

Born-Infeld-Hořava gravity

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We define various Born-Infeld gravity theories in 3+1 dimensions which reduce to Hořava's model at the quadratic level in small curvature expansion. In their exact forms, our actions provide $z \rightarrow \infty$ extensions of Hořava's gravity, but when small curvature expansion is used, they reproduce finite z models, including some half-integer ones.

I. INTRODUCTION

Born-Infeld (BI) [1] type actions appear in physics in various contexts; for example, the simplest one is the relativistic point particle action $I = -m \int dt \sqrt{1 - v^2}$. One pragmatic way of looking at this action is that it restricts $v \leq 1$. Similarly, in electrodynamics to put an upper bound to the electric field, and obtain a finite self-energy for the point charge, electrodynamics can be extended to

$$I = -b^2 \int d^4x \sqrt{-\det \left(g_{\mu\nu} + \frac{1}{b} F_{\mu\nu} \right)}, \quad (1)$$

where b is a dimensionful parameter which sets the scale of the maximum attainable electric field. It is easy to check that at the quadratic order, after dropping a constant term, (1) gives the pure Maxwell theory. It also has the desired properties such as ghost freedom and causal propagation. In string theory, Nambu-Goto action and D-brane actions are of the BI type. For a nice account of BI theories see [2]. It is only natural to consider determinantal actions in gravity theories. In fact, a decade before the BI paper, Eddington [3], using the symmetric connection (not the metric) as the independent field, extended general relativity in the form $I \sim \int d^4x \sqrt{\det [R_{(\mu\nu)}(\Gamma)]}$ which has recently picked up interest [4]. Deser and Gibbons [5], using the metric as the independent field, pondered upon viable BI-type gravity models in four dimensions. They considered the action

$$I = \int d^4x \sqrt{-\det (ag_{\mu\nu} + bR_{\mu\nu} + cX_{\mu\nu})},$$

where $X_{\mu\nu}$ is a “fudge” tensor which should be chosen in such a way that ghosts and tachyons do not appear in the small curvature expansion about the Minkowski or (anti)-de Sitter spaces.

BI-type consistent gravity actions take their most elegant form in three dimensions. Recently [6], it was shown that the action

$$I_{BI} = -\frac{4m^2}{\kappa^2} \int d^3x \left[\sqrt{-\det \left(-\frac{1}{m^2} \mathcal{G} \right)} - \left(\frac{\Lambda}{2m^2} + 1 \right) \sqrt{-g} \right], \quad (2)$$

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with $\mathcal{G}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - m^2g_{\mu\nu}$, at the quadratic level reduces to the new massive gravity (NMG)[7]

$$I_{NMG} = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left[-(R - 2\Lambda) + \frac{1}{m^2} \left(R_{\mu\nu}^2 - \frac{3}{8}R^2 \right) \right],$$

which is the only unitary [7–10], super-renormalizable [11], parity-invariant theory. Conforming to the spirit of the BI actions, for constant curvature spaces, (2) restricts the curvature to be $R_{\mu\nu} \geq -2m^2g_{\mu\nu}$. At the cubic order, (2) yields the action

$$I_{NMG \text{ extended}} = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left[-(R - 2\Lambda) + \frac{1}{m^2} \left(R_{\mu\nu}^2 - \frac{3}{8}R^2 \right) + \frac{2}{3m^4} \left(R^{\mu\nu} R_{\nu}^{\alpha} R_{\alpha\mu} - \frac{9}{8}RR_{\mu\nu}^2 + \frac{17}{64}R^3 \right) + O(R^4) \right], \quad (3)$$

which exactly matches the action obtained by Sinha [12] who used the AdS/CFT conjecture and the existence of a holographic c theorem to find cubic deformations to NMG. It is remarkable that a BI-type gravitational action at the cubic order reproduces a three-dimensional theory in the bulk which is fixed by conformal field theory on the two-dimensional boundary.

Inspired by the success of the BI actions, in this work we will present various extensions of Hořava’s recent nonrelativistic gravity theory [13] to all orders in the curvature. The layout of the paper is as follows: In Sec. II, we propose an extension of Hořava’s gravity by defining a BI-type potential generating action in three dimensions using the detailed-balance principle. In Sec. III, without reference to the detailed-balance principle, we give BI-type extensions of Hořava’s gravity in $3 + 1$ dimensions.

II. BORN-INFELD-HOŘAVA GRAVITY: BI-TYPE POTENTIAL GENERATING ACTION

Since Hořava’s gravity has already been described in many places such as [14], just to fix the notation, using (almost) the form of the action given in [15], we shall briefly recapitulate the essential ingredients. (See [16] for related works.) One starts with the usual ADM [17] decomposition of the four-dimensional space,

$$ds^2 = -N^2 dt^2 + g_{ij} \left(dx^i - N^i dt \right) \left(dx^j - N^j dt \right),$$

where all the involved functions depend on t and x_i . [Note that we do not commit ourselves to the so-called projectable version of Hořava’s gravity for which N depends on t only.] One then assumes different scaling dimensions for time and space: $t \rightarrow b^z t$, $x^i \rightarrow bx^i$. It is expected that in the IR limit $z \rightarrow 1$ and full diffeomorphism invariance is recovered. Once sacred Lorentz invariance is let go, there is no limit to the number of models one can define. Hořava introduced a guiding principle called the “detailed balance” to inherit “reasonable” actions from three dimensions. In short, his specific proposal leads to

$$I_H = \int dt d^3x \sqrt{g} N (\mathcal{L}_K + \mathcal{L}_V),$$

where \mathcal{L}_K and \mathcal{L}_V are kinetic and potential parts, respectively. Not to introduce more than two time derivatives and get hit by the Ostragradski ghosts, the kinetic part is defined as

$$\mathcal{L}_K = \frac{2}{\kappa^2} \left(K_{ij} K^{ij} - \lambda K^2 \right),$$

where $K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ is the extrinsic curvature and λ is a dimensionless coupling constant which hopefully flows to 1 in the IR limit, so that one recovers the kinetic part of the standard Einstein-Hilbert action. The kinetic part of the action is pretty robust, but as for the potential part, one has a great deal of freedom. Specifically, defining $C^{ij} = \epsilon^{ikl} \nabla_k (R^j_l - \frac{1}{4} R \delta_l^j)$ to be the Cotton tensor, the $z = 3$ Hořava theory has the potential:

$$\mathcal{L}_V = \frac{\kappa^2 \mu^2 \Lambda_W (R - 3\Lambda_W)}{8(1-3\lambda)} + \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} R^2 - \frac{\kappa^2}{2w^4} \left(C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left(C^{ij} - \frac{\mu w^2}{2} R^{ij} \right),$$

where *à la* the detailed-balance principle, \mathcal{L}_V comes from the three-dimensional Einstein-Hilbert and the topologically massive gravity (TMG) actions. More concretely, this principle works in the following way, $\mathcal{L}_V = \frac{\kappa^2}{8} E^{ij} G_{ijkl} E^{kl}$, where $E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W_3}{\delta g_{ij}}$. Here, G_{ijkl} is the deformed De Witt metric

$$G_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) + \frac{\lambda}{1-3\lambda} g_{ik} g_{jl}$$

and W_3 is a three-dimensional Euclidean action. [As a side remark, note that in a slightly different context, using the method of steepest descent topologically massive gravity was used to obtain the $C_{ij} C^{ij}$ action to define another nonrelativistic theory which is the Cotton flow theory $\partial_t g_{ij} = C_{ij}$ [18]. The relevance of Cotton and related flows to Hořava's gravity has been recently studied in [19].]

In [20], $z = 4$ Hořava's theory was defined using the NMG as the potential generating action via the detailed-balance principle. As we noted in the Introduction, NMG itself has a consistent BI-type extension. Therefore, our first proposal is to add to TMG the Euclidean BI action to get the potential generating action:

$$W_3 = -4\mu^2 L \int d^3x \left[\sqrt{\det \left(\frac{1}{\mu L} \mathcal{G} \right)} + \left(\frac{\Lambda_W}{2\mu L} - 1 \right) \sqrt{g} \right] + \frac{1}{w^2} \int d^3x \sqrt{g} \epsilon^{ijk} \Gamma_{il}^m \left(\partial_j \Gamma_{km}^l + \frac{2}{3} \Gamma_{jn}^l \Gamma_{km}^n \right), \quad (4)$$

where $\mathcal{G}_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R + \mu L g_{ij}$. This action can be used to obtain the potential at any desired order. In the quadratic order, the result of [20] follows, at the cubic order one should find the equations coming from (3) add the Cotton tensor to find E^{ij} , and using the De Witt metric described as above take the square of E_{ij} to get the potential. If one wants to deal with the exact (that is the $z \rightarrow \infty$) theory, one can use

$$\det A = \frac{1}{6} \left[(\text{Tr} A)^3 - 3 \text{Tr} A \text{Tr} (A^2) + 2 \text{Tr} (A^3) \right],$$

to get

$$\sqrt{\det \left(\frac{1}{m^2} \mathcal{G} \right)} = \sqrt{g} \left(1 - \frac{1}{2m^2} R^{ij} \left[g_{ij} + \frac{1}{m^2} \left(R_{ij} - \frac{1}{2} g_{ij} R \right) - \frac{2}{3m^4} \left(R_{ik} R^k_j - \frac{3}{4} R R_{ij} + \frac{1}{8} g_{ij} R^2 \right) \right] \right)^{1/2},$$

or if one wants to work at a finite z , for example, such as $z = 8$ one can do a small curvature expansion up to $O(A^5)$,

$$[\det(1+A)]^{1/2} = 1 + \frac{1}{2} \text{Tr} A - \frac{1}{4} \text{Tr} (A^2) + \frac{1}{8} (\text{Tr} A)^2 + \frac{1}{6} \text{Tr} (A^3) - \frac{1}{8} \text{Tr} A \text{Tr} (A^2) + \frac{1}{48} (\text{Tr} A)^3 - \frac{1}{8} \text{Tr} (A^4) + \frac{1}{32} [\text{Tr} (A^2)]^2 + \frac{1}{12} \text{Tr} A \text{Tr} (A^3) - \frac{1}{32} (\text{Tr} A)^2 \text{Tr} (A^2) + \frac{1}{384} (\text{Tr} A)^4.$$

Clearly, this procedure can be extended to any desired order.

III. BORN-INFELD-HOŘAVA GRAVITY: BI-TYPE POTENTIAL

In the above section, we proposed that the potential generating three-dimensional action for Hořava's gravity can be taken to be the gravitational BI action together with the TMG action. This procedure leads to a manageable deformation of Hořava's gravity. In this section, we will propose a more radical extension of Hořava's gravity again in the form of a BI action which will not require a reference to the detailed-balance principle. First, observe that for $\lambda = 1$ the potential part of Hořava's theory reduces to the Euclidean NMG in addition to Cotton parts:

$$\mathcal{L}_V = -\frac{\kappa^2 \mu^2 \Lambda_W}{16} \left[(R - 3\Lambda_W) + \frac{2}{\Lambda_W} \left(R_{ij}^2 - \frac{3}{8} R^2 \right) \right] - \frac{\kappa^2}{2w^4} C_{ij}^2 + \frac{\kappa^2 \mu}{2w^2} C^{ij} R_{ij}. \quad (5)$$

Note that the appearance of NMG in the IR limit of $z = 3$ Hořava's gravity should not be confused with the use of NMG as a potential generating action for the $z = 4$ theory discussed above. Observation of (5) led us to consider the following BI action:

$$I_{BI} = \frac{2}{\kappa^2} \int dt d^3 \mathbf{x} \sqrt{g} N \left(K_{ij} K^{ij} - \lambda K^2 \right) + \frac{1}{b} \int dt d^3 x N \left\{ \sqrt{\det [g_{ij} + a \tilde{R}_{ij} + d g_{ij} R + e C_{ij}]} + \frac{1}{2} \sqrt{g} \right\}, \quad (6)$$

where $\tilde{R}_{ij} \equiv R_{ij} - \frac{1}{3} g_{ij} R$. With the coefficients

$$a = \pm \frac{\sqrt{6\lambda - 2}}{\Lambda_W}, \quad \lambda > \frac{1}{3}, \quad b = \frac{2a^2}{\kappa^2 \mu^2}, \quad d = -\frac{1}{3\Lambda_W}, \quad e = -\frac{2a}{\mu w^2},$$

Hořava's gravity is reproduced in the small curvature expansion at the quadratic level. What is of course remarkable about (6) is that by just considering the metric, the Ricci tensor and scalar and the Cotton tensor and not any other higher derivative tensors, one can reproduce and extend Hořava's gravity to any order. For example, at $O(R^3)$ Hořava's action will be augmented with

$$b\mathcal{L}_{O(R^3)} = \frac{1}{6} \left(a^3 R^{ij} R_{jk} R^k{}_i + 3a^2 e R^{ij} R_{jk} C^k{}_i + 3ae^2 R^{ij} C_{jk} C^k{}_i + e^3 C^{ij} C_{jk} C^k{}_i \right) - \frac{1}{6} \left(a - \frac{3}{4} d \right) R \left(a^2 R_{ij}^2 + 2ae R^{ij} C_{ij} + e^2 C_{ij}^2 \right) + \left(\frac{a^3}{27} - \frac{1}{24} a^2 d - \frac{1}{16} d^3 \right) R^3, \quad (7)$$

which defines a $z = 4.5$ theory in the UV. Beyond this order, the computation gets more cumbersome (see the Appendix).

We stress that the BI action (6) is tailor made to reproduce Hořava's gravity at the quadratic order. Therefore, as long as one considers small curvature expansions at the desired order, Hořava's theory merely receives corrections. For example, the solutions found in [15, 21] will be modified. On the other hand, considering (6) as the *exact* theory without any approximation, one can search for solutions. It is easy to check that spherically symmetric *static* solutions are ruled out. To see this, leaning on *symmetric criticality*, which says that symmetric critical points are critical symmetric points when compact symmetry group is integrated out [22, 23], one inserts the ansatz

$$ds^2 = -N^2(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right),$$

to the action (6): The kinetic part vanishes and variation with respect to $N(r)$ shows that there cannot be a static solution. This result is surprising, but (6) is supposed to define a quantum gravity

action and there is no guarantee that in quantum gravity there will be spherically symmetric *static* solutions. In fact, in Einstein's gravity the classical Schwarzschild solution fails to be static even in the semiclassical approach [24]: It has Hawking radiation.

Finally, by allowing quadratic terms inside the determinant, one can find some nonminimal BI extensions of Hořava's gravity. The most general choice using only the Cotton and the Ricci tensors would be

$$\det \left[g_{ij} + a\tilde{R}_{ij} + dg_{ij}R + eC_{ij} + f \left(RR_{ij} + lg_{ij}R^2 \right) + n_R \left(R_{ik}R^k{}_j + pRg_{ij}R^2_{kl} \right) + n_{RC} \left(R_{ik}C^k{}_j + p_{RC}g_{ij}R_{kl}C^{kl} \right) + n_C \left(C_{ik}C^k{}_j + p_Cg_{ij}C^2_{kl} \right) \right].$$

The requirement that Hořava's gravity be reproduced at the quadratic level is highly restrictive, and the explicit computation shows that not all the terms are allowed. In fact, one is left to choose either the RR_{ij} or the $g_{ij}R^2$ term. Therefore, another extension of Hořava's gravity is

$$I_{\text{BI nonminimal}} = \frac{2}{\kappa^2} \int dt d^3 \mathbf{x} \sqrt{g} N \left(K_{ij}K^{ij} - \lambda K^2 \right) + \frac{1}{b} \int dt d^3 x N \sqrt{\det \left[g_{ij} + a\tilde{R}_{ij} + dg_{ij}R + eC_{ij} + fR_{ij}R \right]}, \quad (8)$$

where

$$a = \pm \frac{2}{\Lambda_W} \sqrt{\frac{3\lambda - 1}{3}}, \quad b = \frac{2a^2}{\kappa^2 \mu^2}, \quad d = -\frac{2}{9\Lambda_W}, \quad e = -\frac{2a}{\mu w^2}, \quad f = \frac{1}{54\Lambda_W^2}.$$

Observe that, as opposed to (6), in (8) the cosmological constant term comes with the correct factors from the determinant and one does not need to add a \sqrt{g} . In principle, this action allows spherically symmetric static solutions. Again small curvature expansion will give deformations of Hořava's theory at any order.

IV. CONCLUSIONS

We proposed, without introducing new parameters, three BI-type extensions, (4), (6), (8), of Hořava's gravity. All these extensions can be used to generate finite z theories, or taken in their exact form they define in a compact way $z \rightarrow \infty$ theories in the UV regimes. Our first proposal (4) uses the detailed-balance principle, and the potential is generated from a three-dimensional BI action that generalizes the NMG and its cubic deformation obtained from AdS/CFT. The other two proposals do not use the detailed-balance principle: the potential is given in terms of a BI-type action. In the literature, both the original Hořava's gravity and its extensions [25, 26] have been questioned regarding their consistency and strong coupling problems [27–29]. The models we have proposed here need to be studied along these lines. Classical solutions of the BI-type theories have usually no or less severe singularities. It would be interesting to see if our actions have nonsingular solutions.

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Appendix: $O(R^4)$ extension of BI-Hořava Gravity

In addition to (7), the following terms will be added to get the $z = 6$ theory at $O(R^4)$:

$$\begin{aligned}
b\mathcal{L}_{O(R^4)} = & -\frac{1}{8} \left(a^4 R^{ij} R_{jk} R^{kl} R_{li} + 4a^3 e R^{ij} R_{jk} R^{kl} C_{li} + 6a^2 e^2 R^{ij} R_{jk} C^{kl} C_{li} \right. \\
& \left. + 4ae^3 R^{ij} C_{jk} C^{kl} C_{li} + e^4 C^{ij} C_{jk} C^{kl} C_{li} \right) \\
& + \frac{1}{12} (2a - 3d) R \left(a^3 R^{ij} R_{jk} R^k{}_i + 3a^2 e R^{ij} R_{jk} C^k{}_i + 3ae^2 R^{ij} C_{jk} C^k{}_i + e^3 C^{ij} C_{jk} C^k{}_i \right) \\
& - \frac{1}{96} \left(9d^2 - 24ad + 10a^2 \right) R^2 \left(a^2 R^2_{ij} + 2ae R^{ij} C_{ij} + e^2 C^2_{ij} \right) \\
& + \frac{1}{32} \left[a^4 \left(R^2_{ij} \right)^2 + 4a^3 e R^2_{ij} R^{kl} C_{kl} + 2a^2 e^2 R^2_{ij} C^2_{kl} \right. \\
& \left. + 4a^2 e^2 \left(R^{ij} C_{ij} \right)^2 + 4ae^3 R^{ij} C_{ij} C^2_{kl} + e^4 \left(C^2_{ij} \right)^2 \right] \\
& + \frac{1}{32} \left(\frac{5a^4}{9} - \frac{16a^3 d}{9} + a^2 d^2 + \frac{3d^4}{4} \right) R^4.
\end{aligned}$$

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