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A note on the Gauss maps of Cayley-free embeddings into spin(7)-manifolds

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We show that a closed, orientable 4-manifold M admits a Cayley-free embedding into flat Spin(7)-manifold $\mathbb{R}^8$ if and only if both the Euler characteristic χ_M and the signature τ_M of M vanish.

1. Introduction

Calibrated geometries, introduced by Harvey and Lawson in [4] are the geometries of minimal submanifolds of a Riemannian manifold (X, g) determined by a special closed differential form ϕ on X, called a *calibration*. These geometries, especially on spaces with special holonomy, have been the focus of research interests of many geometers and physicists due to their strong relations with gauge theories in higher dimensions [10], mirror symmetry [9] and modern string theory in physics [6]. Hence, understanding their structures plays a key role in these theories.

Harvey and Lawson has recently introduced new tools on calibrated manifolds which can be used to study geometry and analysis. On a calibrated manifold (X, ϕ), they define ϕ-plurisubharmonic functions and ϕ-convexity which are analogues of classical plurisubharmonic functions and pseudoconvexity in complex

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analysis. They derive many useful results and extend a number of results in complex geometry to calibrated manifolds (cf. [5]).

The analogues in calibrated geometry of totally real submanifolds (ones free of any complex tangent lines) in complex analysis are called ϕ-free submanifolds and they form a very rich source to construct strictly ϕ-convex manifolds with every topological type allowed. Similar to totally real submanifolds, there is an integer bound on the possible dimensions of ϕ-free submanifolds which is called the *free dimension* and is determined by the calibration ϕ. Furthermore, a strictly ϕ-convex domain has the homotopy type of a CW complex of dimension at most the free dimension of ϕ.

In this paper we work on flat Spin(7)-manifold $\mathbb{R}^8$ with Cayley calibration Φ and investigate the embedding problem of orientable, closed 4-manifolds into $\mathbb{R}^8$ as Cayley-free, which is the only non-trivial case. In [12] we showed that any such manifold with Euler characteristic and signature equal to zero can be embedded into flat G_2-manifold $\mathbb{R}^7$ as co-associative-free, hence the inclusion of this embedding into $\mathbb{R}^8$ is automatically Cayley-free. Here we show that the vanishing of these topological invariants is necessary and sufficient for Cayley-free embeddings into $\mathbb{R}^8$ and prove the following theorem.

Theorem 1.1 *(Main Theorem). A closed, orientable 4-manifold M^4 admits a Cayley-free embedding into $\mathbb{R}^8$ if and only if both the Euler characteristic χ_M and the signature τ_M of M vanish.*

This theorem actually holds for any flat (or locally flat around a point) Spin(7)-manifold. If N^8 is a flat Spin(7)-manifold, then for every point $p \in N$ there is a neighborhood U_p of p which is locally isometric to an open set $V \subseteq \mathbb{R}^8$ and if a closed, orientable 4-manifold M^4 admits a Cayley-free embedding into $\mathbb{R}^8$ then it also admits a Cayley-free embedding into V by using a zooming principle similar to coassociative-free embeddings for flat G_2-manifolds in [12]. Then, M^4 can be embedded into $U_p \subset N^8$ as Cayley-free by using the local isometry. As an example, any closed, orientable 4-manifold M^4 with $\chi_M = \tau_M = 0$ can be embedded into flat torus T^8 as Cayley-free.

2. Background

Let (X, g) be a Riemannian manifold. A closed differential p-form ϕ with $p \geq 1$ on X is called a *calibration* if

$$\phi|_\xi \leq vol|_\xi$$

for any oriented tangent p-plane ξ in T_xX for any $x \in X$. A Riemannian manifold (X, g) with such a differential form ϕ is called a *calibrated manifold*. A p-plane ξ is called a *ϕ-plane* or a *calibrated plane* if $\phi|_\xi = vol|_\xi$ and the set of all ϕ-planes at a point x is called the ϕ-Grassmannian denoted by $G(\phi_x)$ which is a subset of the oriented Grassmannian $G_p^+(T_xX)$ of oriented p-planes in T_xX. Furthermore, the set of all p-planes on X forms a subbundle of the oriented Grassmannian bundle $\tilde{G}_p(X)$ on X.

Each calibration ϕ determines a special geometry of submanifolds. An oriented p-dimensional submanifold M^p of a calibrated manifold (X, ϕ) is called a *calibrated submanifold* or a *ϕ-submanifold* if each $T_xM \subset T_xX$ lies in $G(\phi_x)$. The fundamental observation about these submanifolds, which is also called *the fundamental theorem of calibrated geometry*, is that all ϕ-submanifolds are absolutely volume-minimizing in their homology class.

There are lots of interesting examples of calibrations with rich geometries, especially on manifolds with special holonomy. Kähler, Calabi–Yau, hyper-Kähler, G_2 or Spin(7) manifolds come up with one or more canonical calibrations. In this paper we are just interested in flat Spin(7)-manifold $\mathbb{R}^8$ with Cayley calibration, for other examples of calibrations one may look at the foundational paper [4] by Harvey and Lawson or the recent monograph [6] by Joyce, which is mainly focusing on the manifolds with special holonomy.

We define the Cayley calibration on $\mathbb{R}^8$ by using octonions, or Cayley numbers $\mathbb{O} \cong \mathbb{R}^8$ with the triple cross product $x \times y \times z = \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x))$ for all $x, y, z \in \mathbb{O}$.

Definition 2.1. The differential 4-form $\Phi \in \bigwedge^4(\mathbb{R}^8)^*$ defined as

$$\Phi(x, y, z, w) \equiv \langle x, y \times z \times w\rangle \,\text{for}\, x, y, z, w \in \mathbb{R}^8$$

is called the *Cayley calibration* on $\mathbb{R}^8$.

If we use $(x_1, ..., x_8)$ as coordinates on $\mathbb{R}^8$, then we get

$$\begin{aligned}\Phi = &d\mathbf{x}_{1234} - d\mathbf{x}_{1278} - d\mathbf{x}_{1638} - d\mathbf{x}_{1674} - d\mathbf{x}_{1265} - d\mathbf{x}_{1375} - d\mathbf{x}_{1485} \\ &+ d\mathbf{x}_{5678} - d\mathbf{x}_{5634} - d\mathbf{x}_{5274} - d\mathbf{x}_{5238} + d\mathbf{x}_{3478} + d\mathbf{x}_{2468} + d\mathbf{x}_{2367}\end{aligned}$$

where $d\mathbf{x}_{ijkl}$ denotes the form $dx_i \wedge dx_j \wedge dx_k \wedge dx_l$.

On $\mathbb{R}^8$, Φ-Grassmannian is same everywhere and in this paper we denote it by **CAY** rather than $\mathrm{G}(\Phi) = \{\xi \in G_4^+(\mathbb{R}^8)|\ \Phi|_\xi = vol|_\xi\}$. Any oriented 4-plane in **CAY** is called a *Cayley* plane. Spin(7) group acts on **CAY** transitively with isotropy group $K \equiv SU(2) \times SU(2) \times SU(2)/\mathbb{Z}_2$. Hence, $\mathbf{CAY} \cong Spin(7)/K$ is a 12-dimensional submanifold of the oriented Grassmannian $G_4^+(\mathbb{R}^8)$ (see [4] for details). Moreover, there is a diffeomorphic copy of **CAY** in $G_4^+(\mathbb{R}^8)$ denoted by $\overline{\mathbf{CAY}}$ which is defined as $\overline{\mathbf{CAY}} = \{\xi \in G_4^+(\mathbb{R}^8)| -\xi \in \mathbf{CAY}\}$ i.e. the set of 4-planes obtained by reversing the orientation of Cayley planes.

Now we give some background about ϕ-free submanifolds of calibrated manifolds which can be somehow thought as opposite of calibrated submanifolds.

Let (X, ϕ) be a calibrated manifold. A p-plane ξ is said to be **tangential** to a submanifold $M \subset X$ if $span\xi \subset T_xM$ for some $x \in M$

Definition 2.2. A closed submanifold $M \subset X$ is called ϕ**-free** if there are no ϕ-planes $\xi \in G(\phi)$ which are tangential to M. On a Spin(7)-manifold with Cayley calibration Φ, we call these submanifolds *Cayley-free*.

Each submanifold of dimension strictly less than the degree of ϕ is automatically ϕ-free. In dimension p, locally the generic submanifold is ϕ-free. Depending on the calibrated geometry, this may continue through a range of dimensions greater than p, but there is an upper bound determined by the calibration.

Definition 2.3. The **free dimension** of a calibrated manifold (X, ϕ), denoted by $\mathbf{fd}(\phi)$, is the maximum dimension of a linear subspace in T_xX for $x \in X$ which contains no ϕ-planes. Subspaces with this property are called ϕ-free.

Hence, the dimension of a ϕ-free submanifold can not exceed $\mathbf{fd}(\phi)$. For all well-known calibrations on manifolds with special holonomy this dimension is computed (see [4] or [11]). As an example, if (X, ω) is a Kähler manifold with real dimension $2n$, then $\mathbf{fd}(\omega) = n$. For a Spin(7)-manifold with Cayley calibration Φ we have the following result.

Theorem 2.4. *The free dimension of* Φ*,* $\mathbf{fd}(\Phi)$ *is equal to 4.*

Proof. Let $V^5 \subset \mathbb{R}^8$ be a 5-dimensional subspace. Choose an orthonormal basis $\{x, y, z\}$ for $V^\perp$. Then $W = span\{x, y, z, x \times y \times z\}$ is a Cayley 4-plane. Since Φ is self-dual, $W^\perp \subset V$ is a Cayley plane, too. Hence, V is not Cayley-free. Furthermore, there are 4-dimensional subspaces of $\mathbb{R}^8$ which are not Cayley with respect to any orientation. This shows that $\mathbf{fd}(\Phi) = 4$. □

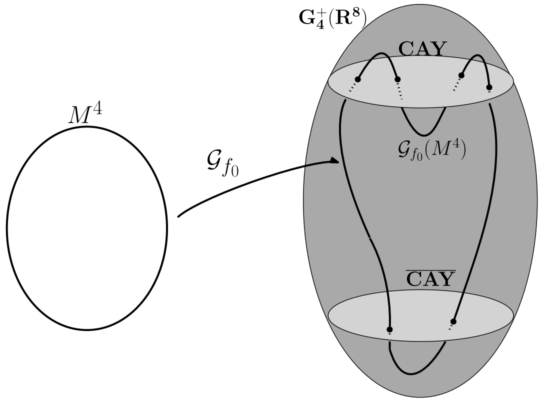

Fig. 1. The Gauss map and intersections.

This result shows that any submanifold with dimension less than 4 or the generic local submanifold with dimension 4 is Cayley-free in a Spin(7)-manifold. Hence, only non-trivial case remaining is the existence of closed 4-dimensional Cayley-free submanifolds, which is studied for flat Spin(7)-manifolds in this paper.

3. Proof of The Main Theorem

Our aim is to show that any embedding of a closed oriented 4-manifold M^4 with $\chi_M = \tau_M = 0$ into $\mathbb{R}^8$ is isotopic to a Cayley-free embedding. Along the proof, it is easy to see that vanishing of χ_M and τ_M is a necessary condition for Cayley-free embeddings.

Let $\tilde{G}_4(\mathbb{R}^8) \cong \mathbb{R}^8 \times G_4^+(\mathbb{R}^8)$ be the Grassmann bundle of flat Spin(7)-manifold $\mathbb{R}^8$ where $G_4^+(\mathbb{R}^8)$ is the Grassmannian of oriented 4-planes in $\mathbb{R}^8$. If $f_0 : M^4 \longrightarrow \mathbb{R}^8$ is an embedding of a closed oriented 4-manifold M^4 into $\mathbb{R}^8$, then there is a corresponding Gauss map $\mathcal{G}_{f_0} : M^4 \longrightarrow G_4^+(\mathbb{R}^8)$. First of all, we look into the intersection of $\mathcal{G}_{f_0}(M^4)$ with **CAY** and $\overline{\mathbf{CAY}}$. If each intersection is empty, then f_0 is automatically a Cayley-free embedding. Otherwise, due to their codimensions in $G_4^+(\mathbb{R}^8)$ generically they intersect at finitely many points. Hence, we can compute the algebraic intersection numbers between them and then try to find conditions on M which may make these numbers equal to zero. Since M^4 and $G_4^+(\mathbb{R}^8)$ are compact and the dimension of $G_4^+(\mathbb{R}^8)$, which is equal to 16, is large enough, by (weak) Whitney Embedding Theorem $\mathcal{G}_{f_0}$ is (smoothly) homotopic to an embedding. Furthermore, by the following result this embedding is isotopic to a smooth embedding which is transversal to **CAY** and $\overline{\mathbf{CAY}}$ in $G_4^+(\mathbb{R}^8)$ since both of **CAY** and $\overline{\mathbf{CAY}}$ are compact.

Theorem 3.1 *([2], §15.4). Let M and N be compact smooth manifolds. Let $f : M \longrightarrow W$ be smooth and let $g_0 : N \longrightarrow W$ be a smooth embedding where W is a smooth manifold. Then, there is an arbitrarily smooth homotopy of g_0 to a smooth embedding $g_1 : N \longrightarrow W$ such that $f \pitchfork g_1$. Indeed, the homotopy can be taken to be smooth and such that each $g_t : N \longrightarrow W$ is an embedding, i.e., it is an isotopy.*

As a result of this, we can consider $\mathcal{G}_{f_0}(M^4)$ as an embedded closed submanifold in $G_4^+(\mathbb{R}^8)$ intersecting with **CAY** and $\overline{\mathbf{CAY}}$ transversally in $G_4^+(\mathbb{R}^8)$. (Fig. 1.)

In order to compute the algebraic intersection numbers we use the following fundamental results proved by Shi and Zhou [8]. Their results give the generators for the free part of the certain homology and cohomology groups of $G_4^+(\mathbb{R}^8)$ with $\mathbb{Z}$ coefficients. These homology or cohomology groups may have a quite amount of torsion parts which can be seen in the case of $G_4^+(\mathbb{R}^7)$ in [1], but since we are just interested in computing the intersection numbers we state and use their results with $\mathbb{R}$ coefficients.

Theorem 3.2 *([8]). Let $E = E(4,8)$ and $F = F(4,8)$ over $G_4^+(\mathbb{R}^8)$ be the canonical vector bundles with fibers generated by vectors of the subspaces and the vectors orthogonal to the subspaces, respectively. Then we have:*

(i) $[G_4^+(\mathbb{R}^7)]$, $[G_3^+(\mathbb{R}^7)]$ and $[\mathbf{CAY}]$ are generators of $H_{12}(G_4^+(\mathbb{R}^8);\mathbb{R})$

(ii) $e(E)$, $e(F)$ and $\frac{1}{2}(p_1(E)+e(E)-e(F))$ are generators of $H^4(G_4^+(\mathbb{R}^8);\mathbb{R})$ and their Poincaré duals are $[G_4^+(\mathbb{R}^7)]$, $[G_3^+(\mathbb{R}^7)]$, $[\mathbf{CAY}]+[G_4^+(\mathbb{R}^7)]-[G_3^+(\mathbb{R}^7)] \in H_{12}(G_4^+(\mathbb{R}^8);\mathbb{R})$, respectively.

(iii) $[G_2^+(\mathbb{R}^4)]$, $[G_1^+(\mathbb{R}^5)]$, and $[G_4^+(\mathbb{R}^5)]$ are generators of $H_4(G_4^+(\mathbb{R}^8);\mathbb{R})$ and the following table shows the value of integration of certain characteristic classes on these generators.

	$[G_2^+(\mathbb{R}^4)]$	$[G_1^+(\mathbb{R}^5)]$	$[G_4^+(\mathbb{R}^5)]$
$e(E)$	0	0	2
$e(F)$	0	2	0
$p_1(E)$	2	0	0

Remark 3.3. If $\mathbb{R}^l \subset \mathbb{R}^m \subset \mathbb{R}^n$ is an inclusion of vector spaces, then we have an inclusion of $G_{(k-l)}^+(\mathbb{R}^m/\mathbb{R}^l) \cong G_{(k-l)}^+(\mathbb{R}^{m-l}) \hookrightarrow G_k^+(\mathbb{R}^n)$. All of the Grassmannians mentioned in Theorem 3.2 are given by inclusions in $G_4^+(\mathbb{R}^8)$. These inclusions are not canonical as they depend on the choice of the chain of subspaces i.e. a flag in $\mathbb{R}^8$. However, they represent the same homology class in the corresponding homology since $SO(8)$ acts on $\mathbb{R}^8$ (and on the chain of subspaces) transitively and the induced action on $G_4^+(\mathbb{R}^8)$ is homotopic to the identity map due to the fact that $SO(8)$ is path-connected.

As a result of Theorem 3.2, we easily obtain that the Poincaré dual of $[\mathbf{CAY}]$ is equal to $\frac{1}{2}(p_1(E)+e(F)-e(E))$. Furthermore, the homology class of the Gauss image of M under f_0 can be written as [8]

$$[\mathcal{G}_{f_0}(M)] = \alpha[G_4^+(\mathbb{R}^5)] + \beta[G_1^+(\mathbb{R}^5)] + \gamma[G_2^+(\mathbb{R}^4)],$$

where $\alpha = \frac{1}{2}\int_M \mathcal{G}_{f_0}{}^* e(E) = \frac{1}{2}\int_M e(TM) = \frac{1}{2}\chi_M$, $\beta = \frac{1}{2}\int_M \mathcal{G}_{f_0}{}^* e(F) = \frac{1}{2}\int_M e(NM)$ and $\gamma = \frac{1}{2}\int_M \mathcal{G}_{f_0}{}^* p_1(E) = \frac{1}{2}\int_M p_1(TM) = \frac{3}{2}\tau_M$. Since f_0 is an embedding into $\mathbb{R}^8$, the Euler class of the normal bundle NM, $e(NM)=0$ (p. 120, [7]), hence we get

$$[\mathcal{G}_{f_0}(M)] = \frac{1}{2}\chi_M[G_4^+(\mathbb{R}^5)] + \frac{3}{2}\tau_M[G_2^+(\mathbb{R}^4)].$$

Lemma 3.4. *The intersection numbers between the classes $[\mathbf{CAY}]$, $[\overline{\mathbf{CAY}}]$ and $[\mathcal{G}_{f_0}(M)]$ are as follows;*

$$[\mathcal{G}_{f_0}(M)] \bullet [\mathbf{CAY}] = \frac{1}{2}(\chi_M - 3\tau_M)$$
$$[\mathcal{G}_{f_0}(M)] \bullet [\overline{\mathbf{CAY}}] = \frac{1}{2}(\chi_M + 3\tau_M)$$

Proof. If $D_G : H_{12}(G_4^+(\mathbb{R}^8);\mathbb{R}) \longrightarrow H^4(G_4^+(\mathbb{R}^8);\mathbb{R})$ is the Poincaré duality (or its inverse) map, then

$$\begin{aligned}[\mathcal{G}_{f_0}(M)] \bullet [\mathbf{CAY}] &= \int_{[\mathcal{G}_{f_0}(M)]} D_G([\mathbf{CAY}]) \\ &= \int_{[\mathcal{G}_{f_0}(M)]} \frac{1}{2}(p_1(E)+e(F)-e(E))\end{aligned}$$

$$= \frac{1}{4}\chi_M \int\limits_{[G_2^+(\mathbb{R}^4)]} p_1(E) - \frac{3}{4}\tau_M \int\limits_{[G_4^+(\mathbb{R}^5)]} e(E)$$

$$= \frac{1}{2}(\chi_M - 3\tau_M)$$

In order to compute the second intersection number we need to find the Poincaré dual of $[\overline{\mathbf{CAY}}]$. Let $\pi : G_4^+(\mathbb{R}^8) \longrightarrow G_4(\mathbb{R}^8)$ be the smooth covering map of the unoriented Grassmannian $G_4(\mathbb{R}^8)$ and $\sigma : G_4^+(\mathbb{R}^8) \longrightarrow G_4^+(\mathbb{R}^8)$, $\xi \to -\xi$ be the deck transformation which maps an oriented 4-plane to the same 4-plane with opposite orientation. Since $G_4(\mathbb{R}^8)$ is orientable, $G_4^+(\mathbb{R}^8)$ is orientable, and σ is orientation-preserving, which implies that $\deg(\sigma) = 1$. Then, the transfer map $\sigma_! = D_G^{-1} \circ \sigma^* \circ D_G$ satisfies $\sigma_* \circ \sigma_! = \deg(\sigma)\mathrm{Id} = \mathrm{Id}$ [2, Prop. 14.1] i.e. the loop in the following diagram gives the identity.

$$\begin{array}{ccc} H_{12}(G_4^+(\mathbb{R}^8);\mathbb{R}) & \xleftarrow{\sigma_*} & H_{12}(G_4^+(\mathbb{R}^8);\mathbb{R}) \\ \downarrow D_G & & \uparrow D_G^{-1} \\ H^4(G_4^+(\mathbb{R}^8);\mathbb{R}) & \xrightarrow{\sigma^*} & H^4(G_4^+(\mathbb{R}^8);\mathbb{R}) \end{array}$$

Now, we have $\sigma \circ \sigma = \mathrm{Id}$ and $\sigma(\mathbf{CAY}) = \overline{\mathbf{CAY}}$, so $\sigma_*^{-1}([\mathbf{CAY}]) = [\overline{\mathbf{CAY}}]$. Then,

$$\begin{aligned} \sigma_* \circ \sigma_!([\mathbf{CAY}]) &= [\mathbf{CAY}] \\ \sigma_!([\mathbf{CAY}]) &= [\overline{\mathbf{CAY}}] \\ \sigma^* \circ D_G([\mathbf{CAY}]) &= D_G([\overline{\mathbf{CAY}}]) \\ \sigma^*(\frac{1}{2}(p_1(E) + e(F) - e(E)) &= D_G([\overline{\mathbf{CAY}}]) \\ \frac{1}{2}(p_1(\sigma^*E) + e(\sigma^*F) - e(\sigma^*E)) &= D_G([\overline{\mathbf{CAY}}]) \\ \frac{1}{2}(p_1(E) - e(F) + e(E)) &= D_G([\overline{\mathbf{CAY}}]) \end{aligned}$$

since for the pull-back bundles we have $\sigma^*E = -E$, $\sigma^*F = -F$ i.e. under the pull-back only the orientations of E and F change.

Hence, in a similar calculation to the one above, we can easily see that $[\mathcal{G}_{f_0}(M)] \bullet [\overline{\mathbf{CAY}}] = \frac{1}{2}(\chi_M + 3\tau_M)$ □

It can be easily seen that both of the algebraic intersection numbers are equal to zero if and only if $\chi_M = \tau_M = 0$ as the equations are linearly independent. Subject to the vanishing of these topological invariants we get the following result.

Lemma 3.5. *If $\chi_M = \tau_M = 0$, then there exists a homotopy $\Psi : [0,1] \times M \longrightarrow G_4^+(\mathbb{R}^8)$ such that $\Psi_0(M) = \mathcal{G}_{f_0}(M)$ and $\Psi_1(M) \cap$ **CAY** $= \emptyset$, $\Psi_1(M) \cap \overline{\mathbf{CAY}} = \emptyset$.*

Proof. By Lemma 3.4 we see that if $\chi_M = \tau_M = 0$, then both of the algebraic intersection numbers $[\mathcal{G}_{f_0}(M)] \bullet [\mathbf{CAY}]$ and $[\mathcal{G}_{f_0}(M)] \bullet [\overline{\mathbf{CAY}}]$ are equal to zero. Furthermore, by Corollary 3.1, the intersections between these submanifolds $\mathcal{G}_{f_0}(M)$, **CAY** and $\overline{\mathbf{CAY}}$ are transversal, so the intersection numbers can be computed by counting the finitely many intersection points with signs. Since the algebraic intersection numbers are zero, both $\mathcal{G}_{f_0}(M) \cap \mathbf{CAY}$ and $\mathcal{G}_{f_0}(M) \cap \overline{\mathbf{CAY}}$ contain the same number of positive intersection points and negative ones. Now, similar to the co-associative case in G_2-manifolds [12], we apply *Whitney trick* to geometrically eliminate algebraically-canceling pairs of intersection points. In order to do this we

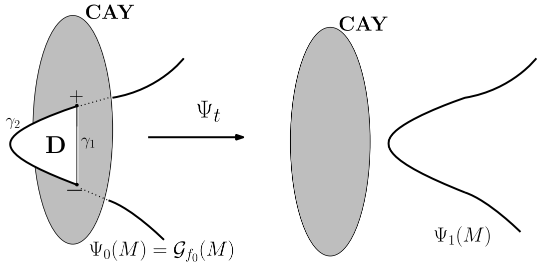

Fig. 2. Whitney trick.

form a *Whitney disk* for each pair of intersection points with opposite signs as follows. Take two paths, γ_1 in **CAY** and γ_2 in $\mathcal{G}_{f_0}(M)$ connecting a pair, together they give a loop in $G_4^+(\mathbb{R}^8)$. $G_4^+(\mathbb{R}^8)$ is simply connected, furthermore, $G_4^+(\mathbb{R}^8) \setminus (\mathcal{G}_{f_0}(M) \cup \mathbf{CAY})$ is also simply connected since codimensions of $\mathcal{G}_{f_0}(M)$ and **CAY** are bigger than 3. Therefore, this loop is homotopically trivial in $G_4^+(\mathbb{R}^8) \setminus (\mathcal{G}_{f_0}(M) \cup \mathbf{CAY})$ and it bounds an embedded disk as the dimension of $G_4^+(\mathbb{R}^8)$ is bigger than 5. By using this *Whitney disk* we push $\mathcal{G}_{f_0}(M)$ past **CAY** until both intersection points are removed (see Fig. 2). By applying this process for each pair in $\mathcal{G}_{f_0}(M) \cap \mathbf{CAY}$ and $\mathcal{G}_{f_0}(M) \cap \overline{\mathbf{CAY}}$ we remove all the intersection points. As a result, the method of Whitney trick gives us a homotopy (actually, an isotopy) $\Psi : [0,1] \times M \longrightarrow G_4^+(\mathbb{R}^8)$ of the Gauss map $\mathcal{G}_{f_0} : M \longrightarrow G_4^+(\mathbb{R}^8)$ such that $\Psi_1(M) \cap \mathbf{CAY} = \emptyset$, $\Psi_1(M) \cap \overline{\mathbf{CAY}} = \emptyset$ as the all the intersections are removed. (Fig. 2.) □

By using the trivial fibration $\pi_2 : \tilde{G}_4(\mathbb{R}^8) \longrightarrow G_4^+(\mathbb{R}^8)$, where $\tilde{G}_4(\mathbb{R}^8) \cong \mathbb{R}^8 \times \tilde{G}_4(\mathbb{R}^8)$, the homotopy Ψ in Lemma 3.5 lifts to a homotopy $\tilde{\Psi} : [0,1] \times M \longrightarrow \tilde{G}_4(\mathbb{R}^8)$ such that $\tilde{\Psi}_1 : M \longrightarrow [\mathbb{R}^8 \times (\mathbf{CAY} \sqcup \overline{\mathbf{CAY}})^c] \subset \tilde{G}_4(\mathbb{R}^8)$. $[\mathbb{R}^8 \times (\mathbf{CAY} \sqcup \overline{\mathbf{CAY}})^c]$ is an open subset of $\tilde{G}_4(\mathbb{R}^8)$ and the corresponding differential relation, in the terminology of h-principle, is exactly equal to $\mathcal{R}_{\Phi\text{-}free} \subset J^1(M, \mathbb{R}^8)$. It is shown in [12], Theorem 3.1 that $\mathcal{R}_{\Phi\text{-}free}$ is ample. Hence, by the following theorem, there is an isotopy of f_0 to a Cayley-free embedding $f_1 : M \longrightarrow \mathbb{R}^8$ when χ_M and τ_M are equal to zero and this finishes the proof of the main theorem.

Theorem 3.6 *([3], §19.4.1). Suppose that $A \subset \widetilde{G}_p(X^n)$ is an open subset and the corresponding open differential relation $\mathcal{R}_A \subset J^1(M, X)$ is ample. Then every embedding $f_0 : M \longrightarrow X$ whose tangential lift i.e. the corresponding Gauss map*

$$\mathcal{G}_{f_0} = \mathcal{G}_0 : M \longrightarrow \widetilde{G}_p(X^n)$$

is homotopic over M to a map $\mathcal{G}_1 : M \longrightarrow A$ can be isotoped to an A-directed embedding $f_1 : M \longrightarrow X$.

For any arbitrary subset $A \subset \widetilde{G}_p(X^n)$, an embedding (or immersion) $f : M^p \longrightarrow X^n$ is called *A-directed* if the corresponding Gauss map sends M into A and the differential relation in $\mathcal{R}_{imm} \subset J^1(M, X)$ corresponding to A-directed immersions is denoted by $\mathcal{R}_A$ [3].

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