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Radiation in Yang-Mills Formulation of Gravity and a Generalized pp-Wave Metric

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Variational methods applied to a quadratic Yang-Mills-type Lagrangian yield two sets of relations interpreted as the field equations and the energy-momentum tensor for the gravitational field. A covariant condition is imposed on the energy-momentum tensor in order for it to represent the radiation field. A generalized pp-wave metric is found to simultaneously satisfy both the field equations and the radiation condition. The result is compared with that of Lichnérowicz.

§1. Introduction

Based on its gauge structure, which was first recognized by Utiyama,[1] alternative formulations of gravity, especially those that are derivable from quadratic Lagrangians of Yang-Mills (YM) type have received substantial interest in the literature,[2)-5)] not simply because the governing equations appear formally analogous to those of other gauge field theories, but mainly because when the gravitational Lagrangian is coupled to matter, the renormalization problems are much less severe.[6]

For massless particles like photons and neutrinos, radiation presents itself if for all observers their respective energy flows in the same direction as that of light. For electromagnetic fields this situation is well known, and neutrino radiation in curved space is also well established.[7,8] It has been shown that any solution to the Einstein-Weyl equations represents neutrino radiation if the energy-momentum tensor of the Weyl field can be expressed in terms of a null four-vector that is colinear to its four-momentum. Therefore, there is no reason not to expect that the gauge quanta of gravitation possess the same conditions to be in a radiative state.

In this article, we begin with a YM-type Lagrangian which is quadratic in the Riemann tensor, and employ Palatini's variational method to derive two sets of relations whose interpretations are given as the field equations and the gravitational energy-momentum tensor. According to Palatini's method, which was first implemented by Stephenson[9] and then elaborated by Fairchild,[2] the connection and the metric are chosen as independent variables of the Lagrangian without making any *a priori* assumption about the relation between them.

We introduce covariant conditions on the energy-momentum tensor in order for it to represent gravitational radiation, and for the solutions we specialize on a generalized form of a metric which represents plane-fronted waves with parallel rays (pp-wave). We compare the radiation criteria with that of Lichnérowicz.

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§2. Radiation in the Yang-Mills theory of gravity

The dynamical equations to be considered here are determined through independent variations of the gauge-invariant action $I = \int_M L$, where M is a four dimensional spacetime endowed with a metric of +2 signature, and the Lagrangian L is

$$L = \sqrt{-g}\, R^{\mu}{}_{\nu\,\rho\sigma}\, R^{\nu}{}_{\mu}{}^{\rho\sigma}. \tag{2·1}$$

Varying with respect to the connection and employing the least-action principle ($\delta I = 0$), we have $\delta L/\delta \Gamma^{\nu}{}_{\alpha\beta} = 0$, which gives

$$\nabla_{\mu} R^{\mu}{}_{\nu\alpha\beta} = 0, \tag{2·2}$$

and the Bianchi identity

$$\nabla_{[\mu} R_{\nu\sigma]\,\alpha\beta} = 0 \tag{2·3}$$

follows from the definition of the Riemann tensor. At this point, we note that, while the variation taken with respect to the connection is set to zero through the implementation of the least-action principle to represent field equations, this principle is not necessarily imposed for variations taken with respect to the metric, as is the case with other gauge field theories. Variation of the action with respect to the metric is defined and interpreted as the energy-momentum tensor of the corresponding field by many authors.[4),2),10)] Therefore, we have the following definition:

$$\delta g^{\mu\nu}\left(\frac{\delta L}{\delta g^{\mu\nu}}\right) \equiv \delta g^{\mu\nu} T_{\mu\nu}. \tag{2·4}$$

Here the tensor $T_{\mu\nu}$ is symmetric and takes the form

$$T_{\mu\nu} = R_{\mu\kappa}{}^{\rho}{}_{\sigma} R_{\nu}{}^{\kappa\sigma}{}_{\rho} - \frac{1}{4} g_{\mu\nu} R^{\kappa}{}_{\tau\rho\sigma} R^{\tau}{}_{\kappa}{}^{\rho\sigma}. \tag{2·5}$$

We define the radiation field as any solution of (2·2), whose energy (2·5) flows for all observers pointwise in the same direction as the velocity of light; that is, $T_{\mu\nu}U^{\nu} \propto l_{\mu}$, where $l_{\mu}l^{\mu} = 0$ for all U^{μ} with $U^{\mu}U_{\mu} = 1$, and "$\propto$" represents proportionality. Therefore the radiation field is represented through a condition on the energy-momentum tensor as

$$T_{\mu\nu} = \rho(x^{\alpha}) l_{\mu} l_{\nu}, \tag{2·6}$$

where $\rho(x^{\alpha}) > 0$ can be considered as the energy density. It immediately follows that the dominant energy conditions $T_{\mu\nu}U^{\mu}U^{\nu} \geq 0$, and $T_{\mu\nu}U^{\mu}$ being non-spacelike, are satisfied. From (2·5) it is seen that $T_{\mu\nu}$ is traceless, and therefore the condition on l_{μ} to be an isotropic vector is already inherent in the description. The trajectories of the vector field l_{μ} are interpreted as gravitational rays.

The criterion for the existence of gravitational radiation proposed by Lichnérowicz is based on an analogy with electromagnetic radiation and imposes algebraic conditions on the Riemann tensor as[11)]

$$l_{[\mu} R_{\nu\sigma]\,\alpha\beta} = 0, \qquad l^{\mu} R_{\mu\nu\,\alpha\beta} = 0, \tag{2·7}$$

with $l_{\mu} \neq 0$ and $R_{\mu\nu\,\alpha\beta} \neq 0$. However, contracting the first expression with $R^{\nu\sigma\kappa\tau}$ and making use of the second expression, it is seen that these conditions imply the vanishing of $T_{\mu\nu}$, and therefore the concept of energy transfer becomes ambiguous.

§3. Generalized pp-wave solutions

To gain insight into the properties of the radiation field we consider a more general form of a pp-wave metric:

$$ds^2 = 2\,du\,dv + dx^2 + dy^2 + 2\,h(v,x,y,u)\,du^2. \tag{3·1}$$

Unlike the function in the metric that represents plane-fronted waves with parallel rays that depends only on x, y and u,[12] here the metric function h depends on all of the coordinates. The null vector field $l^\mu = \delta^\mu{}_1$ is subject to

$$\nabla_\mu\, l^\nu = \alpha_\mu\, l^\nu \tag{3·2}$$

for some vector α_μ. Contracting (3·2) with l^μ yields

$$l^\mu \nabla_\mu\, l^\nu = \kappa\, l^\nu, \tag{3·3}$$

expressing the fact that the trajectories of the vector field l^μ are null geodesics. This vector field can be chosen as $l_\mu = \partial_\mu S$ for some scalar S. It then follows that all of the optical parameters determined by the geodesic null congruence vanish. The function S is a solution of the eikonal equation $g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S = 0$. The hypersurfaces, $S = \mathit{constant}$, represent the gravitational wave fronts and are identical with the characteristics of Einstein's vacuum and Maxwell's equations.[13] Moreover, the function S obeys $g^{\mu\nu}\,\nabla_\mu \nabla_\nu S = 0$, which is the well-known wave equation.

The field equation (2·2) evaluated with (3·1) reduces to:

$$h_{vvv} = 0, \qquad h_{vvx} = 0, \qquad h_{vvy} = 0, \tag{3·4}$$

$$h_{vxy} = 0, \qquad h_{vxx} = 0, \qquad h_{vyy} = 0, \tag{3·5}$$

$$\partial_v\,(h_{xx} + h_{yy} + h_{vu}) = 0, \tag{3·6}$$

$$\partial_x\,(h_{xx} + h_{yy} + h_{vu}) + h_v\,h_{vx} - h_x\,h_{vv} = 0, \tag{3·7}$$

$$\partial_y\,(h_{xx} + h_{yy} + h_{vu}) + h_v\,h_{vy} - h_y\,h_{vv} = 0. \tag{3·8}$$

A closer examination on the components of the energy-momentum tensor and the traceless Einstein tensor $P_{\mu\nu}$ reveals a compact tensorial relation

$$T_{\mu\nu} = -R\,P_{\mu\nu} + \kappa\,l_\mu\,l_\nu, \tag{3·9}$$

where their surviving components are respectively calculated as:

$$\begin{aligned} T_{12} &= -h_{vv}^2, & T_{22} &= h_{vv}^2, \\ T_{23} &= -2\,h_{vv}\,h_{vx}, & T_{33} &= h_{vv}^2, \\ T_{34} &= -2\,h_{vv}\,h_{vy}, & T_{44} &= 2\,(h_{vx}^2 + h_{vy}^2 - h\,h_{vv}^2), \end{aligned} \tag{3·10}$$

and

$$\begin{aligned} P_{14} &= \tfrac{1}{2}h_{vv}, & P_{24} &= \tfrac{1}{2}h_{vx}, & P_{34} &= h_{vy}, \\ P_{22} &= -\tfrac{1}{2}h_{vv}, & P_{33} &= -\tfrac{1}{2}h_{vv}, & P_{44} &= h\,h_{vv} - h_{xx} - h_{yy}. \end{aligned} \tag{3·11}$$

Here, R is the curvature scalar, and it is found to be $R = 2h_{vv}$. Furthermore, $T_{\mu\nu}$ satisfies another relation, which can be expressed in terms of the metric tensor and the principal null vector l_μ:

$$T_{\mu\nu} = \lambda\, g_{\mu\nu} + l_\mu k_\nu + l_\nu k_\mu, \tag{3·12}$$

where the scalars λ, κ and the vector k_μ can easily be determined.

For solely comparative purposes the Einstein tensor, $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, can be calculated. This can also be expressed through a tensorial relation

$$S_{\mu\nu} = -\frac{1}{R}\, T_{\mu\nu} - \frac{1}{4}\, R\, g_{\mu\nu} + \frac{\kappa}{R}\, l_\mu l_\nu, \tag{3·13}$$

whenever $R \neq 0$. We observe that, for this particular metric, the right-hand side is quite different from all known forms of energy appearing in Einstein's equations.

For the metric in (3·1) to represent a radiation field, we first let $h_{vv} = 0$ with $\rho(x^\alpha) > 0$, where

$$\rho(x^\alpha) = 2(h_{vx}^2 + h_{vy}^2). \tag{3·14}$$

Therefore, the metric function h is of the form

$$h = B(x, y, u)\, v \;+\; C(x, y, u), \tag{3·15}$$

where B and C are independent of v, and B is not a function of u alone. Then, as $l_\mu = \delta_\mu^4$, the gravitational energy tensor in (2·5) now takes the appropriate form (2·6).

Field equations impose further conditions on the functions B and C. From (3·5) it is seen that

$$B = \alpha(u)x + \beta(u)y + \gamma(u), \tag{3·16}$$

and so (3·7) and (3·8) reduce to

$$F_x = -B\, B_x, \qquad F_y = -B\, B_y, \tag{3·17}$$

where

$$F = C_{xx} + C_{yy} + B_u. \tag{3·18}$$

These two can be written as

$$\alpha\, F_x - \beta\, F_y = 0. \tag{3·19}$$

The characteristic system associated with this equation admits the first integrals u, $\alpha\, x + \beta\, y + \gamma$ and F, and hence F is of the form $F = F(B, u)$. Thus C is now any solution of

$$C_{xx} + C_{yy} \;=\; F - B_u. \tag{3·20}$$

Therefore, we have found the form of the metric which simultaneously satisfies the field equations and the radiation condition on $T_{\mu\nu}$. With this solution, the scalar ρ in (3·14) remains constant along the geodesic null congruence, and this leads to the conservation of the gravitational energy tensor. At this point, we also note that, since we have assumed $h_{vv} = 0$, the curvature scalar also vanishes, which means that Einstein's equations (3·13) for the radiative solutions of (2·2) are no longer viable.

It is worthwhile to study the algebraic properties of the conformal Weyl tensor with respect to the principal null vector l_μ. With $l^\mu = \delta^\mu{}_1$, the following characterizations for different Petrov types are obtained:

$$\begin{array}{llll} R=0, & h_{vx}^2+h_{vy}^2=0, & h_{xy}^2+(h_{xx}-h_{yy})^2=0 & \Leftrightarrow \text{ type O,} \\ R=0, & h_{vx}^2+h_{vy}^2=0, & h_{xy}^2+(h_{xx}-h_{yy})^2\neq 0 & \Leftrightarrow \text{ type N,} \\ R=0, & h_{vx}^2+h_{vy}^2\neq 0, & & \Leftrightarrow \text{ type III,} \\ R\neq 0, & & & \Leftrightarrow \text{ type II or D.} \end{array} \quad (3{\cdot}21)$$

§4. Conclusion

We have presented a covariant formulation of gravitational radiation based on a second rank tensor, which looks formally like an energy tensor associated with the gravitational field. As a specific example for the illustration of a radiation field, a space-time metric admitting plane waves was considered. The plane wave is strictly parallel (i.e., α in (3·2) is zero) if and only if h is independent of v. Then, this metric becomes identical with that corresponding to pp-waves. This situation represents radiation in the sense of Lichnérowicz. However, we note that in this case, since $T_{\mu\nu} = 0$, the concept of energy transfer becomes ambiguous. The algebraic properties of the Weyl tensor were also studied. Finally, we showed that this metric is never algebraically general, and that radiation corresponds to type III.

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