can define

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} \mathcal{X}$$

which keeps the magnetic field invariant. In order to recover the change in the electromagnetic field due to the new vector potential, we define

$$V' = V - \frac{\partial \mathcal{X}}{\partial t}.$$

It is easy to show that the fields $\mathbf{B}$ and $\mathbf{E}$ are invariant under these transformations which are called *gauge transformations*. We will analyze effects of the gauge transformations classically and quantum mechanically in the next section.

2.2 Dirac Monopole

As we have seen in the previous section, Maxwell equations state that there is no magnetic charge. Although a magnetic monopole has not been observed yet, Dirac [5] assumed the existence of such a magnetic monopole and found a rather interesting result which is called *Dirac quantization condition*. Now assume that there exists a magnetic monopole of charge g (analog of an electric charge) at the origin of a frame in which the monopole is at rest. If we use the standard spherical coordinates, then the magnetic and electric fields of the magnetic monopole is written as

$$\mathbf{B} = \frac{g}{r^2}\hat{\mathbf{e}}_{\mathbf{r}}, \tag{2.5}$$

$$\mathbf{E} = 0. \tag{2.6}$$

Then, on $\mathbb{R}^3 - (0,0,0)$, (2.3) and (2.4) reduce to

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{B} = 0 \tag{2.7}$$

for the magnetic field. Let us analyze the second equation of (2.7). From vector analysis we know that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} V) = 0$ for a twice differentiable, single valued scalar V defined in a simply connected region. Thus we can write the $\mathbf{B}$ field as the gradient of a scalar potential if the region is simply connected.

Simply connectedness is a property related to the topology of a space. Topological spaces can be classified using homotopy groups. Homotopy group can roughly be defined as a tool, which keeps the information about the basic shape (e.g. number of holes, etc) of a topological space. If the base-point preserving maps (which takes a base point in S^n into a base point in the given space) are collected into equivalence classes, then these classes, which are called homotopy classes, form a group called the n^{th} *homotopy group*. Two mappings which can be continuously deformed to each other are called *homotopic* maps. The first homotopy group is called the fundamental group which keeps the information about the closed curves in a topological space. Using this information about the homotopy groups, the simply connectedness can be described as follows: A region is called *simply connected* if every closed curve in this region can be smoothly contracted (shrunk) to a point without leaving the region. One can shrink any closed curve in $\mathbb{R}^3 - (0,0,0)$ without leaving it so it is a simply connected region. In topology this is defined literally as $\pi_1(\mathbb{R}^3 - (0,0,0)) = 0$, which says that the fundamental homotopy group of $\mathbb{R}^3 - (0,0,0)$ is trivial. Since we know that $\mathbb{R}^3 - (0,0,0)$ is simply connected, we can define $\mathbf{B}$ as the gradient of a scalar potential. We have a different situation for the first equation of (2.7). We know that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$. If we imagine a sphere centered at the origin (0, 0, 0) (with radius a) and calculate the magnetic flux on this sphere (whose surface we label as S), then we get

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_S \left(\frac{g}{a^2}\hat{\mathbf{e}}_{\mathbf{r}}\right) \cdot \hat{\mathbf{e}}_{\mathbf{r}} \mathrm{d}S = \frac{g}{a^2} \int_S \mathrm{d}S = \frac{g}{a^2}(4\pi a^2) = 4\pi g \,. \tag{2.8}$$

Now assume that we have a global smooth vector potential on $\mathbb{R}^3 - (0,0,0)$ so that we can write $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. If we divide this sphere into two equal hemispheres as S_+ and S_- (where S_+ and S_- represent the *northern* and *southern* hemispheres) from the equator, then using the Stokes' theorem we get

$$\begin{aligned} \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} &= \int_{S_+} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} + \int_{S_-} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} \\ &= \oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{r} - \oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{r} = 0 \,, \end{aligned} \tag{2.9}$$

where C is the equator curve lying in the region where S_+ and S_- overlap. Note that the minus sign in front of the second integral is due to the path orientation.

By comparing (2.8) and (2.9), we notice that we have different results for the same integral which show us that the assumption we made is not true. Having a simply connected region is not sufficient to write the magnetic field as the curl of a vector potential. Thus we can not have a globally defined smooth vector potential for the given region $\mathbb{R}^3 - (0,0,0)$. The required condition for defining a global vector potential is more restrictive. For that, the second homotopy group of the region in interest should be trivial, i.e. we should have $\pi_2(\mathbb{R}^3 - (0,0,0)) = 0$ in order to have a global vector potential [11]. For this condition to be fulfilled, all the spheres in $\mathbb{R}^3 - (0,0,0)$ must be smoothly contracted (shrunk) to a point without leaving the region. However it is easy to see that any sphere surrounding the origin violates this condition. So $\pi_2(\mathbb{R}^3 - (0,0,0)) \neq 0$. Thus topology prevents us from defining a global vector field. This issue can be solved *partially* by defining a curve (*Dirac string*) which starts at the origin and goes to infinity in an arbitrary direction. Let us consider the complement of such a string. We have the sufficiency condition to have a smooth vector potential on the complement of this string, since all the spheres in this region can be contracted smoothly to a point without leaving the region. Here we said *partially* because, if we define only one Dirac string then we get a smooth vector potential in the complement of this region, but we need at least two Dirac strings to cover the whole $\mathbb{R}^3 - (0,0,0)$. As a result we will have two vector potentials. Consider two strings, both of which start from the origin and proceed in reverse directions ($+z$ and $-z$):

$$\begin{aligned} z_+ &= \big\{(0,0,z) \in \mathbb{R}^3 : z \geq 0\big\}\,, \\ z_- &= \big\{(0,0,z) \in \mathbb{R}^3 : z \leq 0\big\}\,. \end{aligned} \tag{2.10}$$

Let us also define the complements of these strings as $R_+(\mathbb{R}^3 - z_-)$ and $R_-(\mathbb{R}^3 - z_+)$, respectively. Now we need two vector potentials which should satisfy $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B} = g/r^2 \hat{\mathbf{e}}_\mathbf{r}$ on R_+ and separately on R_-. If we compute the radial component of $\boldsymbol{\nabla} \times \mathbf{A}$ in spherical polar coordinates, then we get

$$\mathbf{B}_r = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta \mathbf{A}_\phi) - \frac{\partial}{\partial\phi}(\mathbf{A}_\theta)\right), \tag{2.11}$$

which should be equal to g/r^2 from (2.5). Thus, we can define a vector potential $\mathbf{A}^+$ on R_+, with components

$$\begin{aligned} \mathbf{A}^+_r &= \mathbf{A}^+_\theta = 0\,, \\ \mathbf{A}^+_\phi &= \frac{g}{r\sin\theta}(-\cos\theta)\,. \end{aligned} \tag{2.12}$$

However this function is not analytic at $\theta = 0$. To make it analytic, we can define

$$\mathbf{A}^+_\phi = \frac{g}{r\sin\theta}(1 - \cos\theta)\,. \tag{2.13}$$

Similarly one can define a vector potential on R_- as

$$\mathbf{A}^-_\phi = \frac{-g}{r\sin\theta}(1 + \cos\theta)\,. \tag{2.14}$$

We can't have a global vector potential due to the topological properties of $\mathbb{R}^3 - (0,0,0)$, as seen before. Thus, we have different vector potentials at the intersection of the two domains. The difference of these potentials are

$$\mathbf{A}^+ - \mathbf{A}^- = \frac{2g}{r\sin\theta}\hat{\mathbf{e}}_\phi = \boldsymbol{\nabla}(2g\phi)\,. \tag{2.15}$$

This result is consistent since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} V) = 0$ (for $V = 2g\phi$ here) and the curl of these potentials do give the same vector field $\mathbf{B}$.

2.3 Gauge Transformation and Classical / Quantum Mechanical Effects

In the previous section, we observed that there are two different vector potentials, which differ by a gradient of a scalar, for the magnetic monopole field. Let us see the effect of this difference classically and quantum mechanically. Classically, the force on a particle with charge q in an electromagnetic field is given by the Lorentz formula

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{2.16}$$

Since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} 2g\phi) = 0$, the magnetic field $\mathbf{B}$ is invariant under the transformation expressed in (2.15). Thus both of the vector potentials in (2.13) and (2.14) give the same magnetic field. Since the Lorentz formula depends on $\mathbf{B}$ only, vector potential $\mathbf{A}$ does not have an observable effect classically. However, quantum mechanically, we can no longer think of a classical particle as a point particle. Instead, we consider the particle as an object described by a wave function ψ. The wave function can be found by solving the Schrödinger equation. If there is no electromagnetic field, then the wave function for a particle with charge q, momentum $\mathbf{p}$ and energy E can be found as

$$\psi = |\psi| e^{[i(\mathbf{p}\cdot\mathbf{r} - Et)]} . \tag{2.17}$$

where $\mathbf{r}$ and t denote the position and time, respectively. If there is an electromagnetic field, $\mathbf{p} \to \mathbf{p} - q\mathbf{A}$. Then the wave function is modified as

$$\psi \to \psi e^{(-iq\mathbf{A}\cdot\mathbf{r})} . \tag{2.18}$$

Thus using these two vector potentials, we will only find a phase difference in the wave function.

$$\begin{aligned} &\mathbf{A}^- \to \psi \,, \\ &\mathbf{A}^+ = \mathbf{A}^- + \boldsymbol{\nabla}(2g\phi) \to e^{i2qg\phi}\psi \,. \end{aligned} \tag{2.19}$$

This was not thought to be a problem for a long time because the probability of finding the particle in some region at a given time depends only on $|\psi|^2$. So the phase is not an observable quantity. But Aharonov-Bohm [6] showed that this is not the case if we consider more than one particle. In a double-slit experiment, electrons produce an interference pattern if the slit they pass through is not detected. The experiment that

Aharonov-Bohm proposed was a modification of this experiment in which a small, infinitely long and impenetrable solenoid is introduced behind the walls, between the two slits. The solenoid is so small that the particles do not pass through it, and also the magnetic field is negligible in the region through which the particle passes. Thus the particle is not affected by the magnetic field $\mathbf{B}$ inside the solenoid. Aharonov-Bohm showed that the interference pattern changes in proportion to the flux of the magnetic field inside the solenoid although the electrons move in a region, in the absence of the magnetic field.

The physical interpretation of the wave functions defined by the vector potentials in (2.19), results in an important condition. Since the wave function assigns one value to each point in space, it should be periodic in ϕ. That is, it should be invariant under the transformation $\phi \to \phi + 2\pi$. Thus for the vector potential $\mathbf{A}^-$

$$\psi(\phi) = \psi(\phi + 2\pi). \tag{2.20}$$

The same argument applies to $\mathbf{A}^+$

$$\begin{aligned} e^{i2qg\phi}\psi(\phi) &= e^{i2qg(\phi+2\pi)}\psi(\phi+2\pi)\,, \\ \implies e^{i2qg\phi} &= e^{i2qg(\phi+2\pi)}\,, \\ \implies e^{i4qg\pi} &= 1\,, \\ \implies 4\pi qg &= 2\pi n\,, n \in \mathbb{Z} \\ \implies qg &= \frac{n}{2}\,. \end{aligned} \tag{2.21}$$

The last line shows that if there is a magnetic monopole with charge g, then the electric charge q must be quantized or vice versa. This important result is called the *Dirac quantization condition.*

Returning to the gauge transformation in (2.15), it can be written as

$$\mathbf{A}^-_\phi = \mathbf{A}^+_\phi - \boldsymbol{\nabla}_\phi(2g\phi) = \mathbf{A}^+_\phi - \frac{i}{q}S\boldsymbol{\nabla}_\phi S^{-1}\,, \tag{2.22}$$

where $S = e^{i2qg\phi}$, which shows that in the region of overlap the vector potentials are not the same but related by a gauge transformation S. The gauge transformation can be written in covariant form as [12]

$$\mathbf{A}^-_\mu = \mathbf{A}^+_\mu - \frac{i}{q}S\partial_\mu S^{-1}\,. \tag{2.23}$$

This construction is due to Wu and Yang [13], which is a *fiber bundle formulation of magnetic monopole*. The region $\mathbb{R}^3 - (0,0,0)$ which is isomorphic to $S^2 \times \mathbb{R}^1$ is divided into two overlapping regions where the vector potential is parameterized differently. This is similar to the Mobius strip which can not be parameterized uniquely but can be parameterized by dividing the region into two parts [12]. We need to define the magnetic field in terms of differential forms in order to understand the concept of fiber bundle formulation clearly. The magnetic field of a monopole can be written in Cartesian coordinates as

$$\mathbf{B} = \frac{g}{r^3}\mathbf{r} = \frac{g}{r^3}(x, y, z)\,. \tag{2.24}$$

We can define a 1-form corresponding to this $\mathbf{B}$ field but, since we want to write it as the curl of a vector potential, we will define a 2-form instead. This is because we want to make use of the fact that the exterior derivative of a 1-form is a 2-form with components equal to the components of the curl of the vector corresponding to the 1-form. If we define

$$\mathscr{F} = \mathrm{d}\mathscr{A} = \frac{g}{r^3}(z\,\mathrm{d}x \wedge \mathrm{d}y + y\,\mathrm{d}z \wedge \mathrm{d}x + x\,\mathrm{d}y \wedge \mathrm{d}z)\,, \tag{2.25}$$

then $\mathscr{A}$ can be defined as

$$\mathscr{A}^+ = \frac{g}{r(r+z)}(x\,\mathrm{d}y - y\,\mathrm{d}x) = g(1-\cos\theta)\,\mathrm{d}\phi \tag{2.26}$$

on R_+. Similarly one can define $\mathscr{A}$ on R_- as

$$\mathscr{A}^- = -\frac{g}{r(r-z)}(x\,\mathrm{d}y - y\,\mathrm{d}x) = -g(1+\cos\theta)\,\mathrm{d}\phi\ . \tag{2.27}$$

Thus we find in terms of spherical coordinates

$$\mathscr{F} = \mathrm{d}\mathscr{A}^+ = \mathrm{d}\mathscr{A}^- = g\sin\theta\,\mathrm{d}\phi \wedge \mathrm{d}\theta\ . \tag{2.28}$$

We see that the 1-forms corresponding to the vector potentials do not depend on the coordinate r. So we can regard these 1-forms as ϕ and θ dependent only. Since these 1-forms can be thought of as defined on the $\theta\phi$-plane, one can identify this plane with a sphere S^2. We have seen that the vector potential keeps the phase information ($0 \to 2\pi$ which can be thought as a circle S^1), thus we can regard them as bundle of circles above S^2 which can be written locally as $S^2 \times S^1$. This is the reason that we called the construction as a *fiber bundle formulation of the monopole field*. But there is not a unique way to construct fiber bundles on a sphere. Although we can define fiber bundles which are locally the same as $S^2 \times S^1$ they will not be the same globally. This can be understood if we think of a stack over a circle. A simple stack produces a cylinder but a stack with a 180° twist produces a Mobius Strip which are the same locally but different globally [11]. Since there is not a unique way to construct the fiber bundles, how can we determine the one we should use? The clue that shows which one to choose is related to the Dirac quantization condition [11]. We saw that the magnetic monopole strengths are quantized. Thus, if magnetic monopoles exist, then there is a monopole for each integer. It can be proved that the principal $U(1)$ bundles over S^2 are classified by the elements of the fundamental group of $\pi_1(U(1))$, which are the group of integers ($\mathbb{Z}$). This coincidence suggests that the monopole strength can be used to select the bundle to be used to model it [11]. To get a better understanding of what a principal $S(1)$ or $U(1)$ bundle over S^2 is, we can take the Hopf map as an example. Before analyzing the Hopf map and its properties, we need some information about the stereographic projection and the n-sphere.

2.4 Stereographic Projection and the n-Sphere

2.4.1 Stereographic Projection

The stereographic projection is a mapping which projects a sphere onto a plane. The projection is defined on the entire sphere, except for the projection point. If we define the north pole (0,0,1) as the projection point and define a map $F_N : S^2 - (0, 0, 1) \to \mathbb{R}^2$, then the point $p = (p_1, p_2, p_3)$ on the sphere is projected onto the xy-plane where $z = 0$. To find the projected point on the plane one draws a line which passes through the projection point and the point to be projected. Then the intersection of this line with the projection plane ($z = 0$) gives the projection on the xy-plane (figure 2.1).
If we use the north pole as the projection point, then we find

$$F_N(p_1, p_2, p_3) = (x, y) = \left(\frac{p_1}{1 - p_3}, \frac{p_2}{1 - p_3} \right). \tag{2.29}$$

One can easily find the inverse map $F_N^{-1}(x, y)$ as

$$F_N^{-1}(x, y) = \left(\frac{2x}{\rho^2 + 1}, \frac{2y}{\rho^2 + 1}, \frac{\rho^2 - 1}{\rho^2 + 1} \right), \tag{2.30}$$

where $\rho^2 = x^2 + y^2$.

If we identify the xy-plane with the complex plane by the substitution $z = x + iy$, then we find

$$F_N^{-1}(x, y) = \left(\frac{z + \bar{z}}{z\bar{z} + 1}, \frac{z - \bar{z}}{z\bar{z} + 1}, \frac{z\bar{z} - 1}{z\bar{z} + 1} \right). \tag{2.31}$$

While doing this identification, we can obtain the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ by adding a point at infinity. Thus the extended complex plane is identified to a point independent of the direction you choose and the north pole in S^2 can be projected to this point $\{\infty\}$ under the stereographic map. This procedure is called 1-point compactification [11].

If, instead, we choose the south pole as the projection point, then we get

$$F_S(p_1, p_2, p_3) = \left(\frac{p_1}{1 + p_3}, \frac{p_2}{1 + p_3} \right). \tag{2.32}$$

The inverse map $F_S^{-1}(x, y)$ can be written as

$$F_S^{-1}(x, y) = \left(\frac{2x}{\rho^2 + 1}, \frac{2y}{\rho^2 + 1}, \frac{1 - \rho^2}{\rho^2 + 1} \right), \tag{2.33}$$

$$F_S^{-1}(x, y) = \left(\frac{z + \bar{z}}{z\bar{z} + 1}, \frac{z - \bar{z}}{z\bar{z} + 1}, \frac{1 - z\bar{z}}{z\bar{z} + 1} \right). \tag{2.34}$$

Making a similar mapping for S^3 using (0,0,0,1) as the projection point, we can find that the point $u(u_1, u_2, u_3, u_4)$ is projected onto $\mathbb{R}^3$ by the map $G_N : S^3 - (0, 0, 0, 1) \to \mathbb{R}^3$ as

$$G_N(u_1, u_2, u_3, u_4) = \left(\frac{u_1}{1 - u_4}, \frac{u_2}{1 - u_4}, \frac{u_3}{1 - u_4} \right), \tag{2.35}$$

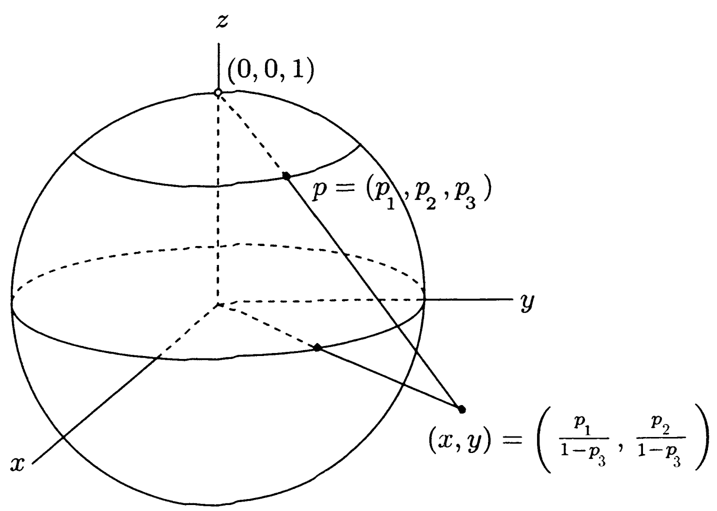

Figure 2.1: Stereographic projection from the north pole.

which has the inverse map $G_N^{-1}(x,y,z)$ as

$$G_N^{-1}(x,y,z) = \left(\frac{2x}{r^2+1}, \frac{2y}{r^2+1}, \frac{2z}{r^2+1}, \frac{r^2-1}{r^2+1}\right), \tag{2.36}$$

where $r^2 = x^2 + y^2 + z^2$.

2.4.2 The n-Sphere

The unit n-sphere is an n-dimensional manifold that can be embedded in $\mathbb{R}^{n+1}$, and defined by

$$S^n = \{p \in \mathbb{R}^{n+1} : \|p\| = 1\}, \tag{2.37}$$

where the norm $\|p\|^2 = p_1^2 + p_2^2 + ...p_n^2$ for S^n. Using the definition (2.37), we can easily see that

$n = 0 \implies S^0 = \{p = (p_1) \in \mathbb{R}^1 : \|p\| = 1\}$ is a pair of points ,
$n = 1 \implies S^1 = \{p = (p_1, p_2) \in \mathbb{R}^2 : \|p\| = 1\}$ is a circle ,
$n = 2 \implies S^2 = \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 : \|p\| = 1\}$ is a 2-sphere ,
$n = 3 \implies S^3 = \{p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4 : \|p\| = 1\}$ is a 3-sphere .

For the case where $n = 3$, since S^3 is embedded in $\mathbb{R}^4$, it is not easy to visualize. Thus, let us first define S^3 in a different way. Instead of identifying S^3 with $\mathbb{R}^4$, we can identify it with $\mathbb{C}^2$

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : \|z_1\|^2 + \|z_2\|^2 = 1\},$$

where $(z_1, z_2) = (u_1 + iu_2, u_3 + iu_4)$ and $\|z_1\|^2 = u_1^2 + u_2^2$. If we write z_1 and z_2 as

$$z_1 = \|z_1\|e^{i\theta_1}, z_2 = \|z_2\|e^{i\theta_2} ,$$

since $\|z_1\|^2 + \|z_2\|^2 = 1$ using the identity $\cos^2 a + \sin^2 a = 1$, then we can find some $\phi \in [0, \frac{\pi}{2}]$ such that

$$\|z_1\| = \cos\phi, \|z_2\| = \sin\phi .$$

We have chosen the interval $[0, \frac{\pi}{2}]$ to make $\|z_1\|$ and $\|z_1\|$ non-negative since they represent the modulus of the respective complex numbers. Thus we can write S^3 as

$$S^3 = \left\{\left(\cos\phi\, e^{i\theta_1}, \sin\phi\, e^{i\theta_2}\right) : 0 \le \phi \le \frac{\pi}{2}, \theta_1, \theta_2 \in \mathbb{R}\right\} .$$

Taking the subset T of S^3 restricted by $\|z_1\| = \|z_2\|$ and using the relation $\|z_1\|^2 + \|z_2\|^2 = 1$, we can find $\|z_1\| = \|z_2\| = \frac{\sqrt{2}}{2}$ which implies $\phi = \frac{\pi}{4}$. Thus

$$T = \left\{ \left(\frac{\sqrt{2}}{2} e^{i\theta_1}, \frac{\sqrt{2}}{2} e^{i\theta_2} \right) : \theta_1, \theta_2 \in \mathbb{R} \right\}.$$

This subset can easily be visualized, since it is just a Cartesian product of two circles, it is a $torus$. Doing similar analysis where $\|z_1\| \leq \|z_2\|$, or vice versa, one can find that S^3 consists of the union of two solid tori [11].

2.5 Hopf Map and Trautman's Proposal

In section 3.2, we will show that one can use the complex valued maps such as $\phi(x, y, z) : \mathbb{R}^3 \to \mathbb{C}$ to construct new solutions for Maxwell equations. We will impose two conditions on these maps and the reason will be clear in section 3.2. The first condition that we will impose is $\lim_{r\to\infty} \phi(x, y, z) = constant$, independent of the direction with which we approach to infinity. This can be achieved by identifying all points at infinity to a single point. Thus the space $\mathbb{R}^3$ is compactified to S^3 which makes the map $\phi(x, y, z) : S^3 \to \mathbb{C}$. The second condition that will be imposed on this map is $\lim_{|z|\to\infty} \phi^{-1}(z)$ does not depend on the direction we approach to infinity in the complex plane. Similarly this condition identifies all the complex numbers at infinity to a single point independent of its argument which results in the compactification $\mathbb{C} \to S^2$. As stated earlier, this is called 1-point compactification. As a result of these two conditions, the map becomes as $\phi : S^3 \to S^2$. Let us now understand the action of the Lie group $U(1)$ on S^3 and the structure constructed by the Hopf map between the spheres S^3 and S^2. Note that the group $U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\}$. Thus $U(1)$ consists of complex numbers having modulus one.

The right action of a Lie group $U(1)$ on S^3 is defined as

$$p \cdot g = (z_1, z_2) \cdot g = (z_1 g, z_2 g),$$

where $p \in S^3$, $g \in U(1)$. The *orbit* of p is defined as the subset $\{p \cdot g : \forall g \in U(1)\}$. It can be proved that the orbit of p under the action of $U(1)$ is a circle inside S^3. Thus, if we have a mapping under which the image of any orbit (image of all the points in the orbit of p) is constant in S^2, then we will have a circle bundle over S^2 as desired.

Keeping the above discussion in mind, let us define the Hopf map as $H : S^3 \to S^2$ by

$$H(z_1, z_2) = F_N^{-1}\left(\frac{z_1}{z_2}\right). \tag{2.38}$$

If we consider the orbit of any point $p \in S^3$ under the action of $U(1)$, then it is obvious that the image of the orbit of p is constant under the map H. That is

$$H(z_1 g, z_2 g) = F_N^{-1}\left(\frac{z_1 g}{z_2 g}\right) = F_N^{-1}\left(\frac{z_1}{z_2}\right) = H(z_1, z_2), \quad \forall g \in U(1).$$

Let $x \in S^2$, then the fiber $H^{-1}(x)$ constructed by the map H above x is defined as the orbit of any (z_1, z_2) satisfying $P(z_1, z_2) = x$.

Definition :

Let X be a manifold and G a Lie group. A *smooth principal bundle over X with structure group G* (or *a smooth G-bundle over X*) consists of a manifold P, a smooth map ϕ of P onto X and a smooth right action $(p, g) \to p \cdot g$ of G on P, if the following conditions are satisfied

1. The action of G on P preserve the fibers of ϕ,

$$\phi(p \cdot g) = \phi(p) \qquad \forall p \in P, \quad \forall g \in G.$$

2. (Local Triviality) For each $x_0 \in X$ there exist an open set V containing x_0 and a diffeormorphism $\Psi : \phi^{-1}(V) \to V \times G$ of the form $\Psi(p) = (\phi(p), \psi(p))$, where $\Psi : \phi^{-1}(V) \to G$ satisfies

$$\psi(p \cdot g) = \psi(p)g \qquad \forall p \in \phi^{-1}(V), \quad \forall g \in G.$$

Using the above definition, it can be shown that the Hopf map provides S^3 with a principal $U(1)$ bundle over S^2 [11]. Now we have reached to the point that we desired. For every distinct point $x \in S^2$, we have a fiber in S^3. It can be proved that the standard projection $S^2 \times U(1) \to S^2$ also satisfies these conditions and it is called the *trivial* $U(1)$ *bundle over* S^2. Any principal $U(1)$ bundle over S^2 can be shown to be locally the same as the trivial bundle. The diffeormorphisms $\Psi : \phi^{-1}(V) \to V \times U(1)$ are called local trivializations of the bundle and V's are called trivializing neighborhoods. The way how these trivializations spliced together on the overlap regions identifies the topological properties of the region [11].

In section 2.2 we have defined two distinct vector potentials for the magnetic monopole field. Although the details of this will not be given in this work, Ehresmann [14] showed that the Lie algebra valued 1-forms constructed from the two vector potentials defines a unique 1-form in S^3. This is called the connection of the map. The exterior derivative of this 1-form is called the curvature of this connection and defines the electromagnetic tensor in S^3. Trautman used Hopf fibration to construct a topological model for the magnetic monopole [7]. We have seen that the Hopf map creates fiber bundles (which are circles) over S^2. The result of this action creates the quotient space S^2. The curvature of the connection defined by the Hopf map is

$$\mathscr{F} = \frac{1}{2} \sin\theta \, \mathrm{d}\phi \wedge \mathrm{d}\theta \ , \tag{2.39}$$

where ϕ and θ represent the Euler angles in spherical coordinates. Since this 2-form is closed ($\mathrm{d}\mathscr{F} = 0$) and also satisfies ($\mathrm{d}^*\mathscr{F} = 0$), where * denotes the *Hodge star* operator that will be explained in section 2.1, it is a solution of Maxwell equations in free space. If we compare (2.39) with (2.28), then we see that the curvature of the connection corresponds to the monopole field with strength $g = \frac{1}{2}$.

CHAPTER 3

TORUS KNOTS

In Chapter 2 we have seen some of the effects of topology on the electromagnetic field in free-space. At the end of the chapter we have seen a solution, proposed by Trautman, for the electromagnetic field by means of a Hopf map. In this chapter we will see another construction, which is called Rañada construction, to generate more general knotted solutions of Maxwell equations.

3.1 Maxwell Equations as Geometric Identities

In this section we will write Maxwell equations in tensorial form and we will give the two Lorentz invariants of the electromagnetic field.

When we write Maxwell equations in free space,we have seen that we can define the magnetic and electric fields, respectively, as

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,, \tag{3.1}$$

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}\,. \tag{3.2}$$

Let us define the 4-vector electromagnetic potential with components

$$A^{\mu} = (V, \mathbf{A})\ . \tag{3.3}$$

We can see from (3.1) and (3.2) that the magnetic and electric field components are the components of a four dimensional curl which can be defined as

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\,, \tag{3.4}$$

where $E_i = F^{i0}$ and $B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}$, with ϵ_{ijk} the totally antisymmetric Levi-Civita symbol. Defining a 1-form $\mathscr{A} = A_\nu \, \mathrm{d}x^\nu$, and taking the exterior derivative of $\mathscr{A}$, we get

$$\begin{aligned}
\mathrm{d}\mathscr{A} &= \mathrm{d}(A_\nu \, \mathrm{d}x^\nu) \\
&= \mathrm{d}A_\nu \wedge \mathrm{d}x^\nu \\
&= (\partial_\mu A_\nu) \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \; .
\end{aligned} \tag{3.5}$$

As can be seen from (3.5) the components of this 2-form is similar to $F_{\mu\nu}$. This suggests us to define a 2-form

$$\mathscr{F} = \frac{1}{2}F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \, . \tag{3.6}$$

Then we have $\mathscr{F} = \mathrm{d}\mathscr{A}$. For any form ω, the *Bianchi identity* states that

$$\mathrm{d}(\mathrm{d}\omega) = \mathrm{d}^2\omega = 0. \tag{3.7}$$

Applying (3.7) to $\mathscr{F}$

$$\mathrm{d}\mathscr{F} = \mathrm{d}(\mathrm{d}\mathscr{A}) = 0 \, . \tag{3.8}$$

Inserting (3.6) in (3.8)

$$\begin{aligned}
\mathrm{d}\mathscr{F} &= \mathrm{d}\left(\frac{1}{2}F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) = \frac{1}{2}(\partial_\beta F_{\mu\nu}) \, \mathrm{d}x^\beta \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0 \\
&\implies \epsilon^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0 \, .
\end{aligned} \tag{3.9}$$

If we write the equations above for each α, then we get Maxwell equations in (2.3). Thus (2.3) follows from geometry. If we had started with

$$\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{C} \, , \tag{3.10}$$

$$\mathbf{B} = \boldsymbol{\nabla} V' + \frac{\partial \mathbf{C}}{\partial t} \tag{3.11}$$

instead, and defined an alternative electromagnetic 4-vector potential as

$$C^\mu = (V', \mathbf{C}) \tag{3.12}$$

by a similar reasoning, then we would have found a dual electromagnetic 2-form

$$\widetilde{\mathscr{F}} = \frac{1}{2}\widetilde{F}_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \; , \tag{3.13}$$

where $B_i = \widetilde{F}^{i0}$ and $E_i = \frac{1}{2}\epsilon_{ijk}\widetilde{F}^{jk}$. Comparing $\widetilde{\mathscr{F}}$ with $\mathscr{F}$ we see that $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ and $\widetilde{\mathscr{F}}$ is the Hodge dual of $\mathscr{F}$. Thus we can write $\widetilde{\mathscr{F}} = {}^*\mathscr{F}$, where * denotes the *Hodge star* or or *duality transformation* operator. In a flat space the Hodge star operator is defined by [12]

$$^*(\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\cdots\wedge\mathrm{d}x^{i_p}) = \frac{1}{(n-p)!}\epsilon_{i_1 i_2\cdots i_p i_{p+1}\cdots i_n}\,\mathrm{d}x^{i_{p+1}}\wedge\mathrm{d}x^{i_{p+2}}\wedge\cdots\wedge\mathrm{d}x^{i_n} \quad (3.14)$$

It can be seen easily that the Hodge star operator converts a *p*-form into a *(n-p)*-form. Thus it converts the electromagnetic 2-form $\mathscr{F}$ into the dual electromagnetic 2-form $\widetilde{\mathscr{F}}$ in the Minkowski spacetime. If we apply (3.7) to $\widetilde{\mathscr{F}}$, then we find

$$\mathrm{d}\widetilde{\mathscr{F}} = \mathrm{d}(\mathrm{d}\mathscr{C}) = 0\,. \quad (3.15)$$

As a result we get the second pair of Maxwell equations (2.4), which also follows from geometry as (2.3).

Now we have two electromagnetic 2-forms dual to each other. It can be shown that there are two invariants, that are left invariant under Lorentz transformations, that can be constructed using these 2-forms. We can define these Lorentz invariant quantities as

$$\mathscr{F}^{\mu\nu}\mathscr{F}_{\mu\nu} = 2(E^2 - B^2)\,, \quad (3.16)$$

$$\mathscr{F}^{\mu\nu}\widetilde{\mathscr{F}}_{\mu\nu} = 4(\mathbf{E}\cdot\mathbf{B})\,. \quad (3.17)$$

If both of the Lorentz invariants are zero for an electromagnetic field, then the field is called a *null field*.

3.2 Darboux Theorem and Clebsch Representation

In the previous section we have defined $\mathscr{F} = \frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Since this 2-form is defined in the four-dimensional Minkowski spacetime ($\mu, \nu = 0, 1, 2, 3$), we need at least four 1-forms to define an electromagnetic field in free space. But we know from (3.8) that $\mathscr{F}$ is a closed 2-form that is $\mathrm{d}\mathscr{F} = 0$. These two properties of the electromagnetic 2-form $\mathscr{F}$ is sufficient to define it as a *symplectic* 2-form.

Darboux Theorem: Let X^{2n} be a real smooth manifold. If ω is symplectic, then for every $p \in X$ there exists a coordinate patch $(U, x^1, ..., x^n, y^1, ..., y^n)$ centered at p such that on U

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i,$$

Due to Darboux theorem, $\mathscr{F}$ can be written locally as $\mathscr{F} = \mathrm{d}q^i \wedge \mathrm{d}p^i$, where $i = 0, 1$. Thus we can write

$$\mathscr{F} = \mathrm{d}q^0 \wedge \mathrm{d}p^0 + \mathrm{d}q^1 \wedge \mathrm{d}p^1 \ . \tag{3.18}$$

The representation in (3.18) is called *Clebsch* representation. The 2-form defined in (3.6) can be expressed in Clebsch representation as follows [1]

$$\begin{aligned}
\mathscr{F} =& \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
=& \frac{1}{2}\left(F_{0i}\, \mathrm{d}x^0 \wedge \mathrm{d}x^i + F_{i0}\, \mathrm{d}x^i \wedge \mathrm{d}x^0 + F_{jk}\, \mathrm{d}x^j \wedge \mathrm{d}x^k\right) \\
=& E_i\, \mathrm{d}x^0 \wedge \mathrm{d}x^i - \epsilon_{ijk} B_i\, \mathrm{d}x^j \wedge \mathrm{d}x^k \\
=& \left(E_1\, \mathrm{d}x^0 + B_3\, \mathrm{d}x^2 - B_2\, \mathrm{d}x^3\right) \wedge \left(\mathrm{d}x^1 + \frac{E_2}{E_1}\mathrm{d}x^2 + \frac{E_3}{E_1}\mathrm{d}x^3\right) \\
&- \left(\frac{\mathbf{E}\cdot\mathbf{B}}{E_1}\mathrm{d}x^2 \wedge \mathrm{d}x^3\right) \\
=& \,\mathrm{d}q^0 \wedge \mathrm{d}p^0 + \mathrm{d}q^1 \wedge \mathrm{d}p^1 \ ,
\end{aligned} \tag{3.19}$$

where (assuming $E_1 \neq 0$)

$$\mathrm{d}q^0 = E_1\, \mathrm{d}x^0 + B_3\, \mathrm{d}x^2 - B_2\, \mathrm{d}x^3 \ , \tag{3.20}$$

$$\mathrm{d}p^0 = \mathrm{d}x^1 + \frac{E_2}{E_1}\mathrm{d}x^2 + \frac{E_3}{E_1}\mathrm{d}x^3 \ , \tag{3.21}$$

$$\mathrm{d}q^1 = -\frac{\mathbf{E}\cdot\mathbf{B}}{E_1}\mathrm{d}x^2 \ , \tag{3.22}$$

$$\mathrm{d}p^1 = \mathrm{d}x^3 \ . \tag{3.23}$$

The determinant of $F_{\mu\nu}$ can easily be calculated as $(\mathbf{E}\cdot\mathbf{B})^2$. If the determinant of the tensor $F_{\mu\nu}$ is *zero*, then the corresponding 2-form can be written as

$$\mathscr{F} = \mathrm{d}q \wedge \mathrm{d}p \ . \tag{3.24}$$

If a 2-form $\mathscr{F}$ can be written as (3.24), then the corresponding field is called a decomposable field. We have seen in (3.18) that we can write any electromagnetic field as a sum of two 2-forms. Thus, the 2-form $\mathscr{F}$ of a non-decomposable electromagnetic field can be written as the sum of two 2-forms corresponding to two decomposable fields. Now assume that we have a decomposable field so that we can represent the corresponding 2-form as in (3.24) globally. Then we have the following identity

$$\mathscr{F} = \mathrm{d}q \wedge \mathrm{d}p \implies \mathbf{B} = \boldsymbol{\nabla} p \times \boldsymbol{\nabla} q\,, \tag{3.25}$$

which shows that we can use the same functions to represent the corresponding magnetic and electric fields.

Proof:

$$q = q(x^\mu) \qquad\qquad p = p(x^\nu), \tag{3.26}$$

$$\mathrm{d}q = \frac{\partial q}{\partial x^\mu}\,\mathrm{d}x^\mu \qquad\qquad \mathrm{d}p = \frac{\partial p}{\partial x^\nu}\,\mathrm{d}x^\nu\,, \tag{3.27}$$

$$\implies \mathrm{d}q \wedge \mathrm{d}p = \frac{\partial q}{\partial x^\mu}\,\mathrm{d}x^\mu \wedge \frac{\partial p}{\partial x^\nu}\,\mathrm{d}x^\nu\,, \tag{3.28}$$

$$= \frac{\partial q}{\partial x^\mu}\frac{\partial p}{\partial x^\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{3.29}$$

$$= \frac{1}{2}\left(\frac{\partial q}{\partial x^\mu}\frac{\partial p}{\partial x^\nu} - \frac{\partial p}{\partial x^\mu}\frac{\partial q}{\partial x^\nu}\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,. \tag{3.30}$$

Defining an antisymmetric tensor

$$F_{\mu\nu} = \partial_\mu q \partial_\nu p - \partial_\mu p \partial_\nu q\,, \tag{3.31}$$

$\mathscr{F}$ can be expressed as

$$F = \frac{1}{2} F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\mu\ . \tag{3.32}$$

We can define a vector field $\mathbf{B}$ with components $B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}$. Multiplying F^{jk} by ϵ_{ijk} and using (3.31)

$$\begin{aligned}
\epsilon_{ijk}F^{jk} &= \epsilon_{ijk}\big(\partial^j q \partial^k p - \partial^j p \partial^k q\big) \\
-2B_i &= \left(\epsilon_{ijk}(-\boldsymbol{\nabla} q)_j(-\boldsymbol{\nabla} p)_k - \epsilon_{ijk}(-\boldsymbol{\nabla} p)_j(-\boldsymbol{\nabla} p)_k\right) \\
&= \left(\boldsymbol{\nabla} q \times \boldsymbol{\nabla} p - \boldsymbol{\nabla} p \times \boldsymbol{\nabla} q\right)_i \\
&= (2\boldsymbol{\nabla} q \times \boldsymbol{\nabla} p)_i \\
\implies \mathbf{B} &= \boldsymbol{\nabla} p \times \boldsymbol{\nabla} q\,.
\end{aligned} \tag{3.33}$$

Defining the $\mathbf{E}$ field components as $E_i = F^{i0}$ and using (3.31), we find the electric field as

$$\mathbf{E} = \frac{\partial q}{\partial t}\boldsymbol{\nabla} p - \frac{\partial p}{\partial t}\boldsymbol{\nabla} q\,. \tag{3.34}$$

It is easy to see that $\mathbf{E} \cdot \mathbf{B} = 0$. This is an expected result since we have started with the decomposable field assumption.

Instead of (3.24), one can start with the dual 2-form

$$\widetilde{\mathscr{F}} = \mathrm{d}r \wedge \mathrm{d}s \ . \tag{3.35}$$

Due to the electromagnetic duality, $\mathrm{d}r \wedge \mathrm{d}s = {}^*(\mathrm{d}q \wedge \mathrm{d}p)$ must be satisfied. If the dual electromagnetic 2-form $\widetilde{\mathscr{F}}$ is used, then using the same method in (3.33), it is easy to show that

$$\begin{aligned} \mathbf{B} &= \frac{\partial r}{\partial t}\boldsymbol{\nabla} s - \frac{\partial s}{\partial t}\boldsymbol{\nabla} r \,, \\ \mathbf{E} &= \boldsymbol{\nabla} r \times \boldsymbol{\nabla} s \,. \end{aligned} \tag{3.36}$$

Comparing (3.36) with (3.33) and (3.34)

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} p \times \boldsymbol{\nabla} q = \frac{\partial r}{\partial t}\boldsymbol{\nabla} s - \frac{\partial s}{\partial t}\boldsymbol{\nabla} r \,, \\ \mathbf{E} &= \boldsymbol{\nabla} r \times \boldsymbol{\nabla} s = \frac{\partial q}{\partial t}\boldsymbol{\nabla} p - \frac{\partial p}{\partial t}\boldsymbol{\nabla} q \,. \end{aligned} \tag{3.37}$$

Before going further, we should note an important point. We know that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and using vector identities, it is easy to show that $\boldsymbol{\nabla} \times (p\boldsymbol{\nabla} q) = \boldsymbol{\nabla} p \times \boldsymbol{\nabla} q = \mathbf{B}$. Similarly $\boldsymbol{\nabla} \times (q\boldsymbol{\nabla} p) = -\boldsymbol{\nabla} p \times \boldsymbol{\nabla} q = \mathbf{B}$. Thus we can define a smooth vector potential $\mathbf{A}$ as a linear combination of $(p\boldsymbol{\nabla} q)$ and $(p\boldsymbol{\nabla} q)$ if the functions p an q are single valued. Comparing this linear combination with (3.34), we see that $\mathbf{A}$ // $\mathbf{E}$ and $\mathbf{A} \cdot \mathbf{B} = 0$. We will see in section 3.4 that the magnetic helicity is defined as $h_m = \int \mathrm{d}^3 r(\mathbf{A} \cdot \mathbf{B})$. The magnetic helicity is clearly zero for this case since the dot product vanishes. Thus, the magnetic helicity of a decomposable field, written globally as in (3.33) and (3.34), is non-zero if p or q is not well-defined at some points in $\mathbb{R}^3$ [1].

If we return to (3.33), then we see that we can write the magnetic field as the cross product of the gradients of two potentials. These potentials are called the Euler potentials of the magnetic field and they provide the Clebsch representation of the field. Since the gradient of a real function is perpendicular to its level curves, the points where these potentials take a constant value determine the magnetic surfaces. Thus the magnetic lines are given by the equations $q = k_1$ and $p = k_2$. Now we have two real equations. Instead of two real equations, we can also combine them to give a single complex equation. Let us define a complex valued potential ϕ using the real valued potentials q and p. Of course there are infinitely many ways to construct a complex potential using the real potentials but let us use a simple one. We will see the reason for this choice in section 3.3. Let us define

$$q := q(\phi\bar{\phi}) = q(|\phi|^2) \qquad\qquad p := p\left(\frac{\phi}{\bar{\phi}}\right) = p(\psi)\,,$$

where

$$\phi := |\phi|e^{i\beta} \qquad\qquad \beta := \frac{\ln\psi}{2i}\,.$$

Then we have

$$\begin{aligned}
\nabla p &= \left((\nabla\phi)\frac{1}{\bar{\phi}} - (\nabla\bar{\phi})\frac{\phi}{(\bar{\phi})^2}\right)\frac{\partial p}{\partial\psi}\,,\\
\nabla q &= \big((\nabla\phi)\bar{\phi} + (\nabla\bar{\phi})\phi\big)\frac{\partial q}{\partial(|\phi|^2)}\,.\\
\implies \mathbf{B} = \nabla p \times \nabla q &= \big(\psi\big(\nabla\phi\times\nabla\bar{\phi}\big) - \psi\big(\nabla\bar{\phi}\times\nabla\phi\big)\big)\frac{\partial q}{\partial(|\phi|^2)}\frac{\partial p}{\partial\psi}\\
&= \big(2\psi\big(\nabla\phi\times\nabla\bar{\phi}\big)\big)\frac{\partial q}{\partial(|\phi|^2)}\frac{\partial p}{\partial\psi}\,.
\end{aligned} \tag{3.38}$$

Now we need the explicit dependencies of q and p in order to calculate the cross product in terms of ϕ and $\bar{\phi}$ only. It is not apparent at this point how to choose these functions but it will be clear in section 3.3. Since we have the ψ term as a coefficient, we can get rid of it by defining

$$p := \frac{1}{4\pi i}\ln\psi = \frac{\beta}{2\pi}\,.$$

Let us also define

$$q := \frac{1}{1+|\phi|^2}\,,$$

Inserting q and p into (3.38), we get

$$\mathbf{B}(\mathbf{r},t) = \frac{\sqrt{a}}{2\pi i}\frac{\nabla\phi\times\nabla\bar{\phi}}{\big(1+\phi\bar{\phi}\big)^2}\,, \tag{3.39}$$

where a is inserted as a normalizing constant [9]. It is a pure number in natural units but in SI units it can be written as $a\hbar c$ in order to have the right dimensions. Since the magnetic field components are defined as $B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}$, we can define

$$F_{ij} = \frac{\sqrt{a}}{2\pi i}\frac{\partial_i\bar{\phi}\partial_j\phi - \partial_j\bar{\phi}\partial_i\phi}{\big(1+\phi\bar{\phi}\big)^2}\,.$$

Using the covariance of the electromagnetic field, the electromagnetic tensor can be defined as

$$F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu \bar{\phi} \partial_\nu \phi - \partial_\nu \bar{\phi} \partial_\mu \phi}{\left(1 + \phi\bar{\phi}\right)^2} .$$

Defining $E_i = F^{i0}$, one finds

$$\mathbf{E}(\mathbf{r}, t) = \frac{\sqrt{a}}{2\pi i} \frac{\partial_0 \bar{\phi} \boldsymbol{\nabla} \phi - \partial_0 \phi \boldsymbol{\nabla} \bar{\phi}}{\left(1 + \phi\bar{\phi}\right)^2} . \tag{3.40}$$

Now we have a way to create electromagnetic solutions by using complex potentials. Note also that the two real equations $q = k_1$ and $p = k_2$ can be combined to give a complex one as follows

$$p = k_2 = \frac{\beta}{2\pi} , \qquad\qquad q = k_1 = \frac{1}{1 + |\phi|^2} , \tag{3.41}$$

$$\implies \phi_0 = \phi_{k_1 k_2} = \sqrt{\frac{1 - k_1}{k_1}} e^{i 2\pi k_2} . \tag{3.42}$$

Thus each k_1,k_2 pair, which defines the level curves of the potential $\phi(t, x, y, z) = \phi_0$, corresponds to the magnetic field lines.

Using the same method for the dual electromagnetic 2-form $\widetilde{\mathscr{F}} = \mathrm{d}r \wedge \mathrm{d}s$ and defining the map θ instead of ϕ, one can find

$$\mathbf{E}(\mathbf{r}, t) = \frac{\sqrt{a}}{2\pi i} \frac{\boldsymbol{\nabla} \bar{\theta} \times \boldsymbol{\nabla} \theta}{\left(1 + \theta\bar{\theta}\right)^2} , \tag{3.43}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\sqrt{a}}{2\pi i} \frac{\partial_0 \bar{\theta} \boldsymbol{\nabla} \theta - \partial_0 \theta \boldsymbol{\nabla} \bar{\theta}}{\left(1 + \theta\bar{\theta}\right)^2} . \tag{3.44}$$

Similarly each $k_1^{'}$,$k_2^{'}$ pair, which defines the level curves of the potential $\theta(t, x, y, z) = \theta_0$, corresponds to the electric field lines.

Since the electromagnetic field also satisfies the electromagnetic duality condition mentioned in section 3.1, pairs of the solutions (3.39) - (3.44) and (3.40) - (3.43) must agree. Thus

$$\frac{\boldsymbol{\nabla} \phi \times \boldsymbol{\nabla} \bar{\phi}}{\left(1 + \phi\bar{\phi}\right)^2} = \frac{\partial_0 \bar{\theta} \boldsymbol{\nabla} \theta - \partial_0 \theta \boldsymbol{\nabla} \bar{\theta}}{\left(1 + \theta\bar{\theta}\right)^2} , \tag{3.45}$$

$$\frac{\boldsymbol{\nabla} \bar{\theta} \times \boldsymbol{\nabla} \theta}{\left(1 + \theta\bar{\theta}\right)^2} = \frac{\partial_0 \bar{\phi} \boldsymbol{\nabla} \phi - \partial_0 \phi \boldsymbol{\nabla} \bar{\phi}}{\left(1 + \phi\bar{\phi}\right)^2} . \tag{3.46}$$

This construction is due to Rañada [9]. Although one can use arbitrary complex potentials for the construction, we can impose two conditions for the electromagnetic fields [10]. The first condition is related to the finiteness of the energy of the field. If the total energy is finite, then we should have $\lim_{r\to\infty}\mathbf{B} = 0$ and $\lim_{r\to\infty}\mathbf{E} = 0$. Although there are also other ways, we can use a simple way to achieve this in our construction. Since we have $\boldsymbol{\nabla}$ operator in the definition of magnetic and electric fields, these fields approach zero in the regions where the potential takes constant value in all directions. Thus one can assume that ϕ and θ take a constant value at infinity independent of the direction that we choose. This condition can be achieved by 1-point compactification of $\mathbb{R}^3$ to S^3. The second condition is to take the inverse images of ϕ and θ at infinity $(\phi^{-1}(\infty), \theta^{-1}(\infty))$ independent of the direction approaching to infinity in $\mathbb{C}$, which compactifies $\mathbb{C}$ to S^2. As a result one can use the maps between S^3 and S^2 to construct Rañada fields. Due to the rich topological structure, it is appealing to use this kind of mapping in the construction of Rañada fields. Before using these maps, let us understand the topological structure that these maps possess.

3.3 Hopf Index and Magnetic Helicity

In section 2.4 for the Hopf map, we have seen that the inverse image of a point in S^2 is a circle in S^3. If we consider a smooth map $f : S^3 \to S^2$, according to the Hopf theorem [15], the inverse images of two distinct points $(z_1, z_2 : z_1 \neq z_2)$ in S^2 are two disjoint closed curves $(f^{-1}(z_1), f^{-1}(z_2))$ in S^3. The linking number l of these curves is equal to the number of times that one of these curves cuts a surface bounded by the other one. Note that we can continuously move from one pair to another pair $(z_1, z_2 \to z_1^{'}, z_2^{'})$, and if the linking number should change, then the curves must tie or untie each other. If this is the case, then the curves should intersect at a specific point which means our map takes the intersection point in S^3 to two different points in S^2, which is not possible. Thus the linking number l, which is independent of the points, is a property of the map. It can be shown that the set of maps can be classified in homotopy classes labeled by the Hopf index, which is isomorphic to $\mathbb{Z}$. If the level curves of a map $\phi = \phi_0$ has m disjoint connected components, then the multiplicity of the map is said to be m. Thus two sets, corresponding to two distinct level curves of a map, has m^2 intersections, each one having l linkings. Then the Hopf index of a map can be written in terms of the linking number as $H(\phi) = lm^2$ [10].

Let us start with the Hopf map as an example. Hopf defined a map with Hopf index $H(\phi) = 1$ as

$$\phi(x, y, z) = \frac{2(x + iy)}{2z + i(r^2 - 1)} = \frac{C_1}{C_2}, \tag{3.47}$$

where $r^2 = x^2 + y^2 + z^2$.

This map can be used to generate new maps having different Hopf indices. Let

$$\phi^{(n,m)} = \frac{(C_1)^{(n)}}{(C_2)^{(m)}} = \frac{(\rho_1 e^{i\alpha})^{(n)}}{(\rho_2 e^{i\beta})^{(m)}} = \frac{\rho_1}{\rho_2} \frac{e^{in\alpha}}{e^{im\beta}}, \tag{3.48}$$

where n, m are positive integers. The notation $C_1^{(n)}$ means that the modulus of C_1 is not affected but its phase is multiplied by the power n. It can be shown that the Hopf index of this map is $H(\phi) = nm$. It is easy to see that the change in the modulus does not affect the linking number and the Hopf index. As a result, two maps differing by modulus only, such as $\phi_1 = \rho e^{in\theta}$ and $\phi_2 = \rho^n e^{in\theta}$, are homotopic maps with the same Hopf index. Thus it is customary to keep the modulus constant and multiply the phase only as in (3.48).

There is a relation between the Hopf index and the magnetic helicity. Let us understand how the Hopf index is related to the mapping. Helicity of a magnetic field $\mathbf{B}$ is defined as

$$h(\mathbf{B}, D) := \int_D \mathrm{d}^3 r \, (\mathbf{A} \cdot \mathbf{B}), \tag{3.49}$$

where $\mathbf{A}$ is defined as in (3.1) and D denotes the region where the helicity is calculated. Since the curl of a vector measures the vector's rotation around a point, the helicity measures how much the vector rotates around itself times its magnitude.

Rañada explained a very important property of the helicity in his paper called "On the magnetic helicity" [16]. It is shown that the magnetic helicity is equal to the linking number of any pair of magnetic lines times the square of the total strength of the field

$$h(\mathbf{B}, D) = n\gamma^2, \tag{3.50}$$

where γ is the total strength of the field. We can see from (3.50) that the helicity is zero iff the linking number is zero which shows that the magnetic helicity is a result of a non-trivial topological configuration.

Consider the volume form $\mathscr{V}$ in S^2. Since it is a top-dimensional form we know that the exterior derivative of the volume form vanishes

$$\mathrm{d}\mathscr{V} = 0\,. \tag{3.51}$$

Let f^* denote the pullback by the map f. It can be proved that the pullback commutes with the differential [17], that is

$$\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)\,, \tag{3.52}$$

where ω denotes an arbitrary form in S^2. If we pullback the volume form by the map f and use the equations (3.51) and (3.52), then we get

$$\mathrm{d}(f^*\mathscr{V}) = f^*(\mathrm{d}\mathscr{V}) = 0\,. \tag{3.53}$$

So by pulling back the volume form $S^2 \to S^3$, we get a closed 2-form in S^3. The cohomological properties of S^3 guarantees that $f^*\mathscr{V}$ is also exact so that we can find a 1-from g satisfying

$$\mathrm{d}g = f^*\mathscr{V}.$$

Whitehead proved that [18] the Hopf index can be calculated as an integral in S^3 as follows

$$H(f) = \int_{S^3} g \wedge f^*\mathscr{V}\,. \tag{3.54}$$

If we define a 1-form as $g = -a_i \mathrm{d}x^i$ (the minus sign is introduced for later convenience)

$$\begin{aligned} f^*\mathscr{V} = \mathrm{d}g &= -(b_1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + b_2 \mathrm{d}x^3 \wedge \mathrm{d}x^1 + b_3 \mathrm{d}x^1 \wedge \mathrm{d}x^2) \\ \implies g \wedge f^*\mathscr{V} &= (a_1 b_1 + a_2 b_2 + a_3 b_3)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,, \end{aligned} \tag{3.55}$$

where

$$\begin{aligned} b_1 &= \left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3\,, \\ b_2 &= \left(\frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1}\right)\mathrm{d}x^3 \wedge \mathrm{d}x^1\,, \\ b_3 &= \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2\,. \end{aligned}$$

Defining two vectors $\mathbf{a}$ and $\mathbf{b}$ with components $(-a_i)$ and $(-b_i)$ respectively, we get

$$g \wedge f^*\mathscr{V} = (\mathbf{a} \cdot \mathbf{b})\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, \tag{3.56}$$

where $\mathbf{b} = \boldsymbol{\nabla} \times \mathbf{a}$.

Writing the integral (3.54) in R^3, the Hopf index will be

$$H(f) = \int \mathrm{d}^3 r \,(\mathbf{a} \cdot \mathbf{b}) \,, \tag{3.57}$$

where the vector $\mathbf{b}$ is called the Whitehead vector of the map f. If we compare (3.57) with (3.49) we can see the similarity. The Hopf index $H(f)$ of the map f is equal to the helicity of the Whitehead vector. As is shown in (3.66), the Whitehead vector $\mathbf{b} = \mathbf{B}/\sqrt{a}$ so the vector potential $\mathbf{A}/\sqrt{a} = \mathbf{a}$. Thus, if one uses a map $\phi : S^3 \to S^2$ to construct a magnetic field, then the helicity of the resulting magnetic field can be found as $h_m = aH(\phi)$. Similarly if a map $\theta : S^3 \to S^2$ is used to construct an electric field, then the helicity of resulting electric field is $h_e = aH(\theta)$.

Let us find the Whitehead vector which corresponds to the resulting magnetic field in terms of f explicitly. Let us write the normalized volume form $\mathscr{V}$ in Cartesian coordinates

$$\mathscr{V} = \frac{1}{4\pi} \frac{\mathrm{d}p_1 \wedge \mathrm{d}p_2}{p_3} \,. \tag{3.58}$$

Using the mapping in (2.31), we can write $\mathscr{V}$ in terms of the complex plane coordinates

$$\mathscr{V} = \frac{1}{2\pi i} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{1 + z\bar{z}} \,. \tag{3.59}$$

Thus

$$f^*\mathscr{V} = \frac{1}{2\pi i} \frac{\mathrm{d}f \wedge \mathrm{d}\bar{f}}{1 + f\bar{f}} \,. \tag{3.60}$$

This is the pullback of the volume form in S^3. Writing the map f as a function of coordinates x^i

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \implies \mathrm{d}f \wedge \mathrm{d}\bar{f} = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \wedge \frac{\partial f}{\partial x^j}\,\mathrm{d}x^j \,, \tag{3.61}$$

$$= \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \,, \tag{3.62}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} - \frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i}\right)\mathrm{d}x^i \wedge \mathrm{d}x^j \,. \tag{3.63}$$

Note that we have used the antisymmetry of the wedge product ($\mathrm{d}x^i \wedge \mathrm{d}x^j = -\,\mathrm{d}x^j \wedge \mathrm{d}x^i$). Now we can write (3.60) in terms of Cartesian coordinates

$$f^*\mathscr{V} = \frac{1}{4\pi i}\frac{\partial_i f \partial_j \bar{f} - \partial_j \bar{f} \partial_i f}{(1+f\bar{f})^2}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \ . \tag{3.64}$$

Defining an antisymmetric tensor

$$f_{ij} = \frac{1}{2\pi i}\frac{\partial_i f \partial_j \bar{f} - \partial_j \bar{f} \partial_i f}{(1+f\bar{f})^2} \ ,$$

$f^*\mathscr{V}$ can be expressed as

$$f^*\mathscr{V} = \frac{1}{2} f_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \ . \tag{3.65}$$

Comparing (3.65) with (3.55) we find that we can define $\mathbf{b}_i = -\frac{1}{2}\epsilon_{ijk} f^{jk}$. Thus

$$\mathbf{b} = \frac{1}{2\pi i}\frac{\boldsymbol{\nabla} f \times \boldsymbol{\nabla} \bar{f}}{(1+f\bar{f})^2} \ . \tag{3.66}$$

This expression is similar to the one that is found in (3.39) and (3.43) without the normalizing constant a. This is the reason that Rañada constructed the electromagnetic field solutions using complex potentials.

3.4 Particle Interpretation of Electromagnetic Helicity

As shown in the previous section, the helicity and the Hopf index of the map which is used to construct the field are strongly related and given by the equations

$$h_m(\mathbf{B}, D) = \int_D \mathrm{d}^3 r \mathbf{A}\cdot\mathbf{B} = aH(\phi) \ , \quad h_e(\mathbf{E}, D) = \int_D \mathrm{d}^3 r \mathbf{C}\cdot\mathbf{E} = aH(\theta) \ , \tag{3.67}$$

where $\mathbf{B}$ (and $\mathbf{E}$) is assumed to be parallel to the surface bounding the region D which is shown as ∂_D [16]. This relation combined with the particle helicity operator defined in quantum electrodynamics has a remarkable interpretation of the electromagnetic helicity. Let us first understand the time evolution of the magnetic or electric helicity. First note that using Maxwell equations we have

$$\int_D \mathrm{d}^3 r \left(\frac{\partial \mathbf{A}}{\partial t}\cdot\mathbf{B}\right) = \int_D \mathrm{d}^3 r(-\mathbf{E}\cdot\mathbf{B} - (\boldsymbol{\nabla} V)\cdot\mathbf{B}) \tag{3.68}$$

$$= \int_D \mathrm{d}^3 r(-\mathbf{E}\cdot\mathbf{B} - \boldsymbol{\nabla}\cdot(V\mathbf{B}) + V(\boldsymbol{\nabla}\cdot\mathbf{B})) \tag{3.69}$$

$$= \int_D \mathrm{d}^3 r(-\mathbf{E}\cdot\mathbf{B}) - \int_{\partial_D} (V\mathbf{B})\cdot\mathbf{n}\mathrm{d}S \tag{3.70}$$

$$= \int_D \mathrm{d}^3 r(-\mathbf{E}\cdot\mathbf{B}) \ , \tag{3.71}$$

where we have used the assumption that $\mathbf{B}$ is parallel to the surface ∂_D, so that $\mathbf{B}\cdot\mathbf{n} = 0$. Now, taking the time derivative of the magnetic helicity

$$\begin{aligned}
\frac{d}{dt}h_m(\mathbf{B}, D) =& \int_D \mathrm{d}^3 r \left(\frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} + \mathbf{A} \cdot \left(\boldsymbol{\nabla} \times \frac{\partial \mathbf{A}}{\partial t} \right) \right) \\
=& \int_D \mathrm{d}^3 r \left(- \mathbf{E} \cdot \mathbf{B} - \boldsymbol{\nabla} \cdot \left(\mathbf{A} \times \frac{\partial \mathbf{A}}{\partial t} \right) + (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \frac{\partial \mathbf{A}}{\partial t} \right) \\
=& - 2 \int_D \mathrm{d}^3 r \mathbf{E} \cdot \mathbf{B} - \int_{\partial D} \left(\mathbf{A} \times \frac{\partial \mathbf{A}}{\partial t} \right) \cdot \mathbf{n} \mathrm{d}S \\
=& - 2 \int_D \mathrm{d}^3 r \mathbf{E} \cdot \mathbf{B} \,,
\end{aligned} \tag{3.72}$$

where the system is assumed to be closed so that at the last step we have used $\frac{\partial \mathbf{A}}{\partial t}\big|_{\partial D} = 0$. This result shows that the conservation of the magnetic helicity depends on $(\mathbf{E} \cdot \mathbf{B})$. Thus the magnetic helicity is conserved for null fields. A similar calculation using the closed system assumption for the electric helicity results in

$$\begin{aligned}
\frac{d}{dt}h_e(\mathbf{E}, D) =& \int_D \mathrm{d}^3 r \left(\frac{\partial \mathbf{C}}{\partial t} \cdot \mathbf{E} + \mathbf{C} \cdot \left(\boldsymbol{\nabla} \times \frac{\partial \mathbf{C}}{\partial t} \right) \right) \\
=& \int_D \mathrm{d}^3 r \left(\mathbf{B} \cdot \mathbf{E} - \boldsymbol{\nabla} \cdot \left(\mathbf{C} \times \frac{\partial \mathbf{C}}{\partial t} \right) + (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \frac{\partial \mathbf{C}}{\partial t} \right) \\
=& 2 \int_D \mathrm{d}^3 r \mathbf{E} \cdot \mathbf{B} - \int_{\partial D} \left(\mathbf{C} \times \frac{\partial \mathbf{C}}{\partial t} \right) \cdot \mathbf{n} \mathrm{d}S \\
=& 2 \int_D \mathrm{d}^3 r \mathbf{E} \cdot \mathbf{B} \,.
\end{aligned} \tag{3.73}$$

Using the magnetic and electric helicities (h_m, h_e) one can define the electromagnetic helicity $h_{em} = (h_m + h_e)/2$. Thus

$$\frac{dh_{em}}{dt} = \frac{d}{dt}h_m + \frac{d}{dt}h_e = 0\,. \tag{3.74}$$

Obviously the electromagnetic helicity is constant and is independent of $\mathbf{E} \cdot \mathbf{B}$. If the circularly polarized waves are used for the decomposition of the electromagnetic field, then the electromagnetic helicity h_{em} can be written as [19]

$$h_{em} = \int \mathrm{d}^3 k (\bar{a}_R(\mathbf{k}) a_R(\mathbf{k}) - \bar{a}_L(\mathbf{k}) a_L(\mathbf{k}))\,, \tag{3.75}$$

where $a_R(\mathbf{k})$ and $a_L(\mathbf{k})$ denote the right and left circularly polarized components, respectively. These components are defined as the annihilation operators for photons (whereas the barred ones are defined as creation operators) in QED. Defining

$$N_R = \int \mathrm{d}^3 k \bar{a}_R(\mathbf{k}) a_R(\mathbf{k})\,, \qquad N_L = \int \mathrm{d}^3 k \bar{a}_L(\mathbf{k}) a_L(\mathbf{k})\,, \tag{3.76}$$

as the number operators for the right and left handed photons, (3.75) gives the difference between the number of right-handed and left-handed photons. Thus we can write (3.75) as

$$h_{em} = N_R - N_L\,, \tag{3.77}$$

In physical units ($\hbar \neq 1$ and $c \neq 1$) (3.77) would be $h_{em} = \hbar c(N_R - N_L)$, which shows that the helicity is the classical limit of the difference between right-handed and left-handed photons. Since the left-hand side of (3.77) is related to the electromagnetic wave whereas the right-hand side is related to the particle helicity, (3.77) shows the relation between the particle and wave aspects of the electromagnetic helicity. Using (3.67), (3.77) can be written as

$$h_{em} = \frac{h_m + h_e}{2} = \frac{aH(\phi) + aH(\theta)}{2} = N_R - N_L\,. \tag{3.78}$$

But we know that the Hopf index is related to the linking number and the linking number is zero if the topology is trivial. Thus, if we have a trivial topology, this shows that the classical value corresponding to the number of right-handed photons is equal to the number of left-handed photons [10].

There is another important observation about the magnetic and electric helicities. It can be proved that [10] $h_m = h_e$ for null fields, yielding

$$h_{em} = h_m = h_e = N_R - N_L\,. \tag{3.79}$$

Thus, one should be careful while constructing a null field since the helicities of the magnetic and electric fields should be the same. If we consider the relation of the helicity and the Hopf index, the previous discussion imposes that the maps we used to construct the fields should have the same Hopf index for the magnetic and electric fields.

3.5 Rañada Construction using the Hopf Map

In section 3.3 we showed that the complex potentials can be used to find new solutions of electromagnetic fields. In this section we will first find the initial magnetic and electric fields using the complex potentials. Then we will use Fourier transformation techniques in order to get the time-dependent solutions. Lastly we will analyze some properties of the knotted solutions.

3.5.1 Initial conditions for the knotted fields

In this subsection we will find the initial values for the magnetic and electric fields. Since the components of these fields require separate calculations and the calculations involved are similar to each other, we will do it for a single component only. The other components can be evaluated using similar arguments.

Let at $t = 0$, the magnetic and electric lines are given by the level curves of complex functions ϕ_{t_0} and θ_{t_0}, respectively. Using (3.39) and (3.43), we define

$$\mathbf{B}(\mathbf{r},t)\Big|_{t=0} = \mathbf{B}(\mathbf{r},0) = \frac{\sqrt{a}}{2\pi i}\frac{\boldsymbol{\nabla}\phi_{t_0} \times \boldsymbol{\nabla}\bar{\phi}_{t_0}}{\left(1+\phi_{t_0}\bar{\phi}_{t_0}\right)^2}, \tag{3.80}$$

$$\mathbf{E}(\mathbf{r},t)\Big|_{t=0} = \mathbf{E}(\mathbf{r},0) = \frac{\sqrt{a}}{2\pi i}\frac{\boldsymbol{\nabla}\bar{\theta}_{t_0} \times \boldsymbol{\nabla}\theta_{t_0}}{\left(1+\theta_{t_0}\bar{\theta}_{t_0}\right)^2} \tag{3.81}$$

define the initial values of the electromagnetic field where $\phi_{t_0} = \phi(\mathbf{r},0)$ and $\theta_{t_0} = \theta(\mathbf{r},0)$. As explained before due to the rich topological structure, we will use the Hopf map for the construction. In section 2.4 the Hopf map is defined as (2.38) where $z_1 = u_1 + iu_2$ and $z_2 = u_3 + iu_4$. It is also shown that the inverse stereographic projection can be used to compactify R^3 to S^3 by the map (2.36). As mentioned in [10], it is convenient to work with dimensionless coordinates which makes it possible to use this construction for different length scales. Thus, the dimensionless coordinates are defined as

$$T = \frac{t}{L_0}, \qquad X = \frac{x}{L_0}, \qquad Y = \frac{y}{L_0}, \qquad Z = \frac{z}{L_0}, \tag{3.82}$$

where L_0 is a constant defining the characteristic size with dimensions of length. Thus

$$\sqrt{X^2+Y^2+Z^2} = \sqrt{\frac{x^2+y^2+z^2}{{L_0}^2}} = \frac{r}{L_0} := R. \tag{3.83}$$

Using dimensionless coordinates

$$\frac{z_1}{z_2} = \frac{u_1+iu_2}{u_3+iu_4} = \frac{\frac{2X}{R^2+1}+i\frac{2Y}{R^2+1}}{\frac{2Z}{R^2+1}+i\frac{R^2-1}{R^2+1}} = \frac{2(X+iY)}{2Z+i(R^2-1)}. \tag{3.84}$$

Clearly this is a mapping from S^3 to $\mathbb{C}$. Using an inverse stereographic projection (2.31), one can also write the coordinates in S^2 as defined in the Hopf map in (2.38). This is not important for our discussion now. But we need to find a second map as in

(3.84), since we need two maps $(\phi_{t_0}, \theta_{t_0})$ to construct the magnetic and electric fields separately. But these maps could not be independent since we should satisfy the null field condition ($\mathbf{E} \cdot \mathbf{B} = 0$). This condition can be written as

$$(\boldsymbol{\nabla}\theta_{t_0} \times \boldsymbol{\nabla}\bar{\theta}_{t_0}) \cdot (\boldsymbol{\nabla}\phi_{t_0} \times \boldsymbol{\nabla}\bar{\phi}_{t_0}) = 0\,.$$

Since for each point in S^2(or in $\mathbb{C}$ using the stereographic projection) there is a fibre in S^3, the null field condition is satisfied if the fibrations of these maps are orthogonal in S^3. As shown in [10] two other maps which satisfy the orthogonality condition could be generated by changing the maps as $(X, Y, Z) \to (Y, Z, X)$ and $(X, Y, Z) \to (Z, X, Y)$. Thus, we define

$$\phi_{t_0} = \frac{2(X + iY)}{2Z + i(R^2 - 1)}\,, \tag{3.85}$$

$$\theta_{t_0} = \frac{2(Y + iZ)}{2X + i(R^2 - 1)}\,, \tag{3.86}$$

$$\psi_{t_0} = \frac{2(Z + iX)}{2Y + i(R^2 - 1)}\,. \tag{3.87}$$

It can be shown that all three maps are orthogonal to each other and that they have the same Hopf index $H(\phi) = H(\theta) = H(\psi) = 1$. Thus one can use any two of these maps to construct the electromagnetic field. Since the Poynting vector is defined as ($\mathbf{E} \times \mathbf{B}$), the fibers of the third one is tangent to the Poynting vector everywhere and describes the energy flux.

For our construction we will use the map ϕ for building the magnetic field and θ for building the electric field. To get a wider solution set having different Hopf indices and helicities, we will use the powers of these maps. Let

$$\begin{aligned}
\phi_{t_0} &= \frac{[2(X + iY)]^{(n)}}{[2Z + i(R^2 - 1)]^{(m)}} = \frac{(\rho_1 e^{i\alpha})^{(n)}}{(\rho_2 e^{i\beta})^{(m)}} = \frac{\rho_1 e^{in\alpha}}{\rho_2 e^{im\beta}} = \frac{\rho_1}{\rho_2} e^{i(n\alpha - m\beta)} = \frac{\rho_1}{\rho_2}\lambda_\phi, \\
\bar{\phi}_{t_0} &= \frac{\rho_1}{\rho_2}(\lambda_\phi)^{-1},
\end{aligned} \tag{3.88}$$

where

$$\begin{aligned}
{\rho_1}^2 &= 4(X^2 + Y^2), & {\rho_2}^2 &= 4Z^2 + (R^2 - 1)^2, \\
R^2 &= X^2 + Y^2 + Z^2, & \lambda_\phi &= e^{i(n\alpha - m\beta)}.
\end{aligned}$$

Similarly, for the electric field we define

$$\begin{aligned}\theta_{t_0} &= \frac{[2(Y+iZ)]^{(l)}}{[2X+i(R^2-1)]^{(s)}} = \frac{(\rho_1^{'}e^{i\alpha^{'}})^{(l)}}{(\rho_2^{'}e^{i\beta^{'}})^{(s)}} = \frac{\rho_1^{'}e^{il\alpha^{'}}}{\rho_2^{'}e^{is\beta^{'}}} = \frac{\rho_1^{'}}{\rho_2^{'}}e^{i(l\alpha^{'}-s\beta^{'})} = \frac{\rho_1^{'}}{\rho_2^{'}}\lambda_\theta, \\ \bar{\theta}_{t_0} &= \frac{\rho_1^{'}}{\rho_2^{'}}(\lambda_\theta)^{-1},\end{aligned} \tag{3.89}$$

where

$$\begin{aligned}({\rho_1}^{'})^2 &= 4(Y^2+Z^2), & ({\rho_2}^{'})^2 &= 4X^2+(R^2-1)^2, \\ R^2 &= X^2+Y^2+Z^2, & \lambda_\theta &= e^{i(l\alpha^{'}-s\beta^{'})}.\end{aligned}$$

Note that, $(n < 2m)$ and $(l < 2s)$ must be satisfied for the value of these maps at infinity not to depend on the direction, as imposed before.

Inserting (3.88) into (3.80) and (3.89) into (3.81), the magnetic and electric fields at $t = 0$ can be found. Let us find the first component of the magnetic field,

$$\begin{aligned}B_1(\mathbf{r},0) =& \frac{\sqrt{a}}{2\pi i}\frac{\partial_z\bar{\phi}_{t_0}\partial_y\phi_{t_0} - \partial_y\bar{\phi}_{t_0}\partial_z\phi_{t_0}}{(1+\bar{\phi}_{t_0}\phi_{t_0})^2} \\ =& \frac{\sqrt{a}(\rho_2)^4}{2\pi i(r^2+1)^4}\left[\left(\frac{\partial}{\partial z}\left(\frac{\rho_1}{\rho_2}\right)\lambda^{-1} + \frac{\rho_1}{\rho_2}\left(\frac{\partial}{\partial z}\lambda^{-1}\right)\right)\left(\frac{\partial}{\partial y}\left(\frac{\rho_1}{\rho_2}\right)\lambda + \frac{\rho_1}{\rho_2}\left(\frac{\partial}{\partial y}\lambda\right)\right)\right. \\ &\left. - \left(\frac{\partial}{\partial y}\left(\frac{\rho_1}{\rho_2}\right)\lambda^{-1} + \frac{\rho_1}{\rho_2}\left(\frac{\partial}{\partial y}\lambda^{-1}\right)\right)\left(\frac{\partial}{\partial z}\left(\frac{\rho_1}{\rho_2}\right)\lambda + \frac{\rho_1}{\rho_2}\left(\frac{\partial}{\partial z}\lambda\right)\right)\right] \\ =& \frac{\sqrt{a}(\rho_2)^4}{2\pi i(r^2+1)^4}\left[\frac{\partial}{\partial z}\left(\frac{\rho_1}{\rho_2}\right)\lambda^{-1}\frac{\rho_1}{\rho_2}\left(\frac{\partial}{\partial y}\lambda\right) - \frac{\rho_1}{\rho_2}\lambda^{-1}\left(\frac{\partial}{\partial z}\lambda\right)\frac{\partial}{\partial y}\left(\frac{\rho_1}{\rho_2}\right)\right. \\ &\left. - \frac{\partial}{\partial y}\left(\frac{\rho_1}{\rho_2}\right)\lambda^{-1}\frac{\rho_1}{\rho_2}\left(\frac{\partial}{\partial z}\lambda\right) + \frac{\rho_1}{\rho_2}\lambda^{-1}\left(\frac{\partial}{\partial y}\lambda\right)\frac{\partial}{\partial z}\left(\frac{\rho_1}{\rho_2}\right)\right] \\ =& \frac{\sqrt{a}(\rho_2)^4}{2\pi i(r^2+1)^4}\frac{2\rho_1}{\rho_2}\lambda^{-1}\left[\frac{\partial}{\partial z}\left(\frac{\rho_1}{\rho_2}\right)\frac{\partial}{\partial y}\lambda - \frac{\partial}{\partial y}\left(\frac{\rho_1}{\rho_2}\right)\frac{\partial}{\partial z}\lambda\right] \\ =& \frac{\sqrt{a}(\rho_2)^4}{2\pi i(r^2+1)^4}\frac{2\rho_1}{\rho_2}\left[\frac{\partial}{\partial z}\left(\frac{\rho_1}{\rho_2}\right)\frac{\partial}{\partial y}(n\alpha-m\beta)i - \frac{\partial}{\partial y}\left(\frac{\rho_1}{\rho_2}\right)\frac{\partial}{\partial z}(n\alpha-m\beta)i\right].\end{aligned} \tag{3.90}$$

Now we need the following partial derivatives,

$$\frac{\partial \rho_1}{\partial z} = 0\,, \qquad \frac{\partial \rho_1}{\partial y} = \frac{4Y}{\rho_1 L_0}\,,$$
$$\frac{\partial \rho_2}{\partial z} = \frac{2Z(R^2+1)}{\rho_2 L_0}\,, \qquad \frac{\partial \rho_2}{\partial y} = \frac{2Y(R^2-1)}{\rho_2 L_0}\,,$$
$$\frac{\partial \alpha}{\partial y} = \frac{4X}{{\rho_1}^2 L_0}\,, \qquad \frac{\partial \alpha}{\partial z} = 0\,,$$
$$\frac{\partial \beta}{\partial y} = \frac{4YZ}{{\rho_2}^2 L_0}\,, \qquad \frac{\partial \beta}{\partial z} = \frac{4Z^2+2(R^2-1)}{{\rho_2}^2 L_0}\,.$$

Inserting these partial derivatives into (3.90) and simplifying, one gets

$$B_1(\mathbf{r},0) = \frac{8\sqrt{a}}{\pi L_0^2(1+R^2)^3}(mY - nXZ)\,. \tag{3.91}$$

Doing similar calculations for the other magnetic field components and the electric field components, one gets

$$\mathbf{B}(\mathbf{r},0) = \frac{8\sqrt{a}}{\pi L_0^2(1+R^2)^3}\left(mY - nXZ, -mX - nYZ, n\frac{X^2+Y^2-Z^2-1}{2}\right), \tag{3.92}$$

$$\mathbf{E}(\mathbf{r},0) = \frac{8\sqrt{a}}{\pi L_0^2(1+R^2)^3}\left(l\frac{X^2-Y^2-Z^2+1}{2}, lXY - sZ, lXZ + sY\right). \tag{3.93}$$

3.5.2 Time dependent knotted fields

We have the initial values for the magnetic and electric fields and we can obtain the time dependent expressions using Fourier transform techniques [20]. We need to evaluate some integrals for this purpose. Since the integrals involved are similar to each other, we will show the calculations for a single component only. The other components can be evaluated using similar arguments.

Let us use plane wave solutions for field decomposition

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\big(\mathbf{B}_+(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{B}_-(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\big)\,, \tag{3.94}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\big(\mathbf{E}_+(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E}_-(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\big) \tag{3.95}$$

where $\omega^2 = \mathbf{k}^2$. Then the initial fields at $t = 0$ can be decomposed as

$$\begin{aligned}\mathbf{B}(\mathbf{r},0) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\big(\mathbf{B}_+(\mathbf{k}) + \mathbf{B}_-(\mathbf{k})\big)e^{-i\mathbf{k}\cdot\mathbf{r}}\,, \\ \mathbf{E}(\mathbf{r},0) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\big(\mathbf{E}_+(\mathbf{k}) + \mathbf{E}_-(\mathbf{k})\big)e^{-i\mathbf{k}\cdot\mathbf{r}}\,.\end{aligned} \tag{3.96}$$

Defining

$$\begin{aligned}\mathbf{B}'(\mathbf{k}) &= \mathbf{B}_+(\mathbf{k}) + \mathbf{B}_-(\mathbf{k})\,, \\ \mathbf{E}'(\mathbf{k}) &= \mathbf{E}_+(\mathbf{k}) + \mathbf{E}_-(\mathbf{k})\,,\end{aligned} \tag{3.97}$$

(3.96) can be written as

$$\begin{aligned}\mathbf{B}(\mathbf{r},0) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\mathbf{B}'(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}\,, \\ \mathbf{E}(\mathbf{r},0) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\mathbf{E}'(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}\,.\end{aligned} \tag{3.98}$$

So the inverse transforms are

$$\begin{aligned}\mathbf{B}'(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3r\mathbf{B}(\mathbf{r},0)e^{i\mathbf{k}\cdot\mathbf{r}}\,, \\ \mathbf{E}'(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3r\mathbf{E}(\mathbf{r},0)e^{i\mathbf{k}\cdot\mathbf{r}}\,.\end{aligned} \tag{3.99}$$

Inserting (3.94) and (3.95) into the second line of (2.3), we obtain

$$\Big(\boldsymbol{\nabla}\times\big(\mathbf{E}_+(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E}_-(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\big)\Big)\Big|_{t=0} = \\ -\frac{\partial}{\partial t}\big(\mathbf{B}_+(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{B}_-(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\big)\Big|_{t=0}$$

$$-i\mathbf{k}e^{-i(\mathbf{k}\cdot\mathbf{r})}\times(\mathbf{E}_+(\mathbf{k}) + \mathbf{E}_-(\mathbf{k})) = -i\omega e^{-i(\mathbf{k}\cdot\mathbf{r})}\big(\mathbf{B}_+(\mathbf{k}) + \mathbf{B}_-(\mathbf{k})\big) \tag{3.100}$$

$$\mathbf{k}\times\mathbf{E}'(\mathbf{k}) = k(\mathbf{B}_+(\mathbf{k}) - \mathbf{B}_-(\mathbf{k}))\,. \tag{3.101}$$

Similarly inserting (3.95) into (2.4), we get

$$\mathbf{k}\times\mathbf{B}'(\mathbf{k}) = k(\mathbf{E}_+(\mathbf{k}) - \mathbf{E}_-(\mathbf{k}))\,. \tag{3.102}$$

We need $\mathbf{B}_+(\mathbf{k}), \mathbf{B}_-(\mathbf{k}), \mathbf{E}_+(\mathbf{k})$ and $\mathbf{E}_-(\mathbf{k})$ to find the time dependent fields. Using (3.97), (3.101) and (3.102), we obtain

$$\begin{aligned}\mathbf{B}_+(\mathbf{k}) &= \frac{1}{2}\big(\mathbf{B}'(\mathbf{k}) + \frac{\mathbf{k}}{k}\times\mathbf{E}'(\mathbf{k})\big)\,, \\ \mathbf{B}_-(\mathbf{k}) &= \frac{1}{2}\big(\mathbf{B}'(\mathbf{k}) - \frac{\mathbf{k}}{k}\times\mathbf{E}'(\mathbf{k})\big)\,, \\ \mathbf{E}_+(\mathbf{k}) &= \frac{1}{2}\big(\mathbf{E}'(\mathbf{k}) - \frac{\mathbf{k}}{k}\times\mathbf{B}'(\mathbf{k})\big)\,, \\ \mathbf{E}_-(\mathbf{k}) &= \frac{1}{2}\big(\mathbf{E}'(\mathbf{k}) + \frac{\mathbf{k}}{k}\times\mathbf{B}'(\mathbf{k})\big)\,.\end{aligned} \tag{3.103}$$

Inserting the expressions in (3.103) into (3.94) and (3.95), we get

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\big(\mathbf{B}^{'}(\mathbf{k})\cos\omega t + i\frac{\mathbf{k}}{k}\times\mathbf{E}^{'}(\mathbf{k})\sin\omega t\big)e^{-i\mathbf{k}\cdot\mathbf{r}}\,,\\ \mathbf{E}(\mathbf{r},t) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\big(\mathbf{E}^{'}(\mathbf{k})\cos\omega t - i\frac{\mathbf{k}}{k}\times\mathbf{B}^{'}(\mathbf{k})\sin\omega t\big)e^{-i\mathbf{k}\cdot\mathbf{r}}\,.\end{aligned} \tag{3.104}$$

Now we need to evaluate the integrals in (3.99) and insert the results into (3.104). Inserting the initial conditions (3.81) and (3.80) into (3.99) and using the dimensionless coordinates defined as

$$k_x = \frac{K_x}{L_0}\,, \qquad k_y = \frac{K_y}{L_0}\,, \qquad k_z = \frac{K_z}{L_0}\,, \tag{3.105}$$

$$\sqrt{k_x^2+k_y^2+k_z^2} = \sqrt{\frac{K_x^2+K_y^2+K_z^2}{{L_0}^2}} = \frac{K}{L_0} = k = \omega\,, \tag{3.106}$$

we get

$$\begin{aligned}\mathbf{B}^{'}(\mathbf{k}) &= \frac{8\sqrt{a}L_0}{(2\pi)^{5/2}}\int \mathrm{d}^3R e^{i\mathbf{K}\cdot\mathbf{R}}\\ &\qquad\left[\frac{1}{(1+R^2)^3}\left(mY-nXZ,-mX-nYZ,n\frac{X^2+Y^2-Z^2-1}{2}\right)\right]\\ \mathbf{E}^{'}(\mathbf{k}) &= \frac{8\sqrt{a}L_0}{(2\pi)^{5/2}}\int \mathrm{d}^3R e^{i\mathbf{K}\cdot\mathbf{R}}\\ &\qquad\left[\frac{1}{(1+R^2)^3}\left(l\frac{X^2-Y^2-Z^2+1}{2},lXY-sZ,lXZ+sY\right)\right].\end{aligned} \tag{3.107}$$

Before evaluating these integrals, it will be useful to observe that

$$\partial_{K_i}\frac{e^{i\mathbf{K}\cdot\mathbf{R}}}{(1+R^2)^3} = iX_i\frac{e^{i\mathbf{K}\cdot\mathbf{R}}}{(1+R^2)^3}\,. \tag{3.108}$$

So let us first evaluate the integral

$$\int \mathrm{d}^3R\frac{e^{i\mathbf{K}\cdot\mathbf{R}}}{(1+R^2)^3}\,. \tag{3.109}$$

We can rotate the R-space so that the Z-axis is aligned along $\mathbf{K}$ and evaluate the integral in spherical coordinates

$$
\begin{aligned}
\iiint \mathrm{d}R\,\mathrm{d}\theta\,\mathrm{d}\phi\,\frac{R^2e^{iKR\cos\theta}\sin\theta}{(1+R^2)^3} &= 2\pi\iint \mathrm{d}R\,\mathrm{d}\theta\,\frac{R^2e^{iKR\cos\theta}\sin\theta}{(1+R^2)^3}\\
&= 2\pi\int \mathrm{d}R\left[\frac{R^2}{(1+R^2)^3}\int \mathrm{d}\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{e^{iKR\cos\theta}}{-iKR}\right)\right]\\
&= 4\pi\int \mathrm{d}R\,\frac{R^2\sin(KR)}{KR(1+r^2)^3}\\
&= \frac{4\pi}{K}\int \mathrm{d}R\,\frac{R\sin(KR)}{(1+R^2)^3}\\
&= \frac{1}{4}\pi^2e^{-K}(K+1)\,.
\end{aligned}
\tag{3.110}
$$

Note that the last step is evaluated using calculus of residues. Now we can calculate the integral in (3.107) by using the result of (3.108) and (3.110). Let us evaluate the third component of the magnetic field defined in (3.107)

$$
\begin{aligned}
B_3'(\mathbf{k}) =&\frac{8\sqrt{a}L_0}{(2\pi)^{5/2}}\int \mathrm{d}^3R\frac{1}{(1+R^2)^3}\frac{n(X^2+Y^2-Z^2-1)}{2}e^{i\mathbf{K}\cdot\mathbf{R}}\\
=&\frac{8\sqrt{a}L_0}{(2\pi)^{5/2}}n\bigg(\int \mathrm{d}^3R\frac{X^2e^{i\mathbf{K}\cdot\mathbf{R}}}{(1+R^2)^3}+\int \mathrm{d}^3R\frac{Y^2e^{i\mathbf{K}\cdot\mathbf{R}}}{(1+R^2)^3}\\
&-\int \mathrm{d}^3R\frac{Z^2e^{i\mathbf{K}\cdot\mathbf{R}}}{(1+R^2)^3}-\int \mathrm{d}^3R\frac{e^{i\mathbf{K}\cdot\mathbf{R}}}{(1+R^2)^3}\bigg)
\end{aligned}
\tag{3.111}
$$

We have already evaluated the last integral in (3.110). Let us evaluate the other integrals using the result of (3.110)

$$
\begin{aligned}
\int \mathrm{d}^3RX^2e^{i\mathbf{K}\cdot\mathbf{R}} &= -\frac{\partial^2}{\partial K_x^2}\int \mathrm{d}^3Re^{i\mathbf{K}\cdot\mathbf{R}}\\
&= -\frac{\partial^2\big(\frac{1}{4}\pi^2e^{-K}(K+1)\big)}{\partial K_x^2}\\
&= \frac{1}{4}\pi^2\frac{e^{-K}(K-K_x^2)}{K}\,.
\end{aligned}
\tag{3.112}
$$

Similarly

$$
\int \mathrm{d}^3RY^2e^{i\mathbf{K}\cdot\mathbf{R}} = \frac{1}{4}\pi^2\frac{e^{-K}(K-K_y^2)}{k}\,, \tag{3.113}
$$

$$
\int \mathrm{d}^3RZ^2e^{i\mathbf{K}\cdot\mathbf{R}} = \frac{1}{4}\pi^2\frac{e^{-K}(K-K_z^2)}{K}\,. \tag{3.114}
$$

Thus

$$B_3^{'}(\mathbf{k}) = \frac{8\sqrt{a}L_0}{(2\pi)^{5/2}}\bigg(\frac{1}{4}\pi^2 e^{-K}\bigg(\frac{(K-K_x^2)}{K} + \frac{(K-K_y^2)}{K} - \frac{(K-K_z^2)}{K} - (K+1)\bigg) \tag{3.115}$$

$$= \frac{L_0\sqrt{a}e^{-K}}{\sqrt{2\pi}}\Big[\frac{n}{K}(-K_x^2 - K_y^2)\Big]. \tag{3.116}$$

We can find all three components using the same procedure. Then we get the Fourier transforms of the initial magnetic and electric fields as

$$\begin{aligned}\mathbf{B}^{'}(\mathbf{k}) &= \frac{L_0\sqrt{a}e^{-K}}{\sqrt{2\pi}}\Big[\frac{n}{K}(K_xK_z, K_yK_z, -K_x^2 - K_y^2) + im(K_y, -K_x, 0)\Big],\\ \mathbf{E}^{'}(\mathbf{k}) &= \frac{L_0\sqrt{a}e^{-K}}{\sqrt{2\pi}}\Bigg[\frac{l}{K}(K_y^2 + K_z^2, -K_xK_y, -K_xK_z) + is(0, -K_z, K_y)\Bigg].\end{aligned} \tag{3.117}$$

We also need to compute $\frac{\mathbf{k}}{k} \times (\mathbf{B}_0(\mathbf{k}))$, $\frac{\mathbf{k}}{k} \times (\mathbf{E}_0(\mathbf{k}))$ and insert them into Eq.(3.104) to find the desired fields. These can be found easily as

$$\begin{aligned}\frac{\mathbf{k}}{k} \times \mathbf{B}^{'}(\mathbf{k}) &= \frac{L_0\sqrt{a}e^{-K}}{\sqrt{2\pi}}\Big[n(-K_y, K_x, 0) + i\frac{m}{K}(K_xK_z, K_yK_z, -K_x^2 - K_y^2)\Big],\\ \frac{\mathbf{k}}{k} \times \mathbf{E}^{'}(\mathbf{k}) &= \frac{L_0\sqrt{a}e^{-K}}{\sqrt{2\pi}}\Big[\frac{s}{K}(-K_y^2 - K_z^2, K_xKy, K_xKz) + il(0, K_z, -K_y)\Big].\end{aligned} \tag{3.118}$$

Inserting (3.117) and (3.118) into (3.104), we can find the magnetic and electric fields as

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \frac{\sqrt{a}}{4\pi^2 L_0^2}\int \mathrm{d}^3 K e^{-i\mathbf{K}\cdot\mathbf{R}}\\ &\quad\Bigg[\cos(KT)\frac{e^{-K}}{K}\Big(\frac{n}{K}(K_xK_z, K_yK_z, -K_x^2 - K_y^2) + im(K_y, -Kx, 0)\Big)\\ &\quad + i\sin(KT)\Big(\frac{s}{K}(-K_y^2 - K_z^2, K_xKy, K_xKz) + il(0, K_z, -K_y)\Big)\Bigg],\\ \mathbf{E}(\mathbf{r}, t) &= \frac{\sqrt{a}}{4\pi^2 L_0^2}\int \mathrm{d}^3 K e^{-i\mathbf{K}\cdot\mathbf{R}}\\ &\quad\Bigg[\cos(KT)\frac{e^{-K}}{K}\Bigg(\frac{l}{K}(K_y^2 + K_z^2, -K_xK_y, -K_xK_z) + is(0, -K_z, K_y)\Bigg)\\ &\quad - i\sin(KT)\Big(n(-K_y, K_x, 0) + i\frac{m}{K}(K_xK_z, K_yK_z, -K_x^2 - K_y^2)\Big)\Bigg].\end{aligned} \tag{3.119}$$

Now we have a similar case as in (3.108)

$$\partial_{X_i}\left(\int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\cos(KT)\frac{e^{-K}}{K}\right) = -iK_x \int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\cos(KT)\frac{e^{-K}}{K}\,. \tag{3.120}$$

So we can evaluate the integrals listed below and later obtain the required expressions using the observation (3.120)

$$\begin{aligned}
&\frac{\sqrt{a}}{4\pi^2 {L_0}^2}\int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\cos(KT)\frac{e^{-K}}{K}\,,\\
&\frac{\sqrt{a}}{4\pi^2 {L_0}^2}\int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\cos(KT)e^{-K}\,,\\
&\frac{\sqrt{a}}{4\pi^2 {L_0}^2}\int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\sin(KT)\frac{e^{-K}}{K}\,,\\
&\frac{\sqrt{a}}{4\pi^2 {L_0}^2}\int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\sin(KT)e^{-K}\,.
\end{aligned} \tag{3.121}$$

Let us calculate the second one since it is a bit harder than the others. Similar to the integral (3.109), we can rotate the K-space so that $\mathbf{K}_z$ is aligned along $\mathbf{R}$ and evaluate the integral in spherical coordinates

$$\begin{aligned}
&\frac{\sqrt{a}}{4\pi^2 {L_0}^2}\iiint \mathrm{d}K\,\mathrm{d}\theta\,\mathrm{d}\phi e^{-iKR\cos\theta}\cos(KT)e^{-K}K^2\sin\theta\\
&\qquad = \frac{\sqrt{a}}{2\pi {L_0}^2}\int \mathrm{d}K\cos(KT)e^{-K}K^2\int \mathrm{d}\theta\, e^{-iKR\cos\theta}\sin\theta\\
&\qquad = \frac{\sqrt{a}}{\pi {L_0}^2}\int \mathrm{d}K\cos(KT)e^{-K}K\frac{\sin(KR)}{R}\\
&\qquad = \frac{\sqrt{a}}{\pi {L_0}^2 r}\int \mathrm{d}K\,\frac{\sin[(R+T)K]+\sin[(R-T)K]}{2}e^{-K}K\\
&\qquad = \frac{\sqrt{a}}{2\pi {L_0}^2 R}\left(\frac{2(R-T)}{[(R-T)^2+1]^2}+\frac{2(R+T)}{[(R+T)^2+1]^2}\right).
\end{aligned} \tag{3.122}$$

After some algebra, we get

$$\frac{\sqrt{a}}{2\pi {L_0}^2}\left(\frac{A^2-T^2+2AT^2}{(A^2+T^2)^2}\right) \tag{3.123}$$

where $A = \frac{R^2-T^2+1}{2}$. Similar calculations give

$$\begin{aligned}
\frac{\sqrt{a}}{4\pi^2{L_0}^2}\int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\cos(KT)\frac{e^{-K}}{K} &= \frac{\sqrt{a}}{2\pi{L_0}^2}\left(\frac{A}{A^2+T^2}\right),\\
\frac{\sqrt{a}}{4\pi^2{L_0}^2}\int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\cos(KT)e^{-K} &= \frac{\sqrt{a}}{2\pi{L_0}^2}\left(\frac{A^2-T^2+2AT^2}{(A^2+T^2)^2}\right),\\
\frac{\sqrt{a}}{4\pi^2{L_0}^2}\int \mathrm{d}^3K e^{-i\mathbf{K}\cdot\mathbf{R}}\sin(KT)\frac{e^{-K}}{K} &= \frac{\sqrt{a}}{2\pi{L_0}^2}\left(\frac{T}{A^2+T^2}\right),\\
\frac{\sqrt{a}}{4\pi^2{L_0}^2}\int \mathrm{d}^3k e^{-i\mathbf{K}\cdot\mathbf{R}}\sin(KT)e^{-K} &= \frac{\sqrt{a}}{2\pi{L_0}^2}\left(\frac{-A^2T+2AT+T^3}{(A^2+T^2)^2}\right).
\end{aligned} \tag{3.124}$$

Using the results in (3.124) and (3.120), the magnetic and electric fields can be written as

$$\mathbf{B}(\mathbf{r},t) = \frac{\sqrt{a}}{\pi{L_0}^2}\frac{Q\mathbf{H}_1+P\mathbf{H}_2}{(A^2+T^2)^3}, \tag{3.125}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{\sqrt{a}}{\pi{L_0}^2}\frac{Q\mathbf{H}_4-P\mathbf{H}_3}{(A^2+T^2)^3}. \tag{3.126}$$

where the vectors $\mathbf{H}_u$ ($u = 1,2,3,4$) are defined as

$$\begin{aligned}
\mathbf{H}_1 &= \Big(-nXZ+mY+sT, -nYZ-mX-lTZ,\\
&\quad n\frac{X^2+Y^2-Z^2+T^2-1}{2}+lTY\Big),\\
\mathbf{H}_2 &= \Big(s\frac{X^2-Y^2-Z^2-T^2+1}{2}-mTY, sXY-lZ+mTX,\\
&\quad sXZ+lY+nT\Big),\\
\mathbf{H}_3 &= \Big(-mXZ+nY+lT, -mYZ-nX-sTZ,\\
&\quad m\frac{X^2+Y^2-Z^2+T^2-1}{2}+sTY\Big),\\
\mathbf{H}_4 &= \Big(l\frac{X^2-Y^2-Z^2-T^2+1}{2}-nTY, lXY-sZ+nTX,\\
&\quad lXZ+sY+mT\Big),
\end{aligned}$$

and A, P, Q are defined as

$$A = \frac{R^2 - T^2 + 1}{2}\,,$$
$$P = T(T^2 - 3A^2)\,,$$
$$Q = A(A^2 - 3T^2)\,.$$

The first topologically nontrivial solution constructed by Rañada [9] was based on Hopf fibration with $n = m = l = s = 1$ and it is called *electromagnetic Hopfion*. For the Hopfion

$$\begin{aligned} n = m = l = s = 1 &\implies \mathbf{H}_1 = \mathbf{H_3}, \mathbf{H}_2 = \mathbf{H_4}\,, \\ &\implies \mathbf{B}(\mathbf{r}, t) = \frac{\sqrt{a}}{\pi {L_0}^2} \frac{Q\mathbf{H}_1 + P\mathbf{H}_2}{(A^2 + T^2)^3}\,, \\ &\implies \mathbf{E}(\mathbf{r}, t) = \frac{\sqrt{a}}{\pi {L_0}^2} \frac{Q\mathbf{H}_2 - P\mathbf{H}_1}{(A^2 + T^2)^3}\,. \end{aligned} \tag{3.127}$$

3.5.3 Interpretation of the knotted solutions and their properties

It can be shown [21] that the magnetic and electric fields of the Hopfion, found in (3.127), can be obtained by using the time dependent maps

$$\phi = \frac{(AX - TZ) + i(AY + T(A-1))}{(AZ + TX) + i(A(A-1) - TY)}\,, \tag{3.128}$$
$$\theta = \frac{(AY + T(A-1)) + i(AZ + TX)}{(AX - TZ) + i(A(A-1) - TY)}\,. \tag{3.129}$$

Note that we have started with the initial conditions defined by the maps ϕ_{t_0} and θ_{t_0}. Then applying Fourier transform techniques we obtained the time dependent fields and we see that these fields can be obtained directly, using the time dependent maps defined in (3.128). If we define the coordinates u_μ as

$$u_1 = \frac{AX - TZ}{(A^2 + T^2)}\,, \tag{3.130}$$
$$u_2 = \frac{AY + T(A-1)}{(A^2 + T^2)}\,, \tag{3.131}$$
$$u_3 = \frac{AZ + TX}{(A^2 + T^2)}\,, \tag{3.132}$$
$$u_4 = \frac{A(A-1) - TY}{(A^2 + T^2)}\,, \tag{3.133}$$

then we can rewrite the maps in (3.128) as

$$\phi = \frac{u_1 + iu_2}{u_3 + iu_4}\,, \tag{3.134}$$

$$\theta = \frac{u_2 + iu_3}{u_1 + iu_4}\,. \tag{3.135}$$

Note that the coordinates u_μ satisfy $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$. Thus these can be considered as the coordinates in S^3. At $t = 0$, these coordinates correspond to the coordinates obtained by the stereographic projection but they deform in time. Thus, they can be regarded as the coordinates of a time-dependent stereographic projection [1].

As shown in section 3.2, one can find the Euler potentials corresponding to the complex potentials or vice versa. Since the level curves of the Euler potentials define the magnetic surfaces, the intersection of these surfaces gives the magnetic field lines. The Euler potentials of the Hopfion can be written as [1]

$$\begin{aligned}
\alpha_1(\mathbf{r},t) &= \frac{(AZ+TX)^2 + (A(A-1)-TY)^2}{(A^2+T^2)^2}\,,\\
\alpha_2(\mathbf{r},t) &= \frac{1}{2\pi}\arctan\\
&\quad \left(\frac{(A^2-T^2)(YZ-(A-1)X) + 2AT(XY+(A-1)Z)}{(A^2-T^2)(XZ+(A-1)Y) + AT((A-1)^2+X^2-Y^2-Z^2)}\right),\\
\beta_1(\mathbf{r},t) &= \frac{(AX-TZ)^2 + (A(A-1)-TY)^2}{(A^2+T^2)^2}\,,\\
\beta_2(\mathbf{r},t) &= \frac{1}{2\pi}\arctan\\
&\quad \left(\frac{(A^2-T^2)(XZ-(A-1)Y) + AT(X^2+Y^2-Z^2-(A-1)^2)}{(A^2-T^2)(XY+(A-1)Z) + 2AT((A-1)X-YZ)}\right).
\end{aligned}$$

In terms of these potentials, the magnetic and electric fields are defined as

$$\mathbf{B}(\mathbf{r},t) = \sqrt{a}\,\nabla\alpha_1 \times \nabla\alpha_2\,, \tag{3.136}$$

$$\mathbf{E}(\mathbf{r},t) = \sqrt{a}\,\nabla\beta_1 \times \nabla\beta_2\,. \tag{3.137}$$

Thus, $\alpha_1 = k_1$ and $\alpha_2 = k_2$ define the magnetic field lines for each k_1, k_2 pair. Similarly $\beta_1 = k_1^{'}$ and $\beta_2 = k_2^{'}$ define the electric field lines. For the magnetic field at $t = 0$, the Euler potentials of the Hopfion are

$$\begin{aligned}
\alpha_1(\mathbf{r},0) &= \frac{4Z^2 + (R^2-1)^2}{(R^2+1)^2}\,,\\
\alpha_2(\mathbf{r},0) &= \frac{1}{2\pi}\arctan\left(\frac{2YZ - X(R^2-1)}{2XZ + Y(R^2-1)}\right).
\end{aligned}$$

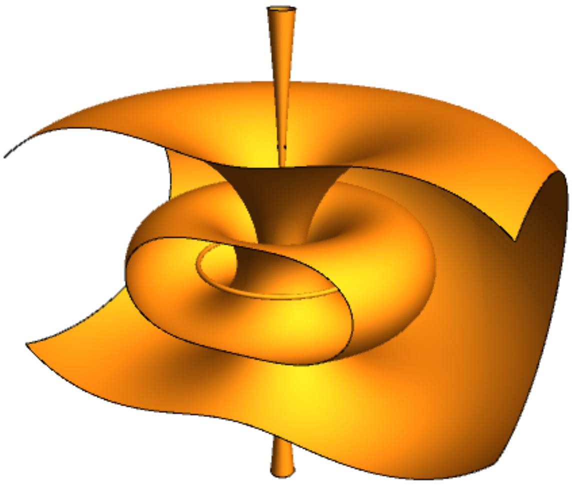

Figure 3.1: Magnetic surfaces of Hopfion at $T = 0$ for $\alpha_1(\mathbf{r}, t) = k_1$ where $k_1 = 0.001, 0.3, 0.75, 0.999$. As k_1 approaches to 1, the radius of the torus goes to infinity. As k_1 approaches to 0, the torus shrinks to a circle.

As can be seen in figure 3.1, at $t = 0$, the magnetic surfaces corresponding to the Hopfion for $\alpha_1 = constant$ are tori. The surfaces given by $\alpha_2 = constant$ are not tori and more complicated then the $\alpha_1 = constant$ surfaces as shown in figure 3.2. The intersection of $\alpha_1 = constant$ surfaces with $\alpha_2 = constant$ surfaces gives the

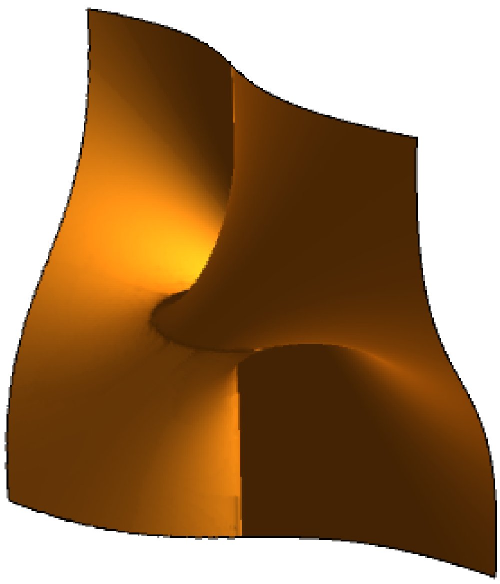

Figure 3.2: Magnetic surfaces of Hopfion at $T = 0$ for $\alpha_2(\mathbf{r}, t) = k_2$ where $k_2 = 0$.

magnetic lines. If we imagine the intersection of these surfaces as depicted in figure 3.3, then we see that they will be closed curves on the surface of a torus, which are topologically equivalent to circles. Since the Hopf index for the Hopfion is equal to 1, the magnetic field lines are circles, linked to each other only once. The set of all these circles is called *Hopf fibration* [15].

If we consider the general case, we see that the field lines are knotted curves lying on a torus. These knots are defined by two coprime integers (n, m), such that the curve winds around a circle in the torus n times and the curve winds the line through the hole of the torus m times [1]. Some of these torus knots can be seen in figure 3.4.

Since we have (n, m) and (l, s) torus knot solutions for the electromagnetic field, we can analyze the properties of this class. Our initial construction was based on decomposable fields that is $\mathbf{E} \cdot \mathbf{B} = 0$. Let us see if this is valid for all knotted

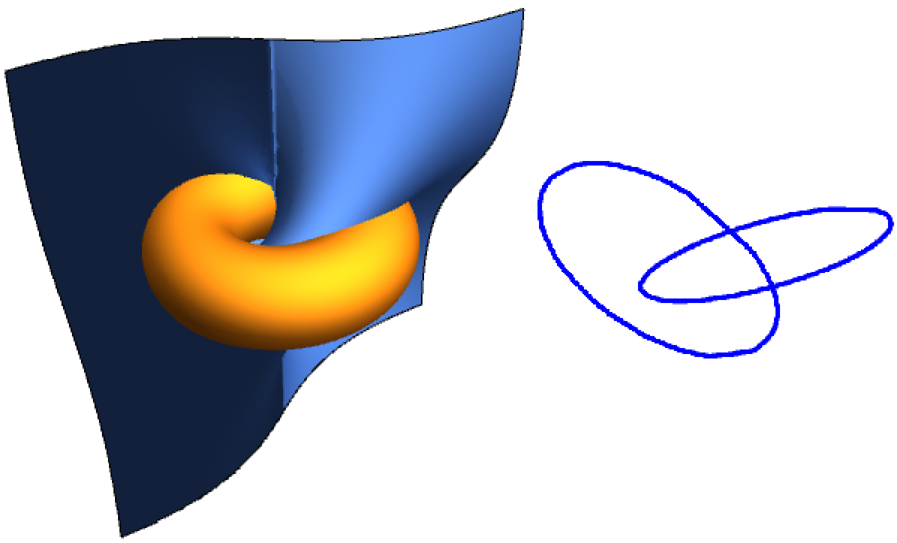

Figure 3.3: Magnetic surfaces of Hopfion at $T = 0$ for $\alpha_1(\mathbf{r}, t) = 0.3$ and $\alpha_2(\mathbf{r}, t) = 0$ (left). The intersection of these surfaces, which are topologically equivalent to circles, gives the magnetic field lines (right).

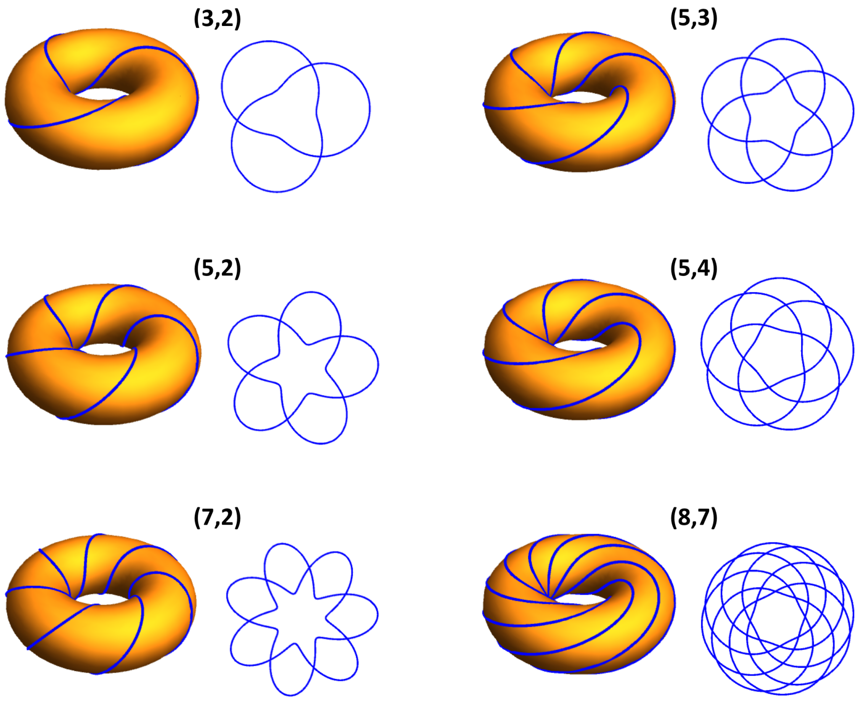

Figure 3.4: (n, m) torus knots lying on the surface of a torus. Each (n, m) knot is shown on a torus (left) and alone (right).

solutions. Ignoring the coefficients, Lorentz invariants can be computed as

$$\begin{aligned}\mathbf{E}\cdot\mathbf{B} = \frac{a}{\pi^2 L_0^4}\frac{1}{(A^2+T^2)^4}\Big((ms-nl)(A^2+T^2)+ \\ (ml-ns)(A^2+T^2)(1-A)Y \\ +2(ls-mn)AT(T^2-A^2)\Big), \end{aligned} \tag{3.138}$$

$$\begin{aligned}E^2-B^2 = \frac{a}{\pi^2 L_0^4}\frac{1}{(A^2+T^2)^4}\Big((n^2-m^2)(A^2+T^2)(X^2+Y^2) \\ +(s^2-l^2)(A^2+T^2)(Y^2+Z^2) \\ +4(m^2-s^2)A^2T^2-(n^2-l^2)(A^2-T^2)^2\Big). \end{aligned} \tag{3.139}$$

It is easy to see that if all the positive numbers n, m, l, s are equal as in the Hopfion case, then we get $\mathbf{E}\cdot\mathbf{B} = 0$ and $E^2 - B^2 = 0$, thus the resulting field is a null-field. If this condition is not satisfied, then we have a non-null field. This is consistent with the observation (3.79). As a result, changing the (n, m), (l, s) values, we get a wider class of knotted solutions for Maxwell equations which also covers the non-null fields.

If we compute the helicities of these knotted solutions

$$h_m(t) = \int \mathrm{d}^3 r \mathbf{A} \cdot \mathbf{B} = \frac{a}{2}\left((nm + ls) + (nm - ls)\frac{1 - 6T^2 + T^4}{(1 + T^2)^4}\right), \tag{3.140}$$

$$h_e(t) = \int \mathrm{d}^3 r \mathbf{C} \cdot \mathbf{E} = \frac{a}{2}\left((nm + ls) - (nm - ls)\frac{1 - 6T^2 + T^4}{(1 + T^2)^4}\right). \tag{3.141}$$

At the end of the section 3.4 we have seen that the Hopf indices of the two maps, used to construct the magnetic and electric fields, should be the same for null fields. Thus it is easy to understand that one should have $mn = ls$, which results in $h_m = h_e = a(mn)$ for a null field as expected in (3.79). If we return to the general case, the electromagnetic helicity can be written as

$$h_{em} = \frac{h_m + h_e}{2} = a(nm + ls)\,. \tag{3.142}$$

We see that h_{em} is constant and does not change in time as we found in (3.74). If we evaluate the magnetic and electric helicities as time goes to infinity, then we find

$$\lim_{t\to\infty} h_m(t) = \lim_{t\to\infty} h_e(t) = \frac{a}{2}(nm + ls)\,. \tag{3.143}$$

We see that if the helicities of the magnetic and electric fields are not equal initially, then they approach to the same value, the half of the electromagnetic helicity, in time. Since we know from (3.142) that the sum of the helicities is constant, this means that there is a helicity exchange between the magnetic and electric fields as in figure 3.5. This is called exchange of helicities mechanism and was first studied in [22].

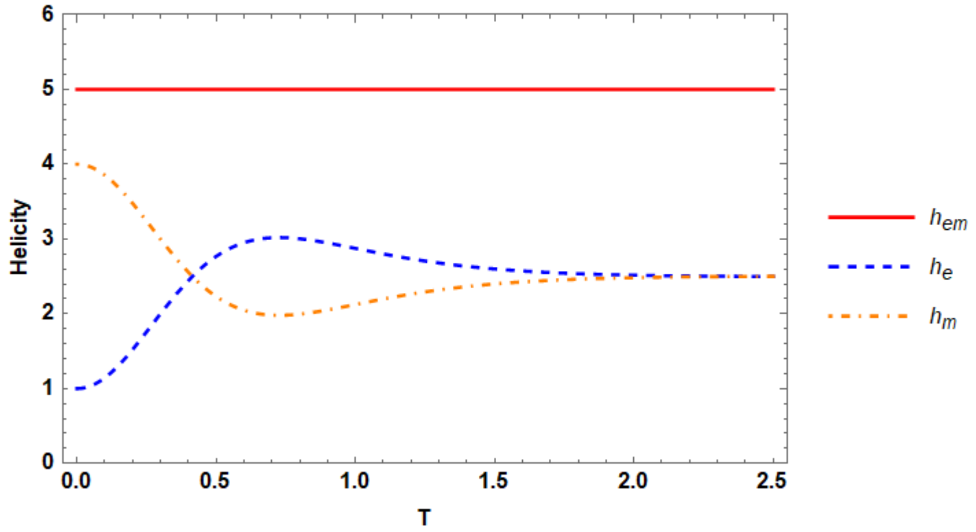

Figure 3.5: Evolution of the magnetic and electric helicities for $n = m = 2$ and $l = s = 1$, where the helicities are represented in units of a. The dashed line with dots represents the magnetic helicity and the dashed line represents the electric helicity. At $t = 0$, $h_m = 4$ and $h_e = 1$. As time passes, we can see the helicity exchange between the magnetic and electric fields. However, the total electromagnetic helicity, which is represented by the solid line, is equal to 5 and it does not change in time.

CHAPTER 4

CONCLUSIONS

In this work we have reviewed the set of torus knotted electromagnetic fields which is proposed first by Rañada [9] and developed further by Arrayás, Bouwmeester and Trueba [1]. Rañada construction is based on two complex scalars potentials ϕ and θ. By imposing two conditions on these potentials, they are converted into maps from S^3 to S^2. At some initial time $t = 0$, the magnetic and electric lines are given by the level curves of complex maps ϕ_{t_0} and θ_{t_0}, respectively. Due to the topological properties, Rañada used Hopf map for these complex maps. The first topologically nontrivial solution constructed by Rañada [9] was based on Hopf fibration with $n = m = l = s = 1$ and it is called *electromagnetic Hopfion*. For this particular case the time dependent complex maps and the corresponding Euler potentials can be found. Using the Euler potentials [1] of the Hopfion, we have analyzed the configuration of magnetic field lines at t = 0. For the general case, magnetic (electric) lines are (n, m) torus knots ((l, s) for electric field) and are linked to other magnetic (electric) lines. It is shown that if $n = m = l = s$ is satisfied, then the field is a null field and the magnetic helicity is equal to the electric helicity initially, and does not change in time. If $n = m = l = s$ is not satisfied then the field is a non-null field and the initial helicities are not equal to each other generally. If the initial helicities are not the same, then it is shown that there is a helicity exchange between the magnetic and electric fields. Due to the helicity exchange mechanism we see that the topological structure of these field lines is not preserved. Since the helicity is related to the difference of classical values corresponding to the number of right-handed and left handed photons, a change in the helicity can be seen as an indication of creation or annihilation of photons. Thus the process of helicity exchange and the change of the topological structure should be understood in detail and it is an open problem.

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