Chern-Simons Modified General Relativity: Conserved charges

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We construct the conserved charges (mass and angular momentum) of the Chern-Simons modified General Relativity in asymptotically flat and Anti-de Sitter (AdS) spacetimes. Our definition is based on background Killing symmetries and reduces to the known expressions in the proper limits.

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I. INTRODUCTION

Soon after General Relativity (GR) was written, various modifications of it- such as adding higher curvature terms based on the invariants of the Riemann tensor- or adding scalar fields appeared. In the beginning, these attempts usually arose from pure academic interest and eventually tried to answer the question of how unique GR was. The simplest modification of GR, adding a cosmological constant to the action, was done by Einstein himself (although his reason of bringing in a cosmological constant was more physical than of academic curiosity.) More recently, modifying gravity at various energy scales has become, one can safely say, a necessity. In the solar system, for which we have plenty of experimental tests, GR works perfectly well and so there is no compelling reason to modify it. But at very short and large distances, we know that GR cannot be the whole story and hence follows the various modifications and generalizations.

In the literature, most of the modifications of GR do not change the symmetries, such as parity invariance *etc.*. In this paper, we shall be interested in a more recent, rather unconventional yet very intriguing modification introduced by Jackiw and Pi [1]. These authors define a new *four* dimensional, symmetric tensor which is analogous to the *three* dimensional Cotton tensor (or Chern-Simons term in the action) which gave rise to the celebrated Deser-Jackiw-Templeton‘s theory of Topologically Massive Gravity (TMG) [2]. TMG and its abelian and non-abelian cousins (TM electrodynamics and TM Yang-Mills theory)have been much studied: But, since these are three dimensional theories, they do not shed much light on the problems of the four dimensional world, they are more relevant to the planar condensed matter systems. On the other hand, upgrading the Chern-Simons term to four dimensions could actually yield interesting physics; such as Lorentz violating gravity and so on. In fact, in the context of Maxwell‘s theory, such a modification was done some time ago by Carroll *et. al* [3] who obtained a Chern-Simons modified electrodynamics which apparently was falsified by measurements of light from distant galaxies [3].

In any matter-coupled gravity theory, that is based on Riemannian geometry, field equations relate at *each point* in space a covariantly conserved symmetric tensor, obtained from

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the metric tensor and its derivatives, to the local energy momentum tensor $\tau_{\mu\nu}(x)$ of matter fields. Since in a diffeomorphism invariant theory, the coordinates x^μ could be traded with some new ones, one cannot define a meaningful (that is gauge-invariant) local energy expression. But for certain spacetimes one can define total mass and angular momenta such as the Arnowitt-Deser-Misner (ADM) [4] mass in asymptotically flat spaces or the Abbott-Deser (AD) [5] mass for a asymptotically Anti-de Sitter (AdS) spaces. Mass and angular momentum in this construction is given in terms of asymptotic geometry and background Killing vectors. This approach has been quite useful and was carried out in detail for various higher derivative gravity models beyond (cosmological) general relativity: For example conserved charges (in any coordinates !) of higher curvature models were defined in [6] and TMG‘s charges were defined in [7]. These definitions were employed [8] to correctly compute the mass and angular momenta of the recently found D-dimensional Kerr-AdS black holes [9] and the supersymmetric solution of the TMG [10]. For a rigorous definition of asymptotically AdS spaces and a canonical construction of charges, we refer the reader to [11]. In this paper, we will construct the conserved charges of the Chern-Simons (CS) modified GR along the lines developed in the above mentioned works. As we shall see, it is a somewhat non-trivial problem and a brute-force approach of getting the charges in generic coordinates for both asymptotically flat and asymptotically AdS spaces is most certainly bound to fail unless some simplification techniques that proved very handy in [6, 7] are employed.

The outline of the paper is as follows: we first review the basics of the CS modified GR and recall some recently found facts about this model and then we briefly summarize how conserved charges, arising from background Killing vectors, can be constructed in generic gravity theories. Finally we find a general formula for Killing charges in the CS modified (cosmological) GR. Our formula is generic enough to give the mass of the asymptotically flat and asymptotically AdS spacetimes as for the angular momentum, we only provide the formula for asymptotically flat backgrounds.

II. CS MODIFIED GR

Here we introduce and quote the very basics of the model and refer the reader to the original paper [1] and a nice recent review [12] or to [13]. [This latter work has constructed a Papapetrou type pseudo energy-momentum *tensor* which is *not* what we are about to construct here. What we will find is a background conserved, coordinate independent total mass (as a component of a four vector, not a tensor) and the angular momentum in asymptotically AdS or asymptotically flat spaces.] On the left-hand side of the field equations, we have the usual Einstein tensor plus a new symmetric tensor:

$$G_{\mu\nu} + C_{\mu\nu} = 8\pi G \tau_{\mu\nu} \,, \tag{1}$$

where the “Cotton” term reads

$$C^{\mu\nu} = -\frac{1}{2\sqrt{-g}}\Big(v_\sigma (\epsilon^{\sigma\mu\alpha\beta}\nabla_\alpha R^\nu{}_\beta + \epsilon^{\sigma\nu\alpha\beta}\nabla_\alpha R^\mu{}_\beta) + \nabla_\sigma v_\tau \left(*R^{\tau\mu\sigma\nu} + *R^{\tau\nu\sigma\mu}\right)\Big) \,. \tag{2}$$

The dual Riemann tensor is defined as

$$* R^\tau{}_\sigma{}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R^\tau{}_{\sigma\alpha\beta}, \tag{3}$$

and the four-vector v_σ, coined as the *embedding coordinate*, is taken to be timelike and *constant*: $v_\sigma = (\frac{1}{\mu}, 0, 0, 0)$ to carry out the analogy with the well-known three dimensional TMG [1].

Of course, one would like to derive field equations from an action, which in this case is quite simple and well-known

$$S = \frac{1}{16\pi G}\int d^4x\left(\sqrt{-g}R + \frac{1}{4}\theta(x) * RR\right) . \tag{4}$$

For constant θ, the second term does not contribute to the field equations since it can be written as a boundary term. For $\theta(x) = v_\sigma x^\sigma$, minimization yields the field equations (1). This is actually a somewhat unconventional theory since there appears a non-dynamical external 'field'. [See [14, 15] which discuss how a Lorentz-violating term such as above can be radiatively generated in quantum theory.] Of course, we could consider adding a kinetic term to $\theta(x)$ and not restrict it to the above form. Then, we would have an axion field coupled to gravity which apparently can be obtained from string theory [16]. (See also the Appendix of [17] for a nice derivation of this.) For our purposes, we shall take $\theta(x)$ to be a non-dynamical external field, since getting conserved charges in the dynamical model is not too different from what we shall do below.

There are at least two important problems with (1). First of all, unlike GR, these equations are incomplete in the sense that the covariant divergence of the added piece is non-zero and one has an extra constraint:

$$\nabla_\mu C^{\mu\nu} = \frac{1}{8\sqrt{-g}} v^\nu * RR. \tag{5}$$

So, for consistency solutions of this model should satisfy $*RR = 0$ and therefore the usual Kerr metric does not solve the field equations but the Schwarzschild black hole does [1]. Perturbation theory around the Schwarzschild solution ruled out small angular momentum for time-like v_σ recently [18].[Thus, one still needs to look for a Kerr type rotating solution for this theory.] We will see if and how the constraint (5) will play a role in the construction of conserved charges. Second problem is that there is an explicit function $\theta(x)$ which seems to ruin our hope of finding conserved mass and angular momentum. Of course, for a generic external field, the theory will have no conserved charges. But, if the function is chosen as above (namely, $\theta(x) = t/\mu$) then, both mass and angular momentum will be conserved, since constant shifts in time ($t \to t + t_0$) are still symmetries of the theory (owing to the fact that $*RR$ is a boundary term) and pure spatial rotational symmetries also survive since $\theta(x)$ was chosen to be time like and independent of $\vec{x}$.

As promised above, we now add a cosmological constant and consider

$$G_{\mu\nu} + C_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \tau_{\mu\nu} . \tag{6}$$

Note that even if we are interested in the $\Lambda = 0$ theory, for the sake of computations, it is actually better to start with (6) since then, one relatively easily ends up with conserved charges in generic coordinates as opposed to the Cartesian coordinates. Even before declaring what v_σ is, it is easy to check that global AdS

$$\bar{R}_{\mu\alpha\nu\beta} = \frac{\Lambda}{3}\left(\bar{g}_{\mu\nu}\,\bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta}\,\bar{g}_{\alpha\nu}\right), \quad \bar{R}_{\mu\nu} = \Lambda\,\bar{g}_{\mu\nu} \quad ,$$

is a vacuum ($\tau_{\mu\nu}=0$) solution since we have $*\bar{R}^{\tau}{}_{\sigma}{}^{\mu\nu}=\frac{\Lambda}{3}\epsilon^{\mu\nu\tau}{}_{\sigma}$. (Note I will denote all the vacuum, or the background quantities with a bar.) Although, I will not prove here, it is straightforward yet somewhat lengthy to show that AdS-Schwarzschild black hole is also a solution to our full theory.

III. KILLING CHARGES

The general formalism of constructing ordinarily conserved charges (as opposed to covariantly conserved ones) based on background Killing vectors was given in [6]. Here we recapitulate some of the material which we shall use. We will be working with a $(-,+,+,+)$ metric and our sign conventions are $[\nabla_\mu,\nabla_\nu]V_\lambda\equiv R_{\mu\nu\lambda}{}^{\sigma}V_\sigma$ and $R_{\mu\nu}=R_{\mu\lambda\nu}{}^{\lambda}$.

Let $\bar{g}_{\mu\nu}$ denote the background metric for the AdS (or flat space) that solves our full equations without $\tau_{\mu\nu}$ and let

$$g_{\mu\nu}\equiv\bar{g}_{\mu\nu}+h_{\mu\nu}\,, \tag{7}$$

solve the equations with a non-vanishing matter density. This equation is exact and $h_{\mu\nu}$ need not be small everywhere but we require that at infinity, away from the sources, it goes to zero. (Otherwise, one cannot consider $\bar{g}_{\mu\nu}$ as a background spacetime with zero charges and $h_{\mu\nu}$ as a perturbation outside a compact region, whose charge is measured with respect to the background.)

Let $T_{\mu\nu}(h)$ denote the matter $\tau_{\mu\nu}$ plus all the higher power terms in h. Linearizing the full equations we have:

$$T_{\mu\nu}(h)=R^{L}{}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}R^{L}-\Lambda h_{\mu\nu}+C^{L}{}_{\mu\nu}\,. \tag{8}$$

Some new notation will help us later, let us introduce

$$\mathcal{G}_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda\,g_{\mu\nu}\,, \tag{9}$$

whose linear part reads as

$$\mathcal{G}_{\mu\nu}{}^{L}=R^{L}{}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}R^{L}-\Lambda h_{\mu\nu}\,. \tag{10}$$

As can be checked, this linearized tensor is background covariantly conserved: $\bar{\nabla}_\mu\mathcal{G}^{\mu\nu}{}_{L}=0$. So, in order to now have $\bar{\nabla}_\mu T^{\mu\nu}{}_{L}=0$, we should have $\bar{\nabla}_\mu C^{\mu\nu}{}_{L}=0$. To see that this holds, we can linearize the consistency condition (5) which upon explicit use of

$$(R^{\mu}{}_{\alpha\beta\nu})_L=\bar{\nabla}_\beta\,\delta\Gamma^{\mu}{}_{\alpha\nu}-\bar{\nabla}_\nu\,\delta\Gamma^{\mu}{}_{\alpha\beta}, \tag{11}$$

and

$$(*R^{\tau\mu\sigma\nu})_L=\frac{1}{2}\epsilon^{\sigma\nu\alpha\beta}\bar{g}^{\mu\rho}(R^{\tau}{}_{\rho\alpha\beta})_L-\frac{\Lambda}{3}\epsilon^{\sigma\nu\tau}{}_{\rho}\,h^{\rho\mu}\,, \tag{12}$$

gives

$$\bar{\nabla}_\mu C^{\mu\nu}{}_{L}=0\,. \tag{13}$$

Thus, at the linearized level, the consistency condition (5) does *not* put any constraint on the metric. This fact is vital in constructing conserved charges. Let us explain how: if we have background symmetries as generated by background Killing vectors $\bar{\xi}^{I}_{\mu}$

$$\bar{\nabla}_\mu\xi^{I}_{\nu}+\bar{\nabla}_\nu\xi^{I}_{\mu}=0, \tag{14}$$

where I refers to different Killing vectors such as mass and angular momentum (I shall suppress this index in what follows): we can find partially conserved four-currents

$$\sqrt{-\bar{g}}\bar{\nabla}_\mu\Big(T^{\mu\nu}\,\bar{\xi}_\nu\Big)=\partial_\mu\Big(\sqrt{-\bar{g}}T^{\mu\nu}\,\bar{\xi}_\nu\Big)=0. \tag{15}$$

As usual, upon integration over the spatial 3-space and assuming that the three vector current falls of sufficiently fast at infinity, we will get conserved charges for the $\mu = 0$ component, which up to a normalization reads

$$Q^0=\int d^3\,x\sqrt{-\bar{g}}T^{0\nu}\bar{\xi}_\nu\,. \tag{16}$$

Given background symmetries, and a divergence-free energy-momentum tensor of matter fields, this approach guarantees the existence of the charges but getting them explicitly as surface integrals is a different issue which we shall work out below. [The reader might wonder why we have to convert the bulk integrals to surface integrals: In principle, we can of course work with the volume integrals but since in the bulk, there usually will be singularities of the spacetime, explicit computation of charges could become somewhat tricky. We can give the classical Maxwell theory as a simple analogy: volume integral of the divergence of the electric field has Dirac Delta functions coming from the points where the source charges are located. Hence we use the surface integral of the electric field (the Gauss law) to count the charges.]

We have to write $\sqrt{-\bar{g}}_\mu T^{\mu\nu}\,\bar{\xi}_\nu$ as a surface integral at spatial infinity. The cosmological Einstein part was taken care of in [6] or in a different notation earlier in [5]. Here we quote the final result

$$\begin{aligned}2\sqrt{\bar{g}}\,\mathcal{G}_L^{\mu\nu}\,\bar{\xi}_\nu \;=\; &\sqrt{\bar{g}}\bar{\nabla}_\rho\left(\bar{\xi}_\nu\,\bar{\nabla}^\mu\,h^{\rho\nu}-\bar{\xi}_\nu\,\bar{\nabla}^\rho\,h^{\mu\nu}+\bar{\xi}^\mu\,\bar{\nabla}^\rho\,h-\bar{\xi}^\rho\,\bar{\nabla}^\mu\,h\right.\\ &\left.+h^{\mu\nu}\,\bar{\nabla}^\rho\,\bar{\xi}_\nu-h^{\rho\nu}\,\bar{\nabla}^\mu\,\bar{\xi}_\nu+\bar{\xi}^\rho\,\bar{\nabla}_\nu\,h^{\mu\nu}-\bar{\xi}^\mu\,\bar{\nabla}_\nu\,h^{\rho\nu}+h\,\bar{\nabla}^\mu\,\bar{\xi}^\rho\right),\end{aligned} \tag{17}$$

where $h=\bar{g}^{\mu\nu}h_{\mu\nu}$. To get this expression one makes use of the linearized Ricci tensor

$$R^L_{\mu\nu}=\frac{1}{2}(-\bar{\Box}\,h_{\mu\nu}-\bar{\nabla}_\mu\,\bar{\nabla}_\nu\,h+\bar{\nabla}^\sigma\,\bar{\nabla}_\nu\,h_{\sigma\mu}+\bar{\nabla}^\sigma\,\bar{\nabla}_\mu\,h_{\sigma\nu})\,,$$

and the linearized Ricci scalar

$$R^L\equiv(R_{\mu\nu}\,g^{\mu\nu})^L=R^L_{\mu\nu}\,\bar{g}^{\mu\nu}-\Lambda\,h=-\bar{\Box}\,h+\bar{\nabla}_\mu\,\bar{\nabla}_\nu\,\bar{h}^{\mu\nu}-\Lambda\,h\,.$$

Let us now concentrate on the new four-dimensional Cotton part. But first, we get a great deal of simplification if we re-write the Cotton term (2) in the following form using (9)

$$C^{\mu\nu}=-\frac{1}{2\sqrt{-g}}\Big(v_\sigma(\epsilon^{\sigma\mu\alpha\beta}\nabla_\alpha\mathcal{G}^\nu{}_\beta+\epsilon^{\sigma\nu\alpha\beta}\nabla_\alpha\mathcal{G}^\mu{}_\beta)+\nabla_\sigma v_\tau\,(*R^{\tau\mu\sigma\nu}+*R^{\tau\nu\sigma\mu})\Big). \tag{18}$$

One should observe the nice cancellation in the $\frac{1}{2}g_{\mu\nu}R$ parts. The linearization of (18) about AdS yields

$$-2\sqrt{-\bar{g}}C^{\mu\nu}_L=v_\sigma\epsilon^{\sigma\mu\alpha\beta}\bar{g}^{\nu\rho}\bar{\nabla}_\alpha\mathcal{G}^L{}_{\rho\beta}+v_\sigma\epsilon^{\sigma\nu\alpha\beta}\bar{g}^{\mu\rho}\bar{\nabla}_\alpha\mathcal{G}^L{}_{\rho\beta}+\bar{\nabla}_\sigma v_\tau(*R^{\tau\mu\sigma\nu}{}_L+*R^{\tau\nu\sigma\mu}{}_L)\,. \tag{19}$$

At this point, it is better to separate the last term involving the linearized dual Riemann tensors from the first two terms. One can immediately see that, for flat backgrounds in the Cartesian coordinates, the last term will not contribute to the charges, since $\bar{\nabla}_\sigma v_\tau = \partial_\sigma v_\tau = 0$. For AdS backgrounds, on the other hand, the discussion bifurcates: the last term does not contribute to mass but it does contribute to the angular momentum. Thus in what follows, we will first discuss mass of asymptotically AdS spaces and then mass and angular momentum of asymptotically flat spaces. One can of course consider angular momentum in AdS backgrounds, but we will not do it here.

Mass (energy) in AdS

Let us first show that for the time-like Killing vector $\bar{\xi}^\mu = (-1, 0, 0, 0)$, which is related to the conserved energy, the terms involving dual Riemann tensors in (19) do not contribute to $\sqrt{-\bar{g}} C^{00}{}_L \bar{\xi}_0$. We have

$$\bar{\xi}_0 \bar{\nabla}_\sigma v_\tau (*R^{\tau 0 \sigma \nu}{}_L + *R^{\tau \nu \sigma 0}{}_L) = -\bar{\xi}_0 \bar{\Gamma}^0{}_{\sigma\tau} v_0 \epsilon^{\sigma 0 \alpha\beta} \bar{g}^{\rho 0} (R^\tau{}_{\rho\alpha\beta})_L \tag{20}$$

In the commonly used coordinates, the background AdS metric

$$ds^2 = -(1 - \frac{\Lambda}{3} r^2) dt^2 + (1 - \frac{\Lambda}{3} r^2)^{-1} dr^2 + r^2 d\Omega_2 \,, \tag{21}$$

has non-vanishing $\bar{\Gamma}^0{}_{0r}$ and its symmetric partner. Given this and the fact that linearized Riemann tensor obeys the algebraic symmetries of the full Riemann tensor, we see that (20) does not contribute to the energy in AdS or flat backgrounds. on the other hand, the remaining non-trivial part can be expressed as (let us keep $\bar{\xi}_\nu$ for now and choose it to be time-like at the very end)

$$\begin{aligned} -2\sqrt{-\bar{g}} C^{\mu\nu}{}_L \bar{\xi}_\nu \;=\; & \bar{\nabla}_\alpha \left(v_\sigma \bar{\xi}^\rho \epsilon^{\sigma\mu\alpha\beta} \mathcal{G}^L{}_{\rho\beta} + v_\sigma \bar{\xi}_\nu \epsilon^{\sigma\nu\alpha\beta} \mathcal{G}^{L\,\mu}{}_\beta + v_\sigma \bar{\xi}_\nu \epsilon^{\sigma\mu\nu\beta} \mathcal{G}^{L\,\alpha}{}_\beta \right) \\ & - v_\sigma \epsilon^{\sigma\nu\alpha\beta} \mathcal{G}^{L\,\mu}{}_\beta \bar{\nabla}_\alpha \bar{\xi}_\nu - v_\sigma \epsilon^{\sigma\mu\alpha\beta} \mathcal{G}^L{}_{\rho\beta} \bar{\nabla}_\alpha \bar{\xi}^\rho - v_\sigma \epsilon^{\sigma\mu\nu\beta} \mathcal{G}^{L\,\alpha}{}_\beta \bar{\nabla}_\alpha \bar{\xi}_\nu \,. \end{aligned}$$

The last two terms cancel each other due to the fact that $\bar{\xi}^\mu$ is a Killing vector. Hence one is left with

$$\begin{aligned} -2\sqrt{-\bar{g}} C^{\mu\nu}{}_L \bar{\xi}_\nu \;=\; & \bar{\nabla}_\alpha \left(v_\sigma \bar{\xi}^\rho \epsilon^{\sigma\mu\alpha\beta} \mathcal{G}^L{}_{\rho\beta} + v_\sigma \bar{\xi}_\nu \epsilon^{\sigma\nu\alpha\beta} \mathcal{G}^{L\,\mu}{}_\beta + v_\sigma \bar{\xi}_\nu \epsilon^{\sigma\mu\nu\beta} \mathcal{G}^{L\,\alpha}{}_\beta \right) \\ & - v_\sigma \epsilon^{\sigma\nu\alpha\beta} \mathcal{G}^{L\,\mu}{}_\beta \bar{\nabla}_\alpha \bar{\xi}_\nu \end{aligned} \tag{22}$$

The first line, with an anti-symmetric tensor density inside the covariant derivative, is in the desired boundary form. The final piece can be handled in the following way: define

$$\bar{\Xi}^\mu \equiv \frac{1}{\sqrt{-g}} v_\sigma \, \epsilon^{\sigma\nu\alpha\mu} \bar{\nabla}_\alpha \bar{\xi}_\nu \tag{23}$$

then the last term in (22) becomes

$$-\sqrt{-\bar{g}} \mathcal{G}^{L\,\mu}{}_\beta \bar{\Xi}_\beta \tag{24}$$

which resembles the conserved charges in the pure cosmological Einstein theory except we still need to show that $\bar{\Xi}_\beta$ is a Killing vector. In fact,

$$\bar{\nabla}_\alpha \bar{\Xi}_\beta + \bar{\nabla}_\beta \bar{\Xi}_\alpha = 0 \tag{25}$$

follows once we employ a basic identity $\bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\Xi}_\nu = \bar{R}^\mu{}_{\nu\beta\alpha} \bar{\Xi}_\mu$. Thus the last term in (22) can also be written as a boundary term as in cosmological Einstein theory); we just need to replace $\bar{\xi}_\nu$ with $\bar{\Xi}_\nu$ in (17).

Finally let us summarize the mass formula in Chern-Simons Modified GR for asymptotically AdS (or in fact asymptotically flat spaces as well.) It is more convenient to keep a covariant looking expression [One can easily choose $\xi^\nu = (-1,0,0,0)$.]

$$E = \frac{1}{16\pi G} \oint_{S^2} dS_i \left(\mathcal{Q}_E^{0i}(\bar{\xi}) + \frac{1}{2}\, \mathcal{Q}_E^{0i}(\bar{\Xi}) - \frac{1}{2}\, \mathcal{Q}_C^{0i}(\bar{\xi}) \right) , \qquad (26)$$

where

$$\begin{aligned} \mathcal{Q}_E^{\mu i}(\bar{\xi}) \;&\equiv\; \sqrt{-\bar{g}}\, \big(\bar{\xi}_\nu\, \bar{\nabla}^\mu\, h^{i\nu} - \bar{\xi}_\nu\, \bar{\nabla}^i\, h^{\mu\nu} + \bar{\xi}^\mu\, \bar{\nabla}^i\, h - \bar{\xi}^i\, \bar{\nabla}^\mu\, h \\ &\qquad + h^{\mu\nu}\, \bar{\nabla}^i\, \bar{\xi}_\nu - h^{i\nu}\, \bar{\nabla}^\mu\, \bar{\xi}_\nu + \bar{\xi}^i\, \bar{\nabla}_\nu\, h^{\mu\nu} - \bar{\xi}^\mu\, \bar{\nabla}_\nu\, h^{i\nu} + h\, \bar{\nabla}^\mu\, \bar{\xi}^i \big) \,, \qquad (27) \\ \mathcal{Q}_C^{\mu i}(\bar{\xi}) \;&\equiv\; v_\sigma \bar{\xi}^\rho \epsilon^{\sigma\mu i\beta} \mathcal{G}^L{}_{\rho\beta} + v_\sigma \bar{\xi}_\nu \epsilon^{\sigma\nu i\beta} \mathcal{G}^{L\,\mu}{}_\beta + v_\sigma \bar{\xi}_\nu \epsilon^{\sigma\mu\nu\beta} \mathcal{G}^{L\,i}{}_\beta \qquad (28) \end{aligned}$$

[The integration should be carried over a sphere at spatial infinity. So, de Sitter (as opposed to AdS), with a cosmological horizon, is ruled out [5, 6].] This result resembles to what we have found for the three dimensional TMG [7], but of course, now we have an additional background vector.

It is important to check the background gauge invariance of our definition, which in our case follows easily. Consider small diffeomophisms generated by ζ_μ, such that

$$\delta_\zeta h_{\mu\nu} = \bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu \,. \qquad (29)$$

One can show that (see the second paper in [6], though there is an important typo in that discussion), $\mathcal{G}^L{}_{\mu\nu}$ is (gauge) invariant under these transformations. Therefore, Einstein and Cotton parts are separately gauge invariant in our energy expression.

Now that we have the mass formula in our hand we can find the mass of the known solutions. Unfortunately, in this model, the only solution we have is the AdS-Schwarzschild black hole, whose metric reads

$$ds^2 = -(1 - \frac{r_0}{r} - \frac{\Lambda}{3} r^2) dt^2 + (1 - \frac{r_0}{r} - \frac{\Lambda}{3} r^2)^{-1} dr^2 + r^2 d\Omega_2 \,. \qquad (30)$$

Explicit computation of the mass of this metric using our formula (26) is actually straightforward: $\bar{\Xi}^\mu$ vanishes so $\mathcal{Q}_E(\bar{\Xi})$ term does not contribute. on the other hand, $\mathcal{Q}_C(\bar{\xi})$ term vanishes for various reasons (such as symmetry and $\mathcal{G}^{L\,\mu}{}_\beta$ being zero for this Einstein space at infinity). From the first part, we get $E = M$, if we identify $r_0 = 2GM$ as usual. This result is consistent and expected, as the metric received no correction from the Cotton part. But, if a non-trivial solution to the full CS modified GR is found, our formula will give its mass. In the three dimensional theory, for rotating solutions in AdS, one can see how the Cotton term modifies the mass and angular momentum [8, 10].

Angular Momentum and energy in flat space

In flat space (and in Cartesian coordinates), our task is somewhat simple: we can give a unified formula for both mass and angular momentum. All the covariant derivatives become ordinary derivatives. Without going into further details let us write the final answer

$$Q^0(\bar{\xi}_\mu) = \frac{1}{16\pi G} \oint_{S^2} dS_i \Big(\bar{\xi}_0 (\partial_j h^{ij} - \partial^i h^j{}_j) + \bar{\xi}^i \partial_j h^{0j} - \bar{\xi}_j \partial^i h^{0j} + \frac{\mu}{2} \epsilon^{ijk} \bar{\xi}_j \mathcal{G}^{L\,0}{}_k \Big) \,. \qquad (31)$$

For $\bar{\xi}_0 = (1,0,0,0)$, $Q^0(\bar{\xi}_0) = E$ and and the formula reduces to the usual ADM one. In the case of the angular momentum, $\bar{\xi}_i = (0,0,0,1)$ and $Q^0(\bar{\xi}_i) = J$: There *is* a contribution from the Cotton part. Currently, since we do not know any rotating solution in this model, we cannot give an explicit example.

IV. CONCLUSIONS

We have constructed the conserved energy of the Chern-Simons modified General Relativity [1] for asymptotically flat and asymptotically AdS spaces. We also provided the angular momentum for asymptotically flat spaces. Our construction follows [5] and [6, 7] and can be generalized to the models with additional higher curvature terms. The theory we have considered has an external field which depends on time and breaks some part of the Lorentz group. But constant time translations and spatial rotations are still symmetries of the model which allowed us to define the total mass and angular momentum of the spacetimes that solve the field equations. Whether or not CS modified GR is physical or not is not yet clear since the novel predictions of the theory have been related to weak gravity regime where the current experiments are of no help. Two such examples are : The theory predicts parity-violating gravitational waves [1] and parity-violating interaction between gravitoelectric and gravito-magnetic fields [17]

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