



# Is there a novel Einstein–Gauss–Bonnet theory in four dimensions?

Metin Gürses<sup>1,a</sup>, Tahsin Çağrı Şişman<sup>2,b</sup>, Bayram Tekin<sup>3,c</sup>

<sup>1</sup> Department of Mathematics, Faculty of Sciences, Bilkent University, 06800 Ankara, Turkey

<sup>2</sup> Department of Astronautical Engineering, University of Turkish Aeronautical Association, 06790 Ankara, Turkey

<sup>3</sup> Department of Physics, Middle East Technical University, 06800 Ankara, Turkey

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**Abstract** No! We show that the field equations of Einstein–Gauss–Bonnet theory defined in generic  $D > 4$  dimensions split into two parts one of which always remains higher dimensional, and hence the theory does not have a non-trivial limit to  $D = 4$ . Therefore, the recently introduced four-dimensional, novel, Einstein–Gauss–Bonnet theory does not admit an *intrinsically* four-dimensional definition, in terms of metric only, as such it does not exist in four dimensions. The solutions (the spacetime, the metric) always remain  $D > 4$  dimensional. As there is no canonical choice of 4 spacetime dimensions out of  $D$  dimensions for generic metrics, the theory is not well defined in four dimensions.

## 1 Introduction

Recently a *four-dimensional* Einstein–Gauss–Bonnet theory that is claimed to propagate only a massless spin-2 graviton was introduced as the  $D \rightarrow 4$  limit in [1] with the action

$$I = \int d^D x \sqrt{-g} \left[ \frac{1}{\kappa} (R - 2\Lambda_0) + \frac{\alpha}{D-4} (R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2) \right], \quad (1)$$

of which the field equations are [2,3]

$$\frac{1}{\kappa} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_0 g_{\mu\nu} \right) + \frac{\alpha}{D-4} \mathcal{H}_{\mu\nu} = 0 \quad (2)$$

where the “Gauss–Bonnet tensor” reads

$$\mathcal{H}_{\mu\nu} = 2 \left[ RR_{\mu\nu} - 2R_{\mu\alpha\nu\beta} R^{\alpha\beta} + R_{\mu\alpha\beta\sigma} R_{\nu}{}^{\alpha\beta\sigma} - 2R_{\mu\alpha} R_{\nu}{}^{\alpha} \right. \\ \left. - \frac{1}{4} g_{\mu\nu} (R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2) \right]. \quad (3)$$

For  $D > 4$ , (2) is the well-known Einstein–Gauss–Bonnet theory which has been studied in the literature in great detail. On the other hand, for  $D = 4$ , the  $\mathcal{H}_{\mu\nu}$  tensor vanishes *identically* and hence, as per common knowledge, the field equations (2) reduces to the Einstein’s theory. This is because in four dimensions, the Gauss–Bonnet combination  $\mathcal{G} := R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2$  can be written as  $\mathcal{G} = \epsilon_{\mu\nu\alpha\beta} \epsilon^{\mu\nu\sigma\rho} R^{\alpha\beta\gamma\lambda} R_{\gamma\lambda\sigma\rho}$  and yields a topological action, i.e. the Euler number which is independent of the metric  $g_{\mu\nu}$ . This was the state of affairs until the paper [1] implicitly asked the question “how does the  $\mathcal{H}_{\mu\nu}$  tensor go to zero as  $D \rightarrow 4$ ?”. The answer is very interesting: because if it goes to zero in the following way

$$\mathcal{H}_{\mu\nu} = (D-4) \mathcal{Y}_{\mu\nu}, \quad (4)$$

where  $\mathcal{Y}_{\mu\nu}$  is a new tensor to be found, then the authors of [1] argue that in the  $D \rightarrow 4$  limit, the field equations (2) define a four-dimensional theory in the limit. So namely, the suggested four-dimensional theory would be the following theory in source-free case:

$$\lim_{D \rightarrow 4} \left[ \frac{1}{\kappa} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_0 g_{\mu\nu} \right) + \frac{\alpha}{D-4} \mathcal{H}_{\mu\nu} \right] = 0. \quad (5)$$

Let us try to understand what the suggested theory is. As there is no intrinsically defined four-dimensional covariant tensor that the Gauss–Bonnet tensor reduces to; namely,  $\mathcal{Y}_{\mu\nu}$

<sup>a</sup> e-mail: gurses@fen.bilkent.edu.tr

<sup>b</sup> e-mail: tahsin.c.sisman@gmail.com

<sup>c</sup> e-mail: btekin@metu.edu.tr (corresponding author)

in (4) does not exist as guaranteed by the Lovelock theorem [4–6] and as will be shown below, the theory must be defined as a limit. Thus, to compute anything in this theory, say the perturbative particle content, the maximally symmetric vacua, the black hole solutions, or any solution with or without a symmetry, one must do the computation in  $D$  dimensions and then take the  $D \rightarrow 4$  limit. Surely, for some components of the metric such as the spherically symmetric metric, due to the nature of the Gauss–Bonnet tensor, this limit might make sense. But, at the level of the solutions, namely at the level of the full metric, this limit makes no sense at all. For example, assume that there is a solution to the theory given locally with the  $D$  dimensional metric  $g_{\mu\nu}$  say which has no isometries. Then, as we need to take the  $D \rightarrow 4$  limit, which dimensions or coordinates do we dispose of, is there a canonical prescription? The answer is no! Even for spherically symmetric solutions of Boulware and Deser [7], studied so far, we do not have the right to dispose any dimension we choose.

What we have just stated is actually at the foundations of defining a gravity theory in the Riemannian geometry context. The Riemannian geometry depends on the number of dimensions, in defining a classical gravity theory based on geometry one first fixes the number of dimensions to be some  $D$ ; and as this number changes, the theory changes. There is no sensible limiting procedure as defined by (5); there is of course compactification, dimensional reduction etc where all the spacetime dimensions still survive albeit not in equal sizes generically.

The layout of the paper is as follows: in Sect. 2, we recast the  $D$ -dimensional Gauss–Bonnet tensor using the Weyl tensor in such a way that it naturally splits into two parts. One part has a formal  $D \rightarrow 4$  limit, while the other part is always higher dimensional. In Sect. 3, we give another proof that the theory is non-trivial only for  $D > 4$  using the first-order formalism with the vielbein and the spin-connection. In Sect. 4, we give an explicit example in the form of a direct-product metric where the role of the higher dimensional part is apparent.

### 2 $D \rightarrow 4$ limit of the field equations

To further lay out our ideas, and to show that there is no four-dimensional definition of the theory, let us recast the Gauss–Bonnet tensor, in such a way that we can see the limiting behaviors. For this purpose, the Weyl tensor,

$$C_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} - \frac{2}{(D-2)}(g_{\mu[\nu}R_{\beta]\alpha} - g_{\alpha[\nu}R_{\beta]\mu}) + \frac{2}{(D-1)(D-2)}Rg_{\mu[\nu}g_{\beta]\alpha}, \tag{6}$$

becomes rather useful. Using Appendix A of [8], the Gauss–Bonnet tensor in  $D$  dimensions can be split as

$$\mathcal{H}_{\mu\nu} = 2(\mathcal{L}_{\mu\nu} + \mathcal{Z}_{\mu\nu}), \tag{7}$$

where the first term does not have an explicit coefficient related to the number of dimensions and is given as

$$\mathcal{L}_{\mu\nu} := C_{\mu\alpha\beta\sigma}C_{\nu}^{\alpha\beta\sigma} - \frac{1}{4}g_{\mu\nu}C_{\alpha\beta\rho\sigma}C^{\alpha\beta\rho\sigma}, \tag{8}$$

which we shall name as the Lanczos–Bach tensor, and the other part carries explicit coefficients regarding the number of dimensions:

$$\mathcal{Z}_{\mu\nu} := \frac{(D-4)(D-3)}{(D-1)(D-2)} \left[ -\frac{2(D-1)}{(D-3)}C_{\mu\rho\nu\sigma}R^{\rho\sigma} - \frac{2(D-1)}{(D-2)}R_{\mu\rho}R_{\nu}^{\rho} + \frac{D}{(D-2)}R_{\mu\nu}R + \frac{1}{(D-2)}g_{\mu\nu} \left( (D-1)R_{\rho\sigma}R^{\rho\sigma} - \frac{D+2}{4}R^2 \right) \right], \tag{9}$$

where we kept all the factors to see how the limiting procedure might work. With the  $2/(D-4)$  factor, the second part nicely reduces to a tensor  $\mathcal{S}_{\mu\nu}$  which is finite in the  $D \rightarrow 4$  limit;

$$\mathcal{S}_{\mu\nu} := \frac{2}{D-4}\mathcal{Z}_{\mu\nu}. \tag{10}$$

But, the first part is rather non-trivial. In  $D = 4$  dimensions we have the Lanczos–Bach identity [6,9] for any smooth metric;

$$C_{\mu\alpha\beta\sigma}C_{\nu}^{\alpha\beta\sigma} = \frac{1}{4}g_{\mu\nu}C_{\alpha\beta\rho\sigma}C^{\alpha\beta\rho\sigma} \text{ for all metrics in } D = 4. \tag{11}$$

Thus, a cursory look might suggest that one might naively assume the Lanczos–Bach identity in four dimensions and set  $\mathcal{L}_{\mu\nu} = 0$  in the  $D \rightarrow 4$  limit yielding a finite intrinsically four dimensional description of the Gauss–Bonnet tensor as

$$\lim_{D \rightarrow 4} \left( \frac{1}{D-4}\mathcal{H}_{\mu\nu} \right) = \frac{1}{3} \left[ -6C_{\mu\rho\nu\sigma}R^{\rho\sigma} - 3R_{\mu\rho}R_{\nu}^{\rho} + 2R_{\mu\nu}R + \frac{3}{2}g_{\mu\nu} \left( R_{\rho\sigma}R^{\rho\sigma} - \frac{1}{2}R^2 \right) \right]. \tag{12}$$

where the right-hand side is  $\mathcal{S}_{\mu\nu}$ , given in (10), calculated at  $D = 4$ . But this is a red-herring! The  $\mathcal{H}$  tensor or the  $\mathcal{S}$  tensor does not obey the Bianchi identity

$$\nabla^{\mu}\mathcal{S}_{\mu\nu} \neq 0. \tag{13}$$

Therefore, without further assumptions, it cannot be used in the description of a four dimensional theory. Then, this begs the question: How does the  $\mathcal{L}_{\mu\nu}$  tensor go to zero in the  $D \rightarrow 4$  limit, that is

$$\lim_{D \rightarrow 4} \left[ \frac{1}{D-4} \left( C_{\mu\alpha\beta\sigma} C_{\nu}^{\alpha\beta\sigma} - \frac{1}{4} g_{\mu\nu} C_{\alpha\beta\rho\sigma} C^{\alpha\beta\rho\sigma} \right) \right] =? \tag{14}$$

To save the Bianchi identity,  $\mathcal{L}_{\mu\nu}$  should have the form

$$\frac{2}{D-4} \mathcal{L}_{\mu\nu} = \mathcal{T}_{\mu\nu} \text{ for } D \neq 4. \tag{15}$$

If this is the case, then there is a discontinuity for the Gauss–Bonnet tensor as

$$\frac{1}{D-4} \mathcal{H}_{\mu\nu} = \begin{cases} \mathcal{T}_{\mu\nu} + \mathcal{S}_{\mu\nu}, & \text{for } D \neq 4, \\ 0, & \text{for } D = 4. \end{cases} \tag{16}$$

Then, in the  $D \rightarrow 4$  limit, the Gauss–Bonnet tensor with an  $\alpha/D - 4$  factor becomes

$$\lim_{D \rightarrow 4} \left( \frac{1}{D-4} \mathcal{H}_{\mu\nu} \right) = \mathcal{T}_{\mu\nu} + \mathcal{S}_{\mu\nu}, \tag{17}$$

that is the Gauss–Bonnet tensor is not continuous in  $D$  at  $D = 4$ . This discontinuity in the Gauss–Bonnet tensor introduces a problem: Let  $g_{\mu\nu}^D$  be the solution of the field equations for  $D > 4$ , and  $g_{\mu\nu}^{\text{lim}}$  is the solution of the limiting field equations (17); then

$$\lim_{D \rightarrow 4} g_{\mu\nu}^D \neq g_{\mu\nu}^{\text{lim}}, \tag{18}$$

in general.

Incidentally, the  $\mathcal{L}_{\mu\nu}$  tensor is related to the trace of the  $D$  dimensional extension of the Bel–Robinson tensor given in [10] which reads

$$\begin{aligned} \mathcal{B}_{\alpha\beta\lambda\mu} &= C_{\alpha\rho\lambda\sigma} C_{\beta}^{\rho}{}_{\mu}{}^{\sigma} + C_{\alpha\rho\mu\sigma} C_{\beta}^{\rho}{}_{\lambda}{}^{\sigma} \\ &\quad - \frac{1}{2} g_{\alpha\beta} C_{\rho\nu\lambda\sigma} C^{\rho\nu}{}_{\mu}{}^{\sigma} \\ &\quad - \frac{1}{2} g_{\lambda\mu} C_{\alpha\rho\sigma\nu} C_{\beta}^{\rho\sigma\nu} + \frac{1}{8} g_{\alpha\beta} g_{\lambda\mu} C_{\rho\nu\sigma\eta} C^{\rho\nu\sigma\eta}, \end{aligned} \tag{19}$$

and one has

$$g^{\lambda\mu} \mathcal{B}_{\alpha\beta\lambda\mu} = \frac{D-4}{2} \mathcal{L}_{\alpha\beta}. \tag{20}$$

To summarize, in this section, we have shown that there is a part of the Gauss–Bonnet tensor  $\mathcal{H}_{\mu\nu}$  which is always higher dimensional even though one part of the tensor can be made finite with the procedure of dividing by  $1/(D-4)$  and

then formally assuming that the remaining indices are four dimensional. The all important point here is that if one bluntly drops the extra dimensional part (which we called  $\mathcal{L}_{\mu\nu}$ ), then the Bianchi identity is not satisfied for the remaining four-dimensional theory. Thus, one either has all the dimensions, or one has four dimensions without the benefit of the Bianchi identity. If one chooses the second option, one cannot couple the four-dimensional theory to conserved matter fields; or one must impose the Bianchi identity on-shell for the solutions.

### 3 The field equations in first-order formulation

The authors of [1] argued that in the first-order formulation of the Gauss–Bonnet theory a  $(D-4)$  factor arises in the field equations, and this factor can be canceled by introducing the  $\alpha/(D-4)$  factor in the action. This claim needs to be scrutinized carefully as we do here. Let us just consider the Gauss–Bonnet part of the action without any factors or coefficients. Then, we have the  $D$ -dimensional action in terms of the vielbein 1-form  $e^a$  and the curvature 2-form  $R^{ab} := d\omega^{ab} + \omega^{ac} \wedge \omega_c^b$ ;

$$I_{GB} = \int_{\mathcal{M}_D} \epsilon_{a_1 a_2 \dots a_D} R^{a_1 a_2} \wedge R^{a_3 a_4} \wedge e^{a_5} \wedge e^{a_6} \dots \wedge e^{a_D}, \tag{21}$$

where the Latin indices refer to the tangent frame. Then, varying the action with respect to the spin connection yields zero in the zero torsion case; and the rest of the field equations are obtained by varying with respect to the vielbein. At this stage the discussion bifurcates:<sup>1</sup> Assume that  $D = 4$ , then the action reduces to  $\int_{\mathcal{M}_4} \epsilon_{a_1 a_2 a_3 a_4} R^{a_1 a_2} \wedge R^{a_3 a_4}$  where there is no vierbein left and one has

$$\delta_{e^a} \int_{\mathcal{M}_4} \epsilon_{a_1 a_2 a_3 a_4} R^{a_1 a_2} \wedge R^{a_3 a_4} = 0, \quad D = 4. \tag{22}$$

On the other hand, for generic  $D > 4$  dimensions, variation with respect to the vielbein yields the field equation as a  $(D-1)$ -form

$$\begin{aligned} \mathcal{E}_{aD} &= (D-4) \epsilon_{a_1 a_2 \dots a_D} R^{a_1 a_2} \wedge R^{a_3 a_4} \wedge e^{a_5} \wedge e^{a_6} \\ &\quad \dots \wedge e^{a_{D-1}} \quad D > 4. \end{aligned} \tag{23}$$

Clearly the  $(D-4)$  factor arises, but it does so *only* in  $D$  dimensions: one cannot simply multiply with a  $\alpha/(D-4)$  and take the  $D \rightarrow 4$  limit! In fact, starting from the last equation, one can get the second order, metric form of the Gauss–Bonnet tensor  $\mathcal{H}_{\mu\nu}$ , and in the process, one sees the role played by this  $(D-4)$  factor. To do so, instead of the

<sup>1</sup> For  $D \leq 3$  the action vanishes identically and no further discussion is needed.

tangent frame indices we can recast it in terms of the space-time indices as which can be written as

$$\mathcal{E}_v = \frac{(D-4)}{4} \epsilon_{\mu_1 \mu_2 \dots \mu_{D-1} \nu} R^{\mu_1 \mu_2}_{\sigma_1 \sigma_2} R^{\mu_3 \mu_4}_{\sigma_3 \sigma_4} dx^{\sigma_1} \dots \times \wedge dx^{\sigma_4} \wedge dx^{\mu_5} \dots \wedge dx^{\mu_{D-1}}. \tag{24}$$

This is really a covariant vector-valued  $(D-1)$ -form, and the Hodge dual of this  $(D-1)$ -form is a 1-form; and since we have

$$\begin{aligned} * (dx^{\sigma_1} \dots \wedge dx^{\sigma_4} \wedge dx^{\mu_5} \dots \wedge dx^{\mu_{D-1}}) \\ = \epsilon^{\sigma_1 \dots \sigma_4 \mu_5 \dots \mu_{D-1}}{}_{\mu_D} dx^{\mu_D}, \end{aligned} \tag{25}$$

the 1-form field equations read

$$*\mathcal{E}_v = \frac{(D-4)}{4} \epsilon_{\mu_1 \mu_2 \dots \mu_{D-1} \nu} \epsilon^{\sigma_1 \dots \sigma_4 \mu_5 \dots \mu_{D-1}}{}_{\mu_D} R^{\mu_1 \mu_2}_{\sigma_1 \sigma_2} R^{\mu_3 \mu_4}_{\sigma_3 \sigma_4} dx^{\mu_D}, \tag{26}$$

from which we define the rank-2 tensor  $\mathcal{E}_{\nu\alpha}$  as

$$*\mathcal{E}_v =: \mathcal{E}_{\nu\alpha} dx^\alpha. \tag{27}$$

Explicitly one has

$$\mathcal{E}_{\nu\alpha} = \frac{(D-4)}{4} \epsilon_{\mu_1 \mu_2 \dots \mu_{D-1} \nu} \epsilon^{\sigma_1 \dots \sigma_4 \mu_5 \dots \mu_{D-1}}{}_{\alpha} R^{\mu_1 \mu_2}_{\sigma_1 \sigma_2} R^{\mu_3 \mu_4}_{\sigma_3 \sigma_4}, \tag{28}$$

which can be further reduced with the help of the identity

$$\epsilon_{\mu_1 \mu_2 \dots \mu_{D-1} \nu} \epsilon^{\sigma_1 \dots \sigma_4 \mu_5 \dots \mu_{D-1}}{}_{\alpha} = -(D-5)! g_{\beta\alpha} \delta_{\mu_1 \dots \mu_4 \nu}^{\sigma_1 \dots \sigma_4 \beta}, \tag{29}$$

where we used the generalized Kronecker delta. So, we have

$$\begin{aligned} \mathcal{E}_{\nu\alpha} &= -\frac{(D-4)}{4} (D-5)! g_{\beta\alpha} \delta_{\mu_1 \dots \mu_4 \nu}^{\sigma_1 \dots \sigma_4 \beta} R^{\mu_1 \mu_2}_{\sigma_1 \sigma_2} R^{\mu_3 \mu_4}_{\sigma_3 \sigma_4} \\ &= -\frac{(D-4)!}{4} g_{\beta\alpha} \delta_{\mu_1 \dots \mu_4 \nu}^{\sigma_1 \dots \sigma_4 \beta} R^{\mu_1 \mu_2}_{\sigma_1 \sigma_2} R^{\mu_3 \mu_4}_{\sigma_3 \sigma_4}. \end{aligned} \tag{30}$$

Observe that the  $(D-4)$  factor turned into  $(D-4)!$  which does not vanish for  $D=4$ . Since one also has

$$g_{\beta\alpha} \delta_{\mu_1 \dots \mu_4 \nu}^{\sigma_1 \dots \sigma_4 \beta} R^{\mu_1 \mu_2}_{\sigma_1 \sigma_2} R^{\mu_3 \mu_4}_{\sigma_3 \sigma_4} = -8\mathcal{H}_{\nu\alpha}, \tag{31}$$

where  $\mathcal{H}_{\nu\alpha}$  is the Gauss–Bonnet tensor we defined above, we get

$$\mathcal{E}_{\nu\mu} = 2(D-4)! \mathcal{H}_{\nu\alpha} \tag{32}$$

Thus, the moral of the story is that one either has an explicit  $(D-4)$  factor in front of the field equations when they are written in terms of the vielbeins and the spin connection where the dimensionality of the spacetime is explicitly  $D > 4$  as counted by the number of vielbeins; or, one does not

have an explicit  $(D-4)$  factor in the field equations in the metric formulation. There is no other option. In the metric formulation, we have shown in the previous section that a  $(D-4)$  does not arise for generic metrics in all parts of the field equations.

### 4 Direct-product spacetimes

To see the alluded problems explicitly in an example, let us consider the direct-product spacetimes for which the  $D$ -dimensional metric has the form

$$ds^2 = g_{AB} dx^A dx^B = g_{\alpha\beta} (x^\mu) dx^\alpha dx^\beta + g_{ab} (x^c) dx^a dx^b, \tag{33}$$

where  $A, B = 1, 2, \dots, D$ ;  $\alpha, \beta = 1, 2, 3, 4$ ; and  $a, b = 5, 6, \dots, D$ . Here,  $g_{\alpha\beta}$  depends only on the four-dimensional coordinates  $x^\mu$ , and  $g_{ab}$  depends only on the extra dimensional coordinates  $x^c$ . Then, for the Christoffel connection,

$$\Gamma_{BC}^A = \frac{1}{2} g^{AE} (\partial_B g_{EC} + \partial_C g_{EB} - \partial_E g_{BC}), \tag{34}$$

it is easy to show that the only nonzero parts are

$${}_D \Gamma_{\beta\mu}^\alpha = {}_4 \Gamma_{\beta\mu}^\alpha = \frac{1}{2} g^{\alpha\epsilon} (\partial_\beta g_{\mu\epsilon} + \partial_\mu g_{\beta\epsilon} - \partial_\epsilon g_{\beta\mu}), \tag{35}$$

$${}_D \Gamma_{bc}^a = {}_d \Gamma_{bc}^a = \frac{1}{2} g^{ae} (\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc}), \tag{36}$$

where the subindex  $d$  denotes the  $(D-4)$ -dimensional. Due to this property, we have the following nonzero components of the Riemann tensor,  $R_{BCE}^A$ , and Ricci tensor,  $R_{AB}$ ;

$$\begin{aligned} {}_D R^\alpha{}_{\beta\mu\epsilon} &= {}_4 R^\alpha{}_{\beta\mu\epsilon}, \quad {}_D R^a{}_{bce} = {}_d R^a{}_{bce}, \\ {}_D R_{\alpha\beta} &= {}_4 R_{\alpha\beta}, \quad {}_D R_{ab} = {}_d R_{ab}. \end{aligned} \tag{37}$$

In addition, the scalar curvature splits as

$${}_D R = {}_4 R + {}_d R. \tag{38}$$

The nonzero components of the Weyl tensor  $C_{ABEF}$  are

$$\begin{aligned} {}_D C_{\alpha\beta\epsilon\nu} &= {}_4 C_{\alpha\beta\epsilon\nu} + \frac{(D-4)}{(D-2)} (g_{\alpha[\epsilon} {}_4 R_{\nu]\beta} - g_{\beta[\epsilon} {}_4 R_{\nu]\alpha}) \\ &\quad - \frac{(D-4)(D+1)}{3(D-1)(D-2)} {}_4 R g_{\alpha[\epsilon} g_{\nu]\beta} \\ &\quad + \frac{2}{(D-1)(D-2)} {}_d R g_{\alpha[\epsilon} g_{\nu]\beta}, \\ {}_D C_{abef} &= {}_d C_{abef} + \frac{8}{(D-2)(D-6)} \\ &\quad \times (g_{a[e} {}_d R_{f]b} - g_{b[e} {}_d R_{f]a}) \end{aligned} \tag{39}$$

$$\begin{aligned}
 & - \frac{8(2D-7)}{(D-1)(D-2)(D-5)(D-6)} dRg_{a[egf]b} \\
 & + \frac{2}{(D-1)(D-2)} {}_4Rg_{a[egf]b}, \tag{40}
 \end{aligned}$$

$$\begin{aligned}
 {}_D C_{\alpha\beta\alpha\beta} &= \frac{1}{(D-2)} (g_{\alpha\beta} dR_{ab} + g_{ab} {}_4R_{\alpha\beta}) \\
 & - \frac{1}{(D-1)(D-2)} ({}_4R + dR) g_{\alpha\beta} g_{ab}, \tag{41}
 \end{aligned}$$

in addition to  ${}_D C_{\alpha\beta\beta\alpha} = -{}_D C_{b\alpha\beta a} = {}_D C_{b\alpha\alpha\beta} = -{}_D C_{\alpha\beta a\beta}$ .  
 If the  $d$ -dimensional internal space is flat as

$$ds^2 = g_{AB} dx^A dx^B = g_{\alpha\beta} (x^\mu) dx^\alpha dx^\beta + \eta_{ab} dx^a dx^b, \tag{42}$$

then one has

$${}_D R^a{}_{bce} = 0, \quad {}_D R_{ab} = 0, \quad {}_D R = {}_4R, \tag{43}$$

and nonzero components of the Weyl tensor given in (39-41) become

$$\begin{aligned}
 {}_D C_{\alpha\beta\epsilon\nu} &= {}_4C_{\alpha\beta\epsilon\nu} + \frac{(D-4)}{(D-2)} (g_{\alpha[\epsilon} {}_4R_{\nu]\beta} - g_{\beta[\epsilon} {}_4R_{\nu]\alpha}) \\
 & - \frac{(D-4)(D+1)}{3(D-1)(D-2)} {}_4Rg_{\alpha[\epsilon} g_{\nu]\beta}, \tag{44}
 \end{aligned}$$

$${}_D C_{abef} = \frac{2}{(D-1)(D-2)} {}_4R\eta_{a[e}\eta_{f]b}, \tag{45}$$

$${}_D C_{\alpha\beta a\beta} = \frac{1}{(D-2)} \eta_{ab} {}_4R_{\alpha\beta} - \frac{1}{(D-1)(D-2)} {}_4Rg_{\alpha\beta} \eta_{ab}, \tag{46}$$

in addition to  ${}_D C_{\alpha\beta\beta\alpha} = -{}_D C_{b\alpha\beta a} = {}_D C_{b\alpha\alpha\beta} = -{}_D C_{\alpha\beta a\beta}$ .  
 With the above results, let us provide a clear example of where the limit

$$\begin{aligned}
 \lim_{D \rightarrow 4} \mathcal{L}_{AB} &= \lim_{D \rightarrow 4} \left[ \frac{1}{D-4} (C_{AEFG} C_B{}^{EFG} \right. \\
 & \left. - \frac{1}{4} g_{AB} C_{EFGH} C^{EFGH}) \right], \tag{47}
 \end{aligned}$$

fails. Consider the  $\mathcal{L}_{ab}$  components of the Lanczos–Bach tensor,

$$\mathcal{L}_{ab} = C_{aEFG} C_b{}^{EFG} - \frac{1}{4} \eta_{ab} C_{EFGH} C^{EFGH}. \tag{48}$$

These components can be written as

$$\begin{aligned}
 \mathcal{L}_{ab} &= ({}_D C_{aefg} {}_D C_b{}^{efg} + 2 {}_D C_{aef\gamma} {}_D C_b{}^{\epsilon f\gamma}) \\
 & - \frac{1}{4} \eta_{ab} ({}_D C_{\alpha\epsilon\nu\gamma} {}_D C^{\alpha\epsilon\nu\gamma} + {}_D C_{aefg} {}_D C^{aefg} \\
 & + 4 {}_D C_{\alpha\epsilon\gamma f} {}_D C^{\alpha\epsilon\gamma f}). \tag{49}
 \end{aligned}$$

By using (44–46), the  $\mathcal{L}_{ab}$  components of the Lanczos–Bach tensor can be calculated in terms of the four-dimensional and  $d$ -dimensional quantities as

$$\begin{aligned}
 \mathcal{L}_{ab} &= -\frac{1}{4} \eta_{ab} \left( {}_4C_{\epsilon\nu\gamma\eta} {}_4C^{\epsilon\nu\gamma\eta} \right. \\
 & + \frac{2(D^2-6D+4)}{(D-2)^2} {}_4R_{\epsilon\nu} {}_4R^{\epsilon\nu} \\
 & \left. - \frac{(D^3-5D^2+2D-16)}{3(D-1)(D-2)^2} {}_4R^2 \right). \tag{50}
 \end{aligned}$$

Then, the  $D \rightarrow 4$  limit for this term in the form,

$$\begin{aligned}
 \lim_{D \rightarrow 4} \left[ -\frac{\eta_{ab}}{4(D-4)} \left( {}_4C_{\epsilon\nu\gamma\eta} {}_4C^{\epsilon\nu\gamma\eta} \right. \right. \\
 + \frac{2(D^2-6D+4)}{(D-2)^2} {}_4R_{\epsilon\nu} {}_4R^{\epsilon\nu} \\
 \left. \left. - \frac{(D^3-5D^2+2D-16)}{3(D-1)(D-2)^2} {}_4R^2 \right) \right], \tag{51}
 \end{aligned}$$

is undefined, and this fact indicates that in general, there is no proper  $D \rightarrow 4$  limit for the field equations for the direct-product spacetimes.

### 5 Conclusions

Recently [1], contrary to common knowledge and to the Lovelock’s theorem [4–6], a novel *four-dimensional* Einstein–Gauss–Bonnet theory with only a massless spin-2 graviton degree of freedom was suggested to exist. A four-dimensional gravity theory should have four-dimensional equations: here, we have shown that this is not the case. Namely, we have shown that the novel Einstein–Gauss–Bonnet theory in four dimensions does not have an intrinsically four-dimensional description in terms of a covariantly-conserved rank-2 tensor in four dimensions. We have done this by splitting the Gauss–Bonnet tensor (2) into two parts as (7): one is what we called the Lanczos–Bach tensor (8) which is related to the trace of the  $D$ -dimensional Bel–Robinson tensor which does not have an explicit  $(D-4)$  factor, and the other part (9) is a part that has an explicit  $(D-4)$  factor in front. The Lanczos–Bach tensor vanishes identically in four dimensions; however, it cannot be set to identically zero in that dimensions since in the absence of it, the Gauss–Bonnet tensor does not satisfy the Bianchi Identity. Thus, the theory must be defined in  $D > 4$  dimensions to be non-trivial which is in complete agreement with the Lovelock’s theorem. But, once the theory is defined in  $D$  dimensions, it will have all sorts of  $D$  dimensional solutions and in none of these solutions one can simply dispose of  $(D-4)$  dimensions or coordinates as such a discrimination among space-time dimensions simply does not make sense. We gave an

explicit example in the form of a direct product. In the first-order formulation with the vielbein and the spin connection, there is an explicit  $(D - 4)$  factor in front of the field equations, but this factor only arises in  $D > 4$  dimensions not in four dimensions. What we have shown here for the Gauss–Bonnet tensor in its critical  $D = 4$  dimensions is also valid for the other Lovelock tensors in their critical dimensions. As a final remark, let us note that following the  $D \rightarrow 2$  formal limit of Einstein’s gravity as was done in [11], if one carries out a  $D \rightarrow 4$  construction in the EGB theory, one finds [12–14], complementing our arguments, that the ensuing theory is not only a theory of massless spin-2 gravitons but additional scalar fields (of the Horndeski or Galileon type) appear.

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