

AdS-Wave Solutions of $f(\text{Riemann})$ Theories

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We show that the recently found AdS-plane and AdS-spherical wave solutions of quadratic curvature gravity also solve the most general higher derivative theory in D -dimensions. More generally, we show that the field equations of such theories reduce to an equation linear in the Ricci tensor for Kerr-Schild spacetimes having type-N Weyl and traceless Ricci tensors.

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There is a vast literature on the exact solutions of four-dimensional Einstein's gravity. But, as more powers of curvature are added, or computed in a microscopic theory such as string theory, to get a better UV behaved theory, the field equations become highly nontrivial and so solutions are not easy to find. In fact, only a few classes of solutions are known: For example, see [1–4] for solutions in low energy string theory. $\text{AdS}_5 \times S^5$, which played a major role in AdS/CFT, is also expected to be an exact solution of string theory [5]. In this work, we present new asymptotically AdS solutions, which are AdS-plane and AdS-spherical waves, to D -dimensional generic gravity theories based on the Riemann tensor and its arbitrary number of covariant derivatives which are in some sense natural geometric extensions of Einstein's gravity. Certain low energy string theory actions constitute a subclass of this theory once all the non-gravitational fields are turned off [6]. Asymptotically AdS solutions in higher derivative theories are relevant in the context of generic gravity/gauge theory dualities and holography. Here, we shall provide such solutions.

Using a theorem given in [4], we first prove that any spacetime with type-N Weyl and traceless Ricci tensors, where the metric is the Kerr-Schild form, the field equations of the most general higher derivative theory reduce to a linear equation for the traceless Ricci tensor. These spacetimes have constant scalar invariants. Furthermore, using the type-N property of the traceless Ricci tensor in these field equations, we obtain a linear partial differential equation for the metric function V of order $2N$ in the AdS background, where N is related to the number of covariant derivatives in the action of the theory. This result implies that the AdS-wave metrics are universal in the sense defined in [4].

As a special case, the field equations of the theory

which depends on the contractions of the Riemann tensor but not on its derivatives, $f(R_{\rho\sigma}^{\mu\nu})$ theory, are also highly cumbersome, but using our general result for type-N spacetimes under certain assumptions they reduce to those of the quadratic gravity. Then, taking the metric to be in the Kundt subclass of type-N spacetimes we show that AdS-plane wave [7, 8] and AdS-spherical wave [9] solutions of the quadratic gravity theory are also the solutions of the $f(R_{\rho\sigma}^{\mu\nu})$ theory. Log terms arising in the solutions of the quadratic gravity exist also in some $f(R_{\rho\sigma}^{\mu\nu})$ theories corresponding to the generalizations of critical gravity [10, 11]. As an application of our result, we show that any type-N Einstein space ($R_{\mu\nu} = \frac{R}{D}g_{\mu\nu}$) solve the field equations of the Lanczos-Lovelock theory. In addition, for a special choice of the parameters, any spacetime metric having type-N Weyl and traceless Ricci tensors with constant Ricci scalar solve the specific Lanczos-Lovelock theory.

To find exact asymptotically AdS solutions of a generic higher derivative theory, we shall make use of general results of [12–16] which utilize the boost weight formalism.

Type-N Weyl and Traceless Ricci tensors The Weyl tensor of type-N spacetimes have been studied in detail in [12–15]. Let ℓ , \mathbf{n} and $\mathbf{m}^{(i)}$, ($i = 2, \dots, D-1$) be a null tetrad frame with

$$\begin{aligned} \ell_\alpha \ell^\alpha &= n^\alpha n_\alpha = \ell^\alpha m_\alpha^{(i)} = n^\alpha m_\alpha^{(i)} = 0, \\ \ell^\alpha n_\alpha &= 1, \quad m^{(i)\alpha} m_\alpha^{(j)} = \delta_{ij}, \end{aligned} \quad (1)$$

where $\alpha, \beta, \dots = 0, 1, 2, \dots, D-1$ and $i, j = 2, 3, \dots, D-1$. Then spacetime metric takes the form

$$g_{\mu\nu} = \ell_\mu n_\nu + \ell_\nu n_\mu + \delta_{ij} m_\mu^{(i)} m_\nu^{(j)}. \quad (2)$$

The Weyl tensor of type-N spacetimes, expressed in the above frame, where ℓ is the Weyl aligned null direction takes the form

$$C_{\alpha\beta\gamma\delta} = 4\Omega'_{ij} \ell_{\{\alpha} m_{\beta}^{(i)} \ell_\gamma m_{\delta}^{(j)}. \quad (3)$$

which transforms under scale transformations with boost weight -2 .

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Type-N property alone is not sufficient to reduce the field equations of higher derivative gravity theories to a solvable form, we make a further assumption that the spacetime is radiating and the Ricci scalar is constant. For radiating type-N spacetimes, the Ricci tensor has the form

$$R_{\mu\nu} = \rho \ell_\mu \ell_\nu + \frac{R}{D} g_{\mu\nu}, \quad (4)$$

where ρ is a scalar function and R is the Ricci scalar. Taking R to be a constant and using the Bianchi identity, one obtains

$$\nabla^\mu (\rho \ell_\mu \ell_\nu) = 0. \quad (5)$$

Notice that the traceless part

$$S_{\mu\nu} = R_{\mu\nu} - \frac{R}{D} g_{\mu\nu} = \rho \ell_\mu \ell_\nu \quad (6)$$

of the Ricci tensor is of type-N. Following [4], and paraphrasing the statement given in this work we have the following result:

Theorem: *Consider a Kundt spacetime for which the traceless Ricci and the Weyl tensors be of type-N (4), and all scalar invariants be constant. Then, any second rank symmetric tensor constructed by the Riemann tensor and its covariant derivatives is a linear combination of $g_{\mu\nu}$, $S_{\mu\nu}$, and higher orders of $S_{\mu\nu}$ (such as, for example, $\square^n S_{\mu\nu}$).*

The proof of the above statement depends heavily on the boost-weight formalism [4],[19], [18]. In particular, a Kundt spacetime of Ricci and Weyl type N needs to be degenerate Kundt, which further implies, using the same arguments as in [4], that the traceless part of any second rank symmetric tensor must also be of type N. Using the appendix of [4], and the proof of theorem 2.7 in [18], one can now see that the theorem follows.

An immediate consequence of above theorem is as follows:

For the Kundt type of Kerr-Schild metrics where $S_{\mu\nu} = \rho \ell_\mu \ell_\nu$, we get $\square^n S_{\mu\nu} = \ell_\mu \ell_\nu \mathcal{O}^n \rho$ for all $n = 0, 1, 2, \dots$ (we shall give the definition of the operator \mathcal{O} shortly), then the higher orders of $S_{\mu\nu}$ vanish identically. This nice property of this Kerr-Schild metrics leads to the following more general result. The field equations corresponding to the most general action;

$$I = \int d^D x \sqrt{-g} f(g^{\alpha\beta}, R^\mu{}_{\nu\gamma\sigma}, \nabla_\rho R^\mu{}_{\nu\gamma\sigma}, \dots, (\nabla_{\rho_1} \nabla_{\rho_2} \dots \nabla_{\rho_M}) R^\mu{}_{\nu\gamma\sigma}, \dots), \quad (7)$$

are of the form

$$E_{\mu\nu} = e g_{\mu\nu} + \sum_{n=0}^N a_n \square^n S_{\mu\nu} = 0, \quad (8)$$

where e and a_n 's are all constants depending on the form of the action. Hence we reduced the field equations of the most general higher derivative theories to a form linear in the Ricci tensor. Furthermore, the trace part of (8) is

$$e = 0, \quad (9)$$

which gives the relation between the cosmological constant and the parameters of the theory, and the traceless part is

$$\sum_{n=0}^N a_n \square^n (\rho \ell_\mu \ell_\nu) = 0. \quad (10)$$

This is still hard to solve: despite its appearance it is a nonlinear equation. But, we know at least two solutions which are the AdS-plane [7, 8] and the AdS-spherical waves [9]. These waves are of the Kerr-Schild form, $g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V \ell_\mu \ell_\nu$, which belong to the Kundt class of type-N spacetimes [19, 20]. Here, $\bar{g}_{\mu\nu}$ is the AdS background metric and the function V satisfies $\ell^\mu \partial_\mu V = 0$. For these metrics, ρ has the form

$$\rho \equiv \mathcal{O}V = \left[\bar{\square} + 2\xi_\mu \partial^\mu + \frac{1}{2} \xi_\mu \xi^\mu - 2k^2 (D-2) \right] V, \quad (11)$$

where $\bar{\square}$ is the Laplace-Beltrami operator of the AdS background and ξ_μ arises in $\nabla_\mu \ell_\nu = \ell_{(\mu} \xi_{\nu)}$. Because of (11), Eqn. (10) reduces to

$$\sum_{n=0}^N a_n \mathcal{O}^{n+1} V = 0. \quad (12)$$

For $N > 1$, this equation can be factorized as

$$\prod_{n=0}^N (\mathcal{O} + b_n) \mathcal{O}V = 0, \quad (13)$$

where some b_n 's are real and some are complex constants in general (complex b_n 's come in complex conjugate pairs). Then, the most general solution is

$$V = \Re \left(\sum_{i=0}^N V_i \right),$$

where \Re represents the real part and V_i 's solve $(\mathcal{O} + b_i) V_i = 0$, $i = 0, 1, 2, \dots, N$. Solutions of such linear partial differential equations are given in [7] and [9] in the of AdS background. As a conclusion we can say that the AdS- wave metrics found recently [7] and [9] solve the field equations of the most general higher derivative theories. To give explicit exact solutions we have to know the constants b_i , $i = 0, 1, 2, \dots$ in terms of the theory parameters. For this purpose we shall consider below some special cases.

Type-N Spacetimes in Quadratic Gravity – For type-N Weyl and type-N traceless Ricci tensor and (5) the field equations of quadratic gravity [17]

$$\begin{aligned} & \frac{1}{\kappa} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_0 g_{\mu\nu} \right) \\ & + 2\alpha R \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) + (2\alpha + \beta) (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) R \\ & + \beta \square \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + 2\beta \left(R_{\mu\sigma\nu\rho} - \frac{1}{4} g_{\mu\nu} R_{\sigma\rho} \right) R^{\sigma\rho} \\ & + 2\gamma \left[R R_{\mu\nu} - 2 R_{\mu\sigma\nu\rho} R^{\sigma\rho} + R_{\mu\sigma\rho\tau} R_\nu^{\sigma\rho\tau} - 2 R_{\mu\sigma} R_\nu^\sigma \right. \\ & \quad \left. - \frac{1}{4} g_{\mu\nu} (R_{\tau\lambda\sigma\rho}^2 - 4 R_{\sigma\rho}^2 + R^2) \right] = 0, \end{aligned} \quad (14)$$

reduce to the following simplified equations [14, 15],

$$(\beta \square + c) (\rho \ell_\mu \ell_\nu) = 0, \quad (15)$$

and a trace equation that gives a relation between the constant R and the parameters of the theory. Here, c is given as [17]

$$c \equiv \frac{1}{\kappa} + 2R\alpha + \frac{2(D-2)}{D(D-1)} R\beta + \frac{2(D-3)(D-4)}{D(D-1)} R\gamma. \quad (16)$$

Exact solutions, the AdS-wave metrics, of this have been reported recently [7, 8] and [9].

f(R^{μν}_{ρσ}) theory – A case which is more general than the quadratic gravity is the f (Riemann) theory where the action is given by

$$I = \int d^D x \sqrt{-g} f(R_{\rho\sigma}^{\mu\nu}), \quad (17)$$

For this case $N = 1$ and hence (10) reduces to

$$(a \square + b) (\rho \ell_\mu \ell_\nu) = 0, \quad (18)$$

where a and b are constants. For AdS-wave metrics in the Kerr-Schild form, $g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V \ell_\mu \ell_\nu$, the field equations reduce to a fourth order linear partial differential equation with constant coefficients of the form

$$\left(\bar{\square} - \frac{2R}{D(D-1)} - M^2 \right) \left(\bar{\square} - \frac{2R}{D(D-1)} \right) (V \ell_\mu \ell_\nu) = 0, \quad (19)$$

where $\bar{\square}$ is the Laplace-Beltrami operator of the AdS background and

$$M^2 = -\frac{b}{a} - \frac{2R}{D(D-1)}, \quad (20)$$

which corresponds to the mass of the spin-2 excitation in the linearized version of f (Riemann) theory about AdS [21]. For AdS-plane wave, $\bar{g}_{\mu\nu} = -\frac{R}{D(D-1)z^2} \eta_{\mu\nu}$

where $z > 0$ is one of the spatial coordinates, then $\ell_\mu = (1, 1, 0, \dots, 0)$, $\xi_\mu = \frac{2}{z} \delta_\mu^z$ and the most general solution of (19) was given in [7]. In this case ℓ_μ is proportional to the null Killing vector ζ_μ , i.e., $\zeta_\mu = \frac{1}{z} \ell_\mu$. For AdS-spherical wave, $\ell_\mu = \left(1, \frac{x^i}{r}\right)$, $\xi_\mu = -\frac{1}{r} \ell_\mu + \frac{2}{r} \delta_\mu^t + \frac{2}{z} \delta_\mu^z$ where $r^2 = \sum_{i=1}^{D-1} (x^i)^2$ and $x^{D-1} = z$, and the most general solution of (19) was given in [9]. In this case there exists no null Killing vector fields. For $M^2 \neq 0$, V decays sufficiently fast for both AdS-wave solutions and they are asymptotically AdS. When $M^2 = 0$, there are logarithmic solutions which spoil the asymptotically AdS structure in both cases. Let us give two concrete $f(R_{\rho\sigma}^{\mu\nu})$ theories as examples.

Cubic gravity generated by string theory – In [6], the bosonic string has the following effective Lagrangian density at $O[(\alpha')^2]$

$$\begin{aligned} f_{\text{eff}} = & R + \frac{\alpha'}{4} \left(R_{\alpha\beta}^{\mu\nu} R_{\mu\nu}^{\alpha\beta} - 4 R_\nu^\mu R_\mu^\nu + R^2 \right) \\ & + \frac{(\alpha')^2}{24} \left(-2 R^{\mu\alpha\nu\beta} R_{\mu\nu}^{\lambda\gamma} R_{\alpha\gamma\beta\lambda} + R_{\alpha\beta}^{\mu\nu} R_{\mu\nu}^{\gamma\lambda} R_{\gamma\lambda}^{\alpha\beta} \right), \end{aligned} \quad (21)$$

where α' is the usual inverse string tension. Using the results of [22], the field equations of (21) for the AdS-plane and AdS-spherical wave metrics reduce to (19) with a and M^2 given as

$$a = \frac{7\alpha'^2 R}{4D(D-1)}, \quad (22)$$

$$\begin{aligned} M^2 = & -\frac{4D(D-1)}{7\alpha'^2 R} - \frac{2(D-3)(D-4)}{7\alpha'} \\ & + \frac{9D-29}{7D(D-1)} R. \end{aligned} \quad (23)$$

Therefore, the solutions quoted above are the solutions of (21) with this M^2 .

Lanczos-Lovelock theory – Another special case of the f (Riemann) theory is the Lanczos-Lovelock theory given with the Lagrangian density

$$f_{\text{L-L}} = \sum_{n=0}^{\lfloor \frac{D}{2} \rfloor} a_n \delta_{\nu_1 \dots \nu_{2n}}^{\mu_1 \dots \mu_{2n}} \prod_{p=1}^n R_{\mu_{2p-1} \nu_{2p}}^{\nu_{2p-1} \mu_{2p}}, \quad (24)$$

where a_n 's are dimensionful constants, $\lfloor \frac{D}{2} \rfloor$ corresponds to the integer part of $\frac{D}{2}$ and $\delta_{\nu_1 \dots \nu_{2n}}^{\mu_1 \dots \mu_{2n}}$ is the generalized Kronecker delta. In this case, since the constant a in (18) vanishes identically, the field equations reduce to

$$b\rho = 0, \quad (25)$$

where b is calculated in [23] as

$$b = 2(D-3)! \sum_{n=0}^{\lfloor \frac{D}{2} \rfloor} a_n \frac{n(D-2n)}{(D-2n)!} \left[\frac{4\Lambda}{(D-1)(D-2)} \right]^{n-1}. \quad (26)$$

Equation (25) gives two subclasses. First class corresponds to $\rho = 0$ and the solution is type-N Einstein space. The second class corresponds to $b = 0$ which gives a relation between the parameters of the theory. In this subclass, any type-N radiating metric with constant Ricci scalar is an exact solution of the theory. AdS-wave metrics are exact solutions of the Lanczos-Lovelock theory. We note that the Lanczos-Lovelock theory is free of logarithmic solutions.

Conclusion– Type-N radiating spacetimes simplify the field equations of higher derivative theories. This simplification rests on the result: the second rank symmetric tensors constructed by contractions of type-N Weyl, and traceless Ricci tensors and their covariant derivatives re-

duce to a simpler form in general. To find explicit solutions, we showed that one has to consider a subclass that is the Kundt spacetime. Hence, Kerr-Schild metrics reported recently as the solutions of the quadratic gravity, the AdS-plane [7] and AdS-spherical wave metrics [9], which belong to the Kundt subclass of type-N spacetimes solve the field equations of any higher curvature gravity theory exactly. We gave a cubic theory coming from the string theory and the Lanczos-Lovelock theory as explicit examples.

A more detailed version of this work will be communicated elsewhere.

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- [1] G. T. Horowitz and A. R. Steif, Phys. Rev. Lett. **64**, 260 (1990).
 - [2] G. T. Horowitz and A. A. Tseytlin, Phys. Rev. D **51**, 2896 (1995).
 - [3] A. A. Coley, Phys. Rev. Lett. **89**, 281601 (2002).
 - [4] A. A. Coley, G. W. Gibbons, S. Hervik and C. N. Pope, Class. Quant. Grav. **25**, 145017 (2008).
 - [5] T. Banks and M. B. Green, JHEP **9805**, 002 (1998).
 - [6] R. R. Metsaev and A. A. Tseytlin, Phys. Lett. B **185**, 52 (1987).
 - [7] I. Gullu, M. Gurses, T. C. Sisman and B. Tekin, Phys. Rev. D **83**, 084015 (2011).
 - [8] M. Alishahiha and R. Fareghbal, Phys. Rev. D **83**, 084052 (2011).
 - [9] M. Gurses, T. C. Sisman and B. Tekin, Phys. Rev. D **86**, 024009 (2012).
 - [10] H. Lu and C. N. Pope, Phys. Rev. Lett. **106**, 181302 (2011).
 - [11] S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman and B. Tekin, Phys. Rev. **D83**, 061502 (2011).
 - [12] M. Ortaggio, V. Pravda and A. Pravdova, Class. Quantum Grav. **30** 013001, (2013).
 - [13] T. Malek and V. Pravda, Class. Quantum Grav. **28**,125011, (2011).
 - [14] T. Malek and V. Pravda, Phys. Rev. D **84**, 024047 (2011).
 - [15] M. Ortaggio, V. Pravda and A. Pravdova, Phys. Rev. D **82**, 064043 (2010).
 - [16] A. Coley, R. Milson, V. Pravda and A. Pravdova, Class. Quantum Grav. **21** (2004) 5519.
 - [17] S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002); Phys. Rev. **D67**, 084009 (2003).
 - [18] A. Coley, S. Hervik and N. Pelavas, Class. Quant. Grav. **27**, 102001 (2010).
 - [19] A. Coley, S. Hervik and N. Pelavas, Class. Quant. Grav. **23**, 3053-3074 (2006).
 - [20] A. Coley, S. Hervik, G. O. Papadopoulos and N. Pelavas, Class. Quant. Grav. **26**, 105016 (2009).
 - [21] C. Senturk, T. C. Sisman and B. Tekin, Phys. Rev. D **86**, 124030 (2012).
 - [22] T. C. Sisman, I. Gullu and B. Tekin, Class. Quant. Grav. **28**, 195004 (2011).
 - [23] T. C. Sisman, I. Gullu and B. Tekin, Phys. Rev. D **86**, 044041 (2012).