

# Born-Infeld Gravity with a Unique Vacuum and a Massless Graviton

İbrahim Güllü,<sup>1,\*</sup> Tahsin Çağrı Şişman,<sup>2,†</sup> and Bayram Tekin<sup>3,‡</sup>

<sup>1</sup>*Department of Physics,*

*Middle East Technical University, 06800, Ankara, Turkey*

<sup>2</sup>*Department of Astronautical Engineering,*

*University of Turkish Aeronautical Association, 06790 Ankara, Turkey*

<sup>3</sup>*Department of Physics,*

*Middle East Technical University, 06800 Ankara, Turkey*

(Dated: November 7, 2018)

We construct an  $n$ -dimensional Born-Infeld type gravity theory that has the same properties as Einstein's gravity in terms of the vacuum and particle content: Namely, the theory has a unique viable vacuum (maximally symmetric solution) and a single massless unitary spin-2 graviton about this vacuum. The BI gravity, in some sense, is the most natural, minimal generalization of Einstein's gravity with a better UV behavior, and hence, is a potentially viable proposal for low energy quantum gravity. The Gauss-Bonnet combination plays a non-trivial role in the construction of the theory. As an extreme example, we consider the infinite dimensional limit where an interesting exponential gravity arises.

## I. INTRODUCTION

Recently in [1], we have constructed a Born-Infeld type (BI) gravity theory in the metric formulation with the Lagrangian density  $\mathcal{L} = \sqrt{\det(g_{\mu\nu} + \gamma A_{\mu\nu})}$  in  $3 + 1$ -dimensions that has the following properties:

1. The theory is minimal in the sense that the tensor  $A_{\mu\nu}$  is constructed from the powers of the Riemann tensor up to quadratic order, but it does not have the derivatives of the Riemann tensor as in its electrodynamics analogues with the Lagrangian density  $\sqrt{\det(g_{\mu\nu} + b F_{\mu\nu})}$ . It is important to note that linear theories of the form such as  $\sqrt{\det(g_{\mu\nu} + \gamma R_{\mu\nu})}$  do not yield unitary excitations about their maximally symmetric vacua, and hence, one has to include at least the quadratic terms in the curvature [2].
2. The theory reduces to the cosmological Einstein's theory at the lowest order in the small curvature expansion about the flat space or the (anti)-de Sitter [(A)dS], and to the Einstein-Gauss-Bonnet (EGB) theory at the quadratic order.
3. The theory describes only massless gravitons around its flat or (A)dS vacuum at any finite (truncated) order in the curvature expansion or as a full theory (namely, if all powers of curvature are included).
4. The theory has a unique viable vacuum: namely, there is a single maximally symmetric solution for which the massless spin-2 excitation about this solution is unitary.

In the current work, we extend the discussion to  $n$ -dimensional spacetimes where  $n \geq 4$ , and construct BI gravities that have the above properties. [We have studied the  $n = 3$  case in [3, 4]

\*Electronic address: ibrahimgullu2002@gmail.com

†Electronic address: tahsin.c.sisman@gmail.com

‡Electronic address: btekin@metu.edu.tr

which already yields a nice theory at the linear level in the curvature inside the determinant.] Some of the discussion in generic  $n$ -dimensions is similar to the four dimensional case, but as we shall show, there are nontrivial complications beyond four dimensions for a theory to satisfy the above properties.  $n = 3 + 1$  dimension is highly special in the sense that the requirement for the existence of a maximally symmetric vacuum and the requirement for the theory be tree-level unitary are equal only in this particular dimension but yield different constraints in all other dimensions.

General Relativity with its UV and IR problems is at best an effective theory which is expected to be modified. The best scenario is that there exists a quantum theory of gravity, perhaps a theory of strings, from which one constructs low energy gravity theories at any desired order in perturbation theory in powers of curvature which will be of the form

$$I = \int d^4x \left\{ \frac{1}{\kappa} (R - 2\Lambda_0) + \sum_{r=2}^{\infty} a_r (\text{Riem}, \text{Ric}, R, \nabla \text{Riem}, \dots)^r \right\}. \quad (1)$$

In reality, it is extremely complicated to compute the relevant terms in this effective quantum gravity action from the microscopic theory beyond several lower order terms. We, then here, suggest an alternative bottom-up approach and construct effective quantum gravity actions which have the good properties of the cosmological Einstein theory noted above as well as a better UV behavior. Up to now, in the literature, low energy quantum gravity actions have been constructed basically on the principles that they be diffeomorphism invariant, ghost-free, and sometimes supersymmetric. Diffeomorphism invariance is easy to satisfy, and hence, does not much constrain the theory. Ghost-freedom and supersymmetry are hard to satisfy conditions and so there are only a few theories with low powers of curvature,  $R^2$ ,  $R^3$ , and at best  $R^4$ , that satisfy these constraints. Our point of view here comes from the observation that cosmological Einstein's theory has two more crucial properties: *uniqueness of its maximally symmetric vacuum* and *unitarity of its single massless graviton* about this vacuum. Once, more powers of curvature are added to the Einstein-Hilbert action, these two properties are immediately lost [5]. Non-uniqueness of the maximally symmetric vacuum in gravity is highly troublesome since each solution is a spacetime on its own with different asymptotic structures and there would be no way to choose one vacuum over the other [6]. Therefore, as a principle of constructing low energy quantum gravity theories, besides the diffeomorphism invariance, we impose that the theory should have a unique maximally symmetric vacuum about which the only excitation is a unitary spin-2 graviton just like the cosmological Einstein's theory. A priori, these conditions might appear tremendously difficult to satisfy since an action of the form (1) with all possible terms at every order will yield practically intractable expressions. Therefore, we will use the Born-Infeld construction which limits the possible terms as well as fixes the arbitrary numerical factors at each order. The fact that all the desired properties of Einstein's theory can be kept intact in a Born-Infeld type theory is quite remarkable. Especially, the fact that the theory has only a non-ghost massless graviton about a unique vacuum is highly desirable.

Construction of unitary "minimal" BI gravity turned out to be highly non-trivial in four dimensions. For example, the Gauss-Bonnet (GB) term, being a topological invariant, does not change the classical equations of motion in four dimensions, plays a vital role in the BI theory: without the GB term, one cannot build unitary actions of the type described above.

The layout of the paper is as follows: in Section-II, we describe the generic BI theory to be studied and briefly recapitulate the vacua and the spectrum of the EGB theory, show that out of its two possible vacua one of them is unstable due to a ghost massless graviton. In Section-III, we study the vacuum and the particle spectrum about the vacuum for the BI gravity. In Section-IV we discuss the unitarity of BI gravity about the flat space which is also relevant for the unitarity of the theory at  $O(R^2)$  about its (A)dS vacuum. In Section-V, we study the unitarity of the theory in (A)dS backgrounds which requires calculating the effects of all powers of curvature, we

also construct the infinite dimensional BI gravity. We relegate some of the technical parts to the Appendices. For example, in Appendix C, we prove the uniqueness of the viable vacuum.

## II. CONSTRUCTING THE BORN-INFELD ACTION

The theory we shall consider is defined by the action

$$I = \frac{2}{\kappa\gamma} \int d^n x \left[ \sqrt{-\det(g_{\mu\nu} + \gamma A_{\mu\nu})} - (\gamma\Lambda_0 + 1) \sqrt{-g} \right], \quad (2)$$

where  $\gamma$  is a dimensionful parameter (the BI parameter). Defining

$$\mathcal{W}_{\mu\nu} \equiv C_{\mu\rho\alpha\beta} C_{\nu}{}^{\rho\alpha\beta}, \quad \mathcal{W} \equiv g^{\mu\nu} \mathcal{W}_{\mu\nu} \quad (3)$$

where  $C_{\mu\alpha\nu\beta}$  is the Weyl tensor, the most general form of the two tensor  $A_{\mu\nu}$  at quadratic order can be written as

$$\begin{aligned} A_{\mu\nu} = & R_{\mu\nu} + \beta S_{\mu\nu} + \gamma \left( a_1 \mathcal{W}_{\mu\nu} + a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} + a_3 R_{\mu\rho} R_{\nu}^{\rho} + a_4 S_{\mu\rho} S_{\nu}^{\rho} \right) \\ & + \frac{\gamma}{n} g_{\mu\nu} \left( b_1 \mathcal{W} + b_2 R_{\rho\sigma}^2 + b_3 S_{\rho\sigma}^2 \right). \end{aligned} \quad (4)$$

Here and from now on, for brevity we shall denote  $R_{\rho\sigma}^2 = R_{\rho\sigma} R^{\rho\sigma}$ . As mentioned in the Introduction, one has to include in  $A_{\mu\nu}$  at least the quadratic terms to find unitary theories. Here,  $S_{\mu\nu}$  is the traceless-Ricci tensor defined as  $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R$ , and the constants  $\beta$ ,  $a_i$ , and  $b_i$  are dimensionless. Note that we have not included the term  $R_{\mu\nu} S^{\mu\nu}$  since it can be written as

$$R_{\mu\rho} S_{\nu}^{\rho} = \frac{1}{2} R_{\mu\rho} R_{\nu}^{\rho} + \frac{1}{2} S_{\mu\rho} S_{\nu}^{\rho} - \frac{1}{2n} g_{\mu\nu} \left( R_{\rho\sigma}^2 - S_{\rho\sigma}^2 \right). \quad (5)$$

Therefore, all possible linearly independent terms are included in (4). In four dimensions, due to the identity  $\mathcal{W}_{\mu\nu} = \frac{1}{4} g_{\mu\nu} \mathcal{W}$ , one can do away  $a_1$ , but this is not possible in generic  $n$  dimensions. Here, we shall mostly work with the Weyl-traceless-Ricci-Ricci (CSR) basis instead of the Riemann-Ricci-curvature-scalar (RRR) basis. For the advantage of the CSR basis over the RRR basis in the discussion of the spectrum around the constant curvature backgrounds see [1]. Nevertheless, since it is sometimes needed to work in the RRR basis, we give the non-trivial transformations between these two bases in the Appendix A.

It is clear that with 8 dimensionless parameters, the theory (4) is too general to be of much use. So, our task is to first constrain some of these parameters by using physical arguments, which is the main goal of this work. In addition, one should also worry about the appearance of  $\gamma$ : it could be considered as a new parameter of Nature that appears in quantum gravity; and hence, related to the string tension or one could also use the Newton's constant instead of  $\gamma$  if one wants to keep only one single dimensionful parameter in the theory. As discussed in [1], this will be valid as long as the curvature satisfies  $R \ll 1/\ell_p^2$ , where  $\ell_p$  is the Planck length.

Now, let us consider how to constrain the most general action (2): we first require that the theory has a unique maximally symmetric vacuum about which the only excitation is a single massless unitary spin-2 graviton. Since this issue is quite subtle, let us expound on it: the theory in principle could have many maximally symmetric solutions, but only one of them is viable in the sense that excitations about the nonviable vacua are ghost-like while the viable vacuum has the

desired excitation of a unitary massless graviton. These two properties as mentioned above, having a unique vacuum and a massless graviton, are the properties of cosmological Einstein's theory; hence, one may expect that at the free theory level, the general Born-Infeld gravity should have the same properties as Einstein's theory. As we shall show, this is actually a strong condition which cannot be satisfied. But, the weaker condition that the BI theory has the same free-level properties as the EGB theory can be satisfied without any phenomenological difference in four dimensions. Beyond four dimensions, Newton's constant receives corrections due to the cosmological constant.

We will show that the above mentioned constraints reduces the number of dimensionless parameters to four and the  $A_{\mu\nu}$  tensor becomes

$$A_{\mu\nu} = R_{\mu\nu} + \beta S_{\mu\nu} + \gamma \left( a_1 \mathcal{W}_{\mu\nu} + a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} + \frac{\beta+1}{4} R_{\mu\rho} R_{\nu}^{\rho} + a_4 S_{\mu\rho} S_{\nu}^{\rho} \right) + \frac{\gamma}{n} g_{\mu\nu} \left[ \left( \frac{(n-1)^2}{4(n-2)(n-3)} - a_1 \right) \mathcal{W} - \frac{\beta}{4} R_{\rho\sigma}^2 + \left( \frac{\beta(\beta+2)}{2} + \frac{n(4-3n)}{4(n-2)^2} - a_4 \right) S_{\rho\sigma}^2 \right]. \quad (6)$$

Hence, with this  $A_{\mu\nu}$ , the BI gravity given by the action (2) has a single massless unitary graviton about its unique vacuum just like Einstein's theory. To determine the four remaining dimensionless parameters which are not fixed by the condition that the theory has a single unitary massless spin-2 graviton, further conditions such as causality, supersymmetry, the existence of the spherically symmetric solutions and cosmologically viable solutions could be imposed. We will consider these in a separate work. In the absence of further constraints, one can entertain the idea of obtaining simpler theories by setting the constants to particular values. As an example of such a minimal theory, let us consider (6) and set  $\beta = -1$ ,  $a_1 = 0$ ,  $a_2 = 0$ ,  $a_4 = 0$ , which yield a theory without dimensionless parameters and  $A_{\mu\nu}$  can be recast as

$$A_{\mu\nu} = \frac{g_{\mu\nu}}{n} \left( R + \gamma \frac{(n-1)^2}{4(n-2)(n-3)} \chi_{GB} - \gamma \frac{(n-2)}{4n} R^2 \right), \quad (7)$$

where the GB combination is defined as  $\chi_{GB} = R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2$ . Therefore, the full Lagrangian density of this BI theory becomes

$$\mathcal{L} = \frac{2}{\kappa\gamma} \left( \left( 1 + \frac{\gamma}{n} R + \frac{\gamma^2 (n-1)^2}{4n(n-2)(n-3)} \chi_{GB} - \frac{\gamma^2 (n-2)}{4n^2} R^2 \right)^{\frac{n}{2}} - \lambda_0 - 1 \right), \quad (8)$$

where  $\lambda_0 \equiv \gamma\Lambda_0$  is the dimensionless bare cosmological constant. In even dimensions one can get rid off the square root and so this BI gravity becomes a specific  $|\text{Riem}|^n$  theory. Its four dimensional version, which is a  $|\text{Riem}|^4$  was given in [1].

Since the EGB theory will play a major role, let us briefly discuss its properties here. In  $n$  dimensions, the most general quadratic action that describes *only* massless spin-2 excitations around its flat or AdS vacuum is the EGB theory with the Lagrangian

$$\kappa\mathcal{L} = R - 2\Lambda_0 + \gamma\chi_{GB}. \quad (9)$$

Flat space is a vacuum for  $\Lambda_0 = 0$  and there are generically two (A)dS vacua with  $\Lambda_{\pm} = -\frac{1}{4\kappa f} (1 \pm \sqrt{1 + 8\kappa f \Lambda_0})$ , where  $f = \gamma \frac{(n-3)(n-4)}{(n-1)(n-2)}$ . We can rewrite the Lagrangian in terms of the Weyl tensor, the Ricci tensor, and the traceless-Ricci tensor as

$$\kappa\mathcal{L} = R - 2\Lambda_0 + \gamma \left[ \mathcal{W} + \frac{(n-3)(n-2)}{(n-1)} \left( R_{\mu\nu}^2 - \frac{n^2}{(n-2)^2} S_{\mu\nu}^2 \right) \right], \quad (10)$$

where we have used the identity

$$\mathcal{W} = R_{\mu\nu\rho\sigma}^2 - \frac{4}{n-2}R_{\mu\nu}^2 + \frac{2}{(n-1)(n-2)}R^2. \quad (11)$$

To understand the particle content of the EGB theory, one linearizes the field equations about one of its (A)dS vacua to get

$$\frac{1}{\kappa_e}\mathcal{G}_{\mu\nu}^L = 0, \quad (12)$$

where the effective Newton's constant is  $\frac{1}{\kappa_e} = \frac{1}{\kappa} + 4\Lambda f$ , and  $\mathcal{G}_{\mu\nu}^L$  is the linearized Einstein tensor which reduces to  $\mathcal{G}_{\mu\nu}^L = -\frac{1}{2}\left(\bar{\square} - \frac{4\Lambda}{(n-1)(n-2)}\right)h_{\mu\nu}$  for the transverse-traceless perturbations,  $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ . Therefore, (10) describes a unitary massless spin-2 graviton as long as  $\frac{1}{\kappa_e} > 0$ . Let us compare this condition with the condition that there be a maximally symmetric solution. For the latter, one needs

$$1 + 8\kappa f\Lambda_0 \geq 0, \quad (13)$$

for the former one has

$$\frac{1}{\kappa} + 4\Lambda f > 0. \quad (14)$$

We assume  $\kappa > 0$ , therefore once the value of  $\Lambda$  is plugged, the second condition yields

$$\mp \sqrt{1 + 8\kappa f\Lambda_0} > 0, \quad (15)$$

which is not possible for the  $\Lambda_+$  branch; namely, massless spin-2 excitation is ghostlike about this vacuum but unitary for the  $\Lambda_-$  vacuum. The lesson we learn from this exercise is that not all vacua are stable or, in our terminology, viable which will be the case for generic BI gravity that we shall discuss. Note that the  $\Lambda_0 = 0$  case was already noted in [7].

### III. VACUUM AND SPECTRUM OF THE BI THEORY

For generic gravity theories of the Born-Infeld type, it is a cumbersome task to find the vacua and the particle spectrum about any of the vacuum using the conventional techniques such as finding the field equations and linearizing them. As we discussed in [1, 8] there are short-cuts to these computations. Let us briefly recall these short-cuts here for the sake of completeness: Consider a generic action of the form

$$I = \int d^n x \sqrt{-g} f\left(R_{\alpha\beta}^{\mu\nu}\right), \quad (16)$$

where  $f$  is a smooth function of its argument. [Note that the function  $f$  could depend on any arbitrary covariant derivative of the Riemann tensor which will not alter the following discussion of finding the vacuum, but it will of course change the discussion of the particle spectrum about the vacuum.] In order to find the maximally symmetric solutions to this theory one constructs the so called ‘‘equivalent linear action’’ (ELA),  $I_{\text{ELA}}$ , that has the same vacua as (16):

$$I_{\text{ELA}} = \frac{1}{\kappa_l} \int d^n x \sqrt{-g} (R - 2\Lambda_{0,l}). \quad (17)$$

Here,  $l$  denotes the ELA values and one has

$$\frac{1}{\kappa_l} = \zeta, \quad \frac{\Lambda_{0,l}}{\kappa_l} = -\frac{1}{2}\bar{f} + \frac{n\Lambda}{n-2}\zeta, \quad (18)$$

where  $\zeta$  is defined as

$$\left[ \frac{\partial f}{\partial R_{\rho\sigma}^{\lambda\nu}} \right]_{\bar{R}_{\rho\sigma}^{\mu\lambda}} R_{\rho\sigma}^{\lambda\nu} \equiv \zeta R, \quad (19)$$

and the barred quantities are evaluated at the maximally symmetric vacuum given as

$$\bar{R}_{\rho\sigma}^{\mu\lambda} = \frac{2\Lambda}{(n-1)(n-2)} \left( \delta_{\rho}^{\mu} \delta_{\sigma}^{\lambda} - \delta_{\sigma}^{\mu} \delta_{\rho}^{\lambda} \right), \quad (20)$$

for example,  $\bar{f} \equiv f(\bar{R}_{\rho\sigma}^{\mu\lambda})$ . From (17), one sets  $\Lambda = \Lambda_{0,l}$  which then reduces (18) to a compact expression

$$\Lambda = \frac{n-2}{4\zeta} \bar{f}. \quad (21)$$

Solutions of this equation are the possible vacua of the theory. Once the vacua are found, one can consider fluctuations about these vacua. This amounts to finding the  $O(h_{\mu\nu}^2)$  action which is usually very complicated. Again, fortunately, there is a similar short-cut method given in [10] which relies on finding an “equivalent quadratic curvature action (EQCA)” that has the same propagator structure as (16). Since these matters are discussed at length in our previous paper [1], in what follows we shall only quote the final results.

### A. Determining the Vacua of the BI Theory

The equivalent linearized action of the BI theory given in (2) is

$$\kappa \mathcal{L}_{\text{ELA}} = \frac{1}{\kappa_l} \left( R - \frac{2}{\gamma} \lambda_{0,l} \right), \quad (22)$$

where we have defined a dimensionless Newton’s constant  $\kappa_l$  and a dimensionless cosmological parameter  $\lambda_{0,l} \equiv \gamma \Lambda_{0,l}$  which can be found as

$$\kappa_l = \frac{\bar{b}^{\frac{(2-n)}{2}}}{1 + \frac{2\gamma\bar{R}}{n}c}, \quad \lambda_{0,l} = \kappa_l \left( 1 + \lambda_0 - \bar{b}^{\frac{n}{2}} \right) + \frac{\gamma\bar{R}}{2}, \quad \bar{b} \equiv 1 + \frac{\gamma\bar{R}}{n} + \left( \frac{\gamma\bar{R}}{n} \right)^2 c, \quad (23)$$

here we have defined  $c \equiv a_3 + b_2$ . Then, since the vacua of  $\mathcal{L}_{\text{ELA}}$  is given by the equation

$$\lambda = \lambda_{0,l}, \quad (24)$$

with the use of  $\gamma\bar{R} = \frac{2n\lambda}{n-2}$ , one arrives at the algebraic equation that determines the possible maximally symmetric vacua

$$(\lambda_0 + 1) \left( cx^2 + x + 1 \right)^{\frac{2-n}{2}} + cx^2 - 1 = 0, \quad (25)$$

where  $x \equiv \frac{2\lambda}{n-2}$  and  $\lambda_0 \neq -1$ . This will be one of the main equations that we shall study in constraining the theory. For generic  $n$  dimensions, the equation cannot be solved explicitly, but this is not required: we will show that there is a unique solution consistent with the unitarity of the theory. Namely, there is an interval for  $\lambda_0$  which will yield a single real  $\lambda$  consistent with the unitarity of the theory. This is by itself a rather remarkable result since a priori (25) could have many distinct real solutions which will correspond to possible universes out of which ours cannot be identified on the basis of energy comparison. The possible solutions of (25) is a somewhat technical analysis for which we devote Appendix-C to it.

#### IV. UNITARITY AROUND FLAT BACKGROUNDS

In principle, BI gravity is valid in both weak and strong gravity regimes. Once the BI action is considered as an expansion in curvature, that is in  $\gamma R$ , depending on how well the inequality  $\gamma R \ll 1$  is satisfied, the relevant number of terms in the curvature expansion of the BI theory changes. Thus, depending on the strength of gravitational field under investigation, a truncated version of the curvature expansion of the BI action can serve as an effective theory in that gravitational regime.

Now, considering BI theory is the main description of gravity, a natural question is the unitarity of the excitations about the flat background. To have such a unitarity analysis for the whole BI theory or its any truncated order, one only needs to consider the terms up to  $O(R^2)$  in the curvature expansion since the higher order terms do not have any contribution to the free theory, that is the  $O(h_{\mu\nu}^2)$  action about flat backgrounds. For the flat space  $\lambda = 0$  so we must set  $\lambda_0 = 0$  as is clear from (25). Then, expanding (2) up to  $O(R^2)$  with the help of the Taylor series expansion

$$\left(\det(1+M)\right)^{1/2} = 1 + \frac{1}{2}\text{Tr}M + \frac{1}{8}(\text{Tr}M)^2 - \frac{1}{4}\text{Tr}(M^2) + O(M^3), \quad (26)$$

one arrives at

$$\begin{aligned} \kappa\mathcal{L}_{O(R^2)} = & R + \gamma(a_1 + b_1)\mathcal{W} + \gamma\left(a_3 + b_2 - \frac{1}{2} + \frac{n}{4}\right)R_{\mu\rho}^2 \\ & + \gamma\left(a_4 + b_3 - \frac{n}{4} - \frac{\beta(\beta+2)}{2}\right)S_{\mu\rho}^2. \end{aligned} \quad (27)$$

There are two possibilities now: one can either demand that the quadratic terms vanish and one arrives at the Einstein theory or the quadratic terms combine into the Gauss-Bonnet form yielding the EGB theory. As we discussed before, these are the only theories that have massless spin-2 gravitons and therefore both of these possibilities must be separately analyzed. It will turn out that both of these possibilities are viable as far as the flat space unitarity is concerned but in what follows we will see that only the EGB reduction will yield a viable theory when unitarity around the (A)dS space is studied. But, that discussion requires the contributions of all the possible  $O(R^k)$  terms with  $k = 1, 2, 3, \dots, \infty$ .

##### A. Reduction to the Einstein theory

We will reduce (27) to

$$\kappa\mathcal{L} = R, \quad (28)$$

which requires the elimination of the quadratic terms which is possible if the following identifications are made

$$a_1 = -b_1, \quad c = \frac{1}{2} - \frac{n}{4}, \quad a_4 = \frac{\beta(\beta+2)}{2} + \frac{n}{4} - b_3, \quad (29)$$

reducing the  $A_{\mu\nu}$  tensor to a 5 parameter theory

$$\begin{aligned}
A_{\mu\nu} = & R_{\mu\nu} + \beta S_{\mu\nu} + \gamma \left( -b_1 \mathcal{W}_{\mu\nu} + a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} \right) \\
& + \gamma \left[ \left( \frac{1}{2} - \frac{n}{4} - b_2 \right) R_{\mu\rho} R_{\nu}^{\rho} + \left( \frac{\beta(\beta+2)}{2} + \frac{n}{4} - b_3 \right) S_{\mu\rho} S_{\nu}^{\rho} \right] \\
& + \frac{\gamma}{n} g_{\mu\nu} \left( b_1 \mathcal{W} + b_2 R_{\rho\sigma}^2 + b_3 S_{\rho\sigma}^2 \right). \tag{30}
\end{aligned}$$

Equation (29) constitute the constraints of the theory to be unitary about its flat vacuum. As expected these constraints are not very restrictive. But as we mentioned above, if one requires the  $O(R^2)$  expansion of the BI theory to be unitary around the (A)dS vacuum, then one has the same set of constraints. Therefore, these constraints will be used later on.

### B. Reduction to the Einstein–Gauss–Bonnet theory

We will compare the (27) with the EGB theory

$$\kappa \mathcal{L} = R + \gamma \left[ \mathcal{W} + \frac{(n-3)(n-2)}{(n-1)} \left( R_{\mu\nu}^2 - \frac{n^2}{(n-2)^2} S_{\mu\nu}^2 \right) \right]. \tag{31}$$

Comparison of (27) and (31) yield the following identifications

$$c = \frac{(n-2)(n-3)}{(n-1)} (a_1 + b_1) + \frac{2-n}{4}, \tag{32}$$

$$a_4 = \frac{\beta(\beta+2)}{2} - \frac{n^2(n-3)}{(n-1)(n-2)} (a_1 + b_1) - b_3 + \frac{n}{4}, \tag{33}$$

eliminating two of the parameters. Let us now turn to the thornier issue of satisfying the tree-level unitarity around the (A)dS backgrounds.

## V. UNITARITY AROUND (A)dS BACKGROUNDS

In (A)dS backgrounds, unlike the flat space case infinitely many terms contribute to the propagator and the free theory for the generic  $n$ -dimensional BI gravity. Therefore, as explained above, we need the equivalent quadratic curvature theory of

$$\kappa \mathcal{L} = \frac{2}{\gamma} \left( \sqrt{\det(\delta_{\nu}^{\rho} + \gamma A_{\nu}^{\rho})} - (\lambda_0 + 1) \right), \tag{34}$$

which can be found after a Taylor series expansion whose details are given in Appendix-B. After a long computation, one finally arrives at

$$\kappa \mathcal{L}_{\text{EQCA}} = \frac{1}{\tilde{\kappa}} \left( R - \frac{2}{\gamma} \tilde{\lambda}_0 + \alpha_1 \mathcal{W} + \alpha_2 R_{\mu\nu}^2 + \alpha_3 S_{\mu\nu}^2 \right). \tag{35}$$



Let us again note the relation between (34) and (35):  $O(h_{\mu\nu})$  and  $O(h_{\mu\nu}^2)$  expansions of (34) and (35) are identical (they differ at  $O(h_{\mu\nu}^3)$ ) granted that the effective parameters, that are  $\tilde{\kappa}$ ,  $\tilde{\lambda}_0$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$ , satisfy certain relations as derived in Appendix-B and reproduced below. The dimensionless Newton's constant and the dimensionless bare cosmological constant of the EQCA theory are given as

$$\frac{1}{\tilde{\kappa}} = \bar{b}^{\frac{(n-4)}{2}} \left( \bar{b} - \lambda \left( 1 + \frac{4\lambda}{n-2} c \right)^2 \right), \quad (36)$$

$$\tilde{\lambda}_0 = \tilde{\kappa} \left( \frac{n\lambda}{2(n-2)} \bar{b}^{\frac{(n-2)}{2}} \left( 1 + \frac{4\lambda}{n-2} c \right) + 1 - \bar{b}^{n/2} + \lambda_0 \right) + \frac{n\lambda}{2(n-2)}, \quad (37)$$

where with  $\gamma\bar{R} = \frac{2n\lambda}{n-2}$ ,  $\bar{b}$  of (23) simply takes the form

$$\bar{b} = 1 + \frac{2\lambda}{n-2} \left( 1 + \frac{2\lambda}{n-2} c \right). \quad (38)$$

The coefficients of the quadratic parts are given as

$$\alpha_1 = \gamma\tilde{\kappa} (a_1 + b_1) \bar{b}^{\frac{(n-2)}{2}}, \quad \alpha_2 = \frac{(n-2)\gamma}{4\lambda} \left[ \tilde{\kappa} \bar{b}^{\frac{(n-2)}{2}} \left( 1 + \frac{4\lambda}{n-2} c \right) - 1 \right], \quad (39)$$

$$\alpha_3 = \frac{(n-2)\gamma}{4\lambda} \left[ \tilde{\kappa} \bar{b}^{\frac{(n-4)}{2}} \left( \bar{b} \left( \frac{4\lambda}{n-2} (a_4 + b_3) - 1 \right) - \frac{2\lambda}{n-2} \left( \frac{4\lambda}{n-2} a_3 + \beta + 1 \right)^2 \right) + 1 \right], \quad (40)$$

With the EQCA, (35), in our hands, we can now study the unitarity of the BI gravity. Once again we have two options: we can demand that this EQCA matches that of cosmological Einstein theory or that of cosmological EGB theory.

### A. Reduction to the Einstein theory

To reduce (35) to the cosmological Einstein theory we must set the quadratic terms to zero:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0. \quad (41)$$

Recalling the constraints coming from the unitarity in flat backgrounds (now these are equivalent to the unitarity of the  $O(R^2)$  theory around (A)dS background)

$$a_1 = -b_1, \quad c = \frac{2-n}{4}, \quad a_4 = \frac{\beta(\beta+2)}{2} + \frac{n}{4} - b_3. \quad (42)$$

It is clear that  $\alpha_1 = 0$  is automatically satisfied. On the other hand,  $\alpha_2 = 0$  gives another constraint on the parameters of the theory as

$$\frac{(n-2)\gamma}{4\lambda} \left[ \tilde{\kappa} \bar{b}^{\frac{(n-2)}{2}} \left( 1 + \frac{4\lambda}{n-2} c \right) - 1 \right] = 0. \quad (43)$$

Since  $c = \frac{2-n}{4}$ , (38) becomes

$$\bar{b} = 1 + \frac{\lambda(2-\lambda)}{n-2}. \quad (44)$$

Plugging this  $\bar{b}$  in (43) yields  $\lambda = 2$ , or  $\lambda = 0$  *independent of the number of dimensions*. Since we already discussed the  $\lambda = 0$  case, let us consider the  $\lambda = 2$  case which yields  $\bar{b} = 1$  and  $\tilde{\kappa} = -1$  which conflicts the unitarity of the theory since the massless particle is a ghost. This basically says that the BI theory cannot have vanishing quadratic terms: hence, reduction to the unitary Einstein's theory is not possible in any dimensions.

## B. Reduction to the Einstein–Gauss–Bonnet theory

To simplify somewhat lengthy discussion, let us first recast the equivalent quadratic action of the BI theory in the EGB form plus additional quadratic curvature terms, after making use of the constraints coming from the (A)dS unitarity of the theory at  $O(R^2)$  which boils down to the unitarity around the flat background. The conditions are (32) and (33). In order to investigate the theory in (A)dS backgrounds, our starting Lagrangian is

$$\kappa \mathcal{L}_{\text{EQCA}} = \frac{1}{\tilde{\kappa}} \left[ R - \frac{2}{\gamma} \tilde{\lambda}_0 + \alpha_1 \left( \mathcal{W} + \frac{(n-3)}{(n-2)(n-1)} \left( (n-2)^2 R_{\mu\nu}^2 - n^2 S_{\mu\nu}^2 \right) \right) + \tilde{\alpha}_2 R_{\mu\nu}^2 + \tilde{\alpha}_3 S_{\mu\nu}^2 \right], \quad (45)$$

where

$$\tilde{\alpha}_2 \equiv \alpha_2 - \frac{(n-2)(n-3)}{(n-1)} \alpha_1, \quad \tilde{\alpha}_3 \equiv \alpha_3 + \frac{(n-3)n^2}{(n-2)(n-1)} \alpha_1. \quad (46)$$

Once again unitarity is achieved by setting  $\tilde{\alpha}_2 = \tilde{\alpha}_3 = 0$  and  $\tilde{\kappa} > 0$ . Using the constraints (32) and (33)  $\tilde{\alpha}_2 = 0$  yields

$$2\tilde{\kappa}\bar{b}^{\frac{(n-2)}{2}}(n-3)(a_1+b_1) = \frac{(n-1)}{2\lambda} \left( \tilde{\kappa}\bar{b}^{\frac{(n-2)}{2}} \left[ \frac{\lambda(4(n-3)(a_1+b_1) - n + 1)}{n-1} + 1 \right] - 1 \right). \quad (47)$$

Inserting  $\tilde{\kappa}$  from (36) (with the assumption  $1/\tilde{\kappa} \neq 0$ ), (47) reduces to

$$\left( a_1 + b_1 - \frac{(n-1)^2}{4(n-3)(n-2)} \right) \left( \frac{n-1}{2(n-3)} + \lambda \left( a_1 + b_1 - \frac{n-1}{4(n-3)} \right) \right) = 0. \quad (48)$$

The discussion bifurcates depending on the vanishing of the two factors. We will study these below. The vanishing of  $\tilde{\alpha}_3$  yields a complicated equation which we do not depict here.

### 1. The two $a_1 + b_1$ cases:

As mentioned above, one has to discuss the two theories, coming from the vanishing of the two factors in (48) separately.

*a. The  $a_1 + b_1 \neq \frac{(n-1)^2}{4(n-2)(n-3)}$  case:* For this case, the second factor in (48) vanishes yielding

$$a_1 + b_1 = \frac{(\lambda-2)(n-1)}{4\lambda(n-3)}. \quad (49)$$

With this result, (32) reduces to

$$c = \frac{2-n}{2\lambda}, \quad (50)$$

and the explicit forms of  $\bar{b}$  and  $\tilde{\kappa}$  can be computed as

$$\bar{b} = 1, \quad \frac{1}{\tilde{\kappa}} = 1 - \lambda, \quad (51)$$

where  $\lambda \neq 1$  and  $\lambda \neq 0$ . First, observe that  $x \equiv \frac{2\lambda}{n-2} = -\frac{1}{c}$ ; therefore,  $cx^2 + x = 0$ , which greatly simplifies the vacuum equation (25) to

$$\lambda_0 + \frac{1}{c} = 0, \quad (52)$$

fixing the cosmological parameter  $\lambda$  as

$$\lambda = \frac{\lambda_0}{2} (n - 2). \quad (53)$$

Note that for  $n = 4$  the effective and the bare cosmological constants are equal. In addition, with (49), the value of  $a_4 + b_3$  can be calculated from (33) as

$$a_4 + b_3 = \frac{\beta(\beta + 2)}{2} + \frac{n(n - \lambda)}{2\lambda(n - 2)}. \quad (54)$$

Plugging all these in (46) and setting  $\tilde{a}_3 = 0$  yields

$$\lambda a_3 \left( \frac{2\lambda a_3}{n - 2} + \beta + 1 \right) = 0. \quad (55)$$

Once again one has to study the  $a_3 = 0$  or  $\frac{2\lambda a_3}{(n-2)} + \beta + 1 = 0$  cases separately for  $\lambda \neq 0$ .

1. The  $a_3 = 0$  case: For  $a_3 = 0$ ,  $b_2$  can be determined from (50) as

$$b_2 = \frac{2 - n}{2\lambda} = c, \quad (56)$$

then the theory is fixed as

$$\begin{aligned} A_{\mu\nu} = & R_{\mu\nu} + \beta S_{\mu\nu} + \gamma (a_1 \mathcal{W}_{\mu\nu} + a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} + a_4 S_{\mu\rho} S_{\nu}^{\rho}) \\ & + \frac{\gamma}{n} g_{\mu\nu} \left[ \left( \frac{((n-2)\lambda_0 - 4)(n-1)}{4\lambda_0(n-2)(n-3)} - a_1 \right) \mathcal{W} - \frac{1}{\lambda_0} R_{\rho\sigma}^2 \right. \\ & \left. + \left( \frac{\beta(\beta+2)}{2} + \frac{n(2n - \lambda_0(n-2))}{2\lambda_0(n-2)^2} - a_4 \right) S_{\rho\sigma}^2 \right], \end{aligned} \quad (57)$$

where we also used (53) to represent the action in terms of the input parameter  $\lambda_0$  instead of the derived parameter  $\lambda$ . With (57), the theory has four arbitrary dimensionless parameters which has all the desired properties that Einstein's theory has.

2. The  $a_3 = -\frac{(n-2)}{2\lambda}(\beta + 1)$  case: For this value  $a_3$ ,  $b_2$  is also determined from (50) as

$$b_2 = \frac{(n-2)}{2\lambda} \beta. \quad (58)$$

Then, the theory becomes

$$\begin{aligned} A_{\mu\nu} = & R_{\mu\nu} + \beta S_{\mu\nu} + \gamma \left[ a_1 \mathcal{W}_{\mu\nu} + a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} - \frac{(\beta+1)}{\lambda_0} R_{\mu\rho} R_{\nu}^{\rho} + a_4 S_{\mu\rho} S_{\nu}^{\rho} \right] \\ & + \frac{\gamma}{n} g_{\mu\nu} \left[ \left( \frac{((n-2)\lambda_0 - 4)(n-1)}{4\lambda_0(n-2)(n-3)} - a_1 \right) \mathcal{W} + \frac{\beta}{\lambda_0} R_{\rho\sigma}^2 \right. \\ & \left. + \left( \frac{\beta(\beta+2)}{2} + \frac{n(2n - \lambda_0(n-2))}{2\lambda_0(n-2)^2} - a_4 \right) S_{\rho\sigma}^2 \right], \end{aligned} \quad (59)$$

where again we used (53). This is again a four parameter theory that has all the desired properties of Einstein's theory. Even though both (57) and (59) provide a healthy extension of cosmological Einstein's theory with a unique vacuum and a massless spin-2 graviton, they both lack the  $\lambda_0 \rightarrow 0$  limit. On the basis of this, we shall provisionally disregard these two theories.

b. The  $a_1 + b_1 = \frac{(n-1)^2}{4(n-2)(n-3)}$  case: From (32), this choice leads to

$$c = \frac{1}{4}, \quad (60)$$

and the explicit forms of  $\bar{b}$  and  $\tilde{\kappa}$  can be computed as

$$\bar{b} = \left(1 + \frac{\lambda}{n-2}\right)^2, \quad (61)$$

$$\frac{1}{\tilde{\kappa}} = (1-\lambda) \left(1 + \frac{\lambda}{n-2}\right)^{n-2}, \quad (62)$$

where  $\lambda \neq 1$  and  $\lambda \neq 2-n$  at which the effective dimensionless Newton's constant vanish. With (60), the vacuum equation reduces to a polynomial equation

$$\frac{\lambda}{n-2} - 1 + (\lambda_0 + 1) \left(\frac{\lambda}{n-2} + 1\right)^{1-n} = 0, \quad (63)$$

which has of course no explicit solutions for arbitrary  $n$ . But, one can show that for given  $n$ , there is a unique viable solution consistent with the unitarity of the theory (see Appendix-C).

For this case, from (54),  $a_4 + b_3$  becomes

$$a_4 + b_3 = \frac{\beta(\beta+2)}{2} + \frac{n(4-3n)}{4(n-2)^2}. \quad (64)$$

Using this value in the second constraint of the Gauss-Bonnet reduction, that is  $\tilde{\alpha}_3 = 0$ , one obtains

$$(\beta+1) \left(1 + \frac{\lambda}{n-2}\right) = \pm \left(\frac{4\lambda a_3}{n-2} + \beta + 1\right). \quad (65)$$

One must consider both of the cases, but the minus sign case will turn out to be a sub-case (when  $\beta = -1$ ) of the plus sign case for which

$$a_3 = \frac{\beta+1}{4}, \quad (66)$$

and,  $b_2 = -\frac{\beta}{4}$ . Then, the  $A_{\mu\nu}$  tensor becomes

$$\begin{aligned} A_{\mu\nu} = & R_{\mu\nu} + \beta S_{\mu\nu} + \gamma \left( a_1 \mathcal{W}_{\mu\nu} + a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} + \frac{\beta+1}{4} R_{\mu\rho} R_{\nu}^{\rho} + a_4 S_{\mu\rho} S_{\nu}^{\rho} \right) \\ & + \frac{\gamma}{n} g_{\mu\nu} \left[ \left( \frac{(n-1)^2}{4(n-2)(n-3)} - a_1 \right) \mathcal{W} - \frac{\beta}{4} R_{\rho\sigma}^2 + \left( \frac{\beta(\beta+2)}{2} + \frac{n(4-3n)}{4(n-2)^2} - a_4 \right) S_{\rho\sigma}^2 \right]. \end{aligned} \quad (67)$$

The BI gravity based on this  $A_{\mu\nu}$ , with four arbitrary dimensionless parameters, satisfies all the nice properties of Einstein's theory: a unique vacuum, a massless unitary spin-2 graviton about this vacuum. In the small curvature expansion it reduces to the EGB theory while with many powers of curvature it has an improved UV behavior. In Section-II, we have discussed possible ways to further reduce the number of arbitrary dimensionless parameters and suggested a possible minimal theory without any such parameters given by the Lagrangian density (8). For the sake of completeness let us note that the effective cosmological parameter of the theory defined by (67) will come from

the solution of (63) consistent with the positivity of the effective Newton's parameter (62) whose details are in Appendix-C. Let us briefly summarize the results of that analysis. Defining

$$C(n) \equiv \left(\frac{n-3}{n-2}\right) \left(\frac{n-1}{n-2}\right)^{n-1} - 1, \quad (68)$$

one has the following conclusions.

- For even dimensions there is a unique viable vacuum,  $\lambda$ , in the region  $-\infty < \lambda < 1$  and  $\lambda \neq 2 - n$  given that  $-\infty < \lambda_0 < C(n)$  and  $\lambda_0 \neq -1$ .
- For odd dimensions there is a unique viable vacuum,  $\lambda$ , in the region  $2 - n < \lambda < 1$  given that  $-1 < \lambda_0 < C(n)$ .

### C. Infinite Dimensional BI Gravity:

Without going into much detail, it is rather amusing to consider the infinite dimensional ( $n \rightarrow \infty$ ) limit, which received a renewed interest in the context of  $\frac{1}{n}$  expansion in general relativity [17]. As  $n \rightarrow \infty$ , our minimal BI Lagrangian (8) becomes an exponential function compactly written as follows

$$\kappa \frac{\gamma}{2} \mathcal{L}_{n \rightarrow \infty} = \exp\left(\frac{\gamma}{2} R + \frac{\gamma^2}{8} (R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2)\right) - \lambda_0 - 1. \quad (69)$$

As we show in the Appendix-C this theory has a unique vacuum with an effective cosmological parameter  $\lambda = \ln(1 + \lambda_0)$  as long as  $-1 < \lambda_0 < e - 1$  and an effective Newton's constant

$$\frac{1}{\kappa_{\text{eff}}} = \frac{1}{\kappa} \left(1 - \ln(1 + \lambda_0)\right) (1 + \lambda_0), \quad (70)$$

which is always positive in the allowed region. Note that as  $n \rightarrow \infty$  for (A)dS one has  $\bar{R}_{\mu\nu\rho\sigma} = O(n^{-2})$ ,  $\bar{R}_{\mu\nu} = O(n^{-1})$ , and  $\gamma\bar{R} = 2\lambda$ .

### D. Conserved Charges in the BI Gravity:

Finally let us briefly comment on the conserved charges (mass and angular momenta) of asymptotically flat and (A)dS solutions in the BI gravity. Since the conserved charges of the generic  $f(R_{\alpha\beta}^{\mu\nu})$  theory was given in [13] based on the formalism of [14–16] with  $\tilde{\kappa}$  given in (62) it is straight forward to see that any conserved total charge of BI gravity is given as

$$Q_{\text{BI}}(\bar{\xi}) = \frac{1}{\tilde{\kappa}} \left(1 + \frac{4\lambda(n-3)(n-4)}{(n-1)(n-2)} \alpha_1\right) Q_{\text{Einstein}}(\bar{\xi}), \quad (71)$$

where  $\tilde{\kappa}$  and  $\alpha_1$  are given in (36) and (39), respectively, and  $\bar{\xi}$  is the background Killing vector which reads  $\bar{\xi}^\mu = (-1, 0, \dots, 0)$  for energy and  $\bar{\xi}^\mu = (0, 0, 0, 1, \dots, 0)$  for angular momenta.  $Q_{\text{Einstein}}(\bar{\xi})$  refers to the charge of the solution in Einstein's theory. Hence, there is a simple relation between the conserved charges of the BI gravity and Einstein's theory. In particular, for asymptotically flat backgrounds they have the same values. For asymptotically (A)dS backgrounds, they differ a numerical factor depending on  $\lambda$  and  $n$ .

For the BI theory defined with (67), the conserved charge expression reads

$$Q_{\text{BI}}(\bar{\xi}) = \left( \frac{1}{\tilde{\kappa}} + \frac{\lambda(n-1)(n-4)}{(n-2)^2} \right) Q_{\text{Einstein}}(\bar{\xi}),$$

where  $\tilde{\kappa}$  is given in (62). It is interesting to note that for  $n = 4$ , the second term drops out and the BI theory has the same conserved charges as Einstein's theory.

## VI. CONCLUSION

Introducing the principle that the low energy quantum gravity has a unique vacuum, that is a unique maximally symmetric solution, and a single massless spin-2 graviton about this vacuum, we have constructed Born-Infeld gravity theories in generic  $n > 3$  dimensions, including  $n \rightarrow \infty$ . In  $n$  dimensions the final theory has still four arbitrary dimensionless parameters whose values could possibly be determined from phenomenological considerations or other theoretical conditions. The main motivation to construct such a theory was to build a model which in principle has infinitely many terms in the curvature invariants, hence improving Einstein's gravity in the UV region, yet still has the two important properties of Einstein's gravity, the uniqueness of the vacuum and the masslessness of the graviton, which are usually lost when Einstein's theory is modified with higher powers of curvature. Therefore, our construction answers the question whether graviton can be kept massless in low energy quantum gravity with a unique vacuum, in the affirmative. A detailed analysis of the theory presented here in terms of its solutions will appear in a separate work. It is interesting to note that  $n = 4$ , the physically most relevant case, has a rather fascinating property: BI gravity is not only unitary as a full theory, but also unitary at every truncated order in the curvature expansion [1], hence in some sense one can consider every truncated order as a separate theory in the strong coupling limit. We are not aware of such a gravity theory. Whether this property is preserved in higher dimensions needs to be studied.

Here, we have employed the metric formulation of BI gravity, for the Palatini formulation, as envisioned by Eddington [18] who introduced the determinantal type gravity based on generalized volume a decade before Born and Infeld studied the electrodynamics version [19], see the recent works [20–22]. See also [23, 24] for various phenomenological properties of other BI type gravities.

We have studied pure gravity without matter, to couple matter one can consider the minimal coupling assumption and add a  $\sqrt{-g}g^{\mu\nu}T_{\mu\nu}$  to the action where  $T_{\mu\nu}$  is the usual energy-momentum tensor of the matter fields. Of course, one can also use non-minimal coupling such as  $A_{\mu\nu} \rightarrow A_{\mu\nu} + \alpha F_{\mu\nu} + \eta \partial_\mu \phi \partial_\nu \phi$  where  $F_{\mu\nu}$  is the field strength of electromagnetism and  $\phi$  is a scalar field.

## VII. ACKNOWLEDGMENT

I. G. and B. T. are supported by the TÜBİTAK grant 113F155. T. C. S. thanks The Centro de Estudios Científicos (CECs) where part of this work was carried out under the support of Fondecyt with grant 3140127. Some of the calculations in this paper were either done or checked with the help of the computer package Cadabra [25, 26].

### Appendix A: Conversions Between CSR Basis and RRR Basis

In this Appendix, we discuss the conversions between the Weyl–traceless–Ricci–Ricci (CSR) basis and the Riemann–Ricci–curvature–scalar (RRR) basis. The  $A_{\mu\nu}$  tensor written in the CSR

basis, that is

$$\begin{aligned}
A_{\mu\nu} = & R_{\mu\nu} + \beta S_{\mu\nu} \\
& + \gamma \left( a_1 C_{\mu\rho\sigma\lambda} C_{\nu}{}^{\rho\sigma\lambda} + a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} + a_3 R_{\mu\rho} R_{\nu}^{\rho} + a_4 S_{\mu\rho} S_{\nu}^{\rho} \right) \\
& + \frac{\gamma}{n} g_{\mu\nu} \left( b_1 C_{\rho\sigma\lambda\gamma}^2 + b_2 R_{\rho\sigma}^2 + b_3 S_{\rho\sigma}^2 \right), \tag{A1}
\end{aligned}$$

can be converted to the RRR basis, that is

$$\begin{aligned}
A_{\mu\nu} = & \left( 1 + \tilde{\beta} \right) R_{\mu\nu} - \frac{\tilde{\beta}}{4} g_{\mu\nu} R + c_1 g_{\mu\nu} R^2 + c_2 R R_{\mu\nu} + c_3 g_{\mu\nu} R_{\rho\sigma}^2 \\
& + c_4 R_{\mu}^{\sigma} R_{\nu\sigma} + c_5 R_{\mu\sigma\nu\rho} R^{\sigma\rho} + c_6 g_{\mu\nu} R_{\rho\sigma\lambda\gamma}^2 + c_7 R_{\mu}{}^{\sigma\rho\tau} R_{\nu\sigma\rho\tau}, \tag{A2}
\end{aligned}$$

by using  $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R$  and the definition of the Weyl tensor in  $n$  dimensions

$$C_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} - \frac{2}{(n-2)} \left( g_{\mu[\nu} R_{\beta]\alpha} - g_{\alpha[\nu} R_{\beta]\mu} \right) + \frac{2}{(n-1)(n-2)} R g_{\mu[\nu} g_{\beta]\alpha}. \tag{A3}$$

The coefficients in (A2) can be found as follows

$$\begin{aligned}
\tilde{\beta} = & \beta, \\
c_1 = & \frac{\gamma}{n^2(n-1)} \left( -\frac{2n^2}{(n-2)^2} a_1 + \frac{n^2}{n-2} a_2 + (n-1) a_4 + \frac{2n}{n-2} b_1 - (n-1) b_3 \right), \\
c_2 = & \gamma \left( \frac{4}{(n-2)^2} a_1 - \frac{n}{(n-1)(n-2)} a_2 - \frac{2}{n} a_4 \right), \\
c_3 = & \frac{\gamma}{n} \left( \frac{2n}{(n-2)^2} a_1 - \frac{n}{n-2} a_2 - \frac{4}{n-2} b_1 + b_2 + b_3 \right), \\
c_4 = & \gamma \left( -\frac{2n}{(n-2)^2} a_1 + \frac{2}{n-2} a_2 + a_3 + a_4 \right), \\
c_5 = & \gamma \left( -\frac{4}{n-2} a_1 + a_2 \right), \quad c_6 = \frac{\gamma}{n} b_1, \quad c_7 = \gamma a_1. \tag{A4}
\end{aligned}$$

Sometimes the inverse transformation from the RRR basis to the CSR basis is also needed; therefore, we shall give it here

$$\begin{aligned}
\beta = & \tilde{\beta}, \quad a_1 = \frac{c_7}{\gamma}, \quad a_2 = \frac{1}{\gamma} \left( c_5 + \frac{4}{n-2} c_7 \right), \\
a_3 = & \frac{1}{\gamma} \left( \frac{n}{2} c_2 + c_4 + \frac{n-2}{2(n-1)} c_5 + \frac{2}{n-1} c_7 \right), \\
a_4 = & \frac{1}{\gamma} \left( -\frac{n}{2} c_2 - \frac{n^2}{2(n-1)(n-2)} c_5 - \frac{2n}{(n-1)(n-2)^2} c_7 \right), \\
b_1 = & \frac{nc_6}{\gamma}, \quad b_2 = \frac{1}{\gamma} \left( n^2 c_1 + \frac{n}{2} c_2 + n c_3 + \frac{n}{2(n-1)} c_5 + \frac{2n}{n-1} c_6 \right) \\
b_3 = & \frac{1}{\gamma} \left( -n^2 c_1 - \frac{n}{2} c_2 + \frac{n^2}{2(n-1)(n-2)} c_5 + \frac{2n^2}{(n-1)(n-2)} c_6 + \frac{2n}{(n-2)^2} c_7 \right). \tag{A5}
\end{aligned}$$

### Appendix B: Computation of the EQCA of BI in (A)dS

Finding the vacuum and the particle spectrum of BI gravity is somewhat tricky because of the contributions of all powers of curvature. Here, basically we recap the essentials of the short-cuts introduced in our earlier works [1, 8, 11] as applied to the present context. This boils down to finding an equivalent quadratic curvature action (EQCA) that has the same vacuum and particle spectrum as the BI gravity and that theory follows from the Taylor series expansion.

$$\begin{aligned}
\kappa\mathcal{L}_{\text{EQCA}} = & \frac{2}{\gamma} \left[ \sqrt{\det(\delta_\nu^\rho + \gamma \bar{A}_\nu^\rho)} - (\lambda_0 + 1) \right] \\
& + \left[ \frac{\partial \mathcal{L}}{\partial C_{\alpha\beta}^{\mu\nu}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} C_{\alpha\beta}^{\mu\nu} + \left[ \frac{\partial \mathcal{L}}{\partial S_\nu^\mu} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} S_\nu^\mu + \left[ \frac{\partial \mathcal{L}}{\partial R_\nu^\mu} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} (R_\nu^\mu - \bar{R}_\nu^\mu) \\
& + \frac{1}{2} \left[ \frac{\partial^2 \mathcal{L}}{\partial C_{\alpha\beta}^{\mu\nu} \partial C_{\lambda\tau}^{\eta\theta}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} C_{\alpha\beta}^{\mu\nu} C_{\lambda\tau}^{\eta\theta} + \frac{1}{2} \left[ \frac{\partial^2 \mathcal{L}}{\partial S_\nu^\mu \partial S_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} S_\nu^\mu S_\beta^\alpha \\
& + \frac{1}{2} \left[ \frac{\partial^2 \mathcal{L}}{\partial R_\nu^\mu \partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} (R_\nu^\mu - \bar{R}_\nu^\mu) (R_\beta^\alpha - \bar{R}_\beta^\alpha) \\
& + \left[ \frac{\partial^2 \mathcal{L}}{\partial C_{\alpha\beta}^{\mu\nu} \partial S_\theta^\eta} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} C_{\alpha\beta}^{\mu\nu} S_\theta^\eta + \left[ \frac{\partial^2 \mathcal{L}}{\partial C_{\alpha\beta}^{\mu\nu} \partial R_\theta^\eta} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} C_{\alpha\beta}^{\mu\nu} (R_\theta^\eta - \bar{R}_\theta^\eta) \\
& + \left[ \frac{\partial^2 \mathcal{L}}{\partial S_\nu^\mu \partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} S_\nu^\mu (R_\beta^\alpha - \bar{R}_\beta^\alpha), \tag{B1}
\end{aligned}$$

where the bracketed and barred quantities denote the maximally symmetric background values for the corresponding expressions. Note that the background values of the Weyl and the traceless Ricci scalar vanish which was the main reason to work in the CRS basis. Let us compute the terms of (B1) separately. One can show that

$$\left[ \frac{\partial^2 A_\sigma^\rho}{\partial C_{\alpha\beta}^{\mu\nu} \partial C_{\lambda\tau}^{\eta\theta}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} = \gamma a_1 \delta_\sigma^\alpha \delta_\theta^\beta \delta_\mu^\lambda \delta_\nu^\tau \delta_\eta^\rho + \gamma \delta_\eta^\alpha \delta_\theta^\beta \delta_\nu^\tau \left( a_1 \delta_\sigma^\lambda \delta_\mu^\rho + \frac{2b_1}{n} \delta_\mu^\lambda \delta_\sigma^\rho \right). \tag{B2}$$

The other derivative terms can be calculated as

$$\left[ \frac{\partial^2 A_\sigma^\rho}{\partial C_{\alpha\beta}^{\mu\nu} \partial R_\theta^\eta} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} = \gamma a_2 \delta_\eta^\beta \delta_\nu^\theta \delta_\sigma^\alpha \delta_\mu^\rho, \quad \left[ \frac{\partial^2 A_\sigma^\rho}{\partial S_\nu^\mu \partial S_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} = \gamma a_4 \left( \delta_\sigma^\nu \delta_\mu^\beta \delta_\alpha^\rho + \delta_\alpha^\nu \delta_\sigma^\beta \delta_\mu^\rho \right) + \frac{2\gamma b_3}{n} \delta_\alpha^\nu \delta_\mu^\beta \delta_\sigma^\rho, \tag{B3}$$

$$\left[ \frac{\partial^2 A_\sigma^\rho}{\partial R_\nu^\mu \partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} = \gamma a_3 \left( \delta_\sigma^\nu \delta_\mu^\beta \delta_\alpha^\rho + \delta_\alpha^\nu \delta_\sigma^\beta \delta_\mu^\rho \right) + \frac{2\gamma b_2}{n} \delta_\alpha^\nu \delta_\mu^\beta \delta_\sigma^\rho. \tag{B4}$$

It is also obvious that  $\frac{\partial^2 A_\sigma^\rho}{\partial C_{\alpha\beta}^{\mu\nu} \partial S_\theta^\eta} = 0$ ,  $\frac{\partial^2 A_\sigma^\rho}{\partial S_\nu^\mu \partial R_\beta^\alpha} = 0$ . Using these results with

$$\partial^2 \left( \sqrt{\det(\delta_\nu^\rho + \gamma A_\nu^\rho)} \right) = \frac{\gamma}{2} \sqrt{\det(\delta_\nu^\rho + \gamma A_\nu^\rho)} \left[ B_\gamma^\lambda \partial^2 A_\lambda^\gamma - \gamma B_\theta^\lambda B_\gamma^\tau (\partial A_\tau^\theta) \partial A_\lambda^\gamma + \frac{\gamma}{2} (B_\gamma^\lambda \partial A_\lambda^\gamma)^2 \right], \tag{B5}$$



where  $B_\gamma^\lambda$  represents the inverse of the matrix  $(\delta_\gamma^\lambda + A_\gamma^\lambda)$  and for the differential of  $B$  we use  $\partial B = -B(\partial A)B$ . The second order contributions to the EQCA can be expressed as

$$\begin{aligned} \left[ \frac{\partial^2 \mathcal{L}}{\partial C_{\alpha\beta}^{\mu\nu} \partial C_{\lambda\tau}^{\eta\theta}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} C_{\alpha\beta}^{\mu\nu} C_{\lambda\tau}^{\eta\theta} &= \sqrt{\det(\delta_\nu^\mu + \gamma \bar{A}_\nu^\mu)} \\ &\times \left\{ \bar{B}_\rho^\sigma \left[ \frac{\partial^2 A_\sigma^\rho}{\partial C_{\alpha\beta}^{\mu\nu} \partial C_{\lambda\tau}^{\eta\theta}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} - \gamma \bar{B}_\zeta^\sigma \left[ \frac{\partial A_\epsilon^\zeta}{\partial C_{\lambda\tau}^{\eta\theta}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \bar{B}_\rho^\epsilon \left[ \frac{\partial A_\sigma^\rho}{\partial C_{\alpha\beta}^{\mu\nu}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \right. \\ &\quad \left. + \frac{\gamma}{2} \bar{B}_\rho^\sigma \left[ \frac{\partial A_\sigma^\rho}{\partial C_{\alpha\beta}^{\mu\nu}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \bar{B}_\zeta^\epsilon \left[ \frac{\partial A_\epsilon^\zeta}{\partial C_{\lambda\tau}^{\eta\theta}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \right\} C_{\alpha\beta}^{\mu\nu} C_{\lambda\tau}^{\eta\theta}, \end{aligned} \quad (\text{B6})$$

which then yields

$$\left[ \frac{\partial^2 \mathcal{L}}{\partial C_{\alpha\beta}^{\mu\nu} \partial C_{\lambda\tau}^{\eta\theta}} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} C_{\alpha\beta}^{\mu\nu} C_{\lambda\tau}^{\eta\theta} = \gamma^2 \bar{b}^{\frac{(n-2)}{2}} (a_1 + b_1) C_{\mu\nu\rho\sigma}^2. \quad (\text{B7})$$

where  $\bar{b}^n \equiv \det(\delta_\nu^\beta + \gamma \bar{A}_\nu^\beta)$ . Similarly, one has

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial S_\nu^\mu \partial S_\beta^\alpha} S_\nu^\mu S_\beta^\alpha &= \sqrt{\det(\delta_\nu^\mu + \gamma \bar{A}_\nu^\mu)} \\ &\times \left\{ \bar{B}_\rho^\sigma \left[ \frac{\partial^2 A_\sigma^\rho}{\partial S_\nu^\mu \partial S_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} - \gamma \bar{B}_\zeta^\sigma \left[ \frac{\partial A_\epsilon^\zeta}{\partial S_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \bar{B}_\rho^\epsilon \left[ \frac{\partial A_\sigma^\rho}{\partial S_\nu^\mu} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \right. \\ &\quad \left. + \frac{\gamma}{2} \bar{B}_\rho^\sigma \left[ \frac{\partial A_\sigma^\rho}{\partial S_\nu^\mu} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \bar{B}_\zeta^\epsilon \left[ \frac{\partial A_\epsilon^\zeta}{\partial S_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \right\} S_\nu^\mu S_\beta^\alpha, \end{aligned} \quad (\text{B8})$$

which yields

$$\frac{\partial^2 \mathcal{L}}{\partial S_\nu^\mu \partial S_\beta^\alpha} S_\nu^\mu S_\beta^\alpha = \gamma^2 \bar{b}^{\frac{(n-4)}{2}} \left( -\frac{1}{2} \beta^2 + \bar{b} (a_4 + b_3) \right) S_{\mu\nu}^2. \quad (\text{B9})$$

One also has

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial R_\nu^\mu \partial R_\beta^\alpha} (R_\nu^\mu - \bar{R}_\nu^\mu) (R_\beta^\alpha - \bar{R}_\beta^\alpha) &= \sqrt{\det(\delta_\nu^\mu + \gamma \bar{A}_\nu^\mu)} \\ &\times \left\{ \bar{B}_\rho^\sigma \left[ \frac{\partial^2 A_\sigma^\rho}{\partial R_\nu^\mu \partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} - \gamma \bar{B}_\zeta^\sigma \left[ \frac{\partial A_\epsilon^\zeta}{\partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \bar{B}_\rho^\epsilon \left[ \frac{\partial A_\sigma^\rho}{\partial R_\nu^\mu} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \right. \\ &\quad \left. + \frac{\gamma}{2} \bar{B}_\rho^\sigma \left[ \frac{\partial A_\sigma^\rho}{\partial R_\nu^\mu} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \bar{B}_\zeta^\epsilon \left[ \frac{\partial A_\epsilon^\zeta}{\partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \right\} (R_\nu^\mu - \bar{R}_\nu^\mu) (R_\beta^\alpha - \bar{R}_\beta^\alpha), \end{aligned} \quad (\text{B10})$$

which reduces to

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial R_\nu^\mu \partial R_\beta^\alpha} (R_\nu^\mu - \bar{R}_\nu^\mu) (R_\beta^\alpha - \bar{R}_\beta^\alpha) = & - \left( R - \frac{\bar{R}}{2} - \frac{n}{2\bar{R}} R_{\mu\nu}^2 + \frac{n}{2\bar{R}} S_{\mu\nu}^2 \right) \\ & \times \frac{\gamma^2 \bar{R} \bar{b}^{\frac{(n-4)}{2}}}{2n^3} \left[ 4n^2 \bar{b} (a_3 + b_2) + (n-2) (2\gamma \bar{R} (a_3 + b_2) + n)^2 \right] \\ & - \frac{\gamma^2 \bar{b}^{\frac{n}{2}-2} \left( (2a_3 \gamma \bar{R} + n)^2 - 2n^2 \bar{b} (a_3 + b_2) \right)}{2n^2} S_{\mu\nu}^2 \end{aligned} \quad (\text{B11})$$

where we have used  $R^2 = n (R_{\mu\nu}^2 - S_{\mu\nu}^2)$ . Finally, the cross terms can be computed as

$$\frac{\partial^2 \mathcal{L}}{\partial C_{\alpha\beta}^{\mu\nu} \partial S_\theta^\eta} C_{\alpha\beta}^{\mu\nu} S_\theta^\eta = 0, \quad \frac{\partial^2 \mathcal{L}}{\partial C_{\alpha\beta}^{\mu\nu} \partial R_\theta^\eta} C_{\alpha\beta}^{\mu\nu} (R_\theta^\eta - \bar{R}_\theta^\eta) = 0, \quad (\text{B12})$$

with only non-vanishing term coming from

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial S_\nu^\mu \partial R_\beta^\alpha} S_\nu^\mu (R_\beta^\alpha - \bar{R}_\beta^\alpha) = & \sqrt{\det (\delta_\nu^\mu + \gamma \bar{A}_\nu^\mu)} \\ & \times \left\{ \bar{B}_\rho^\sigma \left[ \frac{\partial^2 A_\sigma^\rho}{\partial S_\nu^\mu \partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} - \gamma \bar{B}_\zeta^\sigma \left[ \frac{\partial A_\sigma^\zeta}{\partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \bar{B}_\rho^\epsilon \left[ \frac{\partial A_\sigma^\rho}{\partial S_\nu^\mu} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \right. \\ & \left. + \frac{\gamma}{2} \bar{B}_\rho^\sigma \left[ \frac{\partial A_\sigma^\rho}{\partial S_\nu^\mu} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \bar{B}_\zeta^\epsilon \left[ \frac{\partial A_\sigma^\zeta}{\partial R_\beta^\alpha} \right]_{\bar{R}_{\rho\sigma}^{\mu\nu}} \right\} S_\nu^\mu (R_\beta^\alpha - \bar{R}_\beta^\alpha), \end{aligned} \quad (\text{B13})$$

giving the result

$$\frac{\partial^2 \mathcal{L}}{\partial S_\nu^\mu \partial R_\beta^\alpha} S_\nu^\mu (R_\beta^\alpha - \bar{R}_\beta^\alpha) = -\frac{1}{2} \gamma^2 \bar{b}^{\frac{(n-4)}{2}} \beta \left( 1 + \frac{2\gamma \bar{R} a_3}{n} \right) S_{\mu\nu}^2. \quad (\text{B14})$$

### Appendix C: Proof of the Uniqueness of the Viable Vacuum

Now, for the theory (67) let us discuss the viable parameter regions (unitarity of the theory together with the existence of a maximally symmetric vacuum). The discussion bifurcates for even and odd dimensions  $n$  which need to be studied separately. Before the finite  $n$  discussion, let us look at the extreme case of  $n \rightarrow \infty$  limit which is relevant to the infinite dimensional BI gravity discussed at the end of Section V. In this limit, (62) becomes

$$\frac{1}{\tilde{\kappa}} = (1 - \lambda) e^\lambda. \quad (\text{C1})$$

The positivity of  $\tilde{\kappa}$ , required for attractive gravity, constrains the effective dimensionless cosmological constant to the interval  $-\infty < \lambda < 1$ . In this limit, the vacuum equation (63) becomes

$$\lambda_0 = e^\lambda - 1, \quad (\text{C2})$$

with the unique solution

$$\lambda = \ln(1 + \lambda_0). \quad (\text{C3})$$

$\lambda$  is in the unitarity region,  $(-\infty, 1)$ , as long as the bare dimensionless cosmological constant satisfies  $-1 < \lambda_0 < e - 1 \approx 1.7$ , so there is a small interval for  $\lambda_0$ .

Let us now turn to the discussion of the finite  $n$  case. In analyzing the existence of the roots of the vacuum equation (63), let us define a new variable

$$y \equiv 1 + \frac{\lambda}{n-2}, \quad (\text{C4})$$

which reduces (63) to

$$y^n - 2y^{n-1} + a = 0, \quad (\text{C5})$$

where  $a \equiv \lambda_0 + 1$ . With this definition (62) becomes

$$\frac{1}{\tilde{\kappa}} = (n-1)y^{n-2} \left( 1 - \frac{n-2}{n-1}y \right). \quad (\text{C6})$$

Our task is to prove that for generic  $n$  dimensions (C5) has at least one real solution consistent with the unitarity of the theory. Surprisingly, it will turn out to be there is only one real solution consistent with the unitarity.

For a given  $\lambda_0$ , solving the algebraic equation (C5) is a simple numerical problem for each given dimension  $n$ ; but, we do not know  $\lambda_0$ , hence, the problem becomes a non-trivial one for  $n \geq 5$ . The canonical way of showing the existence of the roots in a given interval or finding approximate numerical solutions is to construct the so called Sturm chain which we shall do below, but to get a feeling let us analyze the function

$$f(y) \equiv y^n - 2y^{n-1} + a, \quad (\text{C7})$$

whose zeros are the real solutions of the vacuum equation. To get the information on the number of zeros, we need to study the extrema of  $f(y)$ :

$$\frac{df}{dy} = ny^{n-2} \left( y - \frac{2(n-1)}{n} \right), \quad (\text{C8})$$

which is zero at the two critical points  $y = 0$  and  $y = \frac{2(n-1)}{n}$ . The second derivative of the function is

$$\frac{d^2f}{dy^2} = n(n-1)y^{n-3} \left( y - \frac{2(n-2)}{n} \right), \quad (\text{C9})$$

which have the following values at the critical points:

$$\left. \frac{d^2f}{dy^2} \right|_{y=0} = 0, \quad \left. \frac{d^2f}{dy^2} \right|_{y=\frac{2(n-1)}{n}} = n \left( \frac{2(n-1)}{n} \right)^{n-2}, \quad (\text{C10})$$

showing that the first critical point is an inflection point and the second one is a minimum. The value of the function at these points are

$$f(0) = a, \quad f\left(\frac{2(n-1)}{n}\right) = -\left(\frac{2}{n}\right)^n (n-1)^{n-1} + a. \quad (\text{C11})$$

Clearly, depending on the signs of these values and the evenness or the oddness of the number of dimensions, the number of roots can be determined.

Let us now construct the Sturm chain. The Sturm function is defined as

$$f_m(x) = - \left( f_{m-2}(x) - f_{m-1}(x) \left[ \frac{f_{m-2}(x)}{f_{m-1}(x)} \right] \right), \quad (\text{C12})$$

where  $\left[ \frac{P(x)}{Q(x)} \right]$  is a polynomial quotient. In other words,  $f_m(x)$  is the negative of the remainder in the polynomial division  $\frac{f_{m-2}(x)}{f_{m-1}(x)}$ , that is  $f_m(x) = -\text{rem}(f_{m-2}(x), f_{m-1}(x))$ . The zeroth and the first orders in the Sturm chain are the function itself and its derivative:

$$f_0(y) = y^n - 2y^{n-1} + a, \quad (\text{C13})$$

$$f_1(y) = ny^{n-1} - 2(n-1)y^{n-2}. \quad (\text{C14})$$

Then,  $f_2(y) = -\text{rem}(f_0, f_1)$  and  $f_3(y) = -\text{rem}(f_1, f_2)$  can recursively be calculated as

$$f_2(y) = \frac{4(n-1)}{n^2}y^{n-2} - a, \quad (\text{C15})$$

$$f_3(y) = -\frac{n^3a}{4(n-1)}y + \frac{n^2a}{2}. \quad (\text{C16})$$

To find  $f_4(y)$ , one needs the quotient  $\left[ \frac{f_2(x)}{f_3(x)} \right]$  which can be computed as

$$\left[ \frac{f_2(x)}{f_3(x)} \right] = -\frac{4}{n^3a}y^{n-1} \sum_{i=2}^{n-1} \left[ \frac{2(n-1)}{ny} \right]^i, \quad (\text{C17})$$

and  $f_4(y) = -\text{rem}(f_2, f_3)$  becomes

$$f_4(y) = - \left( \frac{2}{n} \right)^n (n-1)^{n-1} + \lambda_0 + 1, \quad (\text{C18})$$

which completes the Sturm chain. We can now use the Sturm Theorem which reads (adapted to our notation) as [27]:

The number of real roots of an algebraic equation ( $f(y) = 0$ ) with real coefficients whose real roots are simple over an interval, the endpoints of which are not roots, is equal to the difference between the number of sign changes of the Sturm chains formed for the interval ends.

To be able to use the Sturm Theorem, we need the relevant interval where the theory is unitary. For this purpose even and odd dimensions must be treated separately.

*a. Even dimensions:  $n = 4 + 2k, k = 0, 1, \dots$*

For unitarity  $\tilde{\kappa} > 0$  must be satisfied which requires  $\lambda < 1$  and  $\lambda \neq 2 - n$  in even dimensions as is clear from (C6). Therefore, for even dimensions, the relevant interval is  $y \in \left( -\infty, \frac{n-1}{n-2} \right) - \{0\}$ . There is a corresponding viable interval of  $\lambda_0$  and to find this, let us show that

$$\lambda_0 = 2y^{n-1} - y^n - 1, \quad (\text{C19})$$

is a monotonically increasing function of  $y$  with a positive derivative in the unitarity region:

$$\frac{d\lambda_0}{dy} = -ny^{n-2} \left( y - \frac{2(n-1)}{n} \right). \quad (\text{C20})$$

For even  $n$ ,  $d\lambda_0/dy$  changes sign at  $y = \frac{2(n-1)}{n}$  which is not attained below the upper bound  $y = \frac{n-1}{n-2}$  for  $n \geq 4$ ; hence,  $\frac{d\lambda_0}{dy} > 0$ . The upper bound of  $y$ , that is  $\lambda = 1$ , gives the corresponding upper bound for  $\lambda_0$  as

$$\lambda_0 < C(n) \equiv \left(\frac{n-3}{n-2}\right) \left(\frac{n-1}{n-2}\right)^{n-1} - 1. \quad (\text{C21})$$

The upper bound  $C(n)$  is an increasing sequence since  $C(n)$  involves the multiplication of two increasing sequences:

$$\left(\frac{n-3}{n-2}\right) \left(\frac{n-1}{n-2}\right) = 1 - \frac{1}{(n-2)^2}, \quad (\text{C22})$$

and

$$\left(1 + \frac{1}{n-2}\right)^{n-2}, \quad (\text{C23})$$

which converges to  $e$  as  $n \rightarrow \infty$ . Therefore, as  $n \rightarrow \infty$ ,  $C(n)$  converges to  $e - 1$ .

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$
$-\infty$	$+\infty$	$-\infty$	$+\infty$	$+\infty$	$\lambda_0 - D(n)$
1	$\lambda_0 - C(n)$	$-(n-4) \left(\frac{n-1}{n-2}\right)^{n-1}$	$\frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - \lambda_0 - 1$	$(\lambda_0 + 1) \frac{n^2(n-4)}{4(n-2)}$	$\lambda_0 - D(n)$

Table I: For even  $n$ , the values of the Sturm functions at the endpoints of the unitary interval of  $\lambda$ .

Table 1 summarizes the results. Number of roots depends on the zeros of the expressions in the last row in Table I:

$$\lambda_0 = C(n), \quad \lambda_0 = \frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - 1, \quad \lambda_0 = -1, \quad \lambda_0 = D(n), \quad (\text{C24})$$

where we have defined

$$D(n) \equiv \left(\frac{2}{n}\right)^n (n-1)^{n-1} - 1. \quad (\text{C25})$$

To scan the viable interval of  $\lambda_0$  that is  $\lambda_0 \in (-\infty, C(n)) - \{-1\}$  the order of these zeros is important. Let us prove the relations

$$-1 < C(n) \leq D(n). \quad (\text{C26})$$

First, let us show that

$$C(n) \leq D(n), \quad (\text{C27})$$

where the equality is satisfied for  $n = 4$ . For  $n > 4$ , showing this inequality boils down to proving the following inequality

$$\left(\frac{2}{n}\right)^n > (n-3) \left(\frac{1}{n-2}\right)^n, \quad (\text{C28})$$

since

$$\left(\frac{2}{n}\right)^n > \left(\frac{1}{n-2}\right)^{n-1} > (n-3) \left(\frac{1}{n-2}\right)^n, \quad (\text{C29})$$

it suffices to show

$$\frac{n}{2} \left(\frac{2}{n}\right)^{n-1} > \left(\frac{2}{n}\right)^{n-1} > \left(\frac{1}{n-2}\right)^{n-1}, \quad (\text{C30})$$

since  $n - 2 > \frac{n}{2}$ , (C27) follows. Then, since

$$\frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - 1 \leq C(n) \quad (\text{C31})$$

$$-\frac{(n-4)(n(n-3)+4)}{n^2(n-3)} \leq 0, \quad (\text{C32})$$

where, clearly, equality holds for  $n = 4$  and inequality holds for  $n > 4$ . So,

$$-1 < \frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - 1 \leq C(n) \leq D(n).$$

In four dimensions,  $C(4)$  and  $\left(\frac{2}{n}\right)^n (n-1)^{n-1} - 1$  are both equal to  $\frac{11}{16}$ . This is an important observation since as we discuss below  $D(n)$  represents the bound on the existence of roots, that is for  $\lambda_0 \leq D(n)$  there are two roots for the vacuum field equation. For  $n = 4$ , so if there exist two vacua of the theory, then one of them should be the viable unitary vacuum. On the other hand, for even dimensional theories beyond four dimensions, there may be two roots for the vacuum field equation, but none of them would be in the unitary interval for  $\lambda_0$  values  $C(n) < \lambda_0 < D(n)$ .

In the tables below, for all values of  $\lambda_0$ , the number of roots in the unitarity interval  $\lambda < 1$  is investigated. As a result, for each value of  $\lambda_0$  in the interval  $\lambda_0 < C(n)$ , there is one and only one root for the vacuum equation in the unitary interval of  $\lambda$ , and for  $\lambda_0 > C(n)$ , it is not possible to have a  $\lambda$  value in the unitary interval. The following tables from Table-II to Table-IV depict the sign changes of the Sturm chain for various  $\lambda_0$  intervals proving the uniqueness of the real solution  $\lambda$ .

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$-\infty$	+	-	+	+	-	3
1	-	-	+	-	-	2

Table II:  $\lambda_0 < -1$  case yielding one real root in  $\lambda \in (-\infty, 1)$  interval.

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$-\infty$	+	-	+	+	-	3
1	-	-	+	+	-	2

Table III:  $-1 < \lambda_0 < \frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - 1$  case yielding one real root in  $\lambda \in (-\infty, 1)$  interval.

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$-\infty$	+	-	+	+	-	3
1	-	-	-	+	-	2

Table IV:  $\frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - 1 < \lambda_0 < C(n)$  case yielding one real root in  $\lambda \in (-\infty, 1)$  interval.

The following two tables Table-V and Table-VI show that for the nonunitary interval of  $\lambda_0$  there is no real solution.

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$-\infty$	+	-	+	+	-	3
1	+	-	-	+	-	3

Table V:  $C(n) < \lambda_0 < D(n)$  case yielding no real root in  $\lambda \in (-\infty, 1)$  interval.

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$-\infty$	+	-	+	+	+	2
1	+	-	-	+	+	2

Table VI:  $D(n) < \lambda_0$  case yielding no real root in  $\lambda \in (-\infty, 1)$  interval.

To get a sort of intuitive feeling, this analysis can be done graphically for each  $n$  and a specific viable  $\lambda_0$ : As an example, see Fig-1. One can show that for  $\lambda_0 < C(n)$ , there is one and only one  $\lambda$  value in  $\lambda < 1$  by investigating the asymptotes and extrema of  $f(\lambda)$  whose graph is given in Fig. 1.

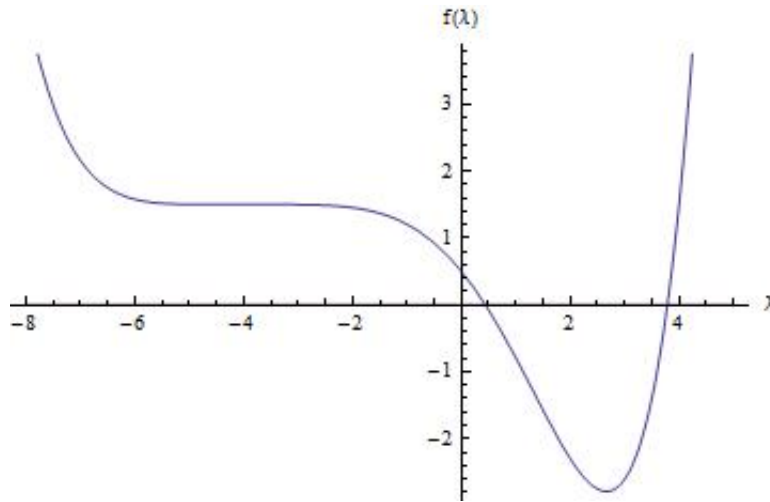


Figure 1: As an example, we chose  $n = 6$ ,  $\lambda_0 = 0.5$ , but the results are generic.

Note that this graph is representative of the generic shape of the  $f(y)$  function for any even  $n$  and for any  $\lambda_0$ . This can be seen as follows:  $\lim_{y \rightarrow \pm\infty} f(y) \rightarrow +\infty$ , the inflection point  $y_{\text{inf}} = 0$  satisfies  $y_{\text{inf}} < y_{\text{min}} = \frac{2(n-1)}{n}$ , finally the value of the function at the inflection point is always larger than the value of the function at  $y_{\text{min}}$ , namely

$$a = f(0) > f\left(\frac{2(n-1)}{n}\right) = -\left(\frac{2}{n}\right)^n (n-1)^{n-1} + a. \quad (\text{C33})$$

Therefore, the equation  $f(y) = 0$  has real root(s) if and only if

$$f\left(\frac{2(n-1)}{n}\right) \leq 0, \quad (\text{C34})$$

when the bound is saturated there is a unique root. This condition requires an upper bound on  $\lambda_0$  as

$$\lambda_0 \leq D(n) = \left(\frac{2}{n}\right)^n (n-1)^{n-1} - 1. \quad (\text{C35})$$

One should compare this bound, coming from the existence of a vacuum, to the bound on  $\lambda_0$ , coming from the unitarity of the theory (C21).

*The conclusion of the above analysis is that for even dimensions, there is a unique viable vacuum,  $\lambda$ , in the region  $-\infty < \lambda < 1$  and  $\lambda \neq 2 - n$  given that  $-\infty < \lambda_0 < C(n)$  and  $\lambda_0 \neq -1$ .*

*b. Odd dimensions:  $n = 5 + 2k$ ,  $k = 0, 1, \dots$*

For unitarity  $\tilde{\kappa} > 0$  must be satisfied which requires  $2 - n < \lambda < 1$  in odd dimensions as is clear from (62). Therefore, for odd dimensions, the relevant interval is  $y \in \left(0, \frac{n-1}{n-2}\right)$ . There is a corresponding viable interval of  $\lambda_0$  since  $\lambda_0 = \lambda_0(y)$ , that is (C19), is a monotonically increasing function of  $y$  as  $d\lambda_0/dy$ , that is (C20), is positive in the unitarity region due to positivity of  $y$  and the same reasoning in the even-dimensional case follows. The lower bound of  $y$  yields a lower bound for  $\lambda_0$ , so unitarity region of  $\lambda$  corresponds to  $-1 < \lambda_0 < C(n)$ . In Table 7, the values of the Sturm functions are given at the endpoints of the unitary interval of  $\lambda$ .

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$
$2 - n$	$\lambda_0 + 1$	0	$-\lambda_0 - 1$	$(\lambda_0 + 1) \frac{n^2}{2}$	$\lambda_0 - D(n)$
1	$\lambda_0 - C(n)$	$-(n-4) \left(\frac{n-1}{n-2}\right)^{n-1}$	$\frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - \lambda_0 - 1$	$(\lambda_0 + 1) \frac{n^2(n-4)}{4(n-2)}$	$\lambda_0 - D(n)$

Table VII: For odd  $n$ , the values of the Sturm functions at the endpoints of the unitary interval of  $\lambda$ .

The number of roots depends on the zeros of the expressions in the second and third rows of the Table VII. In the second row, the only root is  $\lambda_0 = -1$  which was already a root of the third row. The roots in the third row were investigated in the even dimensional case and they can be ordered as

$$-1 < \frac{4(n-2)^2}{n^2(n-3)} \left(C(n) + 1\right) - 1 \leq C(n) \leq D(n). \quad (\text{C36})$$

In the tables below, for all values of  $\lambda_0$ , the number of roots in the unitarity interval  $2 - n < \lambda < 1$  is investigated. As a result, for each value of  $\lambda_0$  in the interval  $-1 < \lambda_0 < C(n)$ , there is one and only one root for the vacuum equation in the unitary interval of  $\lambda$ , and for  $\lambda_0 < -1$  and  $\lambda_0 > C(n)$ , it is not possible to have a  $\lambda$  value in the unitary interval.

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$2 - n$	-	0	+	-	-	2
1	-	-	+	-	-	2

Table VIII:  $\lambda_0 < -1$  case yielding no real root in  $\lambda \in (2 - n, 1)$  interval.



$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$2 - n$	+	0	-	+	-	3
1	-	-	+	+	-	2

Table IX:  $-1 < \lambda_0 < \frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - 1$  case yielding one real root in  $\lambda \in (2 - n, 1)$  interval.

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$2 - n$	+	0	-	+	-	3
1	-	-	-	+	-	2

Table X:  $\frac{4(n-2)^2}{n^2(n-3)} (C(n) + 1) - 1 < \lambda_0 < C(n)$  case yielding one real root in  $\lambda \in (2 - n, 1)$  interval.

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$2 - n$	+	0	-	+	-	3
1	+	-	-	+	-	3

Table XI:  $C(n) < \lambda_0 < D(n)$  case yielding no real root in  $\lambda \in (2 - n, 1)$  interval.

$\lambda$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	# of sign changes
$2 - n$	+	0	-	+	+	2
1	+	-	-	+	+	2

Table XII:  $D(n) < \lambda_0$  case yielding no real root in  $\lambda \in (2 - n, 1)$  interval.

Again, for given  $n$  and  $\lambda_0$  this analysis can be done graphically by investigating the asymptotes and extrema of  $f(\lambda)$  whose graph is given in Fig. 2.

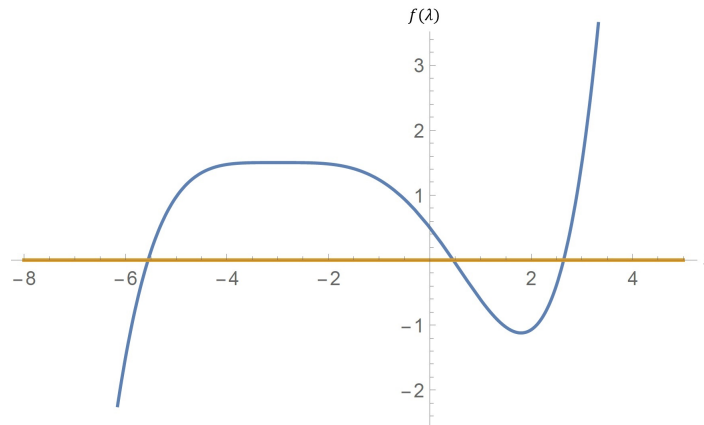


Figure 2:  $n = 5, \lambda_0 = 0.5$

Note that this graph is representative of the generic shape of  $f(y)$  function for any odd  $n$  and for any  $\lambda_0$ . This can be seen as follows:  $\lim_{y \rightarrow \pm\infty} f(y) \rightarrow \pm\infty$ , the inflection point  $y_{\text{inf}} = 0$  satisfies

$y_{\text{inf}} < y_{\text{min}} = \frac{2(n-1)}{n}$ , finally the value of the function at the inflection point is always larger than the value of the function at  $y_{\text{min}}$ , namely

$$a = f(0) > f\left(\frac{2(n-1)}{n}\right) = -\left(\frac{2}{n}\right)^n (n-1)^{n-1} + a.$$

Therefore, the equation  $f(y) = 0$  has one real root for either  $f(y_{\text{inf}}) < 0$  or  $f(y_{\text{min}}) > 0$ , and there are three real roots if  $f(y_{\text{inf}}) > 0$  and  $f(y_{\text{min}}) < 0$ . In addition, there are two real roots for either  $f(y_{\text{inf}}) = 0$  or  $f(y_{\text{min}}) = 0$ . For the one real root case,  $\lambda_0$  should satisfy either

$$f(0) = a < 0 \Rightarrow \lambda_0 < -1,$$

or

$$f\left(\frac{2(n-1)}{n}\right) = -\left(\frac{2}{n}\right)^n (n-1)^{n-1} + a > 0 \Rightarrow \lambda_0 > \left(\frac{2}{n}\right)^n (n-1)^{n-1} - 1 \Rightarrow \lambda_0 > D(n).$$

But, since  $\lambda_0$  should be in the interval  $-1 < \lambda_0 < C(n)$  to have a unitary root, these conditions cannot be satisfied for a unitary theory since  $C(n) \leq D(n)$ . So, for a unitary theory in odd dimensions, there cannot be one real root only. Moving to the case of three real roots which is possible if  $-1 < \lambda_0 < D(n)$  is satisfied. Therefore, if the theory is unitary, that is if  $-1 < \lambda_0 < C(n)$ , then there are always three real roots since  $C(n) \leq D(n)$ , and as  $\lambda$  and  $\lambda_0$  are related to each other with a 1-1 correspondence in the unitary interval, there is always a unique viable vacuum and two nonunitary vacua. Lastly, for the case of two real roots, either  $\lambda_0 = -1$ , which is not possible, or  $\lambda_0 = D(n)$  which cannot be satisfied for an odd dimensional unitary theory. Hence, the case of two real roots does not yield a viable theory.

*The conclusion of the above analysis is that for odd dimensions, there is a unique viable vacuum,  $\lambda$ , in the region  $2 - n < \lambda < 1$  given that  $-1 < \lambda_0 < C(n)$ .*

- 
- [1] I. Gullu, T. C. Sisman and B. Tekin, “Born-Infeld Gravity with a Massless Graviton in Four Dimensions,” Phys. Rev. D **91**, no. 4, 044007 (2015).
  - [2] S. Deser and G. W. Gibbons, “Born-Infeld-Einstein actions?,” Class. Quant. Grav. **15**, L35 (1998).
  - [3] I. Gullu, T. C. Sisman, B. Tekin, “Born-Infeld extension of new massive gravity,” Class. Quant. Grav. **27**, 162001 (2010).
  - [4] I. Gullu, T. C. Sisman and B. Tekin, “c-functions in the Born-Infeld extended New Massive Gravity,” Phys. Rev. D **82**, 024032 (2010).
  - [5] K. S. Stelle, “Renormalization of Higher Derivative Quantum Gravity,” Phys. Rev. D **16**, 953 (1977).
  - [6] I. Gullu, T. C. Sisman and B. Tekin, to appear soon.
  - [7] D. G. Boulware and S. Deser, “String Generated Gravity Models,” Phys. Rev. Lett. **55**, 2656 (1985).
  - [8] I. Gullu, T. C. Sisman and B. Tekin, “Unitarity analysis of general Born-Infeld gravity theories,” Phys. Rev. D **82**, 124023 (2010).
  - [9] T. C. Sisman, “Born-Infeld gravity theories in D-dimensions,” PhD thesis, METU, (2012).
  - [10] A. Hindawi, B. A. Ovrut and D. Waldram, “Nontrivial vacua in higher derivative gravitation,” Phys. Rev. D **53**, 5597 (1996).
  - [11] T. C. Sisman, I. Gullu and B. Tekin, “All unitary cubic curvature gravities in D dimensions,” Class. Quant. Grav. **28**, 195004 (2011).
  - [12] I. Gullu, T. C. Sisman, B. Tekin, “All Bulk and Boundary Unitary Cubic Curvature Theories in Three Dimensions,” Phys. Rev. D **83**, 024033 (2011).
  - [13] C. Senturk, T. C. Sisman and B. Tekin, “Energy and Angular Momentum in Generic F(Riemann) Theories,” Phys. Rev. D **86**, 124030 (2012).
  - [14] L. F. Abbott and S. Deser, “Stability of Gravity with a Cosmological Constant,” Nucl. Phys. B **195**, 76 (1982).

- [15] S. Deser and B. Tekin, “*Gravitational energy in quadratic curvature gravities*,” Phys. Rev. Lett. **89**, 101101 (2002).
- [16] S. Deser and B. Tekin, “*Energy in generic higher curvature gravity theories*,” Phys. Rev. D **67**, 084009 (2003).
- [17] R. Emparan, D. Grumiller and K Tanabe, “*Large-D gravity and low-D strings*,” Phys. Rev. Lett. **110**, 251102 (2013).
- [18] A. Eddington, *The Mathematical Theory of General Relativity* (Cambridge University Press, Cambridge, England, 1924).
- [19] M. Born and L. Infeld, “*Foundations of the new field theory*,” Proc. Roy. Soc. Lond. A **144**, 425 (1934).
- [20] M. Banados and P. G. Ferreira, “*Eddington’s theory of gravity and its progeny*,” Phys. Rev. Lett. **105**, 011101 (2010).
- [21] T. Delsate and J. Steinhoff, “*New insights on the matter-gravity coupling paradigm*,” Phys. Rev. Lett. **109**, 021101 (2012).
- [22] F. Fiorini, “*Nonsingular Promises from Born-Infeld Gravity*,” Phys. Rev. Lett. **111**, 041104 (2013).
- [23] J. B. Jiménez, L. Heisenberg and G. J. Olmo, “*Infrared lessons for ultraviolet gravity: the case of massive gravity and Born-Infeld*,” JCAP **1411**, 004 (2014).
- [24] J. B. Jiménez, L. Heisenberg and G. J. Olmo and C. Ringeval, “*Cascading dust inflation in Born-Infeld gravity*,” arXiv:1509.01188 [gr-qc].
- [25] K. Peeters, “*Cadabra: a field-theory motivated symbolic computer algebra system*”, Comput. Phys. Commun., **176**, 550 (2007).
- [26] K. Peeters, “*Introducing Cadabra: A symbolic computer algebra system for field theory problems*”, arXiv:hep-th/0701238.
- [27] N. B. Conkwright, “*Introduction to the Theory of Equations*,” Ginn and Company, Boston, MA, (1957).