MAXIMAL PAGE CROSSING NUMBER OF EMBEDDED CLOSED
LEGENDRIAN SURFACES IN CLOSED CONTACT 5-MANIFOLDS

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LEGENDRIAN SURFACES IN CLOSED CONTACT 5-MANIFOLDS**

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ABSTRACT

**MAXIMAL PAGE CROSSING NUMBER OF EMBEDDED CLOSED
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The main purpose of this thesis is to introduce a new Legendrian isotopy invariant for any closed orientable Legendrian surface L embedded in a closed contact 5- manifold (M, ξ) which admits an "admissable" open book (B, f) (supporting ξ) for L. We show that to any such L and a fixed page X, one can assign an integer $M\mathcal{P}_X(L)$, called "Relative Maximal Page Crossing Number of L with respect to X", which is invariant under Legendrian isotopies of L. We also show that one can extend this to a page-free invariant, i.e., one can assign an integer $M\mathcal{P}_{(B,f)}(L)$, called "Absolute Maximal Page Crossing Number of L with respect to (B, f)", which is invariant under Legendrian isotopies of L. In particular, this new invariant distinguishes Legendrian surfaces in the standard five-sphere which can not be distinguished by Thurston-Bennequin invariant.

We give definitions of $M\mathcal{P}_X(L)$ and $M\mathcal{P}_{(B,f)}(L)$ and show that the invariants are well defined. Also, we show that they are preserved under Legendrian isotopies of L. Finally, we give an example about these invariants.

ÖZ

KAPALI KONTAKT 5-MANİFOLDLARDA GÖMÜLÜ KAPALI LEGENDRİAN YÜZEYLERİN MAKSİMUM SAYFA GEÇİŞME SAYISI

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Amacımız; bir kapalı, kontakt 5-manifoldun (B, f), open book yapısını kullanarak, her bir kapalı, yönlendirilebilir Legendrian alt yüzeyi için yeni bir Legendrian izotopi değişmezi tanımlamaktır. Böyle bir Legendrian alt manifoldu L ve sabit bir X sayfası için Legendrian izotopiler altında değişmez olan ve bağıl maksimum sayfa geçişme sayısı olarak adlandırılan bir tamsayı atanabileceğini göstermek istiyoruz. Ayrıca, bu değişmezin sayfadan bağımsız yeni bir değişmeze genişletilebileceğini göstermek istiyoruz. Yani böyle bir Legendrian alt manifoldu için Legendrian izotopiler altında değişmez olan ve mutlak maksimum sayfa geçişme sayısı olarak adlandırılan bir tamsayı atanabileceğini göstereceğiz. Bu yeni değişmez Thurston-Bennequin değişmezi tarafından ayırt edilemeyen standart 5-kürelerin Legendrian yüzeylerini ayırt etmeye yardımcı olur.

Bu değişmezlerin tanımını verip iyi tanımlı olduklarını gösterdik. Ayrıca, tanımlanan bu sayıların Legendrian izotopiler altında korunduğunu gösterdik. Son olarak tanımlanan bu değişmezler için bir örnek verdik.

Anahtar Kelimeler: Legendrian yüzey, açık kitap, kontakt yapı, simplektik, maksimum sayfa geçişme sayısı

To my family and fiance

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CHAPTER 1

INTRODUCTION

Contact geometry is the study of an odd dimensional manifolds unlike symplectic geometry which is defined on some even-dimensional manifolds. Contact and symplectic geometry are motivated by the classical mechanics, where one can think either the even-dimensional phase space of a mechanical system or constant-hypersurface which is odd-dimension. Contact geometry has many uses in different areas in physics, like as geometrical optics, classical mechanics, thermodynamics, integrable systems etc. The relation between a contact geometry and a dynamical systems comes from the notion of the Reeb vector field of a contact form in the contact geometry. Contact geometry also has applications to mathematics, especially in low dimensional topology; for example to prove the property P conjecture by Kronheimer and Mrowka [19] and to define invariants of knots by Lenhard Ng [20].

One of the most useful topological tools to study manifolds is open book decompositions. It is relevant to contact structures. Thurston and Winkelnkemper [27] showed that every open book admits a contact structure. Ibort, Martinez-Torres and Presas [1] showed that supporting open books with Weinstein pages always exist for all closed contact 5-manifolds and that monodromies can be assumed to be symplecto-morphisms of the pages. Moreover, there is one-to-one correspondence between the set of supporting open books which arise from the work of Ibort, Martinez-Torres and Presas up to positive stabilizations and the set of supported contact structures up to isotopies. These results are given by a famous theorem of Giroux [14] for $n \geq 5$. By this theorem, contact geometry can be studied from a topological viewpoint and we can see contact structures as topological objects. Etnyre and Ozbagci [12] defined new invariants of contact structures in terms of open book decompositions.

The most interesting subspaces of a contact manifold are its Legendrian submanifolds. In fact, Legendrian submanifolds are counterpart to Lagrangian submanifolds of symplectic manifolds. Legendrian submanifolds are very hard objects. Because there are infinitely many Legendrian isotopy classes of embeddings which are smoothly isotopic. One can use relative contact homology which provides invariants of Legendrian submanifolds to distinguish different Legendrian submanifolds that are smoothly isotopic.

Legendrian knots inside a contact 3-manifold is the simplest example of Legendrian submanifolds. Legendrian knots are very important. Because we can learn geometry and topology of the underlying manifold with their contribution. Many inequivalent Legendrian knots can be distinguished by their Thurston-Bennequin invariants and rotation number, which are together known as the classical invariants of Legendrian knots. Y. Kanda used Legendrian knots to distinguish contact structures in [18] and L. Rudolph used Legendrian knots to detect topological properties of knots in the [22]. Many examples are known about Legendrian non-isotopic knots with the same classical invariants. Also, in higher dimensions, there are interesting examples of non-trivial Legendrian knots when the contact manifold has more topology on it [9]. The Thurston-Bennequin invariant of any Legendrian surface of a contact manifold measures the twisting of contact structure around this Legendrian surface [11] . This invariant was originally defined by Bennequin [3] and independently, Thurston when $n = 1$ and generalized to higher dimensions by Tabachnikov [26].

To distinguish two or more topological objects by using some invariants has always been very important. Hence the notion of invariant sounds good. For this reason, many mathematicians would like to discover some new invariant. Any orientable Legendrian submanifold in any contact manifold comes with a canonical contact framing, called *Thurston-Bennequin framing*. More precisely, if $L^n \subset (M^{2n+1}, \xi)$ is an orientable Legendrian submanifold, then its contact framing is determined by a smooth vector field which is every transverse to $\xi|_L$. If we further assume that L is null-homologous (i.e., if $L = \partial C$ for some $(n + 1)$-chain $C \subset M$), then we can compare the contact framing on L with the one determined by C, and so one can identify it with an integer $tb(L)$ called *Thurston-Bennequin number* of L. In the past two decades new Legendrian isotopy invariants have been defined and studied (see for

instance, [23], [25], [24]) due to insufficiency of $tb(L)$ in distinguishing non-isotopic Legendrian submanifolds in certain cases. Most of these new invariants are based on differential graded algebras and very difficult to compute.

Returning back to dimension five, it has been known (see [24], for instance) that $tb(L)$ can not distinguish Legendrian surfaces in the standard contact $\mathbb{R}^5$ or $\mathbb{S}^5$ which are smoothly (but not Legendrian) isotopic. The reason for this is that $tb(L)$ coincides with a topological invariant for these cases, i.e., it does not carry any information about the Legendrian embedding of L into $\mathbb{R}^5$ or $\mathbb{S}^5$. On the other hand, the new invariants introduced in this thesis can distinguish such Legendrian surfaces by means of classical computations relatively more visual and simpler than those used in computing other Legendrian isotopy invariants. For this reason, my thesis problem comes to the light. We define two new Legendrian isotopy invariants for any closed orientable Legendrian surface in closed contact 5-manifolds using "essentially intersecting" pages of "admissable" open book decompositions.

In Chapter 2, we give a review of background information on contact 5-manifolds, symplectic manifolds, almost complex manifolds, Liouville manifolds, Weinstein manifolds, Stein manifolds, open book decompositions, Legendrian submanifolds, Thurston-Bennequin invariant and handle decompositions of Stein surfaces.

In Chapter 3, we define a new invariant explicitly: For a given closed, orientable Legendrian surface L in a closed contact 5-manifold, we consider an "admissable" open book decomposition for L with a page $X = X_0$ whose double $D(X) = X_0 \cup_\partial X_1$ "essetially intersects" L. (Here X_1 is the dual page of X_0). One may assume that L and the binding intersect transversely. Then for any L' which is Legendrian isotopic to L and transverse to $D(X)$, we consider all the components of the link of transverse intersection of L' and $D(X)$. Sketch the Legendrian arcs constructing these components in the Stein diagrams of X_0 and X_1, and calculate Thurston-Bennequin number of each of these arcs. By summing all of these numbers, we obtain an integer called the "page crossing number" of L'. Finally, maximum of these numbers (among all such L's) give us a number which we will call "Relative Maximal Page Crossing Number of L with respect to X".

In Chapter 4, we will study two kinds of Legendrian isotopies and prove the following

theorem:

Theorem 1.0.1 *One can associate an integer $M\mathcal{P}_X(L)$, called "Relative Maximal Page Crossing Number of L with respect to X", which is invariant under Legendrian isotopies of L.*

In Chapter 5, using the relative invariant, we show that under an additional assumption on admissable open books, one can also define an absolute invariant which is independent of pages of the open book at hand. Namely, assuming that an admissable open book (B, f) for L also "essentially intersects" L, we prove the following theorem:

Theorem 1.0.2 *One can associate an integer $M\mathcal{P}_{(B,f)}(L)$, called "Abolute Maximal Page Crossing Number of L with respect to (B, f)", which is invariant under Legendrian isotopies of L.*

In the last chapter, we give a concrete example for the invariants.

CHAPTER 2

BACKGROUND

In this chapter, we will recall some basic definitions and facts about contact/symplectic geometry and topology in dimension five. In section 2.1, we will define contact structures and give some examples that will be used throughout the thesis. After discussing symplectic manifolds and almost complex manifolds in section 2.2, we will define Liouville, Weinstein and Stein manifold. Open book decompositions and their relations with contact structures will be considered in section 2.4. After we give an overview of Legendrian submanifolds and Thurston-Bennequin invariant, we will recall handle decompositions of Stein domains and their relations with contact surgery diagrams in section 2.6 and we will finish this chapter.

2.1 Contact 5-Manifolds

Let us start with defining a contact structure on (necessarily) odd dimensional manifolds. Due to the nature of this thesis, most of the time we restrict our attention on dimension five. More discussions and details about contact structures can be found, for instance, in [10] and [15].

Definition 2.1.1 *A pair (M^5, ξ) is called a **contact manifold** where M is a smooth manifold and $\xi^4 \subset TM$ is totally non-integrable 4-plane field distrubution on M. A **contact structure** on M is the distrubution ξ where ξ is said to be **co-oriented** if it is the kernel of a globally defined 1-form α. α has the property $\alpha \wedge (d\alpha)^2 \neq 0$. Such α is called a **contact form** on M.*

Since $\alpha \wedge (d\alpha)^2 \neq 0$, a contact manifold M^5 must be oriented. So $\alpha \wedge (d\alpha)^2 \neq 0$ defines the given orientation on M.

Remark Observe that α is a contact form if $\alpha \wedge (d\alpha)^2$ is a **volume form** on M. That is, $\alpha \wedge (d\alpha)^2$ is a nowhere vanishing top dimensional differential form. If α is globally defined, then it is called **co-oriented**. α can be defined locally or globally. If $\xi = Ker(\alpha)$ and $\alpha \wedge (d\alpha)^2 > 0$, then ξ is called a **co-oriented positive contact structure**. In general, we assume that ξ is co-oriented positive contact structure.

Remark 1) The condition $\alpha \wedge (d\alpha)^2 \neq 0$ is independent of the specific choice of α.
2) The 2-form $d\alpha_p$ is nondegenerate on ξ_p for all $p \in M$.

Now we will restrict ourself to contact 5-manifolds and give some examples.

Example 2.1.1 *Consider the standard Cartesian coordinates (x_1, y_1, x_2, y_2, z) in $\mathbb{R}^5$ and 1-form $\alpha_1 = dz + x_1 dy_1 + x_2 dy_2$. Since $\alpha_1 \wedge (d\alpha_1)^2 = (dz + x_1 dy_1 + x_2 dy_2) \wedge (dx_1 \wedge dy_1 + dx_2 \wedge dy_2)^2 = 2dz \wedge dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \neq 0$, $\xi_1 = ker(\alpha_1)$ is a contact structure on $\mathbb{R}^5$. This ξ_1 is called the standard contact structure on $\mathbb{R}^5$.*
See Figure 2.1 for the 3-dimensional case.
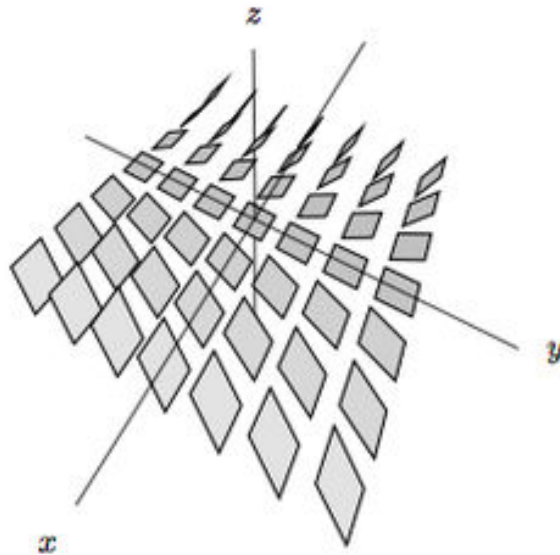
Figure 2.1: The standard contact structure $ker(dz + xdy)$ on $\mathbb{R}^3$.

Example 2.1.2 *Similarly, consider $\mathbb{R}^5$ with the standard coordinates (x_1, y_1, x_2, y_2, z) and the 1-form $\alpha_2 = dz + x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$. $\xi_2 = ker(\alpha_2)$ is also a contact structure on $\mathbb{R}^5$.*

Example 2.1.3 *Let S^5 be the unit 5-sphere in $\mathbb{R}^6$ with standard coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ and 1-form $\alpha_3 = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 + x_3 dy_3 - y_3 dx_3$, which is restricted to S^5. This contact form defines a standard contact structure on S^5. So we have $\xi_3 = ker(\alpha_3)$.*

Example 2.1.4 *Consider $\mathbb{R}^5$ with the polar coordinates (r_j, ϕ_j), in the (x_j, y_j) plane $j = 1, 2$. The following 1-form is a contact form on $\mathbb{R}^5$.*

$$\alpha_4 = dz + r_1^2 d\phi_1 + r_2^2 d\phi_2 = dz + x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$$

This contact form defines a contact structure ξ_4 on $\mathbb{R}^5$. That is $\xi_4 = ker(\alpha_4)$.

Example 2.1.5 *Consider $\mathbb{R}^5$, endowed with cartesian coordinates (x_1, y_1, x_2, y_2, z) and the 1-form $\alpha_5 = dz - y_1 dx_1 - y_2 dx_2$. This contact form defines a contact structure on $\mathbb{R}^5$.*

Remark In fact, this contact form is not different from the contact forms $\alpha_1, \alpha_2, \alpha_4, \alpha_5$.

The following definitions describe the equivalence of contact structures and forms.

Definition 2.1.2 *A diffeomorphism $\psi : (M_1, \xi_1 = ker(\alpha_1)) \rightarrow (M_2, \xi_2 = ker(\alpha_2))$ between two contact 5-manifolds is called **contactomorphism** if its differential $Tf : TM_1 \rightarrow TM_2$ maps the contact structure ξ_1 on M_1 to the contact structure ξ_2 on M_2, i.e. if there is a function $\lambda : M_1 \rightarrow \mathbb{R} \smallsetminus \{0\}$ with $f^*\alpha_2 = \lambda \alpha_1$. Two contact 5-manifolds (M_1, ξ_1) and (M_2, ξ_2) are said to be **contactomorphic** if there exists a contactomorphism between them. That is, $\psi_*(\xi_1) = \xi_2$.*

Definition 2.1.3 *Two contact structures ξ_1 and ξ_2 on a 5-manifold M are **isotopic** if there is a contactomorphism $\psi : (M, \xi_1) \rightarrow (M, \xi_2)$ such that ψ is isotopic to the identity. On the other hand, two contact structures ξ_1 and ξ_2 on a 5-manifold M are called **homotopic** if they are homotopic as tangent plane distributions.*

Two different contact structures can be homotopic but not isotopic. Hence, classification of contact structures is made up to isotopy.

Example 2.1.6 *The contact manifolds* $(\mathbb{R}^5, \xi_1 = ker(\alpha_1))$ *and* $(\mathbb{R}^5, \xi_2 = ker(\alpha_2))$ *from the first two examples are contactomorphic. An explicit contactomorphism f is given by*

$$f(\boldsymbol{x}, \boldsymbol{y}, z) = \left(\frac{\boldsymbol{x+y}}{2}, \frac{\boldsymbol{y\text{-}x}}{2}, \frac{z + \boldsymbol{xy}}{2} \right)$$

where $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$ *and* $\boldsymbol{xy} = x_1 y_1 + x_2 y_2$. *Clearly,* $f^* \alpha_2 = \alpha_1$.

For completeness, we prove:

Proposition 2.1.4 *For any point $p \in S^5$, the two contact manifolds $(S^5 \backslash \{p\}, \xi_3)$ and $(\mathbb{R}^5, \xi_2)$ are contactomorphic.*

Proof By using Geiges' proof, we can give a following proof for S^5 and $\mathbb{R}^5$:

$$\xi_3 = Ker(\alpha_3) = Ker(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 + x_3 dy_3 - y_3 dx_3)$$
$$\xi_2 = Ker(\alpha_2) = Ker(dw + u_1 dv_1 - v_1 du_1 + u_2 dv_2 - v_2 du_2) \text{ or}$$
$$\xi_2 = Ker(\alpha_2) = Ker(dw + r_1^2 d\theta_1 + r_2^2 d\theta_2)$$

Here (r_1, θ_1) is the polar coordinates in the (u_1, v_1)-plane and (r_2, θ_2) is the polar coordinates in the (u_2, v_2)-plane. Let's choose the point $p = (0, 0, 0, 0, 0, +1)$. Suppose that $\psi : S^5 \backslash \{p\} \longrightarrow \mathbb{R}^5$ is the stereographic projection from p. Also assume that $(x_1, y_1, x_2, y_2, x_3, y_3)$ and (u_1, v_1, u_2, v_2, w) are the coordinates in S^5 and $\mathbb{R}^5$, respectively. This map is given by

$$u_1 = \frac{x_1}{1 - y_3}, \qquad v_1 = \frac{y_1}{1 - y_3}, \qquad u_2 = \frac{x_2}{1 - y_3},$$
$$v_2 = \frac{y_2}{1 - y_3}, \qquad w = \frac{x_3}{1 - y_3}$$

Inverse map of ψ is given by

$$\psi^{-1} : \mathbb{R}^5 \longrightarrow S^5 \backslash \{p\}$$

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$$x_1 = \lambda u_1, \qquad y_1 = \lambda v_1, \qquad x_2 = \lambda u_2, \qquad y_2 = \lambda v_2$$
$$x_3 = \lambda w, \qquad y_3 = 1 - \lambda$$

with $\lambda = \dfrac{2}{1 + w^2 + u_1^2 + v_1^2 + u_2^2 + v_2^2}$. So we have

$$\psi^{-1}(u_1, v_1, u_2, v_2, w) = (\lambda u_1, \lambda v_1, \lambda u_2, \lambda v_2, \lambda w, 1 - \lambda)$$

$(\psi^{-1})^*(\alpha_3) = \lambda^2[r_1^2 d\theta_1 + r_2^2 d\theta_2 + wr_1 dr_1 + wr_2 dr_2 + \frac{1}{2}(1 + w^2 - r_1^2 - r_2^2)dw]$.
In fact,

$$(\psi^{-1})^*(\alpha_3) = \lambda^2 \tilde{\alpha}_2$$

where $\tilde{\alpha}_2 = [r_1^2 d\theta_1 + r_2^2 d\theta_2 + wr_1 dr_1 + wr_2 dr_2 + \frac{1}{2}(1 + w^2 - r_1^2 - r_2^2)dw]$. Let's find a diffeomorphism f of $\mathbb{R}^5$ such that $f^*\alpha_2 = \tilde{\alpha}_2$. We can find a suitable f as:

$$f : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$$
$$f(r_1, r_2, \theta_1, \theta_2, w) = (r_1, r_2, \theta_1 - w, \theta_2 - w, \tfrac{1}{2}w(1 + \tfrac{1}{3}w^2 + r_1^2 + r_2^2)).$$

When we calculate $f^*(\alpha_2)$, we find $\tilde{\alpha}_2$. More precisely,

$$f^*(\alpha_2) = f^*(dw + r_1^2 d\theta_1 + r_2^2 d\theta_2) = d(\tfrac{1}{2}w + \tfrac{1}{6}w^3 + \tfrac{1}{2}wr_1^2 + \tfrac{1}{2}wr_2^2) + r_1^2 d(\theta_1 - w) +$$
$$r_2^2 d(\theta_2 - w) = (\tfrac{1}{2} + \tfrac{1}{2}w^2 - \tfrac{1}{2}r_1^2 - \tfrac{1}{2}r_2^2)dw + wr_1 dr_1 + wr_2 dr_2 + r_1^2 d\theta_1 + r_2^2 d\theta_2$$

As a result, $f^*(\alpha_2) = \tilde{\alpha}_2$. Now, we have the following equation:

$$(\psi^{-1})^*(\alpha_3) = \lambda^2 \tilde{\alpha}_2 = \lambda^2 f^*(\alpha_2).$$

If we compose both of this equation by ψ^* from the left side, we can find

$$\alpha_3 = (\lambda^2 f \circ \psi)^*(\alpha_2).$$

Hence $\lambda^2 f \circ \psi$ is a contactomorphism between $(S^5 \backslash \{p\}, \xi_3)$ and $(\mathbb{R}^5, \xi_2)$. □

From the preceding remark and the above proposition, $(\mathbb{R}^5, \xi_1)$, $(\mathbb{R}^5, \xi_2)$, $(\mathbb{R}^5, \xi_4)$, $(\mathbb{R}^5, \xi_5)$ and $(S^5 \backslash \{p\}, \xi_3)$ are contactomorphic. Hence any of these contact structures
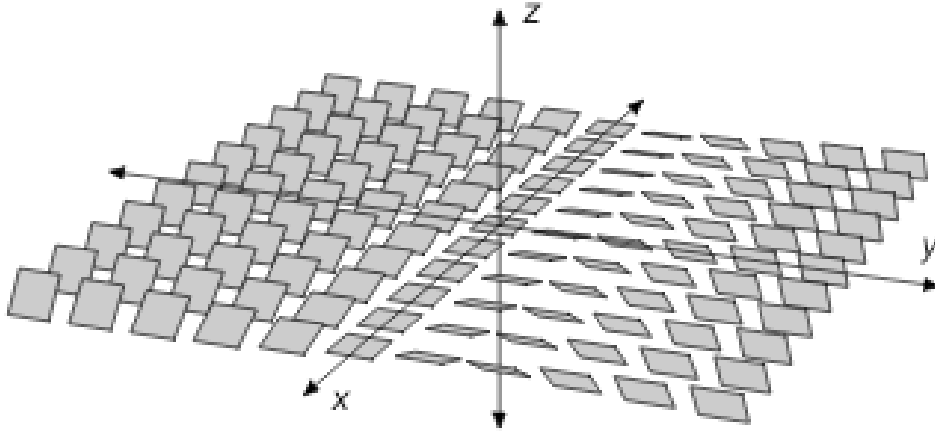
Figure 2.2: The standard tight contact structure on $\mathbb{R}^3$.

ξ_i, $i = 1, 2, 3, 4, 5$ are used as the standard contact structure on $\mathbb{R}^5$ and denoted by ξ_{std}.

Theorem 2.1.5 *(Darboux's Theorem) Let M be a 5-dimensional manifold and x a point on M. Then there is a neighborhood U of x in M such that $(U, \xi \mid_U)$ is contactomorphic to $(\mathbb{R}^5, \xi_{std})$.*

A neighborhood U as in the above theorem is said to be a **Darboux ball**. In dimension three, there are two kinds of contact structures on 3-contact manifolds:

Definition 2.1.6 *If there is an embedded disk D in (M^3, ξ) such that $T_p(\partial D) \subseteq \xi_p$ at every point $p \in \partial D$, then ξ is called an **overtwisted contact structure**. That is, contact plane is tangent to ∂D. D is called an **overtwisted disk**. Otherwise ξ is called a **tight contact structure**.*

Example 2.1.7 *Let $\alpha = dz - ydx$ on $\mathbb{R}^3$ with cartesian coordinates and $\beta = cosrdz + rsinrd\theta$ in $\mathbb{R}^3$ with cylindrical coordinates. Then $Ker(\alpha)$ is the standard tight contact structure on $\mathbb{R}^3$. Also $Ker(\beta)$ is an overtwisted contact structure. These are visualized in Figure 2.2 and 2.3, respectively.*

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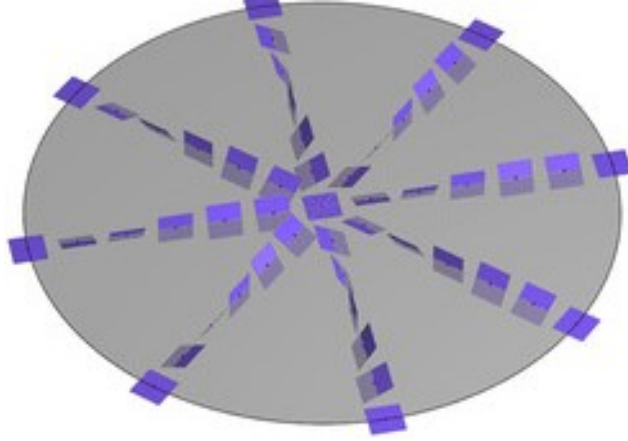
Figure 2.3: An overtwisted contact structure on $\mathbb{R}^3$.

Definition 2.1.7 *Corresponding to a contact form $\alpha \in \Omega^1(M)$, one has the so called* ***Reeb vector field*** *R_α, uniquely defined by the equations*

•$d\alpha(R_\alpha, -) = 0$.

•$\alpha(R_\alpha) = 1$.

Example 2.1.8 *The Reeb vector field of the contact form $\alpha_1 = dz + x_1 dy_1 + x_2 dy_2$ on $\mathbb{R}^5$ is ∂_z.*

A vector field X on a contact manifold is said to be a **contact vector field** if it satisfies

$$\mathcal{L}_X \alpha = f\alpha$$

for some function $f : M \to \mathbb{R}$. That is, if we have a contact vector field, then its flow preserves the contact distrubution.

2.2   Symplectic Manifolds and Almost Complex Manifolds

For more details about definitions and facts given below, we refer the reader [5], [15], and [21].

Definition 2.2.1 *Given a manifold M, a **symplectic structure** on M is a 2-form $\omega \in \Omega^2(M)$ such that*

• *ω is nondegenerate.*

(i. e. $\forall v \in T_x M, v \neq 0, \exists u \in T_x M$ such that $\omega_x(v, u) \neq 0$)

• *ω is a closed form.*

(i. e. $d\omega = 0$)

*The pair (M, ω) is called a **symplectic manifold**.*

Note that any symplectic manifold (M, ω) is even dimensional and oriented. In fact, the first condition of the above definition is equivalent to $\omega^n = \omega \wedge \omega \wedge ... \wedge \omega \neq 0$ if $\dim(M) = 2n$. Also $\omega^n \neq 0$ is called a canonical volume form.

Example 2.2.1 *Consider $\mathbb{R}^{2n}$ with Cartesian coordinates $(x_1, ..., x_n, y_1, ..., y_n)$. The symplectic form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ is the **standard symplectic form** on $\mathbb{R}^{2n}$.*

Definition 2.2.2 *A submanifold X of (M, ω) is called **symplectic** if ω_x restricted to $T_x X$ is symplectic for all $x \in X$ and **isotropic** if ω_x restricted to $T_x X$ is 0 for all $x \in X$. An isotropic submanifold of (M, ω) which has half dimension of M is said to be **Lagrangian**.*

Definition 2.2.3 *A **symplectic vector space** is a finite dimensional real vector space V with a nondegenerate, skew symmetric bilinear form $\omega : V \times V \longrightarrow \mathbb{R}$ such that*

1) $\omega(v, u) = -\omega(u, v)$

2) $\omega(v + cu, t) = \omega(v, t) + c\omega(u, t), \forall c \in \mathbb{R}, v, u, t \in V$

3) $\omega(v, u) = 0, \forall u \in V \Longrightarrow v = 0$

Theorem 2.2.4 *Let (V, ω) be a symplectic vector space of dimension $2n$. Then there exists a basis $\{e_1, f_1, e_2, f_2, ..., e_n, f_n\}$ such that*

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j), \qquad \omega(e_i, f_j) = \delta_{ij}.$$

Definition 2.2.5 *A **symplectomorphism** is a diffeomorphism $\psi : (M, \omega) \to (M', \omega')$ between symplectic manifolds (M, ω) and (M', ω') with $\psi^* \omega' = \omega$.*

Definition 2.2.6 *A **complex structure** on a (real) vector space V is an automorphism $J : V \longrightarrow V$ such that $J^2 = -Id$.*

In fact, a complex structure on V enables us to "multiply $\sqrt{-1}$" on V and thus convert V into a complex vector space.

Definition 2.2.7 *An **almost complex structure** J on a manifold M is an assignment of complex structures J_p on the tangent spaces $T_p M$ which depends smoothly on p. The pair (M, J) is called an **almost complex manifold**.*
In other words, an almost complex structure on M is a $(1,1)$ tensor field $J : TM \to TM$ so that $J^2 = -Id$.

Remark As in the symplectic case, an almost complex manifold must be $2n$ dimensional and orientable. On the other hand, there exists even dimensional orientable manifolds which admit no almost complex structure.

Let (M, ω) be a symplectic manifold and J an almost complex structure on M. Then at each tangent space $T_p M$ we have linear symplectic structure ω_p and linear complex structure J_p.

Definition 2.2.8 *A symplectic structure ω_p is **compatible with** an almost complex structure $J_p : T_p M \to T_p M$ if*
1) $\omega_p(u, v) = \omega_p(Ju, Jv)$ for all $u, v \in T_p M$.
That is, J preserves ω.
*2) $\omega_p(u, J_p u) > 0$ for all nonzero $u \in T_p M$. This means, ω_p is positive on the complex lines. This condition is called the **taming condition**.*
Or equivalently,

$$g_p(u, v) := \omega_p(u, J_p v)$$

is an inner product on $T_p M$.

Definition 2.2.9 *An almost complex structure J on M is **compatible with** a symplectic structure ω on M if at each p, J_p is compatible with ω_p.*
Equivalently, J is compatible with ω if and only if

$$g_p : T_pM \times T_pM \to \mathbb{R}, \ g_p(u,v) := \omega_p(u, J_p v)$$

defines a Riemannian structure on M.

Theorem 2.2.10 *Let (M, ω) be a symplectic manifold, and J_0, J_1 two almost complex structures compatible with ω. Then there is a smooth family J_t, $0 \le t \le 1$, of compatible almost complex structures joining J_0 to J_1.*

The space of compatible complex structures on (V, ω) is denoted by $J(V, \omega)$.

Corollary 2.2.11 *The set of all compatible almost complex structures on a symplectic manifold is path connected.*

Theorem 2.2.12 *The space $J(V, \omega)$ is contractible, and hence non-empty.*

The above theorem is due to Gromov [17] (for a proof see also [13] or [21]) and provides a very useful connection between symplectic and (almost) complex geometry.

Definition 2.2.13 *Let (M, ω) be a symplectic manifold and $f : M \to \mathbb{R}$ a smooth function. f is called a **Hamiltonian** on M. A vector field X on (M, ω) is said to be **symplectic** if $\mathcal{L}_X \omega = 0$. That is, the flow of X preserves ω. Observe that by Cartan's formula, X is symplectic if and only if $i_X \omega$ is closed where i denotes interior product. A vector field X on (M, ω) is called **Hamiltonian** if $i_X \omega$ is exact.*

2.3 Liouville, Weinstein and Stein Manifolds

Now we recall special families of symplectic manifolds in which we are interested. More details about definitions and facts given below can be found in [4] and [21].

Definition 2.3.1 *Let (X, ω) be a compact, symplectic $2n$-manifold such that $\partial X = M_- \sqcup M_+$. $\partial_+ X = M_+$ is said to be **convex boundary** if there exists a **Liouville***

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*vector field Z for ω defined near M_+ such that $\mathcal{L}_Z\omega = \omega$ and Z is pointing outwards along M_+ . Similarly, $\partial_- X = M_-$ is the **concave boundary** if Z is defined near M_- and pointing inwards along M_-. In such a case, $\alpha_\pm = (i_Z\omega)|_{M_\pm}$ is a contact form on $M_\pm$, and (X,ω) is called a **symplectic cobordism** from $(M_-, Ker(\alpha_-))$ to $(M_+, Ker(\alpha_+))$.*

Definition 2.3.2 *A cobordism X between a manifold M and the empty set is called a **filling**. A symplectic cobordism from the empty set to a contact manifold (M,ξ) is called a **strong symplectic filling** (or a **convex filling**). On the other hand, a symplectic cobordism from a contact manifold (M,ξ) to the empty set is called a **concave filling** of (M,ξ). Also a symplectic manifold (X,ω) is said to be a **weak symplectic filling** of a contact manifold (M,ξ) if*

(i) *$\partial X = M$ and the boundary orientation on ∂X and the orientation on M agree, and*

(ii) *$\omega|_\xi$ is nondegenerate, i.e., $\omega|_\xi > 0$.*

*In this case, a contact manifold (M,ξ) is called a **weakly symplectically fillable**.*

Theorem 2.3.3 ([8]) *Any weakly symplectically fillable contact 3-manifold is tight.*

Definition 2.3.4 *A **Liouville cobordism** is a symplectic cobordism (X,ω) from $\partial_- X = M_-$ to $\partial_+ X = M_+$ with a Liouville structure. A **Liouville structure** means that there is a 1-form α on X such that $\omega = d\alpha$ and the ω-dual vector field Z of α is a Liouville vector field for ω transversely pointing inward (resp. outward) along the boundary component $\partial_- X$ (resp. $\partial_+ X$). A Liouville cobordism with $\partial_- X = \varnothing$ is called a **Liouville domain**.*

*When X is an open manifold, if we assume that the flow of Z exists for all times and there exists an exhaustion $X = \bigcup_{k=1}^{\infty} X^k$ by compact domains $X^k \subset X$ such that each $(X^k, \alpha|_{X^k})$ is a Liouville domain with convex boundary $(\partial X^k, \alpha|_{\partial X^k})$ for all $k \geq 1$, then (X,α) is called a **Liouville manifold**. Since ω and Z uniquely determine α (namely, $\alpha = \iota_Z\omega$), one can also use the notation (X, ω, Z) for Liouville cobordisms/domains/manifolds.*

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Definition 2.3.5 *The **core** (or **skeleton**) of the Liouville domain/manifold (X, α) is defined as follows: If $Z^{-t} : V \to V$ denotes the contracting flow of Z, then*

$$Core(X, \alpha) := \bigcup_{k=1}^{\infty} \bigcap_{t>0} Z^{-t}(X^k).$$

By putting more conditions on Liouville manifolds, one can consider the class of Weinstein/Stein manifolds. For these manifolds the core contains all Morse theoritical data as we will observe below. To define Weinstein and Stein manifolds, we need some basic definitions:

Definition 2.3.6 *(i) A vector field Z on a smooth manifold X is said to **gradient-like** for a smooth function $\phi : X \to \mathbb{R}$ if $Z \cdot \phi = \mathcal{L}_Z \phi > 0$ away from the critical point of ϕ.*
*(ii) A real-valued function is said to be **exhausting** if it is proper and bounded from below.*
*(iii) An exhausting function $\phi : X \to \mathbb{R}$ on a symplectic manifold (X, ω) is said to be ω-**convex** if there exists a complete Liouville vector field Z which is gradient-like for ϕ.*
*(iv) Suppose that (X, J) is an almost complex manifold. Then a smooth map $\phi : X \to \mathbb{R}$ is said to be J-**convex** if $\omega_\phi := -d(d\phi \circ J)$ is nondegenerate (i.e., $\omega_\phi(v, Jv) > 0$ for all $v \neq 0$), and so symplectic.*

Definition 2.3.7 *A **Weinstein manifold** (X, ω, Z, ϕ) is a symplectic manifold (X, ω) which admits a ω-convex Morse function $\phi : X \to \mathbb{R}$ whose complete gradient-like Liouville vector field is Z. The triple (ω, Z, ϕ) is called a **Weinstein structure** on X. A **Weinstein cobordism** (X, ω, Z, ϕ) is a Liouville cobordism (X, ω, Z) whose Liouville vector field Z is gradient-like for a Morse function $\phi : X \to \mathbb{R}$ which is constant on the boundary ∂X. A Weinstein cobordism with $\partial_- X = \varnothing$ is called a **Weinstein domain**.*

Any Weinstein manifold (X, ω, Z, ϕ) can be exhausted by Weinstein domains

$$X_k = \{\phi^{-1}(-\infty, d_k]\} \subset X$$

where $\{d_k\}$ is an increasing sequence of regular values of ϕ, and therefore, any Weinstein manifold is a Liouville manifold. In particular, any Weinstein domain is a

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Liouville domain. Also note that any Weinstein domain (X, ω, Z, ϕ) has the convex boundary $(\partial X, Ker((\iota_Z \omega)|_{\partial X}))$.

The following topological characterization of Weinstein domains will be important for us.

Theorem 2.3.8 ([28], see also Lemma 11.13 in [4]) *Any Weinstein domain of dimension $2n$ admits a handle decomposition whose handles have indices at most n.*

Now we give a definition of Stein manifolds. Originally they are defined as the class of manifolds which can be holomorphically embedded into some complex space $\mathbb{C}^N$ for N large enough, and hence they are complex manifolds. In terms of the structure of the present paper, they can be defined as follows:

Definition 2.3.9 *A **Stein manifold** is a triple (X, J, ϕ) where J is an almost complex structure on X and $\phi : X \rightarrow \mathbb{R}$ is an exhausting J-convex Morse function which is also ω_ϕ-convex. A **Stein cobordism** (X, J, ϕ) is a Weinstein cobordism $(X, \omega_\phi, Z, \phi)$. A Stein cobordism with $\partial_- X = \varnothing$ is called a **Stein domain**.*

It is not hard to observe that there is an underlying a Weinstein structure for any given Stein structure. Indeed, it has been shown that the converse is also true:

Theorem 2.3.10 ([4]) *Any Weinstein structure on a manifold X can be deformed to another one which is the underlying Weinstein structure of some Stein structure on X.*

Definition 2.3.11 *A contact manifold (M, ξ) is called **Stein fillable** (or **holomorphically fillable** if there is a Stein domain (X, J, ϕ) such that $\partial X = M$ and $\xi = Ker(-(d\phi \circ J)|M)$.*

It clearly follows from Theorem 2.3.3 that

Theorem 2.3.12 *Any Stein fillable contact 3-manifold is tight.*

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2.4 Open Book Decompositions

Open book decompositions are topological structures and they have a strong relationship with contact structures. More details about definitions and facts given below can be found in [29] and [15]

Definition 2.4.1 *An (**embedded** or **non-abstract**) **open book** (**decomposition**) of a closed $(2n+1)$-manifold M is determined by a pair (B, f) where B is a codimension 2 submanifold with trivial normal bundle and $f : M \smallsetminus B \to S^1$ is a fiber bundle projection such that the normal bundle has a trivialization $B \times D^2$, where the angle coordinate on the disk agrees with the fibration map f. The $(2n-1)$-manifold B is called the **binding** and for any $t_0 \in S^1$, the $2n$-manifold $X = f^{-1}(t_0)$ (a fiber of f) is called a **page** of the open book.*

An alternative definition of an open book decomposition can be given as follows:

Definition 2.4.2 *An open book (B, f) determines an **abstract open book** $(\bar{X}, h)$ where $\bar{X}$ denotes the closure of a page X in M, and $h : \bar{X} \to \bar{X}$ is the self-diffeomorphism (which is identity near the binding $B = \partial \bar{X}$) defined by the time-one map of the flow lines along the S^1-direction. The map h is called the **monodromy** of the open book decomposition.*

In fact, the two notions of open book decomposition are closely related. The difference is that when discussing open books (non-abstract), we can discuss the binding and pages up to isotopy in M, whereas when discussing abstract open books we can only discuss them up to diffeomorphism.

Example 2.4.1 *The 2-dimensional sphere S^2 has an open book decomposition with page a closed interval and trivial monodromy up to isotopy (see Figure 2.4). Furthermore, the open book decomposition of S^2 is unique up to isotopy.*

Example 2.4.2 *The 3-dimensional sphere S^3 has an open book decomposition with page a disc and trivial monodromy. Think S^3 as the one point compactification of $\mathbb{R}^3$*

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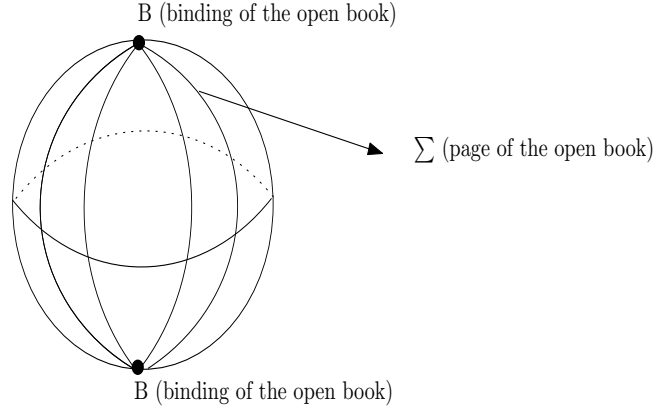
Figure 2.4: The open book decomposition of the 2-sphere.

and extend the binding and pages in the following figure along the z-axis. To be more precise, choose polar coordinates (r, θ) on the xy-plane and define

$$B := \{x = y = 0\} \cup \{\infty\}, \qquad \pi : S^3 \smallsetminus B \to S^1$$

by sending (r, θ, z) to θ, where we identify $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

An abstract open book can be defined by mapping torus as follows (see Figure 2.5): Let Σ be a compact $(n-1)$ manifold with boundary $\partial \Sigma$ and $\phi : \Sigma \to \Sigma$ a self diffeomorphism of Σ such that $\phi \mid_N$=identity on N for some neighborhood N of $\partial \Sigma$. The **mapping torus** of ϕ is denoted by Σ_ϕ and defined by

$$\Sigma_\phi = \Sigma \times [0, 1] / \sim$$

where $\sim$ is the relation $(x, 1) \sim (\phi(x), 0)$ for all $x \in \Sigma$. It is obvious that

$$\partial(\Sigma_\phi) = (\partial \Sigma) \times [0, 1] / (x, 1) \sim (\phi(x), 0) = (\partial \Sigma) \times S^1$$

Take $(x, \theta) \in (\partial \Sigma) \times S^1$. Set

$$M_{(\Sigma, \phi)} = \Sigma_\phi \cup (\partial \Sigma \times D^2) / \sim$$

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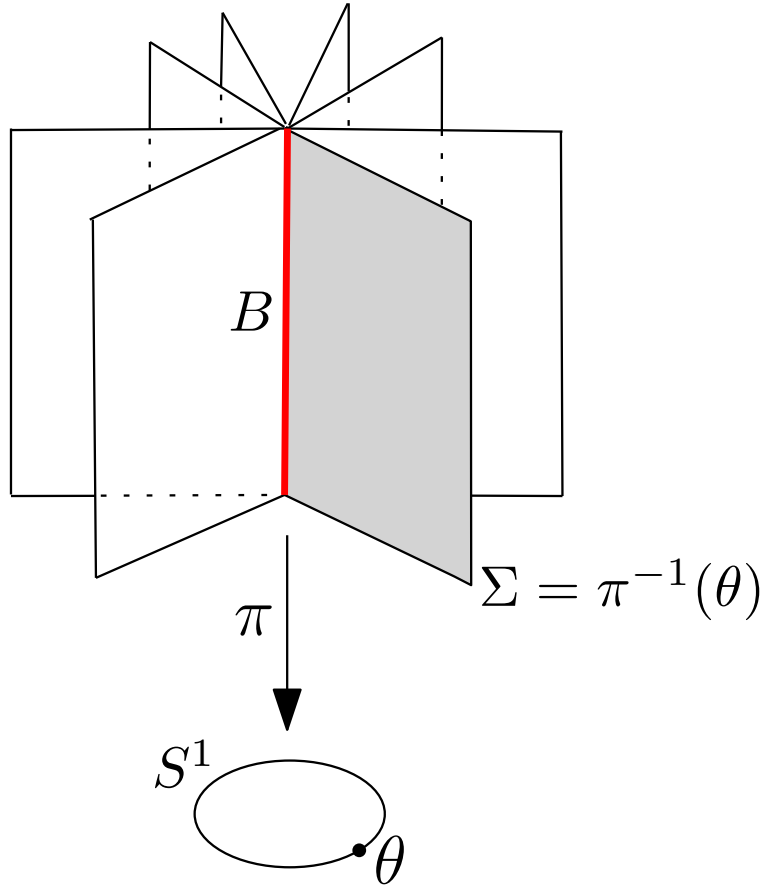
Figure 2.5: An open book decomposition.

We say that (Σ, ϕ) is an **open book decomposition** of a closed n-manifold M if M is diffeomorphic to $M_{(\Sigma,\phi)}$.

Definition 2.4.3 *Two abstract open book decompositions (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are called **equivalent** if there is a diffeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ \phi_2 = \phi_1 \circ h$.*

The following definition is given by Giroux:

Definition 2.4.4 ([14]) *A contact structure ξ on a closed $(2n+1)$-manifold M is said to be **supported by** (or **carried by**, or **compatible with**) an open book (B, f) on M if there exists a contact form α for ξ such that*

(i) $(B, Ker(\alpha|_{TB}))$ *is a contact $(2n-1)$-manifold,*

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(ii) for any $t \in S^1$, the page $(X = f^{-1}(t), d\alpha|_{TX})$ is a symplectic $2n$-manifold, and

(iii) if $\bar{X}$ denotes the closure of a page X in M, then the orientation on $B = \partial \bar{X}$ induced by its contact form $\alpha|_{TB}$ coincides with its orientation as the convex boundary of $(\bar{X}, d\alpha|_{TX})$.

Remark If an open book has Weinstein pages, then each page has its core as any Weinstein manifold is Liouville. The core is not a manifold. In fact, it is a CW-complex, and it carries a topological information of the page and the open book. Furthermore, its dimension is half of the dimension of the page (Theorem 2.3.8). Hence, for our case (i.e., when a page is 4-dimensional), the dimension of the core is 2. We will make use of the following facts:

Proposition 2.4.5 ([14]) *Two contact structures supported by the same open book are isotopic.*

Theorem 2.4.6 ([14]) *Every contact structure on a closed manifold is compatible with some open book decomposition with Weinstein (and so Stein) pages.*

Theorem 2.4.7 *Every oriented contact structure on a closed oriented manifold M^{2n+1} is compatible with some open book decomposition (X, ϕ). Moreover, X may be assumed to be a Weinstein manifold and ϕ a symplectomorphism.*

Remark *From this result, one concludes that there is always open book whose pages are Stein because we know that Weinstein manifolds are Stein by Eliashberg (see [4]).*

We now explain how the page and the monodromy of an open book changes under a certain process called *stabilization*.

Definition 2.4.8 ([14]) *Let $D^n \subset \sum^{2n}$ be an n-dimensional disc embedded into the $2n$-dimensional page of an open book $(\sum, \phi)$ of an odd dimensional manifold M such that D^n meets $\partial \sum$ transversely and exactly in its boundary ∂D^n and such that the normal bundle of ∂D^n in $\partial \sum$ is trivial. Attach an n-handle H to $\sum$ along ∂D^n in such a way that the normal bundle of the sphere $S^n = D^n \cup core(H)$ is isomorphic to $T^* S^n$.*

Then the open book $(\sum \cup H, \phi \circ \tau)$ *is called a **positive stabilization** of* $(\sum, \phi)$, *where* τ *denotes a right-handed Dehn twist along the sphere* S^n. *Similarly, we can also define **negative stabilization** using left-handed Dehn twists instead of right-handed Dehn twists. That is,* $(\sum \cup H, \phi \circ \tau^{-1})$ *is called **negative stabilization** where* τ^{-1} *is a left-handed Dehn twist along the sphere* S^n.

Remark Observe that the original open book $(\sum, \phi)$ and the stabilized open book $(\sum \cup H, \phi \circ \tau)$ give rise (up to diffeomorphism) to the same manifold M. Indeed, the sphere $\partial D^n \subset \partial \sum = B \subset (\sum, \phi)$ is a sphere with trivial normal bundle in M, since the binding B has trivial normal bundle by definition. Attaching handles to each page is equivalent to a surgery along ∂D^n. The manifold M' obtained by that surgery carries the open book structure $(\sum \cup H, \phi)$. Performing the Dehn twist τ along S^n is equivalent to a surgery cancelling the one corresponding to the handle attachment.

Contact structures are geometric objects while open book decompositions are topological objects. Giroux found a relation between them as stated below:

Theorem 2.4.9 ([14]) *Let* M *be a closed* $(2n + 1)$-*manifold. Then there is one to one correspondence between co-oriented contact structures on* M *up to isotopy and open book decompositions of* M *with Weinstein (and so Stein) pages up to positive stabilization.*

This correspondence between oriented contact structures and open book decompositions is called **Giroux Correspondence**. We note that if an open book Weinstein (or Stein) pages, then all of its stabilizations have also Weinstein (or Stein) pages.

Theorem 2.4.10 ([14]) *Two open books with Weinstein (or Stein) pages supporting the same contact manifold* (M, ξ) *are related by positive stabilizations, that is, they have a common positive stabilization.*

2.5   Legendrian Submanifolds and Thurston-Bennequin Invariant

Legendrain submanifolds are the most interesting ones in contact manifolds. Although they are defined in any odd dimensions, we restrict our attention mostly to

dimension five and three. The non-integrability condition of contact 5-manifold ensures that there is no submanifold of dimension greater than or equal to 3 which is tangent to the contact distribution. However, we can find 2-dimensional submanifolds whose tangent spaces lie inside the contact field. Legendrian submanifolds are analogous to Lagrangian submanifolds of symplectic manifolds. The Legendrian knots inside a contact 3-manifold are the simplest example of Legendrian submanifolds. In fact, locally a Legendrian knot is an embedding of an arc (called a *Legendrian arc*) into $\mathbb{R}^3$, which is everywhere tangent to the standard contact structure on $\mathbb{R}^3$.

Definition 2.5.1 *Let (M^5, ξ) be a contact manifold. A submanifold L of (M^5, ξ) is called an **isotropic submanifold** if $T_p L \subset \xi_p$ for all points $p \in L$. An isotropic submanifold of dimension two (an isotropic surface) is called a **Legendrian submanifold** (**Legendrian surface**). (More generally, **Legendrian submanifold** is an embedding of a n-dimensional manifold into $(2n + 1)$-dimensional that is always tangent to the contact hyperplanes.) A **Legendrian embedding** is an embedding $\phi : \Sigma^2 \hookrightarrow (M^5, \xi)$ of a smooth manifold Σ^2 such that the image $L^2 = \phi(\Sigma^2)$ is an embedded Legendrian surface. A smooth 1-parameter family of embedded Legendrian surfaces is called a **Legendrian isotopy**. Two Legendrian submanifolds L, L' are called **Legendrian isotopic** if there is a smooth 1-parameter family L_t, $t \in [0, 1]$, such that $L_0 = L$ and $L_1 = L'$. Equivalently, a Legendrian isotopy is a smooth 1-parameter family $\phi_t : \Sigma^2 \hookrightarrow (M^5, \xi)$ of Legendrian embeddings.*

Indeed all the terms in the above definitions can be restated for Legendrian knots as well. In particular, two Legendrian knots are equivalent if they are isotopic via a family of Legendrian knots. Inequivalent nullhomologous Legendrian knots can be distinguished by their Thurston-Bennequin number and rotation number, which are Legendrian isotopy invariants. The following invariant was originally defined by Bennequin [3] and independently, Thurston when $n = 1$ and generalized to higher dimensions by Tabachnikov [26].

Given an orientable, connected, nullhomologous, Legendrian submanifold L^n in an oriented contact $(2n + 1)$-manifold (M, ξ), we present an invariant $tb(L)$ (see, for instance, [24]), called the ***Thurston-Bennequin number (invariant)*** of L. It is defined

with the help of the linking number as follows:

Pick an orientation on L. Let X be a Reeb vector field for ξ and push L slightly off of itself along X to get another oriented submanifold L' (Legendrian copy of L) disjoint from L. The Thurston-Bennequin number of L is the linking number of L and L', that is, we have

$$tb(L) = lk(L, L')$$

where lk denotes the linking number.

For the linking number, take any $(n+1)$-chain C in M such that $\partial C = L$. Then $lk(L, L')$ equals the algebraic intersection number of C with L'. Intuitively, the Thurston-Bennequin invariant (number) of L measures the twisting of ξ around L. We note that $tb(L)$ is a Legendrian isotopy invariant in any odd dimension.

Remark   (i) Note that $tb(L)$ is independent of the choice of orientation on L since changing it changes also the orientation of L'.

(ii) For any orientable, connected, nullhomologous, Legendrian submanifold L in a contact manifold, we have $tb(L) \in \mathbb{Z}$.

Example 2.5.1 *Consider Legendrian knot L in $(\mathbb{R}^3, \xi_{std})$ where $\xi_{std} = ker(dz + xdy)$. Take its* front projection*, i.e., its projection to the yz-plane. The front projection has no vertical tangencies. Because if the projection had vertical tangency, then $\dfrac{-dz}{dy} = x$ would be equal to $\pm\infty$. L can be approximated by another Legendrian knot for which the projection has only transverse double points and cusp singularities. $\dfrac{\partial}{\partial z}$ is a transverse vector field to the contact plane ξ_{std} in $\mathbb{R}^3$. Hence, we can take a parallel copy L' of L by slightly pushing it in the positive z-direction. Thurston-Bennequin invariant, $tb(L)$ is defined as the twisting of L', the framing of L given by the contact planes relative to L', relative to the framing given by the Seifert surface of L.*

Let L be a Legendrian nullhomologous knot. Its Thurston-Bennequin number can be computed as follows: We have

$$tb(L) = lk(L', L'')$$

where L' and L'' are pushoffs of L with respect to Seifert framing and contact framing, respectively. Contact framing is obtained by pushing L in a direction transverse to ξ, and Seifert framing is obtained by pushing L along a Seifert surface of L. Moreover, if we fix an orientation on L, i.e., if we get an oriented Legendrian knot L, then $tb(L)$ can be computed as:

$$tb(L) = w(L) - \frac{1}{2}c(L)$$

where $w(L)$ is the ***writhe*** of L; i.e., the sum of the signs of the crossings with respect to the following Figure 2.6, and $c(L)$ is the number of cusps in the front projection of L. The ***cusp*** is the singular point in the front diagram of L. Here, $w(L)$ is independent of the choice of the orientation of L while it depends on the projection.

+1   -1
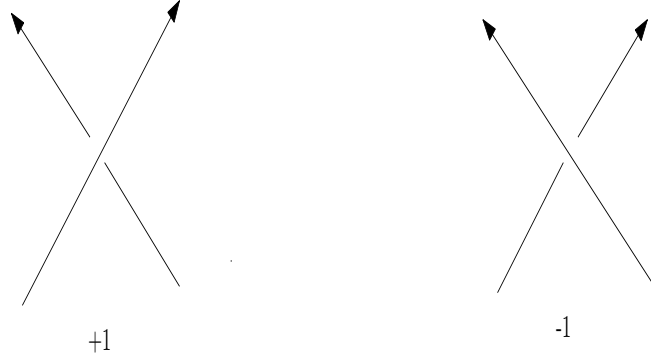
Figure 2.6: Positive and negative crossings.

Definition 2.5.2 *For an oriented Legendrian knot K in $(\mathbb{R}^3, \xi_0 = Ker(dz + xdy))$ (or equivalently in (S^3, ξ_{st})), its **rotation number** $rot(K)$ is defined as*

$$rot(K) = \frac{1}{2}(D - U)$$

where D (resp. U) denotes the number of down (resp. up) cusps in the front projection of K.

Example 2.5.2 *In Figure 2.7 , there are front diagrams of a Legendrian representatives of unknot and trefoil knot. The Thurston-Bennequin number and the rotation number of unknot is -1 and 0, respectively. Similarly, The Thurston-Bennequin number and the rotation number of trefoil knot is 1 and 0, respectively.*
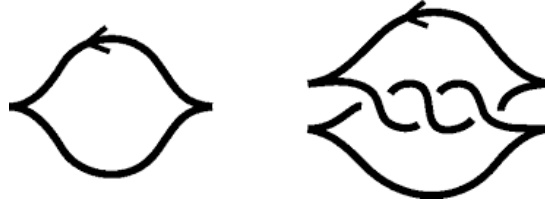
Figure 2.7: A Legendrian unknot (on the left) and a trefoil knot (on the right).

Remark Thurston-Bennequin number and rotation number are known as the classical invariants of Legendrian knots. Note that inequivalent Legendrian knots can be distinguished by these invariants.

Definition 2.5.3 *Let L be an oriented Legendrian knot in the standard contact structure ξ_{std} on $\mathbb{R}^3$. When we modify the front projection of L by adding a down cusp, we get **the positive stabilization** of a Legendrian knot L, which is denoted by $S_+(L)$. In a similar way, when we modify the front projection of L by adding a up cusp, we get **the negative stabilization** of a Legendrian knot L, which is denoted by $S_-(L)$.*
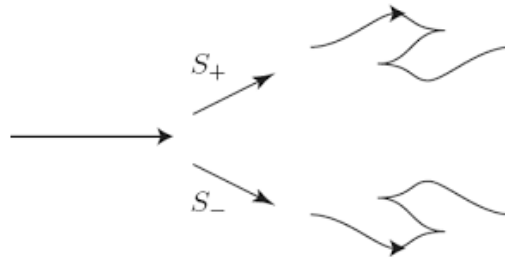
Figure 2.8: The positive and negative stabilizations of L.

Remark For a fixed topological knot type, different Legendrian representatives might have different Thurston Bennequin and rotation numbers. By adding more cusps, one can obtain new Legendrian representatives realizing any pregiven integer as a rotation number. However, although Thurston Bennequin number can be made arbitrarly small, it is not possible to increase forever. That is, stabilizing a Legendrian knot L changes the classical invariants as $tb(S_\mp(L)) = tb(L)-1$ and $rot(S_\mp(L)) = rot(L)\mp1$. There are some relations between the Thurston–Bennequin and the rotation numbers of Legendrian realisations of a given knot type K. One of them is the **Bennequin**

inequality (due to Bennequin and Eliashberg) which holds in any tight contact manifold and for any nullhomologous Legendrian knot. This inequality provides an upper bound for the Thurston Bennequin number for nullhomologous Legendrian knots in tight contact 3-manifolds:

Theorem 2.5.4 (*Bennequin inequality*) *Let K be a Legendrian knot in a tight* 3-*manifold* (M, ξ) *which bounds a surface* $\Sigma \subset M$. *Then*

$$tb(K) + \mid rot(K) \mid \leq -\chi(\Sigma)$$

where $\chi(\Sigma)$ *denotes the Euler characteristic of* Σ.

2.6 Handle Decompositions of Stein Surfaces

Let's recall language of handlebodies. A copy of $D^k \times D^{n-k}$ that is attached to the boundary of an n-manifold along its attaching region $\partial D^k \times D^{n-k}$ is called a ***handle of index*** k or k-***handle***. We begin with a 0-handle, in fact empty set, and attach handles to built a ***(smooth or topological) handlebody***. For the smooth case, we glue each handle by a smooth embedding of its attaching region, after smoothening the resulting corners. These corners can be along $\partial D^k \times D^{n-k}$. One can find the details in [16]. Conversely, starting from a real-valued Morse function on a manifold X, one can obtain its handlebody description which is also referred to as a ***handle decomposition*** of X. In fact, an i-handle is the smooth analogue of an i-cell. Note that every manifold (with or without boundary) is a handlebody. That is, it has an expression as a union of handles. A handlebody decomposition introduce the manifold up to homeomorphism. Furthermore, in dimension 4, they describe the smooth structure if the attaching maps are smooth. Following results will be used:

The phrase "a Stein surface" will refer to a Stein domain of real dimension 4. Pictures of handlebody diagrams of Stein surfaces (***Stein diagrams*** for short) were studied extensively by Gompf [15]. He gave description of 1-handles in the setting of Stein surfaces and a standard form for Legendrian links in $\#nS^1 \times S^2 (=$ Boundary of the 0-handle $\cup$ n 1-handles). From this description, one can define and compute Thurston-Bennequin invariant.

Definition 2.6.1 *A **Legendrian link diagram** in standard form, like in the Figure 2.9, is given by the following way:*

• n 1-handles, showed by horizontal ball pairs

• A collection of n horizontal distinguished segments coresponding to each ball pair

*• A front projection of a generic **Legendrian tangle** (i.e., disjoint union of Legendrian knots and arcs) with endpoints touching the segments.*
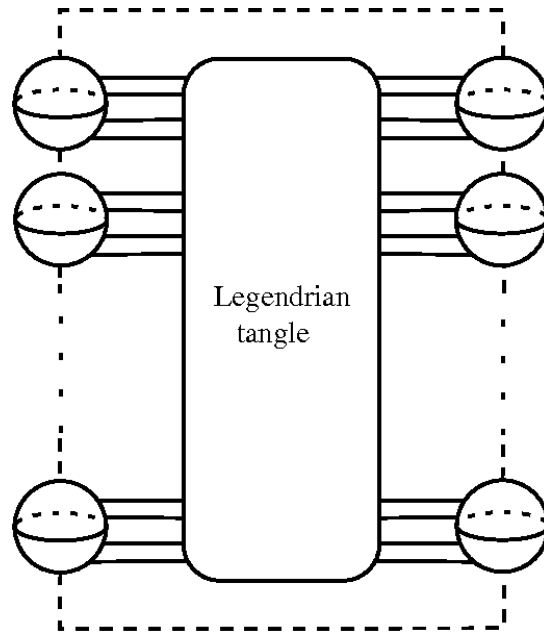
Figure 2.9: A Legendrian link diagram.

Similar to how it is defined for Legendrian knots in the standard contact three-space, the Thurston-Bennequin number of a Legendrian knot K in a boundary of a Stein surface can be defined as

$$tb(K) = w(K) - \frac{1}{2}c(K)$$

with the help of a Legendrian tangle (see [15]). The following result will be used later:

Theorem 2.6.2 ([7], [15]) *An oriented, compact, connected 4-manifold X is a Stein surface if and only if it has a handlebody diagram which formed by a Legendrian link diagram such that 2-handles attached to link components L_i's with framing $tb(L_i)-1$.*

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A band in $(\mathbb{R}^3, \xi_{std})$ is called **_Legendrian band_** if its boundary consists of any Legendrian curve and push-off of it in the z-direction. For any two oriented Legendrian knot L_1 and L_2, the connected sum of the knots by a Legendrian band is called **_Legendrian connected sum_**. Here any band can be used. However, its intersection with the rest of the link must be empty. When the orientation is consistent, this Legendrian sum is called a **_Legendrian handle addition_**. Otherwise we get a **_Legendrian handle substraction_**. In the [6], there are details about this.

Theorem 2.6.3 ([7, 16]) *An oriented* 4*-manifold admits a Stein structure if and only if it is diffeomorphic to the interior of a handlebody whose handles all have index* ≤ 2*, and for which each* 2*-handle is attached along a Legendrian knot with framing obtained from the contact framing by adding one left twist.*

CHAPTER 3

RELATIVE PAGE CROSSING NUMBER

In this chapter, we give a definition of a Legendrian isotopy invariant step by step and also we give a proof of Spider Lemma.

We study Legendrian embeddings of a closed orientable Legendrian submanifold in a closed contact 5-manifold whose contact structure is supported by an open book with Weinstein (and so Stein) pages.

3.1   Spider Lemma

Let $(M^5, \xi = Ker(\alpha))$ be a closed, contact 5-manifold where α is a (global) contact form with the Reeb vector field R that is compatible with an open book (B, f) on M. ξ is a positive contact structure, in other words, $\alpha \wedge (d\alpha)^2 > 0$ with respect to a given orientation on M. Since (M, ξ) is a contact manifold, ξ determines a canonical orientation on M. So M is oriented 5-manifold. B is the binding of an associated abstract open book $OB(X, h)$ where X is the page, $B = \partial X$ and h is the monodromy. Thus, $(B^3, \xi|_B = Ker(\alpha|_B))$ is the convex boundary of each symplectic page $(X^4, d\alpha|_B)$, and so it is a 3-dimensional tight contact (sub)manifold. ($\alpha|_B$ is a contact form on B.) Let L be a closed orientable Legendrian surface of (M, ξ), and so there is a Legendrian embedding $\phi : \Sigma \hookrightarrow (M, \xi)$ such that $\phi(\Sigma) = L$ where Σ is a 2-dimensional surface which determines the topological type of L. For the invariants that we will define, one first need to know that L and B intersect transversely, and the pages are Weinstein. To this end, we define the following class of supporting open books:

Definition 3.1.1 *Let L be a closed, oriented, Legendrian submanifold of a closed contact 5-manifold (M, ξ). An open book (B, f) on M supporting ξ is called an **admissable open book for** L if it has simply-connected Weinstein pages and L intersects B transversely.*

Let us start with showing that one can always find an open book whose binding intersects a given Legendrian surface transversely. For similar arguments, we refer the reader to [2].

Lemma 3.1.2 (*Spider Lemma*) *Let (M, ξ) be a closed contact 5-manifold and (B, f) an open book on M supporting ξ with Weinstein pages. Also let L be a closed orientable Legendrian surface of M. Then there exists an isotopy $(B_t, f_t), t \in [0, 1]$ of open books all of which supporting ξ such that $(B_0, f_0) = (B, f)$, $(B_1, f_1) = (B', f')$, and L intersects B' transversely.*

Proof If L and B transversely intersect, then there is nothing to prove. If they don't intersect transversely, then consider a neighborhood of B in M which can be identified with $B \times D^2$. Nearby generic $B' \subset B \times D^2$ (which is a copy of B) intersects L transversely. Then we can isotope B to B' (and accordingly the pages of the open book (B, f)) using the flow of an appropriate contact vector field compactly supported in $B \times D^2$. (See Figure 3.1.) So, we obtain a family of open books $\{(B_t, f_t)\}$ for M such that $(B_0, f_0) = (B, f)$, $(B_1, f_1) = (B', f')$. Finally, we note that at any time $t \in [0, 1]$ compatibility conditions in Definition 2.4.4 are satisfied by the open book (B_t, f_t) since the isotopy is based on a contact vector field. $\square$

3.2 Definition of Relative Maximal Page Crossing Number

Assuming Spider Lemma have been already applied, we may start with a supporting open book (B, f) such that $L \pitchfork B$ for a given closed, compact, orientable Legendrian surface L in a closed contact 5-manifold M. Since they intersect transversely, we have $dim(L \pitchfork B) = 0$, and so they intersect along a finite number of points. Later we will be interested in minimal geometric intersection. Take any orientation on the Legendrian surface L. If the orientations of L and B are consistent at a transverse
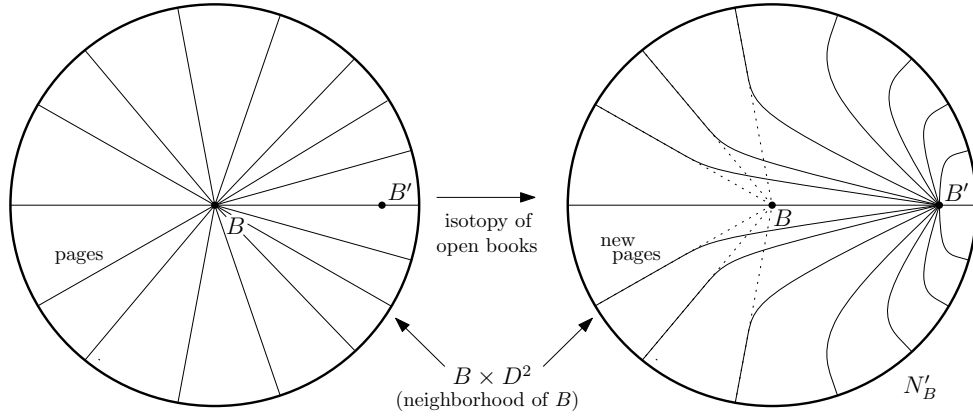
Figure 3.1: Isotoping a given open book so that the new binding B' intersects L transversally.

intersection point, then mark the point with plus $(+)$, otherwise mark the point with minus $(-)$. Continue this procedure until all the intersection points have labelled.

Remark • Since we need an open book structure, M must be closed.

• Since L and B are compact, their intersection consists of finitely many points. Note B is compact because of the open book structure.

• Homology intersection of L and B is 0. This is because B is the boundary of X. So B has zero homology class.

• By Eliashberg, stein fillability implies tightness. Hence the contact structure on B is tight.

From the above remark, the intersection of L and B consists of even number of points. The number of plus points is equal to the number of minus points because homology intersection is 0.

Consider the pages $X_0 = f^{-1}(\theta), X_1 = f^{-1}(\theta + \pi)$ for $\theta \in S^1$. By genericity, we may assume L transversally intersects X_0, X_1 and their common boundary B. Let $D(X) = X_0 \cup_\partial X_1$ be the double of the page X, union of the pages X_0 and X_1 such that $X_0 \cong X \cong X_1$. That is, X_0 and X_1 are dual pages of each other. Clearly, $D(X)$ is closed. $(X_0, d\alpha_0)$ and $(X_1, d\alpha_1)$ are Stein domains. (Recall from the explanation in the background that there exists always an open book whose pages are Stein.) Consider the handle decompositions as in Figure 3.2.
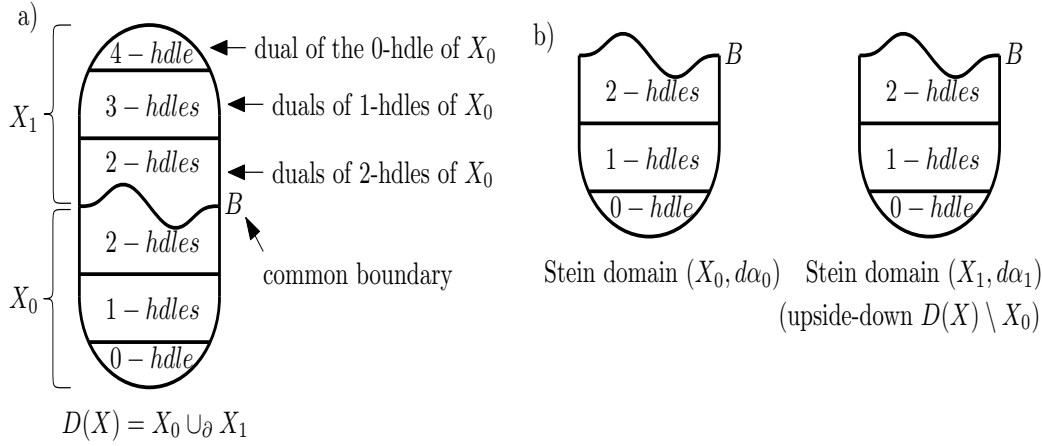
Figure 3.2: a) Handle decomposition of the double $D(X)$, union of the pages X_0 and X_1. b) The Stein domains $(X_0, d\alpha_0)$ and $(X_1, d\alpha_1)$.

Note that $dim(L) = 2$ and $dim(X) = 4$, so $dim(L \pitchfork D(X)) = 1$. That is, L and $D(X)$ intersect at 1-dimensional curves generically. Let's first assume, for simplicity, that $L \pitchfork X$ consists of a single curve K. Also let $L \cap X_0 = k_0$, $L \cap X_1 = k_1$. That is, $K = k_0 \cup k_1$. For k_0, take orientation from plus point to the minus point. Similarly, for k_1, take orientation from minus point to the plus point. See Figure 3.3.

Sketch the Legendrian arcs for k_0 and k_1 in the Stein diagrams of X_0 and X_1, respectively, and calculate Thurston-Bennequin numbers of these two arcs. Summing these two numbers will give us an integer, denoted by $\widetilde{tb}(K)$. In other words, we define

$$\widetilde{tb}(K) := tb(k_0) + tb(k_1).$$

In the general case, the intersection of L and $D(X)$ may consist of finite number of curves, say K_1, K_2, ..., K_r. (Note that K_i's are disjoint by transversality theorem.) That is, Let us assume that

$$L \pitchfork D(X) = \bigsqcup_{i=1}^{r} K_i.$$

Again one can sketch the Legendrian arcs constructing the knot components of the link of the intersection of L with the double $D(X)$ in the Stein diagrams of X_0 and X_1, and therefore, we obtain a diagram in Figure 3.4 describing the transverse intersection $L \pitchfork D(X)$.

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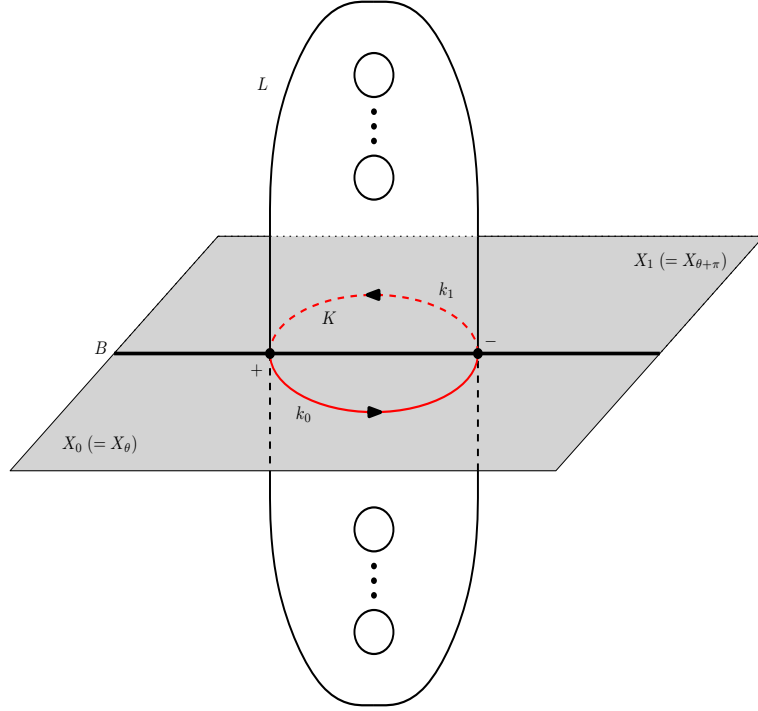
Figure 3.3: Embedded Legendrian surface L intersecting transversally the binding B and a pair of pages X_0 and X_1. A typical knot component $K = k_0 \cup k_1$ of the link of intersection of L with the double $D(X) = X_0 \cup_\partial X_1$.

For all knot components $K_i = k_i^0 \cup k_i^1$, we calculate $\widetilde{tb}(K_i)$ as above. Summing all these together and taking the maximum of such sums by changing L in its Legendrian isotopy class, one can define a number. First, we need some preliminary definitions:

Definition 3.2.1 *Let $L \hookrightarrow (M^5, \xi)$ be a closed orientable Legendrian surface. Fix an admissable open book (B, f) for L. Consider*

$$[L] = \{L' \subset (M, \xi) \mid L' \text{ is Legendrian isotopic to } L\}$$

*This class is called the **Legendrian isotopy class** of L. Fix a page X of the open book (B, f), and L' which is Legendrian isotopic to the L and trasversally intersecting the double $D(X)$. Then the **page crossing number** of L' **with respect to** X is defined as*

$$\mathcal{P}_X(L') = \sum_{i=1}^{r} \widetilde{tb}(K_i).$$

*Lastly, we say that the double $D(X)$ **essentially intersects** L if we have*

$$L' \cap D(X) \neq \varnothing, \quad \forall L' \in [L].$$

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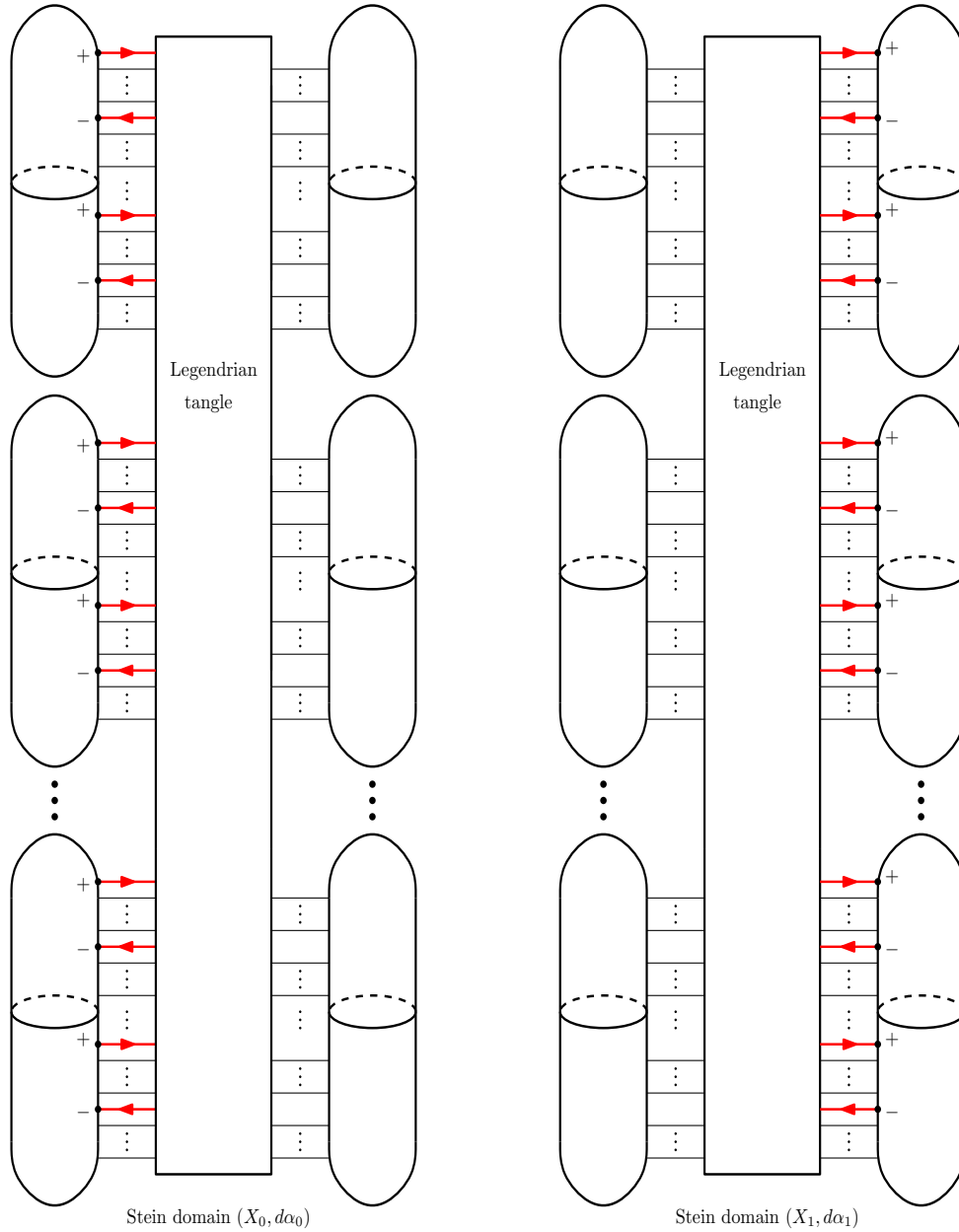
Figure 3.4: Legendrian arcs (in red) constructing the knot components of the link of L with the double $D(X) = X_0 \cup_\partial X_1$.

We are ready to define our first invariant:

Definition 3.2.2 *Let $L \hookrightarrow (M^5, \xi)$ be a closed orientable Legendrian surface. Fix an admissable open book (B, f) for L and a page X of (B, f) such that $D(X)$ essentially intersects L. Then*

$$MP_X(L) \coloneqq Max\{\mathcal{P}_X(L') \mid L' \in [L] \ and \ L' \pitchfork D(X)\}$$

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*is called the **relative maximal page crossing number of** L **with respect to** X.*

Remark We immediately note that (relative maximal) page crossing number can be defined for at least 5-dimensional contact manifolds: By definition of the binding, B is $(2n-1)$-dimensional when M is $(2n+1)$-dimensional, and so $L^n \cap B$ is $(n-2)$-dimensional. Hence, if M is 3-dimensional, then $L \cap B$ must be (-1)-dimensional. And it is clear that this is impossible.

Well-definedness of $M\mathcal{P}_X(L)$ will be discussed in Chapter 4. Until then, $M\mathcal{P}_X(L)$ will be assumed to be well-defined. The following facts indicate that the most practicle way of computing $M\mathcal{P}_X(L)$ is working in the case of geometrically minimal intersection.

Lemma 3.2.3 *Let $K = k_0 \cup k_1$ be a component of the link of intersection of L with the double $D(X) = X_0 \cup_\partial X_1$ constructed using the minimal geometric intersection points of L and B. Suppose γ is an arc on the attaching sphere S of the 1-handle of X_i connecting the boundary points ∂k_i. Then the circle $k_i \cup \gamma$ can not be a homotopically trivial in X_i for each $i = 0, 1$.*

Proof Take $i = 1$ (the case $i = 0$ is similar). The statement of the lemma is equivalent to say that $k_i \cup \gamma$ can not bound a disc in X_1. Suppose there exists such a disk $D \subset X_i$. Then using the flow of a suitable contact vector field (compactly supported in a neighborhood of D in M which is indeed some Darboux ball $\mathbb{D}^5$), we can Legendrian isotope L until the arc k_1 disappears (i.e., the whole k_1 is transformed into X_0). This means that the $\pm$-intersection points corresponding to ∂k_1 is canceling pair. Since in the new Stein pictures, there would be a less number of intersection point, this contradicts to the minimality. (See Figure 3.5.) $\square$

Remark In Lemma 3.2.3, the path γ is chosen away from the points where other knots and arcs meet with S. Also in Figure 3.5, for simplicity, k_1 is drawn with a single left cusp, but more number of cusps are also possible and threated in the same way as long as the disk D exists. When we move k_1, this cusp (and hence the pair of
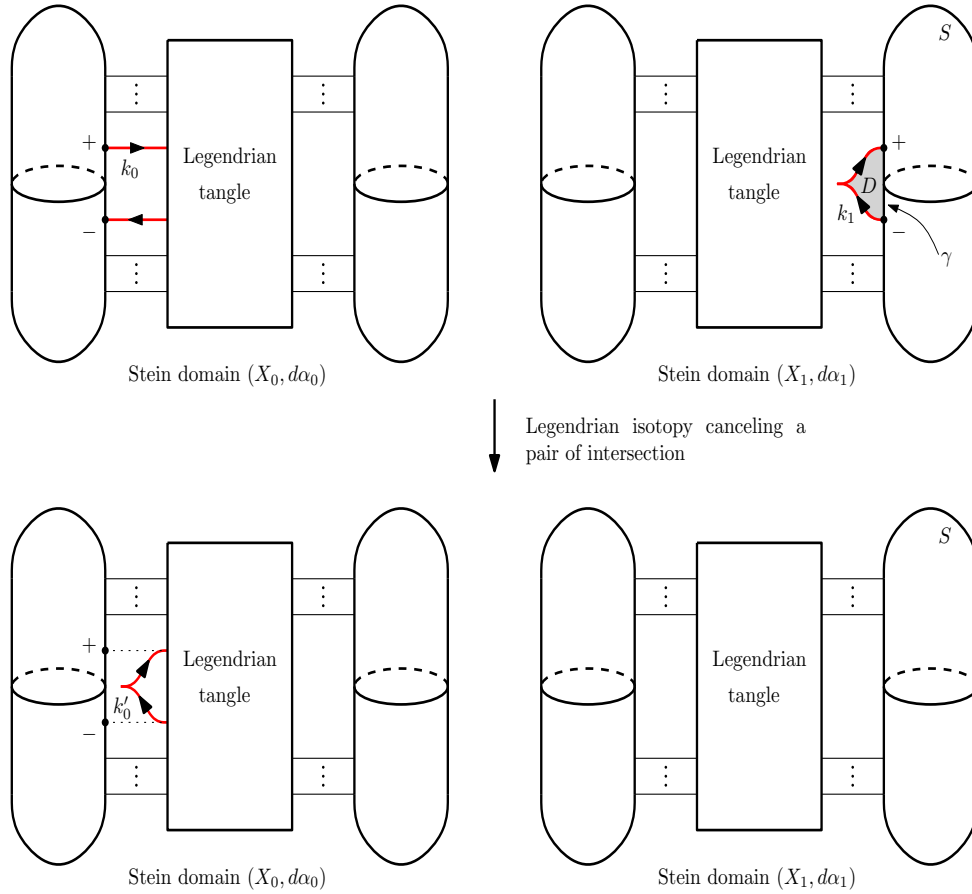
Figure 3.5: Realizing a Legendrian isotopy of L (cancelling a pair of intersection points in $L \cap B$) by isotoping L (through the disk $D \subset X_1$ enclosed by k_1 and the path γ on the attaching sphere S of the 1-handle of X_1 joining the points "+" and "-") in the Stein diagrams $(X_0, d\alpha_0)$ and $(X_1, d\alpha_1)$.

intersection points "+" and "-") will disappear. Note that after such a canceling a pair of intersection, $\widetilde{tb}$ doesn't change. That is,

$$tb(k_0) + tb(k_1) = tb(k_0') + tb(k_1') = tb(k_0')(\text{or} = tb(k_1') \text{ in the case } i = 0).$$

Lemma 3.2.4 *Let $K = k_0 \cup k_1$ be a component of the link of intersection of L with the double $D(X) = X_0 \cup_\partial X_1$ constructed using (not necessarily minimal) geometric intersection points of L and B. Suppose K is homotopically trivial in $D(X)$. Then*

$$\widetilde{tb}(K) = tb(k_0) + tb(k_1) \leq -1.$$

Proof By assumption there exists a disk $D \subset D(X)$ with $K = \partial D$. There are two cases: Either $k_1 = \varnothing$ or $k_1 \neq \varnothing$. If $k_1 = \varnothing$ holds, then $K = k_0$ is a Legendrian unknot inside the Legendrian tangle in the Stein diagram of X_0. Therefore, it can be considered as a Legendrian unknot bounding the disk D inside the Stein fillable (and so tight) boundary ∂X_0. But then Theorem 2.5.4 implies that $\widetilde{tb}(K) \leq -1$. If $k_1 \neq \varnothing$ holds, then this means that $D = D_0 \cup D_1$ where D_0, D_1 are disks in X_0, X_1, respectively, which meet along an arc γ on the attaching spheres of the corresponding 1-handles of X_0 and X_1. Then applying Lemma 3.2.3, one can transform K to K' which lies in X_0. Recall that $\widetilde{tb}(K) = \widetilde{tb}(K')$, that is the number $\widetilde{tb}$ does not change under the move described in the proof of Lemma 3.2.3 (Figure 3.5). Therefore, we are again in the first case above, i.e., $\widetilde{tb}(K) = \widetilde{tb}(K') \leq -1$. $\square$

Lemma 3.2.5 *Let $k_0' \cup k_1'$ be a component of the link of intersection of L with the double $D(X) = X_0 \cup_\partial X_1$ constructed using (not necessarily minimal) geometric intersection points of L and B. If the knot $k_0' \cup k_1'$ is homotopically trivial in both L and the double $D(X)$, then it can be ignored while computing $M\mathcal{P}_X(L)$. That is,*

$$M\mathcal{P}_X(L) > \mathcal{P}_X(L).$$

Proof Suppose there exists such a pair of Legendrian arcs k_0' and k_1' in the Stein diagrams of X_0 and X_1 whose union is homotopically trivial in both L and the double $D(X)$. Therefore, there are disks $D_i \subset X_i$ such that the union $D_0 \cup D_1 \subset D(X)$ (enclosed by $k_0' \cup k_1'$) is not punctured by the rest of $L \cap D(X)$ and the attaching circles of the 2-handles of X_0 and X_1, and also there is a disk $D \subset L$ bounded by $k_0' \cup k_1'$ (Figure 3.6). Then one can get rid of the intersection arcs k_0', k_1' (and so the corresponding intersection points p, q) by isotoping B (and the pages of the open book) in a neighborhood of the 3-disk enclosed by the disk $D \subset L$ and $D_0 \cup D_1$ in M (which is some Darboux ball $\mathbb{D}^5$) using the flow of an appropriate contact vector field compactly supported in $N \cong \mathbb{D}^5$. (See Figure 3.7.)

By Spider Lemma (Lemma 3.1.2) and the genericity, one can think of these isotopy results in a new open book (with the same monodromy) such that X_i transformed to a new page X_i' and B transformed to a new binding B'. Note that this contact isotopy eliminates k_0', k_1'. Now we rewind this isotopy to move all the points inside
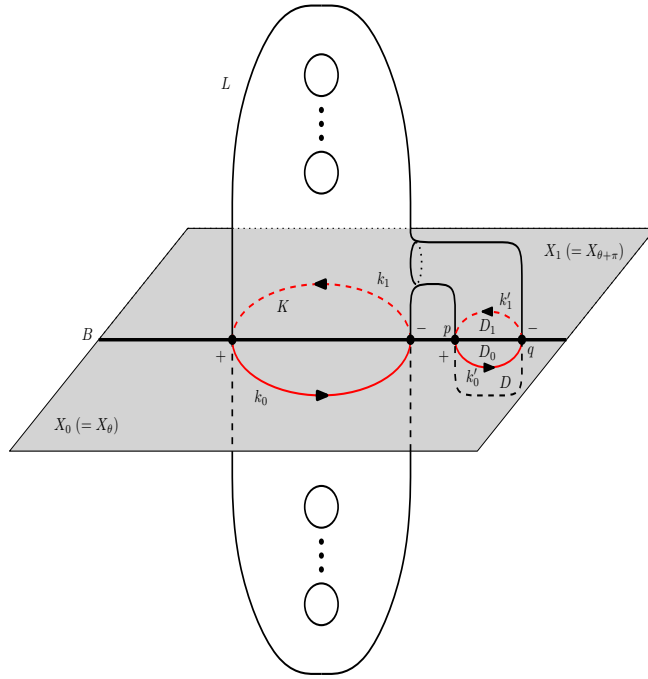
Figure 3.6: A (geometrically) cancelling pair of points (p and q) of intersection between L and the binding B.
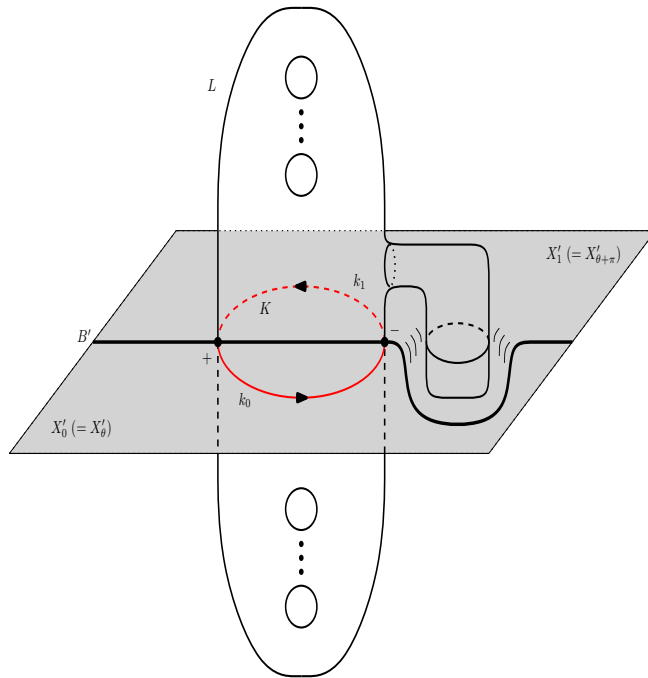
Figure 3.7: Isotoping the binding B and correspondingly all the pages of the open book using the flow of a compactly supported contact vector field.

the Darboux ball $\mathbb{D}^5$ back to the their original positions (at the initial time). While this transform X_0', X_1' and B' back to their original positions, the part of L in $\mathbb{D}^5$ will be pushed further, and we get a Legendrian isotopic copy L' of L which does not intersect B along k_0', k_1'. Since the isotopy is compactly supported near k_0', k_1', the arcs describing L' in the Stein digrams of X_0 and X_1 coinsides with the ones describing L outside the Darboux ball $\mathbb{D}^5$. Therefore, to picture L' in these diagrams, we simply erase the arcs k_0', k_1' from the diagrams, and hence ignore their contributions to $\widetilde{tb}$. That is, we have

$$\widetilde{tb}(L') = \widetilde{tb}(L) - [tb(k_0') + tb(k_1')].$$

On the other hand, by Lemma 3.2.4, we have $tb(k_0') + tb(k_1') < 0$, and so, combining this with the above equality we get

$$\mathcal{P}_X(L') > \mathcal{P}_X(L).$$

Hence, $\mathcal{P}_X(L)$ can not be maximum, and so it is strictyly less than $M\mathcal{P}_X(L)$. □

CHAPTER 4

PROOF OF THEOREM 1.0.1

In this chapter, we will show that the number $MP_X(L)$, which we define in Chapter 3, is preserved under Legendrian isotopies, and also explain why it is well-defined. We give definitions of two types of Legendrian isotopies and deal with $MP_X(L)$ by considering possible types of Legendrian isotopies with respect to a fixed page. First, assuming the number $MP_X(L)$ is well-defined, one can easily observe the following:

Lemma 4.0.1 *The number $MP_X(L)$ is invariant under Legendrian isotopies of L.*

Proof Consider any Legendrian isotopy L_t ($t \in [0, 1]$) between $L = L_0$ and L_1. Let X be a fixed page of an admissable open book (B, f) for L such that $D(X)$ essentially intersects L. Suppose that $L' \in [L]$ is a representative maximizing $\mathcal{P}_X$, that is,

$$MP_X(L) = \mathcal{P}_X(L').$$

Since L_1 is Legendrian isotopic to L, we have $[L_1] = [L]$, that is, their Legendrian isotopy classes are the same. Therefore, L' is maximizing $\mathcal{P}_X$ among all representaives in $[L_1]$ as well, that is,

$$MP_X(L_1) = \mathcal{P}_X(L').$$

Hence, $MP_X(L_1) = \mathcal{P}_X(L') = MP_X(L)$ as required. □

In order to show that $MP_X(L)$ is well-defined, first of all, one needs to understand how $\mathcal{P}_X(L)$ changes under possible types of Legendrian isotopies of L. For a fixed page X, there are two types of Legendrian isotopies of a given Legendrian surface L which are called a *regular isotopy* and an *irregular isotopy*.

Let $L \hookrightarrow (M^5, \xi)$ be a closed orientable Legendrian surface. Take an admissable open book (B, f) for L. Fix a page X of the open book (B, f) such that L is transversely intersecting the double $D(X)$. (By genericity, this is possible.) we define:

Definition 4.0.2 *A **regular isotopy of** L **with respect to** $D(X)$ is a Legendrian isotopy L_t ($t \in [0, 1]$) of $L = L_0$ such that L_t transversely intersects $D(X)$ for all $t \in [0, 1]$.*

Under the assumptions introduced above, we have

Proposition 4.0.3 *The number $\mathcal{P}_X(L)$ is invariant under regular Legendrian isotopies of L with respect to $D(X)$.*

Proof Consider a regular Legendrian isotopy L_t ($t \in [0, 1]$) of $L = L_0$. By definition L_t transversely intersects $D(X)$ for all $t \in [0, 1]$. We need to show that $\mathcal{P}_X(L') = \mathcal{P}_X(L)$ where $L' = L_1$ is the Legendrian copy of L at time $t = 1$.

Let $K = k_0 \cup k_1$ be any knot component in $L \pitchfork D(X)$. Since L_t transversely intersects $D(X)$ for all $t \in [0, 1]$, during the isotopy, K is transformed through knots $K_t \in L_t \pitchfork D(X)$ to a knot component $K' = k_0' \cup k_1' \in L' \pitchfork D(X)$ as depicted in Figure 4.1. (Here we think $K = K_0$, $K' = K_1$.)

Observe that $K_t, t \in [0, 1]$ indeed defines a Legendrian isotopy from K to K' when we consider their arcs to be embedded Legendrian arcs inside Stein diagrams of X_0 and X_1. (See Figure 4.2 for a sample picture.)

Therefore, since their union are isotopic via Legendrian moves, the arcs constructing K and K' satisfies

$$tb(k_0) + tb(k_1) = tb(k_0') + tb(k_1'),$$

and so, $\widetilde{tb}(K) = \widetilde{tb}(K')$. This implies that $\mathcal{P}_X(L) = \mathcal{P}_X(L')$ because each summand of $\mathcal{P}_X(L)$ agrees with the corresponding summand of $\mathcal{P}_X(L')$ by the above discussion.
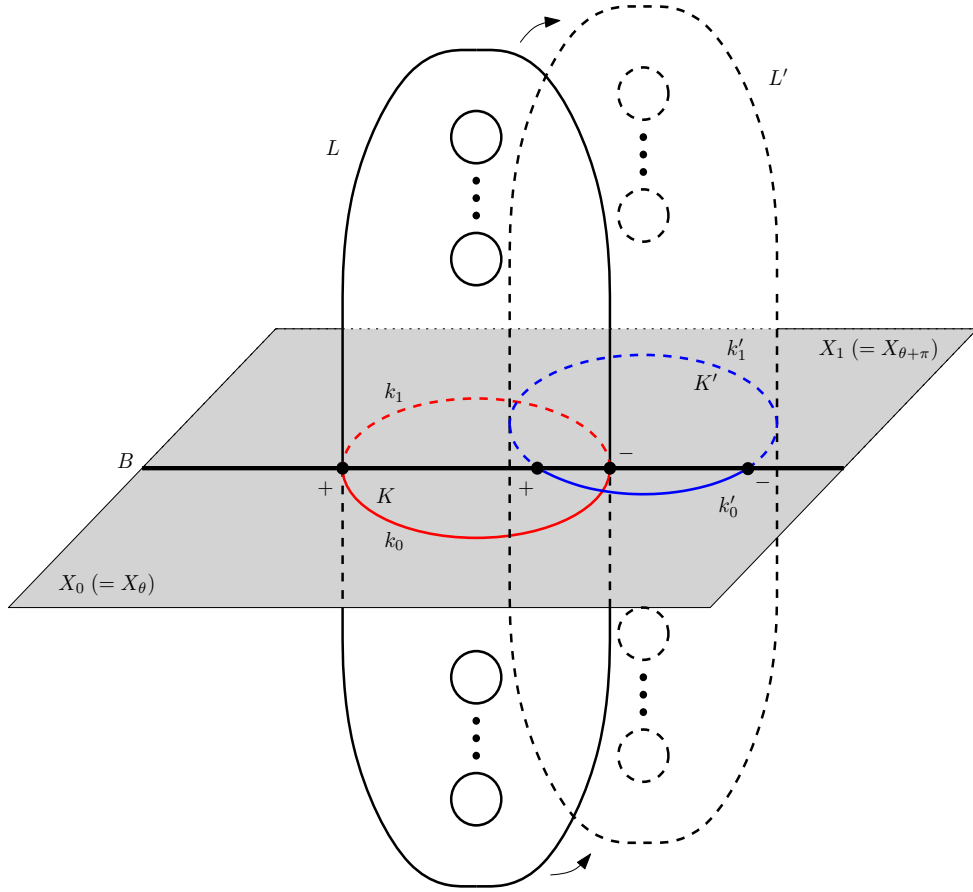
Figure 4.1: A regular Legendrian isotopy L_t taking $L_0 = L$ to another Legendrian $L_1 = L'$ which is still intersecting the double $D(X) = X_0 \cup_\partial X_1$ transversally, but the new points of intersection in $L' \cap B$ are possibly different than the older ones.

Observe that $K_t, t \in [0,1]$ indeed defines a Legendrian isotopy from K to K' when we consider their arcs to be embedded Legendrian arcs inside Stein diagrams of X_0 and X_1. (See Figure 4.2) for a sample picture.)

Therefore, since their union are isotopic via Legendrian moves, the arcs constructing K and K' satisfies

$$tb(k_0) + tb(k_1) = tb(k_0') + tb(k_1'),$$

and so, $\widetilde{tb}(K) = \widetilde{tb}(K')$. This implies that $\mathcal{P}_X(L) = \mathcal{P}_X(L')$ because each summand of $\mathcal{P}_X(L)$ agrees with the corresponding summand of $\mathcal{P}_X(L')$ by the above discussion. $\square$

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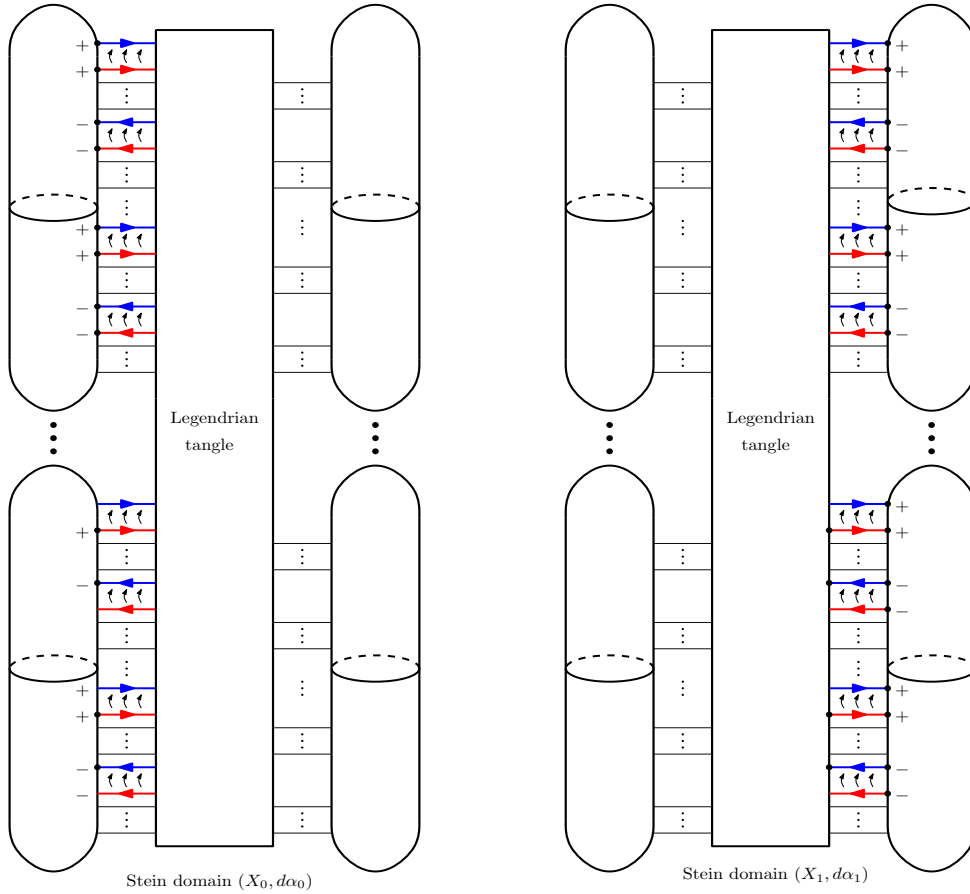
Figure 4.2: Realizing a regular Legendrian isotopy L_t ($t \in [0,1]$) taking $L_0 = L$ to another Legendrian $L_1 = L'$ in the Stein diagrams of X_0 and X_1. The Legendrian arcs (in red) describing $L \cap D(X)$ are Legendrian isotopic to those (in blue) describing $L' \cap D(X)$ through a Legendrian isotopy $K_t = L_t \cap D(X)$, $t \in [0,1]$.

Remark Observe that all the arguments in the proof of Proposition 4.0.3 work whenever we take a Legendrian representative L from the Legendrian isotopy class $[L]$ which transversely intersects the double $D(X)$. In particular, if L (which we start with at the beginning of the proof) is itself maximazing all such possible page crossing numbers, i.e., if

$$M\mathcal{P}_X(L) = \mathcal{P}_X(L),$$

then the same will be also true for L'. As a result, we have $M\mathcal{P}_X(L) = M\mathcal{P}_X(L')$. Hence, this reproves Lemma 4.0.1 in the case of regular Legendrian isotopies with respect to $D(X)$.

Once again let $L \hookrightarrow (M^5, \xi)$ be a closed orientable Legendrian surface. Take an admissable open book (B, f) for L. Fix a page X of the open book (B, f) such that L is transversely intersecting the double $D(X)$. We define:

Definition 4.0.4 *An **irregular isotopy of** L **with respect to** $D(X)$ is a Legendrian isotopy L_t ($t \in [0, 1]$) of $L = L_0$ such that $L' = L_1$ still transversely intersects $D(X)$ but the new intersection set $L' \cap D(X)$ is obtained from $L \cap D(X)$ via a sequence of births or deaths of intersection knots or due to degenerations of knots in $L \cap D(X)$.*

Proposition 4.0.5 *During irregular Legendrian isotopies of L with respect to $D(X)$, there can not be any births or deaths of nontrivial intersection knots with $D(X)$. Moreover, under such isotopies, the number $\mathcal{P}_X(L)$ makes only finite jumps due to births or deaths of unknots and degenerations of knots in $L \cap D(X)$.*

Proof Consider an irregular Legendrian isotopy L_t ($t \in [0, 1]$) of $L = L_0$. By definition, L_t does not transversely intersect $D(X)$ for all $t \in [0, 1]$. But generically almost all intersection will be transverse. After a small perturbation of the isotopy L_t (if necessary) but still calling the resulting isotopy L_t, one may assume that there are numbers $0 < t_0 < t_1 < \cdots < t_r < 1$ so that except finitely many L_{t_i}, $(i = 0, 1, ..., r)$, any other L_t intersects $D(X)$ transversely. Therefore, for the second statement, one needs to show that there exists $N \in \mathbb{N}$ such that

$$|\mathcal{P}_X(L') - \mathcal{P}_X(L)| < N$$

where $L' = L_1$ is the Legendrian copy of L at time $t = 1$.

Let us consider the case when we pass from time $t = 0$ to $t = t_0 + \epsilon$ for $\epsilon < t_1 - t_0$. (the discussion for passing $t = t_i - \epsilon$ to $t = t_i + \epsilon$ is similar.) First of all, comparing to those in $L \cap D(X)$ if there are new unknots (births) in $L_{t_0 + \epsilon} \cap D(X)$ (they necesarrily bound disks in $D(X)$ by admissibility assumption), then these births arise as an unknot K which may (or may not) bound a disk D' in $L_{t_0 + \epsilon}$, but they must bound a disk D in $D(X)$ as depicted in Figure 4.3. The existence of the disk D and Lemma 3.2.4 implies that $\widetilde{tb}$ of all these unknots are negative, and so whenever such an unknot arises, this will decrease the number $\mathcal{P}_X$. Similarly, comparing to those in $L \cap D(X)$
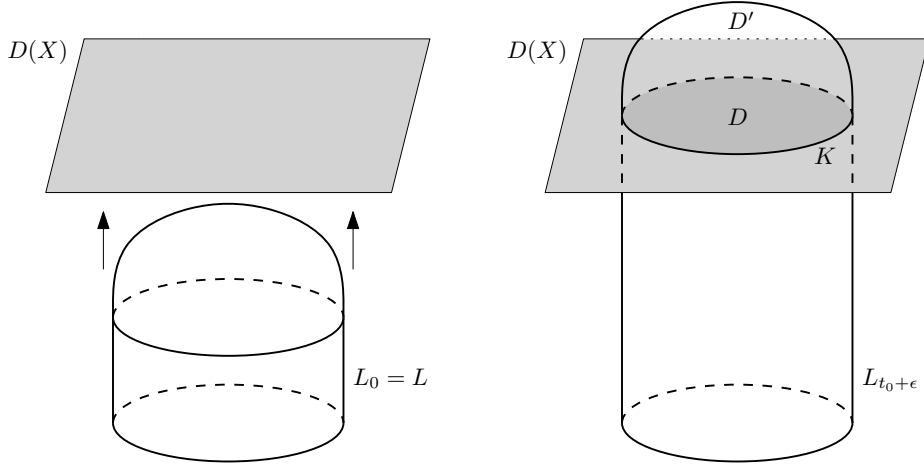
Figure 4.3: A birth of an unknot K during an irregular Legendrian isotopy L_t ($t \in [0,1]$). K bounds a disk D' in $L_{t_0+\epsilon}$ and a disk D in $D(X)$.

if there are missing unknots (deaths) in $L_{t_0+\epsilon} \cap D(X)$ (which were bounding disks in $D(X)$), then these will increase the number $\mathcal{P}_X$.

Note that these births can not be non-trivial knots in $L_{t_0+\epsilon}$ and none of the new knots can link to a knot in $L_{t_0+\epsilon} \cap D(X)$ which was also in $L \cap D(X)$ because otherwise there would be a time $s \in (0, t_0 + \epsilon)$ such that L_s is not an embedding which is a contradiction. Similarly, none of the missing knots (deaths) in $L_{t_0+\epsilon} \cap D(X)$ can be a non-trivial knot and can link to a knot in $L \cap D(X)$ at the time $t = 0$. To sum up, births and deaths in $L_{t_0+\epsilon} \cap D(X)$ can occur only along unknots, say $U_1, .., U_b$ are the births and $U'_1, .., U'_d$ are the deaths. Note the total number of these births and deaths is finite due to smoothness and compactness arguments. Therefore, when passing from $t = 0$ to $t = t_0 + \epsilon$, the change in $\mathcal{P}_X$ due to births and deaths can be at most

$$|\widetilde{tb}(U_1) + \cdots + \widetilde{tb}(U_b) - (\widetilde{tb}(U'_1) + \cdots + \widetilde{tb}(U'_d))|.$$

Next, we will discuss the case when there are degenerations transforming some collection of knots in $L \cap D(X)$ to new ones in $L_{t_0+\epsilon} \cap D(X)$. Degenerations may arise as either unifications or separations which are exactly the opposite of each other, and so it suffices to understand one of them. A typical situation of unification is the following: Suppose that the intersection knots $K_1, K_2 \in L \cap D(X)$ degenerate during the isotopy and a new intersection knot $K \in L' \cap D(X)$ arises while K_1, K_2 disappear (unify) as depicted in Figure 4.4 and Figure 4.5 where for simplicity we assume that there is a single degeneration and take $L' = L_{t_0+\epsilon}$. In the Stein diagrams of X_0 and

X_1, this degeneration and the creation of K correspond to bringing the $+, -$ points together on the attaching spheres of 1-handles, and then taking a Legendrian connect sum of K_1, K_2 along an appropriate Legendrian band. (See Figure 4.5.) We note that such a degeneration may also appear away from the binding, that is, it can occur in the Legendrian tangle of one of the Stein diagrams of either X_0 or X_1.

Observe that during an unification (resp. a separation), the number $\widetilde{tb}$ decreases (resp. increases) by 1. More precisely, in Figure 4.6, some different ways of obtaining a Legendrian connected sum of the knots K_1 and K_2 along appropriate Legendrian bands (in red) are given. Any Legendrian band connecting K_1 and K_2 may arise when K_1, K_2 unify (and a new intersection knot K borns as $K_1 \# K_2$) during an irregular Legendrian isotopy. Eqivalently, any Legendrian band can occur when K separates and decomposes as the disjoint union of K_1, K_2. It is not hard to show that no matter which Legendrian band is used (arises) during a creation (resp. separation) of $K = K_1 \# K_2$), the number $\widetilde{tb}$ always decreases (resp. increases) by 1 because gluing with a Legendrian band always introduces one additional left cusp (see Figure 4.6). That is, the following always holds:

$$\widetilde{tb}(K) = \widetilde{tb}(K_1 \# K_2) = \widetilde{tb}(K_1) + \widetilde{tb}(K_2) - 1.$$

To summarize, when passing from time $t = 0$ to $t = t_0 + \epsilon$, if there are M_u unifications and M_s separations (note the total number of degenerations is again finite by smoothness and compactness arguments), then the change in $\mathcal{P}_X$ due to these degenerations can be at most

$$|M_u - M_s|.$$

Combining with the births and deaths argument above, we conclude that the change in $\mathcal{P}_X$ (when passing from time $t = 0$ to $t = t_0 + \epsilon$) is finite and satisfies

$$|\mathcal{P}_X(L_{t_0+\epsilon}) - \mathcal{P}_X(L)| < N_0 := |\widetilde{tb}(U_1) + \cdots + \widetilde{tb}(U_b) - (\widetilde{tb}(U_1') + \cdots + \widetilde{tb}(U_d'))| + |M_u - M_s|.$$

As a result, repeating the above argument for each t_i with $0 < t_0 < t_1 < \cdots < t_r < 1$, we conclude that during the irregular Legendrian isotopy L_t ($t \in [0, 1]$), the total change in $\mathcal{P}_X$ is finite. More precisely, we have

$$|\mathcal{P}_X(L') - \mathcal{P}_X(L)| = |\mathcal{P}_X(L_1) - \mathcal{P}_X(L_0)| < N$$
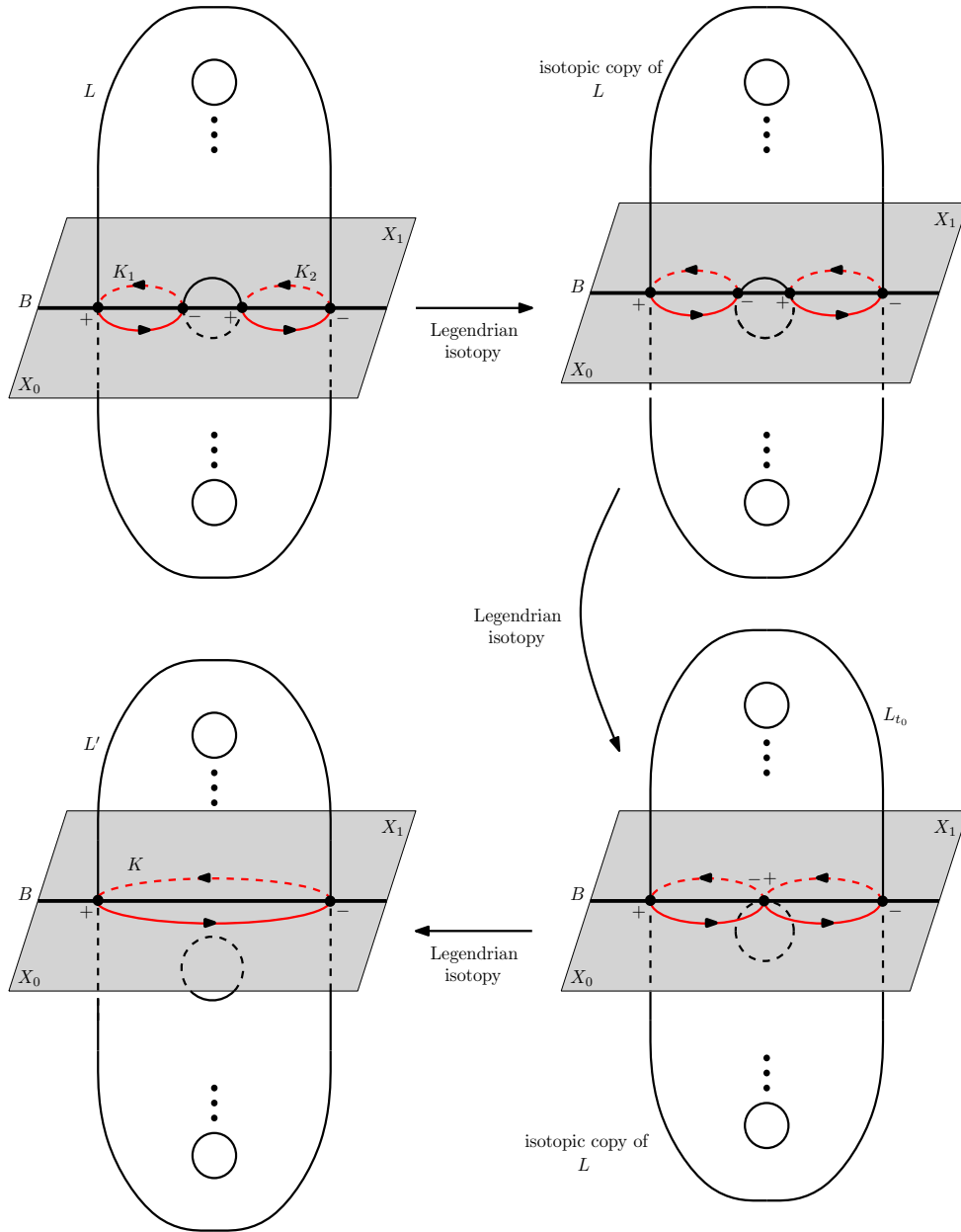
Figure 4.4: A typical degeneration (unification) during an irregular Legendrian isotopy L_t taking $L_0 = L$ to another Legendrian $L_1 = L'$ which also intersects the double $D(X) = X_0 \cup_\partial X_1$ transversally, but the new arcs of intersection in $L' \cap D(X)$ are possibly different than the older ones. Note that traveling in the opposite direction (i.e., from $t = 1$ to $t = 0$) describes a typical separation of K into K_1 and K_2.

where $N := N_0 + \cdots + N_i + \cdots + N_r$. Here, for each $i = 1, ..., r$, the bound N_i is

obtained (similarly to $i = 0$ case above) by analyzing corresponding births/deaths and degenerations occuring when passing from $t = t_i - \epsilon$ to $t = t_i + \epsilon$. □

Theorem 4.0.6 *The number $M\mathcal{P}_X(L)$ is well-defined.*

Proof Recall the definition of relative invariant:

$$M\mathcal{P}_X(L) := Max\left\{\mathcal{P}_X(L') \mid L' \in [L] \text{ and } L' \pitchfork D(X)\right\}$$

where (B, f) is an admissable open book for L and a page X is chosen so that $D(X)$ essentially intersects L. Therefore, for any $L' \in [L]$, the intersection $L' \cap D(X)$ is non-empty which implies that the set

$$\mathcal{P}_X([L]) := \left\{\mathcal{P}_X(L') \mid L' \in [L] \text{ and } L' \pitchfork D(X)\right\} \subset \mathbb{Z}$$

is a non-empty subset. In order to check well-definedness of $M\mathcal{P}_X(L)$, we need to verify that the set $\mathcal{P}_X([L])$ attains its maximum value. That is, one needs to show that there exists $L_{max} \in [L]$ such that

$$Max(\mathcal{P}_X([L])) = \mathcal{P}_X(L_{max}) < \infty.$$

Equivalently, we need to show that the number $\mathcal{P}_X(L)$ can not be made arbitrarly large under Legendrian isotopies of L. By Proposition 4.0.3, $\mathcal{P}_X(L)$ is invariant, and so, can not be made arbitrarly large under regular Legendrian isotopies of L. Therefore, it suffices to consider irregular Legendrian isotopies of L. By Proposition 4.0.5, we know that the jump in $\mathcal{P}_X(L)$ under any irregular isotopy is finite. Consider the subset

$$[L]_{min} \subset [L]$$

of all Legendrian representatives of L in the class $[L]$ which intersects $D(X)$ transversely and minimally. In other words, for any $\widetilde{L} \in [L]_{min}$, the set $\widetilde{L} \pitchfork D(X)$ is a link in $D(X)$ contains no unknot components due to a birth which may arise during an irregular Legendrian isotopy. Clearly, by undoing any such isotopy (if needed) one can get rid of any such unknots (i.e., replacing any birth with the corresponding

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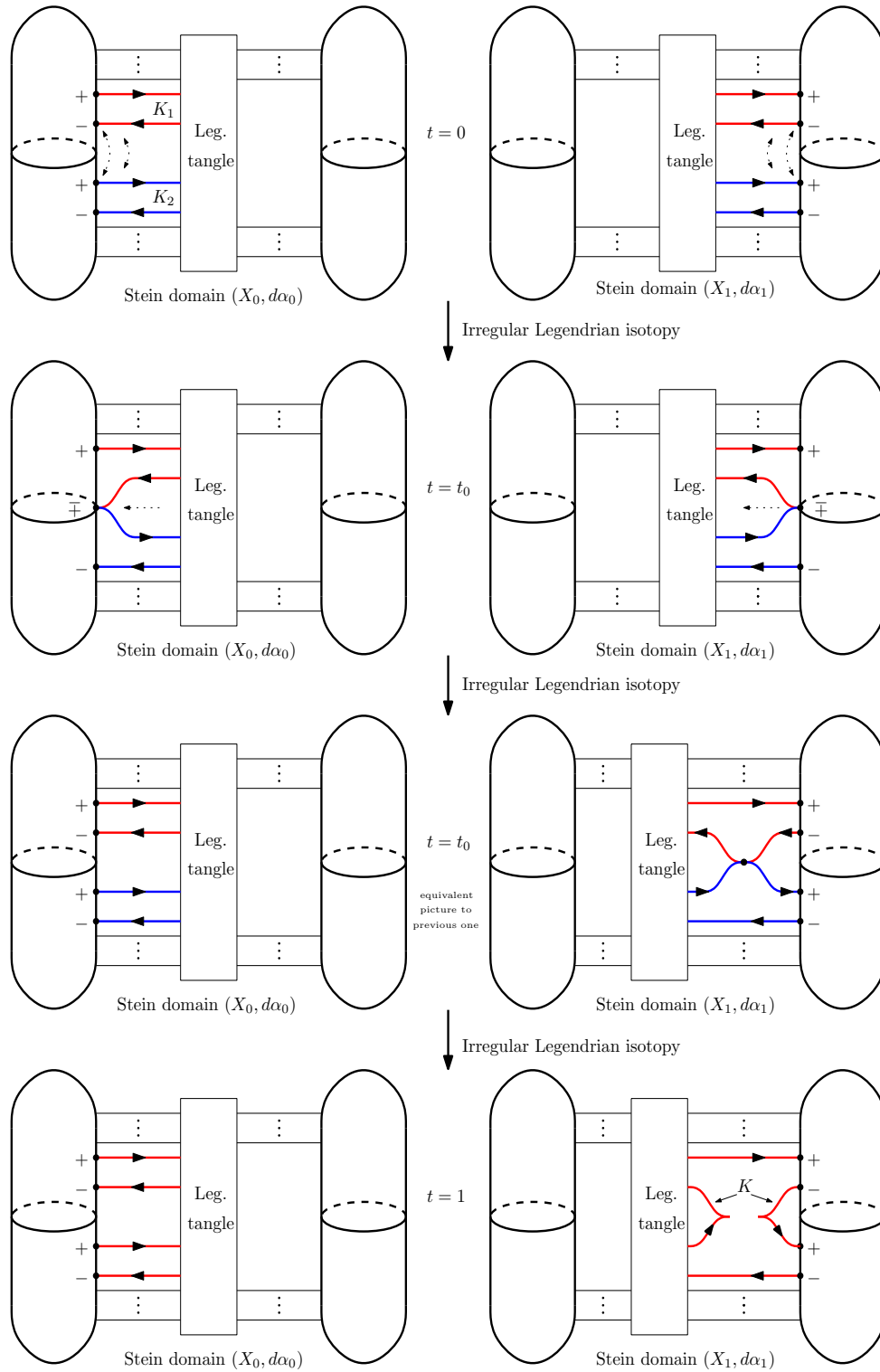
Figure 4.5: Realizing a degeneration (unification) of K_1, $K_2 \in L \cap D(X)$ and the creation of $K \in L' \cap D(X)$ during an irregular Legendrian isotopy L_t, $t \in [0, 1]$. ($L_0 = L, L_1 = L'$ and L_{t_0} is not transverse to $D(X)$.)
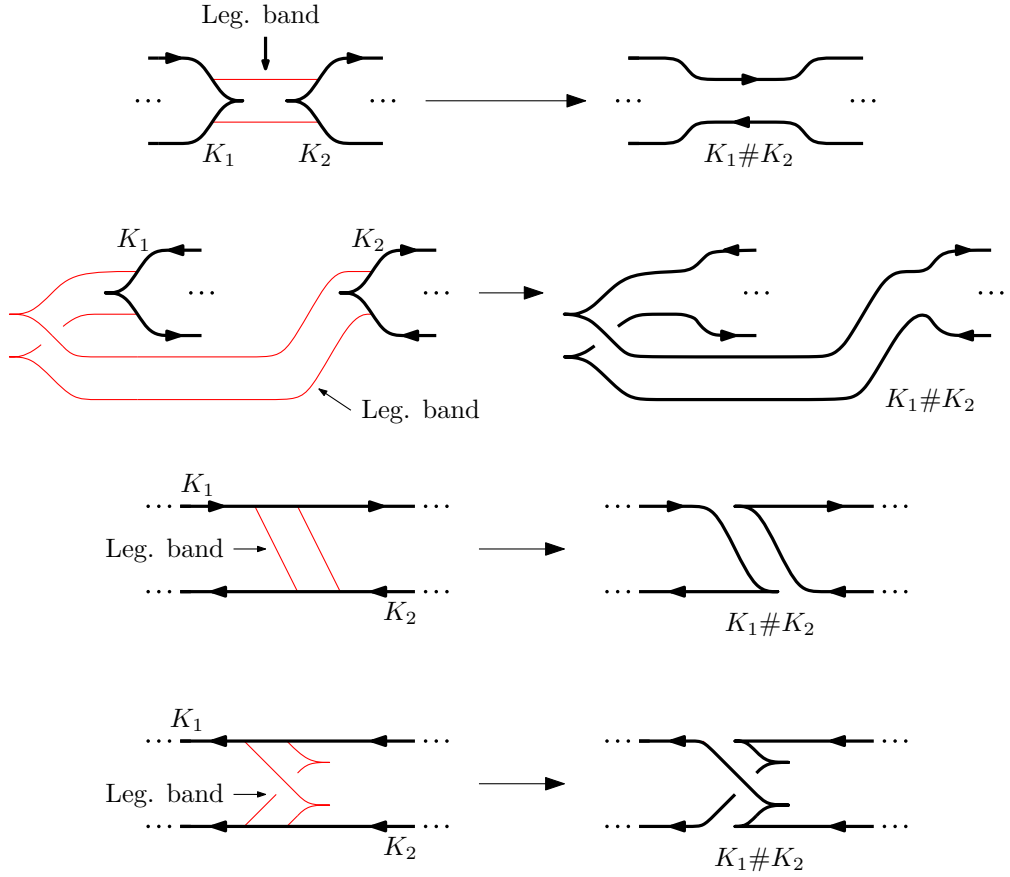
Figure 4.6: Some possible ways (but not all) of obtaining a Legendrian connected sum $K_1 \# K_2$ of the knots K_1 and K_2 along appropriate Legendrian bands (in red).

death), any Legendrian representative $L \in [L]$ intersecting $D(X)$ transversely can be transformed to some $\widetilde{L} \in [L]_{min}$. That is, there is a map

$$\Psi : [L] \longrightarrow [L]_{min}, \quad \Psi(L) = \widetilde{L}.$$

From its construction, it is clear that $\mathcal{P}_X(L') \leq \mathcal{P}_X(\Psi(L'))$ for any $L' \in [L]$ with $L' \pitchfork D(X)$. Therefore, we have $Max(\mathcal{P}_X([L])) = Max(\mathcal{P}_X([L]_{min}))$, and hence, it suffices to focus on the set $[L]_{min}$, i.e., if L_{max} exists, then $L_{max} \in [L]_{min}$. Equivalently, one needs to show that there exists $L_{max} \in [L]_{min}$ such that

$$Max(\mathcal{P}_X([L]_{min})) = \mathcal{P}_X(L_{max}) < \infty.$$

Now inside $[L]_{min}$ consider the subset $[L]^o_{min} \subset [L]_{min}$ which consists of all Legendrian representative of $\widetilde{L} \in [L]_{min}$ such that there exists a knot component K in the

link $\widetilde{L} \cap D(X)$ (of transverse minimal intersections) which separates into two knots K_1 and K_2 (via some irregular Legendrian isotopy with respect to $D(X)$) such that at least one of K_i's (say K_2) is a homotopically nontrivial unknot in $\widetilde{L}$ and does not link to any other components of the resulting link of intersection. Equivalently, K_2 does not bound a disk in $\widetilde{L}$ but it bounds a disk D in $D(X) = X_0 \cup_\partial X_1$ which is not punctured with any other knot in the Stein diagrams of X_0 and X_1. Such a knot component K will be called **_decomposable_**. Given $\widetilde{L} \in [L]^o_{min}$, find all decomposable knots $K \in \widetilde{L} \cap D(X)$ and the corresponding K_2's and D's mentioned above. Composing irregular Legendrian isotopies separating K's into K_1's and K_2's with suitable Legendrian isotopies compactly supported in small neighboorhoods of D's, one can get rid of all these K_2's, and repeating this argument (if necessary) eventually we obtain a Legendrian representative

$$\bar{L} \in [L]_{min} \smallsetminus [L]^o_{min}.$$

Recall that separation of a link component increases $\widetilde{tb}$ by 1, and also erasing a Legendrian unknot (corresponding K_2) from Stein diagrams increases $\widetilde{tb}$ at least by 1. Therefore, for any $\bar{L}$ obtained from $\widetilde{L} \in [L]^o_{min}$ as above, the following always holds:

$$\mathcal{P}_X(\widetilde{L}) < \mathcal{P}_X(\bar{L}).$$

This means that if L_{max} exists, then it must be true that $L_{max} \in [L]_{min} \smallsetminus [L]^o_{min}$. Equivalently, in order to prove the theorem, one needs to show that there exists $L_{max} \in [L]_{min} \smallsetminus [L]^o_{min}$ such that

$$Max(\mathcal{P}_X([L]_{min} \smallsetminus [L]^o_{min})) = \mathcal{P}_X(L_{max}) < \infty.$$

To proceed further, we need a partial order relation on the set of equivalence classes of links in $D(X)$ consisting of all possible intersections of $D(X)$ with elements in $[L]_{min} \smallsetminus [L]^o_{min}$. More precisely, consider the set of links in $D(X)$ defined by

$$\Lambda := \{L \pitchfork D(X) \mid L \in [L]_{min} \smallsetminus [L]^o_{min}\}.$$

As discussed in the earlier sections, every element (link) $K \in \Lambda$ can be realized as the union of collections k_0, k_1 of Legendrian arcs drawn in the Stein diagrams of X_0, X_1,

respectively. We will write $\|K\| = \|K'\|$ and say that two links $K, K' \in \Lambda$ are *isotopy equivalent* if each corresponding collections k_i, k_i' $(i = 0, 1)$ are related via Legendrian Reidemeister moves and their modifications (the ones which does not change tb) for Stein digrams described in [15].

Remark Note that from the definition of page crossing number, for any $L \in [L]_{min} \smallsetminus [L]_{min}^o$, we have

$$\mathcal{P}_X(L) = \widetilde{tb}(L \pitchfork D(X)) = \widetilde{tb}(K).$$

Therefore, showing the existence of an $L_{max} \in [L]_{min} \smallsetminus [L]_{min}^o$ maximizing $\mathcal{P}_X$ is equivalent to showing the existence of a $K_{max} \in \Lambda$ maximizing $\widetilde{tb}$.

Next we will define a partial order relation on the set

$$A := \{ \ \|K\| \ \mid \ K \in \Lambda \ \}.$$

Definition 4.0.7 *Let $K_0, K_1 \in \Lambda$, so there exist $L_0, L_1 \in [L]_{min} \smallsetminus [L]_{min}^o$ so that $K_i = L_i \pitchfork D(X)$. We will write $\|K_0\| \leq \|K_1\|$ if*

(I) There is a regular or an irregular Legendrian isotopy L_t $(t \in [0, 1])$ with respect to $D(X)$ having only separating degenerations such that whenever L_t is transverse to $D(X)$, we have

$$L_t \in [L]_{min} \smallsetminus [L]_{min}^o.$$

(II) $\mathcal{P}_X(L_0) \leq \mathcal{P}_X(L_1)$ (or equivalently, $\widetilde{tb}(K_0) \leq \widetilde{tb}(K_1)$.)

Lemma 4.0.8 *The pair $(A, \leq)$ is a partially ordered set.*

Proof *Reflexivity*: For a given $\|K\| \in A$, consider any representative $K \in \|K\|$ and corresponding $L \in [L]_{min} \smallsetminus [L]_{min}^o$, i.e., $K = L \pitchfork D(X)$. Then one can consider the trivial Legendrian isotopy fixing all the points on L for all time t. The second condition is also clear. Therefore,

$$\|K\| \leq \|K\|.$$

Anti-symmetry: Suppose $\|K_0\| \preceq \|K_1\|$ and $\|K_1\| \preceq \|K_0\|$ for $\|K_0\|, \|K_1\| \in A$. Immediately, we observe $\widetilde{tb}(K_0) \leq \widetilde{tb}(K_1)$ and $\widetilde{tb}(K_1) \leq \widetilde{tb}(K_0)$, and so

$$\widetilde{tb}(K_0) = \widetilde{tb}(K_1).$$

Consider any representatives $K_0 \in \|K_0\|, K_1 \in \|K_1\|$ and the corresponding $L_0, L_1 \in [L]_{min} \smallsetminus [L]_{min}^o$ which are connected via a Legendrian isotopy L_t such that whenever L_t is transverse to $D(X)$, we have $L_t \in [L]_{min} \smallsetminus [L]_{min}^o$.

If L_t is a regular Legendrian isotopy with respect to $D(X)$, then L_t transversely intersects $D(X)$ for all t. In particular, this implies that $L_t \cap D(X)$ is minimal and has no decomposable components for all t because $L_0 \in [L]_{min} \smallsetminus [L]_{min}^o$. Also observe L_t induces an isotopy $K_t := L_t \pitchfork D(X)$ (between K_0 and K_1) whose respective restrictions $K_t \cap X_i$ ($i = 0, 1$) defines Legendrian isotopies between componets of K_0, K_1 in X_0, X_1, respectively. In other words, $\|K_0\| = \|K_t\| = \|K_1\|$, so we are done in this case.

Now suppose L_t is an irregular Legendrian isotopy (of L_0) with respect to $D(X)$ having only separating degenerations. As in the proof of Proposition 4.0.5, suppose there are numbers $0 < t_0 < t_1 < \cdots < t_r < 1$ so that except finitely many $L_{t_i}, (i = 0, 1, ..., r)$, any other L_t is an element of $[L]_{min} \smallsetminus [L]_{min}^o$. By assumption, during L_t no births or deaths can arise, and only degenerations are separations. Recall that separations increase $\mathcal{P}_X$ and so $\widetilde{tb}$ by 1. Therefore, one easily conclude that for any i when passing from $t = t_i - \epsilon$ to $t = t_i + \epsilon$, there can not be any separations of knots in $K_{t_i-\epsilon} := L_{t_i-\epsilon} \pitchfork D(X)$ because otherwise we would have

$$\widetilde{tb}(K_0) \lneqq \widetilde{tb}(K_{t_i+\epsilon}) \leq \widetilde{tb}(K_1).$$

So, L_t must be a regular Legendrian isotopy indeed, and hence $\|K_0\| = \|K_1\|$ as discussed above.

Transivity: Suppose $\|K_0\| \preceq \|K_1\|$ and $\|K_1\| \preceq \|K_2\|$ for $\|K_0\|, \|K_1\|, \|K_2\| \in A$. Immediately, we observe $\widetilde{tb}(K_0) \leq \widetilde{tb}(K_1)$ and $\widetilde{tb}(K_1) \leq \widetilde{tb}(K_2)$, and so

$$\widetilde{tb}(K_0) = \widetilde{tb}(K_2).$$

For each $i = 0, 1, 2$, consider any representative $K_i \in \|K_i\|$ and the corresponding $L_i \in [L]_{min} \smallsetminus [L]^o_{min}$. By assumption, there are Legendrian isotopies L_t from L_0 to L_1 and L'_t from L_1 to L_2 with the prescribed conditions in Definition 4.0.7 part (I). Then one easily concludes that $L'_t \circ L_t$ is a Legendrian isotopy from L_0 to L_2 with the desired properties. Thus, $\|K_0\| \preceq \|K_2\|$.

Returning back to the proof of the theorem, next we will show that every chain in $(A, \preceq)$ has an upper bound in A. To this end, suppose that we are giving a chain

$$\|K_0\| \preceq \|K_1\| \preceq \|K_2\| \preceq \cdots \preceq \|K_i\| \preceq \cdots.$$

Since regular Legendrian isotopies does not change the isotopy equivalence classes, it suffices to consider irregular Legendrian isotopy (with respect to $D(X)$) having only separating degenerations. We need to show that under such isotopies, separations must eventually stop after a finite step, and when it stops the corresponding $\widetilde{tb}$ must be finite.

Let L_i's be Legendrian representatives in $[L]_{min} \smallsetminus [L]^o_{min}$ such that, for each $i \geq 0$, we have $K_i = L_i \pitchfork D(X)$ and L_{i+1} is the image of L_i under an irregular Legendrian isotopy L^t_i satifying the condition (I) of Definition 4.0.7. Suppose the the link K_i consists of r_i knot components. (Recall by compactness there must be finite number of components for each K_i.) By Proposition 4.0.5 and from the assumptions $K_i \in \Lambda$ and $L_i \in [L]_{min} \smallsetminus [L]^o_{min}$, we know that each isotopy L^t_i consists only of finitely many separations, and $\mathcal{P}_X$ (and so $\widetilde{tb}$) has a finite jump (increment) during each L^t_i. That is, we have

$$r_0 < r_1 < r_2 < \cdots < r_i < \cdots$$

with $0 < r_{i+1} - r_i < \infty$, and

$$\widetilde{tb}(K_0) < \widetilde{tb}(K_1) < \widetilde{tb}(K_2) < \cdots < \widetilde{tb}(K_i) < \cdots$$

with $\widetilde{tb}(K_{i+1}) - \widetilde{tb}(K_i) < \infty$.

Now observe that during the separations of any L_i^t, knot components in K_i split into "simpler" knot components (which form the link K_{i+1}) which are still disjointly embedded simple closed curves in the resulting Legendrian suface L_{i+1}. From our choices, knot components in any K_i can not bound disks in L_i and can not be decomposable. Therefore, there must exist some $i_{max} \in \mathbb{N}$ such that we can not proceed further. That is, we have

$$r_0 < r_1 < r_2 < \cdots < r_i < \cdots < r_{i_{max}}$$

where the sequence stops at $r_{i_{max}} < \infty$, and

$$\widetilde{tb}(K_0) < \widetilde{tb}(K_1) < \widetilde{tb}(K_2) < \cdots < \widetilde{tb}(K_i) < \cdots < \widetilde{tb}(K_{i_{max}})$$

where the sequence stops at $\widetilde{tb}(K_{i_{max}}) < \infty$.

Therefore, every chain in $(A, \preceq)$ has an upper bound in A, and hence, by Zorn's Lemma, the partially ordered set $(A, \preceq)$ has at least one maximal element, say $\|K_{max}\| \in A$. Then by the definition of the partial order relation "$\preceq$", for a chosen representative $K_{max} \in \|K_{max}\|$, the number $\widetilde{tb}(K_{max}) < \infty$ (exists) and is the maximum value among all possible values obtained from such links of transverse intersections. Then for a corresponding Legendrian representative, say $L_{max} \in [L]_{min} \setminus [L]_{min}^o$, one obtains $\mathcal{P}_X(L_{max})$ is finite and maximal among all, i.e., the relative invariant $M\mathcal{P}_X(L) = \mathcal{P}_X(L_{max})$ is well-defined.

ABSOLUTE PAGE CROSSING NUMBER AND PROOF OF THEOREM 1.0.2

In this chapter we introduce page-free version of maximal page crossing number and show that this number is well-defined and invariant under Legendrian isotopies of a Legendrian surface L.

Definition 5.0.1 *Let $L \hookrightarrow (M^5, \xi)$ be a closed orientable Legendrian surface. Fix an admissable open book (B, f) for L **essentially intersecting** L which means that the double of every page of (B, f) essentially intersects L. Fix any page X of (B, f). Then*

$$M\mathcal{P}_{(B,f)}(L) \coloneqq M\mathcal{P}_X(L)$$

*is called the **absolute maximal page crossing number of** L **with respect to** (B, f).*

We start with the following fact which will be useful in proving well-definedness of $M\mathcal{P}_{(B,f)}(L)$:

Lemma 5.0.2 *The relative invariant $M\mathcal{P}_X(L)$ does not change under (small) perturbations of the double $D(X)$ transverse to a Legendrian representative in $[L]$ realizing $M\mathcal{P}_X(L)$.*

Proof Suppose that $L' \in [L]$ realizes $M\mathcal{P}_X(L)$. In other words, $D(X)$ transversely intersects the Legendrian isotopic copy L' of L and we have

$$M\mathcal{P}_X(L) = \mathcal{P}_X(L').$$

We want to show that $M\mathcal{P}_X(L) = M\mathcal{P}_{X'}(L)$ for any pair X, X' of pages such that their doubles are isotopic to each other via a 1-parameter family of doubles of pages transverse to L'. Equivalently, need to show that

$$\mathcal{P}_X(L') = \mathcal{P}_{X'}(L').$$

To this end, suppose $X = X_\theta$, $X' = X_{\theta'}$ is such a pair of pages. Let $K = k_0 \cup k_1$ be any knot component of $L' \pitchfork D(X)$. Then as depicted in Figure 5.1 that a new knot component $K' \in L' \pitchfork D(X')$ is (Legendrian) isotopic to the older one K. So, the contribution of K' to $\mathcal{P}_{X'}(L')$ is the same as the contribution of K to the $\mathcal{P}_X(L')$. Thus, the claim follows. $\square$
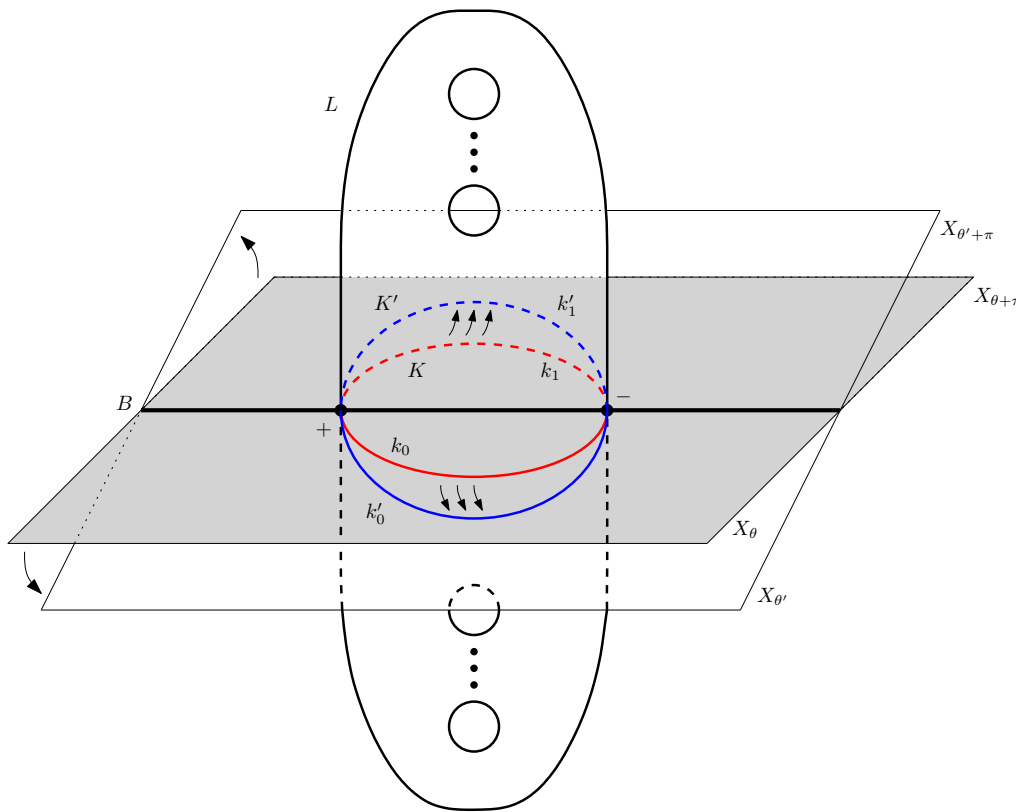
Figure 5.1: Replacing $X_0 = X_\theta$ (resp. $X_1 = X_{\theta+\pi}$) with a nearby (Stein) page $X_{\theta'}$ (resp. $X_{\theta'+\pi}$) which are still intersecting L' transversely. A new knot component $K' = k_0' \cup k_1'$ (isotopic to older one K) of the link of intersection of L' with the new double $X_{\theta'} \cup_\partial X_{\theta'+\pi}$.

Now with the help of the results from previous chapters, one can easily prove the following:

Proposition 5.0.3 *Let $L \hookrightarrow (M^5, \xi)$ be a closed orientable Legendrian surface. Fix an admissable open book (B, f) for L. Then the number $M\mathcal{P}_{(B,f)}(L)$ is well-defined and invariant under Legendrian isotopies of L.*

Proof Recall the definition of $M\mathcal{P}_{(B,f)}(L)$:

$$M\mathcal{P}_{(B,f)}(L) := M\mathcal{P}_X(L)$$

where X is any page of (B, f), or equivalently,

$$M\mathcal{P}_{(B,f)}(L); = M\mathcal{P}_{X_\theta}(L) \text{ for a fixed } \theta \in S^1.$$

By assumption, $D(X_\theta)$ essentially intersects L for all $\theta \in S^1$. For each $\theta \in S^1$, by Lemma 5.0.2, the number $M\mathcal{P}_{X_\theta}(L)$ takes the same value on some small enough neighborhood U_θ of θ in S^1, and so the collection $\{U_\theta \mid \theta \in S^1\}$ is an open cover for S^1. By compactness of S^1, there exists a finite subcover, i.e., there exist $\theta_1, \theta_2, ..., \theta_r \in S^1$ such that

$$S^1 = U_{\theta_1} \cup U_{\theta_2} \cup \cdots \cup U_{\theta_r}.$$

After renaming (if necessary), one may assume that for any two consecutive arcs, we have $U_{\theta_i} \cap U_{\theta_{i+1}} \neq \varnothing$. Since $M\mathcal{P}_{X_{\theta_i}}(L)$ and $M\mathcal{P}_{X_{\theta_{i+1}}}(L)$ take constant values on their domains, they must agree on $U_{\theta_i} \cap U_{\theta_{i+1}}$, and hence $M\mathcal{P}_{X_\theta}(L)$ takes a constant value on $U_{\theta_i} \cup U_{\theta_{i+1}}$. Repeating the argument (by changing i one by one), we conclude that $M\mathcal{P}_{X_\theta}(L)$ takes the same value on the whole S^1. Hence, $M\mathcal{P}_{(B,f)}(L) = M\mathcal{P}_{X_\theta}(L)$ is independent of θ, and, in particular, is well-defined.

Finally, the fact that $M\mathcal{P}_{(B,f)}(L)$ is invariant under Legendrian isotopies just follows from its definition combined with Lemma 4.0.1. $\square$

Hence, Theorem 1.0.2 is proved.

CHAPTER 6

EXAMPLE FOR THE INVARIANTS

Example 6.0.1 *Let $\mathbb{C}^3$ be the complex space with the complex coordinates*

$$(z_1, z_2, z_3) = (r_1, \theta_1, r_2, \theta_2, r_3, \theta_3),$$

where $z_j = r_j e^{i\theta_j} (j = 1, 2, 3)$ are the polar coordinates, and $\mathbb{S}^5$ be the unit 5-sphere in $\mathbb{C}^3$, i.e.,

$$\mathbb{S}^5 = \left\{ (z_1, z_2, z_3) \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \right\}.$$

The restriction of the 1-form (a primitive of the standard symplectic form on $\mathbb{C}^3$)

$$\alpha_{st} = r_1^2 d\theta_1 + r_2^2 d\theta_2 + r_3^2 d\theta_3.$$

on $\mathbb{S}^5$ is a contact form and defines the standard contact structure ξ_5 on $\mathbb{S}^5$. So we have a closed contact manifold $(\mathbb{S}^5, \xi_5 = Ker(\alpha_5))$ where $\alpha_5 := \alpha_{st}|_{\mathbb{S}^5}$.

We will consider an open book supporting ξ_5 which is admissable for a Legendrian surface we pick later inside $(\mathbb{S}^5, \xi_5)$. Consider the standard 3-sphere

$$\mathbb{S}^3 = \left\{ (z_1, z_2, z_3) \in \mathbb{S}^5 \mid z_1 = 0 \right\} \subset \mathbb{S}^5$$

as a contact submanifold of $\mathbb{S}^5$ as follows:

$$(\mathbb{S}^3, \xi_3 = Ker(\alpha_3)) \hookrightarrow (\mathbb{S}^5, \xi_5)$$

where $\alpha_3 = r_2^2 d\theta_2 + r_3^2 d\theta_3 (= \alpha_5|_{\mathbb{S}^3})$ is the contact form (defining ξ_3).

Consider the most standard open book on $\mathbb{S}^5$ which can be explicitly described as follows:

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$$\pi : \mathbb{S}^5 \smallsetminus \mathbb{S}^3 \longrightarrow S^1$$
$$(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3) \longmapsto \theta_1.$$

Note that the standard $\mathbb{S}^3$ is the binding, and a typical page $X_{\theta_1} = \pi^{-1}(\theta_1)$ is an open 4-ball (simply connected and Weinstein). The closure of X_{θ_1} (still denoted by X_{θ_1} for simplicity) can be parametrized by

$$X_{\theta_1} = \pi^{-1}(\theta_1):$$
$$\left\{ \vec{\Gamma} : D^4 \longrightarrow S^5, \quad (\rho_1, \phi_1, \rho_2, \phi_2) \longmapsto \left(\sqrt{1 - \rho_1^2 - \rho_2^2}, \theta_1, \rho_1, \phi_1, \rho_2, \phi_2\right) \right\}.$$

(Clearly, X_{θ_1} is diffeomorphic to D^4, and note $0 \le \rho_1^2 + \rho_2^2 \le 1$.) One can easily check that the embedded open book $(\mathbb{S}^3, \pi)$ on $\mathbb{S}^5$ supports ξ_5 and the corresponding abstract open book is (D^4, id_{D^4}) (with a trivial monodromy).

*Let's now pick a Legendrian surface L inside $(\mathbb{S}^5, \xi_5)$. For a fixed constant k, consider the **Clifford torus** (a well-known and well-studied surface) defined by*

$$L = T_k = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 = |z_2|^2 = |z_3|^2 = \tfrac{1}{3}, \quad \theta_1 + \theta_2 + \theta_3 = k \right\} \subset \mathbb{S}^5$$

(note in polar coordinates we have $r_1^2 = r_2^2 = r_3^2 = \tfrac{1}{3}$.) Clearly T_k is a surface inside $\mathbb{S}^5$. One needs to check that T_k is Legendrian $(\mathbb{S}^5, \xi_5)$. To this end, consider the following parametrization for T_k where φ_1, φ_2 are angular coordinates on an abstract torus T^2:

$$T_k: \quad \vec{\sigma}(\varphi_1, \varphi_2) = \left(\tfrac{1}{\sqrt{3}}, \varphi_1, \tfrac{1}{\sqrt{3}}, \varphi_2, \tfrac{1}{\sqrt{3}}, k - \varphi_1 - \varphi_2 \right) \in \mathbb{S}^5$$

$$\vec{\sigma}_{\varphi_1} = < 0, 1, 0, 0, 0, -1 > = \frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_3}$$

$$\vec{\sigma}_{\varphi_2} = < 0, 0, 0, 1, 0, -1 > = \frac{\partial}{\partial \theta_2} - \frac{\partial}{\partial \theta_3}$$

Then, we easily see that

$$\alpha_5 \big|_{T_k} = \tfrac{1}{3} d\theta_1 + \tfrac{1}{3} d\theta_2 + \tfrac{1}{3} d\theta_3$$

$$\alpha_5 \big|_{T_k} (\vec{\sigma}_{\varphi_1}) = 0 = \alpha_5 \big|_{T_k} (\vec{\sigma}_{\varphi_2}).$$

Therefore, T_k is a Legendrian torus in $(\mathbb{S}^5, \xi_5)$. Let's understand how T_k intersects with the binding $\mathbb{S}^3$ and a typical page $X_{\theta_1} \approx D^4$:

For $T_k \cap \mathbb{S}^3$, we have

$$\mathbb{S}^3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 = 0\} = \{r_2^2 + r_3^2 = 1, r_1 = 0\}.$$

But on T_k, $r_1 = \frac{1}{\sqrt{3}} \neq 0$. Hence, $T_k \cap \mathbb{S}^3 = \varnothing$. In particular, this shows that the binding of the open book $(\mathbb{S}^3, \pi)$ intersects T_k transversely.

For $K_{\theta_1} := T_k \cap X_{\theta_1}$,

$$X_{\theta_1} : \left\{ \vec{\Gamma}(\rho_1, \phi_1, \rho_2, \phi_2) = \left(\sqrt{1 - \rho_1^2 - \rho_2^2}, \theta_1, \rho_1, \phi_1, \rho_2, \phi_2\right) \right\}$$
$$T_k : \vec{\sigma}(\varphi_1, \varphi_2) = \left(\tfrac{1}{\sqrt{3}}, \varphi_1, \tfrac{1}{\sqrt{3}}, \varphi_2, \tfrac{1}{\sqrt{3}}, k - \varphi_1 - \varphi_2 \right).$$

Equating the corresponding coordinates, one gets the equations defining the intersection K_{θ_1}:

$$\sqrt{1 - \rho_1^2 - \rho_2^2} = \tfrac{1}{\sqrt{3}}, \quad \theta_1 = \varphi_1, \quad \rho_1 = \tfrac{1}{\sqrt{3}}, \quad \phi_1 = \varphi_2, \quad \rho_2 = \tfrac{1}{\sqrt{3}}, \quad \phi_2 = k - \varphi_1 - \varphi_2.$$

If we let $\phi_1 = \varphi_2 = \theta$, then we obtain the parametrization of K_{θ_1} given by

$$K_{\theta_1} : \quad \vec{r} : S^1 \to \mathbb{S}^5, \quad \vec{r}(\theta) = \left(\tfrac{1}{\sqrt{3}}, \theta_1, \tfrac{1}{\sqrt{3}}, \theta, \tfrac{1}{\sqrt{3}}, k - \theta_1 - \theta \right).$$

Note that the parameter θ appears in two distinct angular coordinates with opposite signs, and so K_{θ_1} is an embedded unknot in $\mathbb{S}^5$ sitting as a $(1, -1)$-torus knot on the Clifford torus T_k. Note that following the same steps, one can also consider $K_{\theta_1 + \pi} := T_k \cap X_{\theta_1 + \pi}$ which is also a $(1, -1)$-torus knot on the Clifford torus T_k (a paralel copy of K_{θ_1}). Hence, we conclude that the double $D(X) = X_{\theta_1} \cup_\partial X_{\theta_1 + \pi}$ intersects T_k transversely and essentially along the (un)link $D(X) \cap T_k = K_{\theta_1} \sqcup K_{\theta_1 + \pi}$.

Now observe that

$$\vec{r}'(\theta) = <0, 0, 0, 1, 0, -1> = \frac{\partial}{\partial \theta_2} - \frac{\partial}{\partial \theta_3},$$
$$\alpha_3 \mid_{K_{\theta_1}} = \tfrac{1}{3} d\theta_2 + \tfrac{1}{3} d\theta_3,$$

so $\alpha_3 \mid_{K_{\theta_1}} (\vec{r}'(\theta)) = 0$. *Hence, one can think of $K_{\theta_1}, K_{\theta_1+\pi}$ as Legendrian unknots in* $(\mathbb{S}^3, \xi_3)$.

Moreover, we also have

$$\alpha_5 \mid_{X_{\theta_1}} = \Gamma^*(\alpha_5) = d(\underbrace{1 - \rho_1^2 - \rho_2^2}_{=0}) \wedge d\theta_1 + \rho_1^2 d\phi_1 + \rho_2^2 d\phi_2 = \rho_1^2 d\phi_1 + \rho_2^2 d\phi_2, \text{ and so}$$

$$d(\alpha_5 \mid_{X_{\theta_1}}) = d(\rho_1^2 d\phi_1 + \rho_2^2 d\phi_2) = 2\rho_1 d\rho_1 \wedge d\phi_1 + 2\rho_2 d\rho_2 \wedge d\phi_2$$

from which we compute

$$d(\alpha_5 \mid_{X_{\theta_1}}) \mid_{K_{\theta_1}} = d(\alpha_5 \mid_{T_k \cap X_{\theta_1}}) = \tfrac{2}{\sqrt{3}} d\rho_1 \wedge d\phi_1 + \tfrac{2}{\sqrt{3}} d\rho_2 \wedge d\phi_2, \text{ and also}$$

$$(\alpha_5 \mid_{X_{\theta_1}}) \mid_{K_{\theta_1}} = \tfrac{1}{3} d\phi_1 + \tfrac{1}{3} d\phi_2 (= \tfrac{1}{3} d\theta_2 + \tfrac{1}{3} d\theta_3)$$

$$\Rightarrow (\alpha_5 \mid_{X_{\theta_1}}) \mid_{K_{\theta_1}} (\vec{r}'(\theta)) = \tfrac{1}{3} - \tfrac{1}{3} = 0.$$

These veriy that $K_{\theta_1}, K_{\theta_1+\pi}$ are isotropic unknots in every (simply-connected) We-instein (so Stein) page $(X_{\theta_1}, d(\alpha_5 \mid_{X_{\theta_1}}))$, and every page of the open book $(\mathbb{S}^3, \pi)$ essentially intersects T_k. As a result, we conclude that $(\mathbb{S}^3, \pi)$ is an essentially inter-secting admissable open book for the Clifford torus T_k. See Figure 6.1 for a schematic picture for the front projection of T_k.
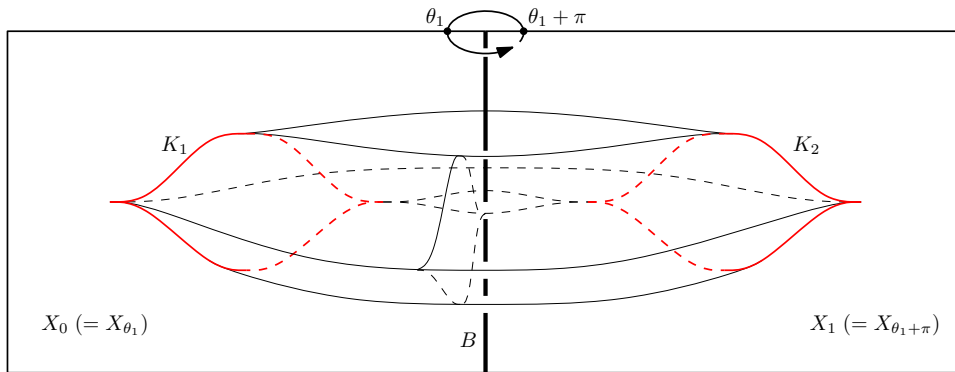
Figure 6.1: A schematic picture of the front projection of the Legendrian Clifford torus $T_k \subset (\mathbb{S}^5, \xi_5)$ onto $\mathbb{R}^3$ with coordinates z, y_1, y_2. The components K_1, K_2 (in red) of the (un)link of intersection of T_k with the double $X_0 \cup_\partial X_1 \approx \mathbb{S}^4$ of a page $X \approx D^4$ of the trivial open book on $\mathbb{S}^5$. (Note: $B \approx \mathbb{S}^3$ and $T_k \pitchfork B = \varnothing$.)

Next, we set $K_1 = K_{\theta_1}, K_2 = K_{\theta_1+\pi}$ and also $X_0 = X_{\theta_1}$ and $X_1 = X_{\theta_1+\pi}$, so that the

(un)link of intersection T_k with the double $D(X) = X_0 \cup_\partial X_1$ is given by

$$T_k \pitchfork D(X) = K_1 \sqcup K_2$$

With respect to the notation introduced in Chapter 3, we have $K_1 = k_1^0 \subset X_0$ (no k_1^1) and $K_2 = k_2^1 \subset X_1$ (no k_2^0). Also recall $T_k \cap \mathbb{S}^3 = \varnothing$. Hence, $T_k \pitchfork D(X)$ is an unlink with two components k_1^0, k_2^1 which can be realized in Stein diagrams of $(X_0, d\alpha_5|_{X_0})$ and $(X_1, d\alpha_5|_{X_1})$ as in Figure 6.2.
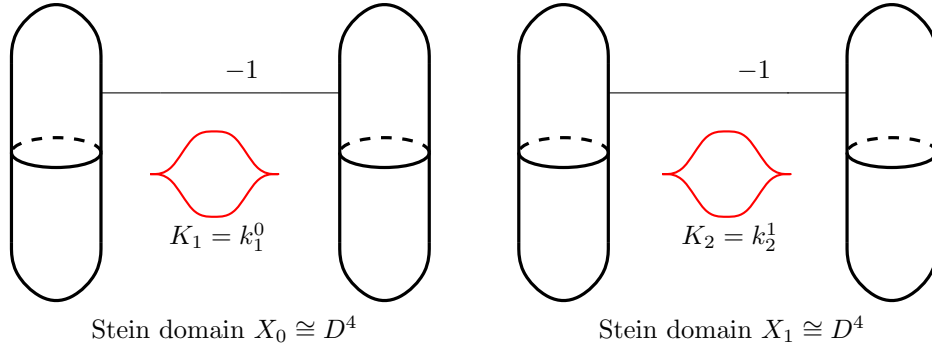

Stein domain $X_0 \cong D^4$ Stein domain $X_1 \cong D^4$

Figure 6.2: Realizing the (un)link of transverse intersection of the Legendrian Clifford torus T_k with the double $X_0 \cup_\partial X_1 \approx \mathbb{S}^4$ of a page $X \approx D^4$ in the Stein diagrams of $(X_0, d\alpha_5|_{X_0})$ and $(X_1, d\alpha_5|_{X_1})$.

Now we can calculate our invariants by using Stein diagrams. Using the notations introduced, we have

$$\widetilde{tb}(K_1) = tb(k_1^0) = -1 \quad and \quad \widetilde{tb}(K_2) = tb(k_2^1) = -1.$$

So, the page crossing number $\mathcal{P}_X(L')$ for any page X of $(\mathbb{S}^3, \pi)$ and any $L' \in [T_k]$ is computed as

$$\mathcal{P}_X(L') = \widetilde{tb}(K_1) + \widetilde{tb}(K_2) = -2.$$

(Because with respect to any page, minimal link of intersection set has always two components as depicted in Figure 6.2.) As a result, the absolute and relative maximal page crossing numbers are computed as

$$M\mathcal{P}_{(S^3, \pi)}(L) = M\mathcal{P}_X(L) = -2.$$

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Remark *Thurston-Bennequin number of any Legendrian torus L (regardless of how it is embedded in $(\mathbb{S}^5, \xi_5)$) is computed as $tb(L) = 0$ since it coincides with a topological invariant (see [24]). So it is not possible to distinguish them using Thurston-Bennequin invariant. On the other hand, since the new invariants defined here keep track Legendrian embeddings, they distinguish not only their smooth embedding type but also Legendrian isotopy types of such embedded Legendrian tori.*

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