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Duff-Inami-Pope-Sezgin-Stelle Bosonic Membrane Equations as an Involutory System

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Using Cartan's geometric formulation of partial diffential equations in the language of exterior differential forms, it is shown that bosonic membrane equations of Duff-Inami-Pope-Sezgin-Stelle (DIPSS) constitute an involutory system. The symmetries of reformulated DIPSS bosonic membrane equations are studied using three forms, elucidating in this way the previous results concerning Lie-point symmetries (Killing symmetries).

§1. Introduction

There are many methods for studying physically and mathematically important solutions of systems of partial differential equations.[1] A systematic method is Cartan's[2] geometric formulation of partial differential equations in the language of exterior differential forms invariant under the isovector fields which generate the symmetry transformations. The basic work on invariance groups of partial differential equations was established by Lie. One can also use Cartan's geometric theory of partial differential equations, which was expounded upon by Slebodzinski.[3]

Recently, Cartan's idea was used to study the infinitesimal symmetries of the second heavenly equations.[4] These symmetries are Killing symmetries of the associated sigma model.[5] Self-dual Einstein equation (SdE) can also be reformulated as a trio of two-forms.[6] Q-Han Park and Husain showed that the SdE can be derived from several two-dimensional sigma models with the gauge group of area preserving diffeomorphisms.[7),8] On the other hand, by Ashtekar's canonical formulation of general relativity, the SdE has been formulated as the Nahm equation.[9]

Our goal in this work is to give a formulation based on Cartan's geometric theory of partial differential equations for the recognition of the geometric structures and to study the isovector fields, which could offer a method to solve the membrane equations from the symmetry data. In §2, a brief description of Cartan's geometric theory of partial differential equations is given in accordance with the notation of Harrison-Estabrook.[10] In §3, a brief description of Duff-Inami-Pope-Sezgin-Stelle[11] bosonic membrane equations is presented. In §4, we give the involutory set of the membrane equations of DIPSS in terms of three forms which make an invariant ideal under the action of the isovector field also generating Lie-point transformations connecting solutions of the membrane equations.

§2. Cartan's theory

Cartan discussed the criteria for the equivalence of a given set of partial differential equations with a closed set of differential forms on an n-dimensional manifold ($n-p$ dependent variables and p independent variables). The latter set is the basis of the differential ideal of the Grassmann algebra of forms on the manifold. An integral manifold of the ideal is a subspace, the extension elements of which annul (give zero values) to all forms in the ideal. In n-dimensional manifold spanned by $x^\mu = x^\mu(y^A)(A = 1, \cdots, p)$, the y^A can be adapted as coordinates in the subspace. We can apply section operators $\lambda^\mu_A = \frac{\partial x^\mu}{\partial y^A}$ to covariant quantities. For example, vector ϕ_μ after sectioning become $\tilde{\phi}_A \equiv \lambda^\mu_A \phi_\mu$. The sectioning of forms is defined similarly. If sectioning a form ϕ gives a form $\tilde{\phi}$ that is identically zero, any extension element of the subspace must annul ϕ.

If an integral manifold of dimensionality p exists, p of the variables vary freely (i.e., may also be adopted as the coordinates in the manifold). Such an integral manifold in the n-space can represent solutions of the original set of partial differential equations. In fact, denoting the $n-p$ variables by z^i and the p variables by x^A, one must be able to re-obtain the original partial differential equations by using substitutions into the set of forms in n-dimensions. The sectioning of these forms into the p-manifold can be given as

$$d\tilde{x}^A = dx^A, \quad d\tilde{z}_i = \frac{\partial z_i}{\partial x^A} dx^A. \tag{1}$$

We impose the independence of variables x^A on the set of forms by requiring the coefficients of the forms dx^A, $dx^A \wedge dx^B$ etc. to vanish.

The integral manifolds of a differential ideal were classified by Cartan. He gave the local algebraic criteria for the existence of general integral manifolds in which the variables and their differentials can be freely chosen. Then, the ideal is said to be in involution with respect to the variables.

We consider a set of forms in the n-space w_j $(i = 1, \cdots)$ where the ith form ω_i is taken to be of degree p_i. We require that the ideal be invariant under the continuous dragging transformation generated by the vector field V by writing

$$L_V \omega_i = \sum_i \tau^j_i \omega_j. \tag{2}$$

The sum includes only forms ω_j for which $p_j \leq p_i$ (thus including ω_i itself). $\tau^j_i$is an arbitrary form of degree $p_i - p_j$. V is a contravariant vector field in the space of n dimensions and they are functions of all n variables. A set of Lie increments $\varepsilon L_V \omega_i$ represents simultaneous infinitesimal changes in the objects ω_i equivalent to the active coordinate change or point transformation

$$x^\mu = x^\mu + \varepsilon V^\mu. \tag{3}$$

The increments of the set of ω_i produce no change in any of the integral manifolds. That is, they preserve the form of the original system of partial differential equations.

To find the invariance group, we expand (2) in terms of basis p_i forms. Then, we equate the coefficients of the basis p_i forms to zero and algebraically eliminate the τ_i^j. This yields a system of linear partial differential equations for the components of V which can usually be solved in a straightforward fashion.

§3. Bosonic membrane equations of DIPSS

A two dimensional closed object sweeps out a three dimensional world volume in the course of its time evolution. The space-time configuration of this world volume is specified by a map $X^\mu(\xi^i)$ from a chart of the world volume with local coordinates ξ^i to a chart of space-time with local coordinates X^μ. The membrane action is given by[11)]

$$I = -\frac{1}{8\pi^2}\int d\xi^3\sqrt{-g}[g^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - 1], \tag{4}$$

where g_{ij} is the auxiliary independent world volume metric whose field equations is

$$g_{ij} = \partial_i X^\mu \partial_j X_\mu. \tag{5}$$

Here $\xi_i = (\tau, \sigma, \rho)(i = 1, 2, 3)$ are the world volume coordinates and X^μ gives the embedding of the world-volume in an n-dimensional Minkowski space-time. Varying (4) with respect to X^μ yields the field equation

$$\partial_i(\sqrt{-g}g^{ij}\partial_j X^\mu) = 0. \tag{6}$$

Owing to the general coordinate invariance on the world volume, the system of equations for the fields $X^\mu(\xi)$ is underdetermined; i.e., there are 3 degrees of freedom that may be specified arbitrarily. The independent degrees of freedom can be analyzed in the Hamiltonian formalism using an ADM[11)] parametrization of the world volume metric. The world volume metric is eliminated entirely, but at the price of nonlinearity in the equations of motion for independent physical degrees of freedom. Further simplifications can be achieved by making a string-type light-cone gauge choice $X^\mu = (X^+, X^-, \underline{X})$, $\dot{\underline{X}} = \frac{\partial}{\partial\tau}\underline{X}$, $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^n)$, $X^- = \frac{1}{\sqrt{2}}(X^0 - X^n)$.

The coordinates X^+ and X^- are eliminated as independent degrees of freedom. But, we have the constraint equation for transverse coordinates $\underline{X}$,

$$\partial_a\underline{X}\cdot\partial_b\dot{\underline{X}} - \partial_b\underline{X}\cdot\partial_a\dot{\underline{X}} = 0. \tag{7}$$

The equation of motion for the transverse coordinates $\underline{X}$ becomes

$$\begin{aligned}\ddot{\underline{X}} = \partial_\sigma[(\partial_\rho\underline{X}\cdot\partial_\rho\underline{X})\partial_\sigma\underline{X} - (\partial_\sigma\underline{X}\cdot\partial_\rho\underline{X})\partial_\rho\underline{X}] \\ +\partial_\rho[(\partial_\sigma\underline{X}\cdot\partial_\sigma\underline{X})\partial_\rho\underline{X} - (\partial_\sigma\underline{X}\cdot\partial_\rho\underline{X})\partial_\sigma\underline{X}].\end{aligned} \tag{8}$$

Instead of the equations above, we consider membrane equations for two dependent variables $\underline{X} = (u, v)$ in two spatial and one temporal dimension t in space time by letting $(\sigma, \rho) = (x, y)$ and $\dot{\underline{X}} = \frac{\partial \underline{X}}{\partial t}$[12)]

$$-u_{tt} + u_{xx}v_y^2 + u_{yy}v_x^2 - v_{xx}u_yv_y - v_{yy}u_xv_x - 2u_{xy}v_xv_y + v_{xy}(u_xv_y + u_yv_x) = 0, \tag{9}$$

$$-v_{tt}+v_{xx}u_y^2+v_{yy}u_x^2-u_{xx}u_yv_y-u_{yy}u_xv_x-2v_{xy}u_xu_y+u_{xy}(u_xv_y+u_yv_x)=0. \quad (10)$$

Using the Lie bracket $\{,\}$ on the algebra of functions on the membrane surface,

$$\{u,v\}_{xy}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}, \quad (11)$$

the equations of motion (9) and (10) for the membrane equations can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_{tt}=\begin{pmatrix} -\{v,\{v,u\}_{xy}\}_{xy} \\ \{u,\{v,u\}_{xy}\}_{xy} \end{pmatrix} \quad (12)$$

with the constraint

$$\{u_t,u\}_{xy}+\{v_t,v\}_{xy}=0. \quad (13)$$

Equation (12) is implied by the reduced version of the self-dual Yang-Mills equations (Nahm equation).[13] Equation (12) can also be cast into the evolutionary form*)

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t}=\partial_t^{-1}\begin{pmatrix} -\{v,\{v,u\}_{xy}\}_{xy} \\ \{u,\{v,u\}_{xy}\}_{xy} \end{pmatrix}. \quad (14)$$

§4. The involutory set of DIPSS membrane equations

In this section, to study the symmetries of DIPSS equations, (9) and (10) are written in an equivalent form. In order to express these equations in differential forms, we define new variables,

$$u_t=m, \quad u_x=n, \quad u_y=p, \quad v_t=o, \quad v_x=r, \quad v_y=s. \quad (15)$$

In the eleven dimensional space of dependent and independent variables $\{x,y,t,m,n,p,u,o,r,s,v\}$, we adopt the basis forms $\{dx,dy,dt,dm,dn,dp,du,do,dr,ds,dv\}$. The first order equations (9), (10) and (15) can be expressed by the following set of eight three-forms

$$\alpha=du\wedge dx\wedge dy-mdx\wedge dy\wedge dt, \quad (16)$$

$$\beta=du\wedge dy\wedge dt-ndx\wedge dy\wedge dt, \quad (17)$$

$$\gamma=-du\wedge dx\wedge dt-pdx\wedge dy\wedge dt, \quad (18)$$

$$\begin{aligned}\delta=&-dm\wedge dx\wedge dy+s^2dn\wedge dy\wedge dt,\\ &+r^2dx\wedge dp\wedge dt-psdr\wedge dy\wedge dt\\ &-nrdx\wedge ds\wedge dt-2rsdx\wedge dn\wedge dt\\ &+nsdx\wedge dr\wedge dt+prdx\wedge dr\wedge dt,\end{aligned} \quad (19)$$

$$\mu=dv\wedge dx\wedge dy-odx\wedge dy\wedge dt, \quad (20)$$

$$\nu=dv\wedge dy\wedge dt-rdx\wedge dy\wedge dt, \quad (21)$$

$$\sigma=-dv\wedge dx\wedge dt-sdx\wedge dy\wedge dt, \quad (22)$$

*) Non Lie point symmetries of the equation (14) and recursion operator for these symmetries are currently being studied.

$$\eta = -do \wedge dx \wedge dy + p^2 dr \wedge dy \wedge dt \\ +n^2 dx \wedge ds \wedge dt - psdn \wedge dy \wedge dt \\ -nrdx \wedge dp \wedge dt - 2rsdx \wedge dr \wedge dt \\ +nsdx \wedge dn \wedge dt + prdx \wedge dn \wedge dt, \tag{23}$$

where d denotes the exterior derivative and $\wedge$ denotes the wedge product. The sectioning of the eight forms using

$$du = u_x dx + u_y dy + u_t dt, \; dv = v_x dx + v_y dy + v_t dt \tag{24}$$

will annul the set of forms

$$\tilde{\alpha} = (u_t - m) dx \wedge dy \wedge dt = 0, \tag{25}$$
$$\tilde{\beta} = (u_x - n) dx \wedge dy \wedge dt = 0, \tag{26}$$
$$\tilde{\gamma} = (u_y - p) dx \wedge dy \wedge dt = 0, \tag{27}$$
$$\tilde{\delta} = (-u_{tt} + u_{xx} v_y^2 + u_{yy} v_x^2 - v_{xx} u_y v_y - v_{yy} u_x v_x \\ -2u_{xy} v_x v_y + v_{xy}(u_x v_y + u_y v_x)) dx \wedge dy \wedge dt = 0, \tag{28}$$
$$\tilde{\mu} = (u_t - m) dx \wedge dy \wedge dt = 0, \tag{29}$$
$$\tilde{\nu} = (u_x - n) dx \wedge dy \wedge dt = 0, \tag{30}$$
$$\tilde{\sigma} = (u_y - p) dx \wedge dy \wedge dt = 0, \tag{31}$$
$$\tilde{\eta} = (-v_{tt} + v_{xx} u_y^2 + v_{yy} u_x^2 - u_{xx} u_y v_y - u_{yy} u_x v_x \\ -2v_{xy} u_x u_y + u_{xy}(u_x v_y + u_y v_x)) dx \wedge dy \wedge dt = 0. \tag{32}$$

In order to assert complete equivalence between Eqs. (9), (10) and the differential forms, the set of forms must be closed. That is, the Lie derivative of the forms must be contained in the ring of forms generated by the set. The differential constraint (13) is not included in the set, because the forms are not closed under the Lie derivative. Thus, Cartan's theorem guarantees that these surface elements fit together to produce a global 3-surface which constitutes a solution manifold. We have

$$L_V \alpha = A_1 \alpha + A_2 \beta + A_3 \gamma + A_4 \delta, \tag{33}$$
$$L_V \beta = B_1 \alpha + B_2 \beta + B_3 \gamma + B_4 \delta, \tag{34}$$
$$L_V \gamma = C_1 \alpha + C_2 \beta + C_3 \gamma + C_4 \delta, \tag{35}$$
$$L_V \delta = D_1 \alpha + D_2 \beta + D_3 \gamma + D_4 \delta, \tag{36}$$
$$L_V \mu = E_1 \mu + E_2 \nu + E_3 \sigma + E_4 \eta, \tag{37}$$
$$L_V \nu = F_1 \mu + F_2 \nu + F_3 \sigma + F_4 \eta, \tag{38}$$
$$L_V \sigma = G_1 \mu + G_2 \nu + G_3 \sigma + G_4 \eta, \tag{39}$$
$$L_V \eta = H_1 \mu + H_2 \nu + H_3 \sigma + H_4 \eta, \tag{40}$$

where A_i, B_i, C_i, D_i, E_i, F_i and G_i are arbitrary zero-forms and the isovector field is given by

$$V = uvx \frac{\partial}{\partial x} + uvy \frac{\partial}{\partial y} + uvt \frac{\partial}{\partial t} + uu \frac{\partial}{\partial u} + vv \frac{\partial}{\partial v}$$

$$+um\frac{\partial}{\partial m}+un\frac{\partial}{\partial n}+up\frac{\partial}{\partial p}+vo\frac{\partial}{\partial o}+vr\frac{\partial}{\partial r}+vs\frac{\partial}{\partial s}, \tag{41}$$

with $uvx = uvx(x,y,t,u,v,m,n,p,o,r,s)$, $uvy = uvy(x,y,t,u,v,m,n,p,o,r,s)$, $\cdots, vs = vs(x,y,t,u,v,m,n,p,o,r,s)$. In order to find $uvx, uvy, \cdots, vs$, a reduce program [14] using the "EXCALC" [15] differential geometry package gives the following vector field:

$$V_1 = (\alpha_1+\alpha_2 x)\frac{\partial}{\partial x}, \tag{42}$$

$$V_2 = (\alpha_3+\alpha_4 y)\frac{\partial}{\partial y}, \tag{43}$$

$$V_3 = (\alpha_5+t(\alpha_2+\alpha_4-\alpha_6))\frac{\partial}{\partial t}, \tag{44}$$

$$V_4 = (\alpha_7+\alpha_8 t+\alpha_6 u)\frac{\partial}{\partial u}, \tag{45}$$

$$V_5 = (\alpha_9+\alpha_{10} t+\alpha_6 v)\frac{\partial}{\partial v}, \tag{46}$$

$$V_6 = (\alpha_{11}+m(2\alpha_6-\alpha_2-\alpha_4))\frac{\partial}{\partial m}, \tag{47}$$

$$V_7 = n(\alpha_6-\alpha_2)\frac{\partial}{\partial n}, \tag{48}$$

$$V_8 = p(\alpha_6-\alpha_4)\frac{\partial}{\partial p}, \tag{49}$$

$$V_9 = (\alpha_{12}+o(2\alpha_6-\alpha_2-\alpha_4))\frac{\partial}{\partial o}, \tag{50}$$

$$V_{10} = r(\alpha_6-\alpha_2)\frac{\partial}{\partial r}, \tag{51}$$

$$V_{11} = s(\alpha_6-\alpha_4)\frac{\partial}{\partial s}, \tag{52}$$

where the α_i are constants. These are scaling and translational invariant transformations. The same vector field was also obtained using the Lie-point symmetry approach of Olver. [12] These symmetries were used to reduce DIPSS membrane equations to two coupled Cauchy-Euler-type ordinary differential equations.

§5. Conclusion

In this work, we have given the involutory set of membrane equations of DIPSS using Cartan's geometric formulation of the partial differential equations in the language of exterior differential forms invariant under the isovector fields which generate the symmetry transformations.

The symmetry group of the membrane equations can also be calculated without ever introducing forms. However, the use of forms enables us to carry out the calculation more quickly, and the recognition of the n-dimensional geometric structures involved may be of great value in discovering special classes of solutions. [16] Using the differential form formulation also elucidates the relations among those field theories in the form of the sigma model (harmonic map) with an infinite-dimensional group.

There are two kinds of symmetries for membrane models, world volume (base manifold) symmetries and space time (configuration manifold) symmetries, bacause we can formulate a membrane as a harmonic map from the base manifold to the configuration manifold.[20] The constraint of the membrane is the generator of the space time symmetries (Poincaré group which includes space-time rotations).

Many partial differential equations (KdV, Burgers, Maxwell equations, $\cdots$) have involutive sets that are either complete or can be completed after the introduction of new forms.[10] It is interesting to note that Élie Cartan proved that Einstein equations form an involutive system. I believe that the involutive sets can be obtained for the partial differential equations of any well-set physical theories.[17)-19] To close this paper, I would like to conclude that although it is not yet complete, it is profitable to use the apparatus of the theory of integral manifolds for a better understanding of the nature of membrane theories.[4]

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References

1) P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1993).
2) E. Cartan, *Les systémes diffèrentielles extèrieurs et leurs applications geometrique* (Herman, Paris, 1945).
3) W. Slebodzinski, *Exterior Forms and Applications* (Polish Scientific Publishers, Warsaw, 1970).
4) C. P. Boyer and J. F. Plebanski, J. Math. Phys. **18** (1997), 1022.
5) S. Başkal, A. Eriş and A. Satır, Phys. Lett. **A196** (1994), 43.
6) T. Ueno, hep-th/9508012.
7) Q-Han Park, Phys. Lett. **A236** (1990), 429; Phys. Lett. **A238** (1990), 287; Int. J. Mod. Phys. **A17** (1992), 1415.
8) V. Husain, Phys. Rev. Lett. **72** (1994), 800.
9) A. Ashtekar, T. Jacobson and L. Smolin, Commun. Math. Phys. **115** (1998), 631.
10) B. K. Harrison and F. B. Estabrook, J. Math. Phys. **12** (1971), 653.
11) M. J. Duff, T. Inami, C. N. Pope, E. Sezgin and K. S. Stelle, Nucl. Phys. **B297** (1988), 515.
12) A. Satır, Int. J. Mod. Phys. **A12** (1997), 1933.
13) H. G. Compeań and J. F. Plebański, Phys. Lett. **A234** (1997), 5.
14) A. C. Hearn, 'REDUCE' User's Manual. Version 3.4. The Rand Corporation, Santa Monica, CA 90406-2138, USA.
15) E. Schrufer, F. W. Hehl and J. D. McCrea, Gen. Relat. Gravit. J. **19** (1987), 197.
16) under active consideration.
17) P. J. Olver, *Equivalence, Invariants and Symmetry* (Cambridge, University Press, 1995).
18) R. B. Gardner, *The method of Equivalence and Its Applications* (Society for industrial and applied mathematics, Philadelphia, Pennysylvania, 1989).

19) R. L. Bryant, S. S. Chern, R. B. Gardner, H. J. Gouldschmit, P. A. Griffiths, *Exterior differential systems* (Springer-Verlag, 1991).
20) N. Sanchez, Harmonic maps in general relativity and quantum field theory, CERN-TH. 4583/86, invited lecture at the Meeting, "Applications harmoniques", CIRM, Luminy, June 1986 (Proccedings, 'Travaux en Cours' Series, Hermann, Paris).