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Research Article

Oscillation Criteria for Second-Order Forced Dynamic Equations with Mixed Nonlinearities

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We obtain new oscillation criteria for second-order forced dynamic equations on time scales containing mixed nonlinearities of the form $(r(t)\Phi_\alpha(x^\Delta))^\Delta + f(t,x^\sigma) = e(t)$, $t \in [t_0,\infty)_{\mathbb{T}}$ with $f(t,x) = q(t)\Phi_\alpha(x) + \sum_{i=1}^n q_i(t)\Phi_{\beta_i}(x)$, $\Phi_*(u) = |u|^{*-1}u$, where $[t_0,\infty)_{\mathbb{T}}$ is a time scale interval with $t_0 \in \mathbb{T}$, the functions $r, q, q_i, e : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ are right-dense continuous with $r > 0$, σ is the forward jump operator, $x^\sigma(t) := x(\sigma(t))$, and $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots \beta_n > 0$. All results obtained are new even for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. In the special case when $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$ our theorems reduce to (Y. G. Sun and J. S. W. Wong, Journal of Mathematical Analysis and Applications. 337 (2007), 549–560). Therefore, our results in particular extend most of the related existing literature from the continuous case to arbitrary time scale.

1. Introduction

Let $\mathbb{T}$ be a time scale which is unbounded above and $t_0 \in \mathbb{T}$ a fixed point. For some basic facts on time scale calculus and dynamic equations on time scales, one may consult the excellent texts by Bohner and Peterson [1, 2].

We consider the second-order forced nonlinear dynamic equations containing mixed nonlinearities of the form

$$\left(r(t)\Phi_\alpha(x^\Delta)\right)^\Delta + f(t,x^\sigma) = e(t), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

with

$$f(t,x) = q(t)\Phi_\alpha(x) + \sum_{i=1}^n q_i(t)\Phi_{\beta_i}(x), \qquad \Phi_*(u) = |u|^{*-1}u, \tag{1.2}$$

where $[t_0,\infty)_{\mathbb{T}}$ denotes a time scale interval, the functions $r,q,q_i,e:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ are right-dense continuous with $r>0$, σ is the forward jump operator, $x^\sigma(t):=x(\sigma(t))$, and

$$\beta_1>\cdots>\beta_m>\alpha>\beta_{m+1}>\cdots\beta_n>0. \tag{1.3}$$

By a proper solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$ we mean a function $x\in C^1_{\rm rd}[t_0,\infty)_{\mathbb{T}}$ which is defined and nontrivial in any neighborhood of infinity and which satisfies (1.1) for all $t\in[t_0,\infty)_{\mathbb{T}}$, where $C^1_{\rm rd}[t_0,\infty)_{\mathbb{T}}$ denotes the set of right-dense continuously differentiable functions from $[t_0,\infty)_{\mathbb{T}}$ to $\mathbb{R}$. As usual, such a solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. The equation is called oscillatory if every proper solution is oscillatory.

In a special case, (1.1) becomes

$$\left(r(t)\Phi_\alpha(x^\Delta)\right)^\Delta+c(t)\Phi_\beta(x^\sigma)=e(t), \tag{1.4}$$

which is called half-linear for $\beta=\alpha$, super-half-linear for $\beta>\alpha$, and sub-half-linear for $0<\beta<\alpha$. If $\mathbb{T}=\mathbb{R}$, (1.4) takes the form

$$\left(r(t)\Phi_\alpha(x')\right)'+c(t)\Phi_\beta(x)=e(t). \tag{1.5}$$

The oscillation of (1.5) has been studied by many authors, the interested reader is referred to the seminal books by Došlý and Řehák [3] and Agarwal et al. [4, 5], where in addition to mainly oscillation theory, the existence, uniqueness, and continuation of solutions are also discussed. In [3], one may also find several results related to the oscillation of (1.4) when $\mathbb{T}=\mathbb{Z}$, that is, for

$$\Delta(r(k)\Phi_\alpha(\Delta x(k)))+c(k)\Phi_\beta(x(k+1))=e(k), \tag{1.6}$$

where Δ is the forward difference operator.

There are several methods in the literature for finding sufficient condition for oscillation of solutions in terms of the functions appearing in the corresponding equation, and almost all such conditions involve integrals or sums on infinite intervals [3–19]. The interval oscillation method is different in a sense that the conditions make use of the information of the functions on a union of intervals rather than on an infinite interval. Following El-Sayed [20], many authors have employed this technique in various works [20–30]. For instance, Sun et al. [26], Wong [28], and Nasr [25] have studied (1.5) when $\alpha=1$ and $\beta\ge 1$, while the case $\alpha=1$ and $0<\beta<1$ is taken into account by Sun and Wong in [16]. The results in [25, 28] have been extended by Sun [27] to superlinear delay differential equations of the form

$$x''(t)+c(t)|x(\tau(t))|^{\beta-1}x(\tau(t))=e(t). \tag{1.7}$$

Further extensions of these results can be found in [30, 31], where the authors have studied some related super-half-linear differential equations with delay and advance arguments.

Recently, there have been also numerous papers on second-order forced dynamic equations on time scales, unifying particularly the discrete and continuous cases and handling many other possibilities. For a sampling of the work done we refer in particular to [6, 8, 9, 12, 13, 22, 32, 33] and the references cited therein. In [22] Anderson and Zafer have extended the above mentioned interval oscillation criteria to second-order forced super-half-linear dynamic equations with delay and advance arguments including

$$\left(r(t)\Phi_\alpha(x^\Delta(t))\right)^\Delta + c(t)\Phi_\beta(x(\tau(t))) = e(t). \tag{1.8}$$

Our motivation in this study stems from the work contained in [34], where the authors have derived interval criteria for oscillation of second-order differential equations with mixed nonlinearities of the form

$$x'' + f(t,x) = e(t), \quad t \geq t_0, \tag{1.9}$$

with

$$f(t,x) = q(t)x + \sum_{i=1}^{m} q_i(t)\Phi_{\beta_i}(x) \tag{1.10}$$

by using a Riccati substitution and an inequality of geometric-arithmetic mean type. As it is indicated in [34], further research on the oscillation of equations of mixed type is necessary as such equations arise in mathematical modeling, for example, in the growth of bacteria population with competitive species. We aim to make a contribution in this direction for a class of more general equations on time scales of the form (1.1) by combining the techniques used in [22, 34]. Notice that when $\alpha = 1$, $r(t) \equiv 1$, and $\mathbb{T} = \mathbb{R}$, (1.1) coincides with (1.9), and therefore our results provide new interval oscillation criteria even for $\mathbb{T} = \mathbb{R}$ when $\alpha \neq 1$. Moreover, for the special case $\mathbb{T} = \mathbb{Z}$ we obtain interval oscillation criteria for difference equations with mixed nonlinearities of the form

$$\Delta(r(k)\Phi_\alpha(\Delta x(k))) + q(k)\Phi_\alpha(x(k+1)) + \sum_{i=1}^{n} q_i(k)\Phi_{\beta_i}(x(k+1)) = e(k), \tag{1.11}$$

for which almost nothing is available in the literature.

2. Lemmas

We need the following preparatory lemmas. The first two lemmas are given by Wong and Sun as a single lemma [34, Lemma 1] for $\alpha = 1$. The proof for the case $\alpha \neq 1$ is exactly the same, in fact one only needs to replace the exponents α_i by β_i/α in their proof. Lemma 2.3 is the well-known Young inequality.

Lemma 2.1. *For any given n-tuple $\{\beta_1, \beta_2, \ldots, \beta_n\}$ satisfying*

$$\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0, \tag{2.1}$$

there corresponds an n*-tuple* $\{\eta_1, \eta_2, \dots, \eta_n\}$ *such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \quad \sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1. \tag{2.2}$$

If $n = 2$ and $m = 1$ (cf. [34] for the case $\alpha = 1$) one may take

$$\eta_1 = \frac{\alpha - \beta_2(1 - \eta_0)}{\beta_1 - \beta_2}, \qquad \eta_2 = \frac{\beta_1(1 - \eta_0) - \alpha}{\beta_1 - \beta_2}, \tag{2.3}$$

where η_0 is any positive number with $\beta_1 \eta_0 < \beta_1 - \alpha$.

Lemma 2.2. *For any given* n*-tuple* $\{\beta_1, \beta_2, \dots, \beta_n\}$ *satisfying*

$$\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0, \tag{2.4}$$

there corresponds an n*-tuple* $\{\eta_1, \eta_2, \dots, \eta_n\}$ *such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \quad \sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{2.5}$$

If $n = 2$ and $m = 1$, it turns out that

$$\eta_1 = \frac{\alpha - \beta_2}{\beta_1 - \beta_2}, \qquad \eta_2 = \frac{\beta_1 - \alpha}{\beta_1 - \beta_2}. \tag{2.6}$$

Lemma 2.3 (Young's Inequality). *If* $p > 1$ *and* $q > 1$ *are conjugate numbers* $(1/p + 1/q = 1)$, *then*

$$\frac{|u|^p}{p} + \frac{|v|^q}{q} \geq |uv|, \quad \forall u, v \in \mathbb{R}, \tag{2.7}$$

and equality holds if and only if $u = |v|^{q-2} v$.

Let $\gamma > \delta$. Put $u = A^{\delta/\gamma}$, $p = \gamma/\delta$, and $v = (B\alpha)^{1-\delta/\gamma}(\gamma - \delta)^{\delta/\gamma - 1}$. It follows from Lemma 2.3 that

$$Ax^{\gamma} + B \geq \gamma \delta^{-\delta/\gamma}(\gamma - \delta)^{(\delta/\gamma)-1} A^{\delta/\gamma} B^{1-\delta/\gamma} x^{\delta} \tag{2.8}$$

for all $A, B, x \geq 0$. Rewriting the above inequality we also have

$$Cx^{\delta} - D \leq \delta^{-\gamma/\delta} \delta(\gamma - \delta)^{(\gamma/\delta)-1} C^{\gamma/\delta} D^{1-\gamma/\delta} x^{\gamma} \tag{2.9}$$

for all $C, x \geq 0$ and $D > 0$.

3. The Main Results

Following [21, 22, 30], denote for $a, b \in [t_0, \infty)_{\mathbb{T}}$ with $a < b$ the admissible set

$$\mathscr{A}(a,b) := \left\{u \in C^1_{\mathrm{rd}}[a,b]_{\mathbb{T}} : u(a) = 0 = u(b),\ u \not\equiv 0\right\}. \tag{3.1}$$

The main results of this paper are contained in the following three theorems. The arguments used in the proofs have common features with the ones developed in [22, 30, 34].

Theorem 3.1. *Suppose that for any given $T \in [t_0, \infty)_{\mathbb{T}}$ there exist subintervals $[a_1, b_1]_{\mathbb{T}}$ and $[a_2, b_2]_{\mathbb{T}}$ of $[T, \infty)_{\mathbb{T}}$ such that*

$$\begin{gathered} q_i(t) \geq 0 \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}},\ (i = 1, 2, \ldots, n), \\ (-1)^k e(t) \geq 0\ (\not\equiv 0) \quad \text{for } t \in [a_k, b_k]_{\mathbb{T}},\ (k = 1, 2). \end{gathered} \tag{3.2}$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.2) *in Lemma 2.1. If there exists a function $u \in \mathscr{A}(a_k, b_k)$, $(k = 1, 2)$, such that*

$$\int_{a_k}^{b_k} \left\{ |u^\sigma(t)|^{\alpha+1} \left[q(t) + \eta |e(t)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(t) \right] - \left|u^\Delta(t)\right|^{\alpha+1} r(t) \right\} \Delta t \geq 0 \tag{3.3}$$

for $k = 1, 2$, where

$$\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \tag{3.4}$$

then (1.1) *is oscillatory.*

Proof. To arrive at a contradiction, let us suppose that x is a nonoscillatory solution of (1.1). First, we assume that $x(t)$ is positive for all $t \in [t_1, \infty)_{\mathbb{T}}$, for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$.

Let $t \in [a_1, b_1]_{\mathbb{T}}$, where $a_1 \in [t_1, \infty)_{\mathbb{T}}$ is sufficiently large. Define

$$w(t) = -r(t) \frac{\Phi_\alpha\left(x^\Delta(t)\right)}{\Phi_\alpha(x(t))}. \tag{3.5}$$

It follows that

$$w^\Delta(t) = \frac{f(t, x^\sigma)}{\Phi_\alpha(x^\sigma(t))} - \frac{e(t)}{\Phi_\alpha(x^\sigma(t))} + \frac{r(t)\Phi\left(x^\Delta(t)\right)(\Phi_\alpha(x(t)))^\Delta}{\Phi_\alpha(x(t))\Phi_\alpha(x^\sigma(t))}, \tag{3.6}$$

and hence

$$w^\Delta(t) = q(t) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i - \alpha}(x^\sigma(t)) + \frac{|e(t)|}{\Phi_\alpha(x^\sigma(t))} + \frac{r(t)\Phi\left(x^\Delta(t)\right)(\Phi_\alpha(x(t)))^\Delta}{\Phi_\alpha(x(t))\Phi_\alpha(x^\sigma(t))}. \tag{3.7}$$

By our assumptions (3.2) we have $q_i(t) \geq 0$ and $e(t) \leq 0$ for $t \in [a_1, b_1]_{\mathbb{T}}$. Set

$$u_i = \frac{1}{\eta_i} q_i(t) \Phi_{\beta_i - \alpha}(x^\sigma(t)), \qquad u_0 = \frac{1}{\eta_0} \frac{|e(t)|}{\Phi_\alpha(x^\sigma(t))}. \tag{3.8}$$

Then (3.7) becomes

$$w^\Delta(t) = q(t) + \sum_{i=0}^{n} \eta_i u_i + \frac{r(t)\Phi(x^\Delta(t))(\Phi_\alpha(x(t)))^\Delta}{\Phi_\alpha(x(t))\Phi_\alpha(x^\sigma(t))}. \tag{3.9}$$

In view of the arithmetic-geometric mean inequality, see [35],

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i}, \tag{3.10}$$

and equality (3.9) we obtain

$$w^\Delta(t) \geq q(t) + \eta |e(t)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(t) + \frac{r(t)\Phi(x^\Delta(t))(\Phi_\alpha(x(t)))^\Delta}{\Phi_\alpha(x(t))\Phi_\alpha(x^\sigma(t))}. \tag{3.11}$$

Multiplying both sides of inequality (3.11) by $|u^\sigma|^{\alpha+1}$ and then using the identity

$$(u\Phi_\alpha(u)w)^\Delta = u^\sigma \Phi_\alpha(u^\sigma) w^\Delta + \left(|u|^{\alpha+1}\right)^\Delta w \tag{3.12}$$

result in

$$(u\Phi_\alpha(u)w)^\Delta \geq |u^\sigma|^{\alpha+1} Q - \left|u^\Delta\right|^{\alpha+1} r + G(u, w), \tag{3.13}$$

where

$$\begin{gathered} Q(t) = q(t) + \eta |e(t)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(t), \\ G(u, w) = \left|u^\Delta\right|^{\alpha+1} r + \left(|u|^{\alpha+1}\right)^\Delta w + |u^\sigma|^{\alpha+1} \frac{r\ \Phi_\alpha(x^\Delta)(\Phi_\alpha(x))^\Delta}{\Phi_\alpha(x)\Phi_\alpha(x^\sigma)}. \end{gathered} \tag{3.14}$$

As demonstrated in [7, 12], we know that $G(u, w) \geq 0$, and that $G(u, w) = 0$ if and only if

$$u^\Delta = \Phi_\alpha^{-1}\left(-\frac{w}{r}\right) u, \tag{3.15}$$

where Φ_α^{-1} stands for the inverse function. In our case, since $1 + \mu\Phi_\alpha^{-1}(-w/r) = x^\sigma/x > 0$, dynamic equation (3.15) has a unique solution satisfying $u(a_1) = 0$. Clearly, the unique solution is $u \equiv 0$. Therefore, $G(u, w) > 0$ on $[a_1, b_1]_{\mathbb{T}}$.

For the benefit of the reader we sketch a proof of the fact that $G(u,w) \geq 0$. Note that if t is a right-dense point, then we may write

$$G(u,w) = \frac{\alpha+1}{\Phi_\alpha^{-1}(r)} \left\{ \frac{|\Phi_\alpha^{-1}(r)u^\Delta|^{\alpha+1}}{\alpha+1} + w\Phi_\alpha(u)\Phi_\alpha^{-1}(r)u^\Delta + \frac{|w\Phi_\alpha(u)|^{(\alpha+1)/\alpha}}{(\alpha+1)/\alpha} \right\}. \tag{3.16}$$

Applying Young's inequality (Lemma 2.3) with

$$p = \alpha+1, \qquad u = \Phi_\alpha^{-1}(r)u^\Delta, \qquad v = w\Phi_\alpha(u), \tag{3.17}$$

we easily see that $G(u,w) \geq 0$ holds. If t is a right-scattered point, then G can be written as a function of $\overline{u} = \mu(t)u^\Delta$ and $\overline{v} = u$ as

$$G(\overline{u},\overline{v}) = \frac{1}{\mu} \left\{ \frac{r}{\mu^\alpha}|\overline{u}|^{\alpha+1} + \frac{wr}{\Phi_\alpha\left(\Phi_\alpha^{-1}(r) + \mu\Phi_\alpha^{-1}(w)\right)}|\overline{u}+\overline{v}|^{\alpha+1} - w\,|\overline{v}|^{\alpha+1} \right\}. \tag{3.18}$$

Using differential calculus, see [7], the result follows.

Now integrating the inequality (3.13) from a_1 to b_1 and using $G(u,w) > 0$ on $[a_1,b_1]_{\mathbb{T}}$ we obtain

$$\int_{a_1}^{b_1} \left\{ |u^\sigma(t)|^{\alpha+1}Q(t) - |u^\Delta(t)|^{\alpha+1}r(t) \right\} \Delta t < 0, \tag{3.19}$$

which of course contradicts (3.3). This completes the proof when $x(t)$ is eventually positive. The proof when $x(t)$ is eventually negative is analogous by repeating the arguments on the interval $[a_2,b_2]_{\mathbb{T}}$ instead of $[a_1,b_1]_{\mathbb{T}}$. □

A close look at the proof of Theorem 3.1 reveals that one cannot take $e(t) \equiv 0$. The following theorem is a substitute in that case.

Theorem 3.2. *Suppose that for any given $T \in [t_0,\infty)_{\mathbb{T}}$ there exists a subinterval $[a_1,b_1]_{\mathbb{T}}$ of $[T,\infty)_{\mathbb{T}}$ such that*

$$q_i(t) \geq 0 \quad \textit{for } t \in [a_1,b_1]_{\mathbb{T}},\ (i = 1,2,\ldots,n). \tag{3.20}$$

Let $\{\eta_1,\eta_2,\ldots,\eta_n\}$ be an n-tuple satisfying (2.5) in Lemma 2.2. If there exists a function $u \in \mathcal{A}(a_1,b_1)$ such that

$$\int_{a_1}^{b_1} \left\{ |u^\sigma(t)|^{\alpha+1}\left[q(t) + \eta\prod_{i=1}^{n} q_i^{\eta_i}(t)\right] - |u^\Delta(t)|^{\alpha+1}r(t) \right\} \Delta t \geq 0, \tag{3.21}$$

where

$$\eta = \prod_{i=1}^{n} \eta_i^{-\eta_i}, \tag{3.22}$$

then (1.1) *with* $e(t) \equiv 0$ *is oscillatory.*

Proof. We proceed as in the proof of Theorem 3.1 to arrive at (3.7) with $e(t) \equiv 0$, that is,

$$w^{\Delta}(t) = q(t) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i-\alpha}(x^{\sigma}(t)) + \frac{r(t)\Phi\left(x^{\Delta}(t)\right)\left(\Phi_{\alpha}(x(t))\right)^{\Delta}}{\Phi_{\alpha}(x(t))\Phi_{\alpha}(x^{\sigma}(t))}. \tag{3.23}$$

Setting

$$u_i = \frac{1}{\eta_i} q_i(t)\Phi_{\beta_i-\alpha}(x^{\sigma}(t)), \tag{3.24}$$

and using again the arithmetic-geometric mean inequality

$$\sum_{i=1}^{n} \eta_i u_i \geq \prod_{i=1}^{n} u_i^{\eta_i}, \tag{3.25}$$

we have

$$w^{\Delta}(t) \geq q(t) + \eta \prod_{i=1}^{n} q_i^{\eta_i}(t) + \frac{r(t)\Phi\left(x^{\Delta}(t)\right)\left(\Phi_{\alpha}(x(t))\right)^{\Delta}}{\Phi_{\alpha}(x(t))\Phi_{\alpha}(x^{\sigma}(t))}. \tag{3.26}$$

The remainder of the proof is the same as that of Theorem 3.1. □

As it is shown in [34] for the sublinear terms case, we can also remove the sign condition imposed on the coefficients of the sub-half-linear terms to obtain interval criterion which is applicable for the case when some or all of the functions $q_i(t)$, $i = m+1, \ldots, n$, are nonpositive. We should note that the sign condition on the coefficients of super-half-linear terms cannot be removed alternatively by the same approach. Furthermore, the function $e(t)$ cannot take the value zero on intervals of interest in this case. We have the following theorem.

Theorem 3.3. *Suppose that for any given* $T \in [t_0, \infty)_{\mathbb{T}}$ *there exist subintervals* $[a_1, b_1]_{\mathbb{T}}$ *and* $[a_2, b_2]_{\mathbb{T}}$ *of* $[T, \infty)_{\mathbb{T}}$ *such that*

$$\begin{gathered} q_i(t) \geq 0 \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}},\ (i = 1, 2, \ldots, m), \\ (-1)^k e(t) > 0 \quad \text{for } t \in [a_k, b_k]_{\mathbb{T}},\ (k = 1, 2). \end{gathered} \tag{3.27}$$

If there exist a function $u \in \mathcal{A}(a_k, b_k)$, $(k = 1, 2)$, *and positive numbers* λ_i *and* μ_i *with*

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1, \tag{3.28}$$

such that

$$\int_{a_k}^{b_k} \left\{ |u^\sigma(t)|^{\alpha+1} \left[q(t) + \sum_{i=1}^{m} P_i(t) - \sum_{i=m+1}^{n} R_i(t) \right] - \left|u^\Delta(t)\right|^{\alpha+1} r(t) \right\} \Delta t \geq 0 \tag{3.29}$$

for $k = 1, 2$, *where*

$$\begin{aligned} P_i(t) &= \beta_i (\beta_i - \alpha)^{\alpha/\beta_i - 1} \alpha^{-\alpha/\beta_i} \lambda_i^{1-\alpha/\beta_i} q_i^{\alpha/\beta_i}(t) |e(t)|^{1-\alpha/\beta_i}, \\ R_i(t) &= \beta_i (\alpha - \beta_i)^{\alpha/\beta_i - 1} \alpha^{-\alpha/\beta_i} \mu_i^{1-\alpha/\beta_i} (-q_i^+)^{\alpha/\beta_i}(t) |e(t)|^{1-\alpha/\beta_i}, \end{aligned} \tag{3.30}$$

with

$$(-q_i)^+(t) = \max\{-q_i(t), 0\}, \tag{3.31}$$

then (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution. We may assume that $x(t)$ is eventually positive on $[a_1, b_1]_{\mathbb{T}}$ when a_1 is sufficiently large. If $x(t)$ is eventually negative, then one can repeat the proof on the interval $[a_2, b_2]_{\mathbb{T}}$. Rewrite (1.1) as follows:

$$\left(r(t) \Phi_\alpha(x^\Delta) \right)^\Delta + q(t) \Phi_\alpha(x^\sigma) + g(t, x^\sigma) = 0, \quad t \in [a_1, b_1]_{\mathbb{T}}, \tag{3.32}$$

with

$$g(t, x) = \sum_{i=1}^{m} \left[q_i(t) x^{\beta_i} + \lambda_i |e(t)| \right] - \sum_{i=m+1}^{n} \left[-q_i(t) x^{\beta_i}(x) - \mu_i |e(t)| \right]. \tag{3.33}$$

Clearly,

$$g(t, x) \geq \sum_{i=1}^{m} \left[q_i(t) x^{\beta_i} + \lambda_i |e(t)| \right] - \sum_{i=m+1}^{n} \left[(-q_i)^+(t) x^{\beta_i} - \mu_i |e(t)| \right], \tag{3.34}$$

where

$$(-q_i)^+(t) = \max\{-q_i(t), 0\}. \tag{3.35}$$

Applying (2.8) and (2.9) to each summation on the right side with

$$\begin{aligned} &A = q_i(t), \qquad B = \lambda_i|e(t)|, \qquad \gamma = \beta_i, \qquad \delta = \alpha, \\ &C = (-q_i)^+(t), \qquad D = \mu_i|e(t)|, \qquad \delta = \beta_i, \qquad \gamma = \alpha, \end{aligned} \tag{3.36}$$

we see that

$$g(t,x) \geq \left[\sum_{i=1}^{m} P_i(t) - \sum_{i=m+1}^{n} R_i(t)\right] x^{\alpha}, \tag{3.37}$$

where

$$\begin{aligned} P_i(t) &= \beta_i(\beta_i - \alpha)^{\alpha/\beta_i - 1}\alpha^{-\alpha/\beta_i}\lambda_i^{1-\alpha/\beta_i} q_i^{\alpha/\beta_i}(t)|e(t)|^{1-\alpha/\beta_i}, \\ R_i(t) &= \left(\frac{\beta_i}{\alpha}\right)\left(\frac{1-\beta_i}{\alpha}\right)^{\alpha/\beta_i - 1} \mu_i^{1-\alpha/\beta_i}(-q_i^+)^{\alpha/\beta_i}(t)|e(t)|^{1-\alpha/\beta_i}. \end{aligned} \tag{3.38}$$

From (3.32) and inequality (3.37) we obtain

$$\left(r(t)\Phi_{\alpha}(x^{\Delta})\right)^{\Delta} + Q(t)\Phi_{\alpha}(x^{\sigma}) \leq 0, \quad t \in [a_1, b_1]_{\mathbb{T}}, \tag{3.39}$$

where

$$Q(t) = q(t) + \sum_{i=1}^{m} P_i(t) - \sum_{i=m+1}^{n} R_i(t). \tag{3.40}$$

Set

$$w(t) = -r(t)\frac{\Phi_{\alpha}(x^{\Delta}(t))}{\Phi_{\alpha}(x(t))}. \tag{3.41}$$

In view of inequality (3.39) it follows that

$$w^{\Delta}(t) \geq Q(t) + \frac{r(t)\Phi(x^{\Delta}(t))(\Phi_{\alpha}(x(t)))^{\Delta}}{\Phi_{\alpha}(x(t))\Phi_{\alpha}(x^{\sigma}(t))}. \tag{3.42}$$

The remainder of the proof is the same as that of Theorem 3.1, hence it is omitted. □

4. Applications

To illustrate the usefulness of the results we state the corresponding theorems for the special cases $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{\mathbb{N}}$, $(q > 1)$. One can easily provide similar results for other specific time scales of interest.

4.1. Differential Equations

Let $\mathbb{T} = \mathbb{R}$, then we have $f^{\Delta} = f'$, $\sigma(t) = t$, and

$$\left(r(t)\Phi_\alpha(x')\right)' + q(t)\Phi_\alpha(x) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x) = e(t), \quad t \in [t_0, \infty), \tag{4.1}$$

where $r, q, q_i, e : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $r > 0$, and $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots \beta_n > 0$. Let $\mathcal{A}_1(a, b) := \{u \in C^1[a, b] : u(a) = 0 = u(b), u \not\equiv 0\}$.

Theorem 4.1. *Suppose that for any given $T \in [t_0, \infty)$ there exist subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, \infty)$ such that*

$$\begin{gathered} q_i(t) \ge 0 \quad \text{for } t \in [a_1, b_1] \cup [a_2, b_2],\ (i = 1, 2, \ldots, n), \\ (-1)^k e(t) \ge 0\ (\not\equiv 0) \quad \text{for } t \in [a_k, b_k],\ (k = 1, 2). \end{gathered} \tag{4.2}$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.2) *in Lemma 2.1. If there exists a function $u \in \mathcal{A}_1(a_k, b_k)$, $(k = 1, 2)$, such that*

$$\int_{a_k}^{b_k} \left\{ |u(t)|^{\alpha+1} \left[q(t) + \eta |e(t)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(t) \right] - |u'(t)|^{\alpha+1} r(t) \right\} dt \ge 0 \tag{4.3}$$

for $k = 1, 2$, where

$$\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \tag{4.4}$$

then (4.1) *is oscillatory.*

Theorem 4.2. *Suppose that for any given $T \in [t_0, \infty)$ there exists a subinterval $[a_1, b_1]$ of $[T, \infty)$ such that*

$$q_i(t) \ge 0 \quad \text{for } t \in [a_1, b_1],\ (i = 1, 2, \ldots, n). \tag{4.5}$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.5) *in Lemma 2.2. If there exists a function $u \in \mathcal{A}_1(a_1, b_1)$ such that*

$$\int_{a_1}^{b_1} \left\{ |u(t)|^{\alpha+1} \left[q(t) + \eta \prod_{i=1}^{n} q_i^{\eta_i}(t) \right] - |u'(t)|^{\alpha+1} r(t) \right\} dt \ge 0, \tag{4.6}$$

where

$$\eta = \prod_{i=1}^{n} \eta_i^{-\eta_i}, \tag{4.7}$$

then (4.1) *with* $e(t) \equiv 0$ *is oscillatory.*

Theorem 4.3. *Suppose that for any given* $T \in [t_0, \infty)$ *there exist subintervals* $[a_1, b_1]$ *and* $[a_2, b_2]$ *of* $[T, \infty)$ *such that*

$$\begin{gathered} q_i(t) \geq 0 \quad \text{for } t \in [a_1, b_1] \cup [a_2, b_2],\ (i = 1, 2, \ldots, m), \\ (-1)^k e(t) > 0 \quad \text{for } t \in [a_k, b_k],\ (k = 1, 2). \end{gathered} \tag{4.8}$$

If there exist a function $u \in \mathcal{A}(a_k, b_k)$, $(k = 1, 2)$, *and positive numbers* λ_i *and* μ_i *with*

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1, \tag{4.9}$$

such that

$$\int_{a_k}^{b_k} \left\{ |u^\sigma(t)|^{\alpha+1} \left[q(t) + \sum_{i=1}^{m} P_i(t) - \sum_{i=m+1}^{n} R_i(t) \right] - |u'(t)|^{\alpha+1} r(t) \right\} dt \geq 0 \tag{4.10}$$

for $k = 1, 2$, *where*

$$\begin{gathered} P_i(t) = \beta_i(\beta_i - \alpha)^{\alpha/\beta_i - 1} \alpha^{-\alpha/\beta_i} \lambda_i^{1-\alpha/\beta_i} q_i^{\alpha/\beta_i}(t) |e(t)|^{1-\alpha/\beta_i}, \\ R_i(t) = \beta_i(\alpha - \beta_i)^{\alpha/\beta_i - 1} \alpha^{-\alpha/\beta_i} \mu_i^{1-\alpha/\beta_i} (-q_i^+)^{\alpha/\beta_i}(t) |e(t)|^{1-\alpha/\beta_i}, \end{gathered} \tag{4.11}$$

with

$$(-q_i)^+(t) = \max\{-q_i(t), 0\}, \tag{4.12}$$

then (4.1) *is oscillatory.*

4.2. Difference Equations

Let $\mathbb{T} = \mathbb{Z}$, then we have $f^\Delta(k) = \Delta f(k) = f(k+1) - f(k)$, $\sigma(k) = k + 1$, and

$$\Delta(r(k)\Phi_\alpha(\Delta x(k))) + q(k)\Phi_\alpha(x(k+1)) + \sum_{i=1}^{n} q_i(k)\Phi_{\beta_i}(x(k+1)) = e(k), \quad k \in [k_0, \infty)_{\mathbb{N}}, \tag{4.13}$$

where $[k_0, \infty)_{\mathbb{N}} = \{k_0, k_0 + 1, k_0 + 2, \ldots\}$, $r, q, q_i, e : [k_0, \infty)_{\mathbb{N}} \to \mathbb{R}$ with $r(k) > 0$, and $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots \beta_n > 0$. Let $[a, b]_{\mathbb{N}} = \{a, a+1, a+2, \ldots, b\}$, and $\mathcal{A}_2(a, b) := \{u : [a, b]_{\mathbb{N}} \to \mathbb{R},\ u(a) = 0 = u(b),\ u \not\equiv 0\}$.

Theorem 4.4. *Suppose that for any given* $K \in [k_0, \infty)_{\mathbb{N}}$ *there exist subintervals* $[a_1, b_1]_{\mathbb{N}}$ *and* $[a_2, b_2]_{\mathbb{N}}$ *of* $[K, \infty)_{\mathbb{N}}$ *such that*

$$\begin{gathered} q_i(j) \geq 0 \quad \text{for } j \in [a_1, b_1]_{\mathbb{N}} \cup [a_2, b_2]_{\mathbb{N}},\ (i = 1, 2, \ldots, n), \\ (-1)^k e(j) \geq 0\, (\not\equiv 0) \quad \text{for } j \in [a_k, b_k]_{\mathbb{N}},\ (k = 1, 2). \end{gathered} \tag{4.14}$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ *be an* n*-tuple satisfying* (2.2) *in Lemma 2.1. If there exists a function* $u \in \mathcal{A}_2(a_k, b_k)$, $(k = 1, 2)$, *such that*

$$\sum_{j=a_k}^{b_k-1} \left\{ |u(j+1)|^{\alpha+1} \left[q(j) + \eta |e(j)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(j) \right] - |\Delta u(j)|^{\alpha+1} r(j) \right\} \geq 0 \tag{4.15}$$

for $k = 1, 2$, *where*

$$\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \tag{4.16}$$

then (4.13) *is oscillatory.*

Theorem 4.5. *Suppose that for any given* $K \in [k_0, \infty)_{\mathbb{N}}$ *there exists a subinterval* $[a_1, b_1]_{\mathbb{N}}$ *of* $[K, \infty)_{\mathbb{N}}$ *such that*

$$q_i(j) \geq 0 \quad \text{for } j \in [a_1, b_1]_{\mathbb{N}},\ (i = 1, 2, \ldots, n). \tag{4.17}$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ *be an* n*-tuple satisfying* (2.5) *in Lemma 2.2. If there exists a function* $u \in \mathcal{A}_2(a_1, b_1)$ *such that*

$$\sum_{j=a_1}^{b_1-1} \left\{ |u(j+1)|^{\alpha+1} \left[q(j) + \eta \prod_{i=1}^{n} q_i^{\eta_i}(j) \right] - |\Delta u(j)|^{\alpha+1} r(j) \right\} \geq 0, \tag{4.18}$$

where

$$\eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \tag{4.19}$$

then (4.13) *with* $e(k) \equiv 0$ *is oscillatory.*

Theorem 4.6. *Suppose that for any given* $K \in [k_0, \infty)_{\mathbb{N}}$ *there exist subintervals* $[a_1, b_1]_{\mathbb{N}}$ *and* $[a_2, b_2]_{\mathbb{N}}$ *of* $[K, \infty)_{\mathbb{N}}$ *such that*

$$\begin{gathered} q_i(j) \geq 0 \quad \text{for } j \in [a_1, b_1]_{\mathbb{N}} \cup [a_2, b_2]_{\mathbb{N}},\ (i = 1, 2, \ldots, m), \\ (-1)^k e(j) > 0 \quad \text{for } j \in [a_k, b_k]_{\mathbb{N}},\ (k = 1, 2). \end{gathered} \tag{4.20}$$

If there exist a function $u \in \mathcal{A}_2(a_k, b_k)$, $(k = 1, 2)$, and positive numbers λ_i and μ_i with

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1, \tag{4.21}$$

such that

$$\sum_{j=a_k}^{b_k-1} \left\{ |u(j+1)|^{\alpha+1} \left[q(j) + \sum_{i=1}^{m} P_i(j) - \sum_{i=m+1}^{n} R_i(j) \right] - |\Delta u(j)|^{\alpha+1} r(j) \right\} \geq 0 \tag{4.22}$$

for $k = 1, 2$, where

$$\begin{gathered} P_i(t) = \beta_i (\beta_i - \alpha)^{\alpha/\beta_i - 1} \alpha^{-\alpha/\beta_i} \lambda_i^{1-\alpha/\beta_i} q_i^{\alpha/\beta_i}(t) |e(t)|^{1-\alpha/\beta_i}, \\ R_i(t) = \beta_i (\alpha - \beta_i)^{\alpha/\beta_i - 1} \alpha^{-\alpha/\beta_i} \mu_i^{1-\alpha/\beta_i} (-q_i^{+})^{\alpha/\beta_i}(t) |e(t)|^{1-\alpha/\beta_i}, \end{gathered} \tag{4.23}$$

with

$$(-q_i)^{+}(t) = \max\{-q_i(t), 0\}, \tag{4.24}$$

then (4.13) *is oscillatory.*

4.3. q-Difference Equations

Let $\mathbb{T} = q^{\mathbb{N}}$ with $q > 1$, then we have $\sigma(t) = qt$, $f^{\Delta}(t) = \Delta_q f(t) = [f(qt) - f(t)]/(qt - t)$, and

$$\Delta_q \left(r(t) \Phi_\alpha \left(\Delta_q x(t) \right) \right) + p(t) \Phi_\alpha \left(x(qt) \right) + \sum_{i=1}^{n} p_i(t) \Phi_{\beta_i} \left(x(qt) \right) = e(t), \quad t \in [t_0, \infty)_q, \tag{4.25}$$

where $[t_0, \infty)_q := \{q^{t_0}, q^{t_0+1}, q^{t_0+2}, \ldots\}$ with $t_0 \in \mathbb{N}$, $r, p, p_i, e : [t_0, \infty)_q \to \mathbb{R}$ with $r(t) > 0$, and $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots \beta_n > 0$. Let $[a, b]_q = \{q^a, q^{a+1}, q^{a+2}, \ldots, q^b\}$ with $a, b \in \mathbb{N}$, and $\mathcal{A}_3(a, b) := \{u : [a, b]_q \to \mathbb{R},\ u(q^a) = 0 = u(q^b),\ u \not\equiv 0\}$.

Theorem 4.7. *Suppose that for any given $T \in [t_0, \infty)_q$ there exist subintervals $[a_1, b_1]_q$ and $[a_2, b_2]_q$ of $[T, \infty)_q$ such that*

$$\begin{gathered} p_i(t) \geq 0 \quad \text{for } t \in [a_1, b_1]_q \cup [a_2, b_2]_q,\ (i = 1, 2, \ldots, n), \\ (-1)^k e(t) \geq 0\ (\not\equiv 0) \quad \text{for } t \in [a_k, b_k]_q,\ (k = 1, 2). \end{gathered} \tag{4.26}$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.2) *in Lemma 2.1. If there exists a function $u \in \mathcal{A}_3(a_k, b_k)$, $(k = 1, 2)$, such that*

$$\sum_{j=a_k}^{b_k-1} q^j \left\{ |u(q^{j+1})|^{\alpha+1} \left[p\left(q^j\right) + \eta |e(q^j)|^{\eta_0} \prod_{i=1}^{n} p_i^{\eta_i}\left(q^j\right) \right] - |\Delta_q u(q^j)|^{\alpha+1} r\left(q^j\right) \right\} \geq 0 \tag{4.27}$$

for $k = 1, 2$, *where*

$$\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \tag{4.28}$$

then (4.25) *is oscillatory.*

Theorem 4.8. *Suppose that for any given* $T \in [t_0, \infty)_q$ *there exists a subinterval* $[a_1, b_1]_q$ *of* $[T, \infty)_q$ *such that*

$$p_i(t) \geq 0 \quad \text{for } t \in [a_1, b_1]_q, \ (i = 1, 2, \ldots, n). \tag{4.29}$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ *be an* n*-tuple satisfying* (2.5) *in Lemma 2.2. If there exists a function* $u \in \mathcal{A}_3(a_1, b_1)$ *such that*

$$\sum_{j=a_1}^{b_1-1} q^j \left\{ |u(q^{j+1})|^{\alpha+1} \left[p\left(q^j\right) + \eta \prod_{i=1}^{n} p_i^{\eta_i}\left(q^j\right) \right] - |\Delta_q u(q^j)|^{\alpha+1} r\left(q^j\right) \right\} \geq 0, \tag{4.30}$$

where

$$\eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \tag{4.31}$$

then (4.25) *with* $e(t) \equiv 0$ *is oscillatory.*

Theorem 4.9. *Suppose that for any given* $T \in [t_0, \infty)_q$ *there exist subintervals* $[a_1, b_1]_q$ *and* $[a_2, b_2]_q$ *of* $[T, \infty)_q$ *such that*

$$\begin{gathered} p_i(t) \geq 0 \quad \text{for } t \in [a_1, b_1]_q \cup [a_2, b_2]_q, \ (i = 1, 2, \ldots, m), \\ (-1)^k e(t) > 0 \quad \text{for } t \in [a_k, b_k]_q, \ (k = 1, 2). \end{gathered} \tag{4.32}$$

If there exist a function $u \in \mathcal{A}_3(a_k, b_k)$, $(k = 1, 2)$, *and positive numbers* λ_i *and* μ_i *with*

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1 \tag{4.33}$$

such that

$$\sum_{j=a_k}^{b_k-1} q^j \left\{ |u(q^{j+1})|^{\alpha+1} \left[p\left(q^j\right) + \sum_{i=1}^{m} P_i\left(q^j\right) - \sum_{i=m+1}^{n} R_i\left(q^j\right) \right] - |\Delta_q u(q^j)|^{\alpha+1} r\left(q^j\right) \right\} \geq 0 \tag{4.34}$$

for $k = 1, 2$, *where*

$$P_i(t) = \beta_i(\beta_i - \alpha)^{\alpha/\beta_i - 1}\alpha^{-\alpha/\beta_i}\lambda_i^{1-\alpha/\beta_i} q_i^{\alpha/\beta_i}(t)|e(t)|^{1-\alpha/\beta_i},$$
$$R_i(t) = \beta_i(\alpha - \beta_i)^{\alpha/\beta_i - 1}\alpha^{-\alpha/\beta_i}\mu_i^{1-\alpha/\beta_i}(-q_i^+)^{\alpha/\beta_i}(t)|e(t)|^{1-\alpha/\beta_i} \tag{4.35}$$

with

$$(-p_i)^+(t) = \max\{-p_i(t), 0\}, \tag{4.36}$$

then (4.25) *is oscillatory.*

5. Examples

We give three simple examples to illustrate the importance of our results. For clarity, we have taken $n = 2$ and $e(t) \equiv 0$. Then,

$$\eta_1 = \frac{\alpha - \beta_2}{\beta_1 - \beta_2}, \quad \eta_2 = \frac{\beta_1 - \alpha}{\beta_1 - \beta_2}, \quad \beta_1 > \alpha > \beta_2 > 0. \tag{5.1}$$

Example 5.1. Consider the constant coefficient differential equation

$$\left(|x'|^{\alpha-1}x'\right)' + a|x|^{\alpha-1}x + b|x|^{\beta_1-1}x + c|x|^{\beta_2-1}x = 0, \quad t \geq 0, \tag{5.2}$$

where $b, c > 0$ and a are real numbers.

Let $u(t) = \sin(t - a_1)$, $a_1 = m$ and $b_1 = m + \pi$, $m \in \mathbb{N}$ is arbitrarily large. Applying Theorem 4.2 we see that every solution of (5.2) is oscillatory if

$$a + \left(\frac{b}{\eta_1}\right)^{\eta_1}\left(\frac{c}{\eta_2}\right)^{\eta_2} \geq 1. \tag{5.3}$$

Example 5.2. Consider the constant coefficient difference equation

$$\Delta\left(|\Delta x(k)|^{\alpha-1}\Delta x(k)\right) + a|x(k+1)|^{\alpha-1}x(k+1) + b|x(k+1)|^{\beta_1-1}x(k+1)$$
$$+ c|x(k+1)|^{\beta_2-1}x(k+1) = 0, \quad k \geq 1, \tag{5.4}$$

where $b, c > 0$ and a are real numbers.

Let $u(j) = 1 - (-1)^j$, and $a_1 = 2m$ and $b_1 = 2m + 2$, $m \in \mathbb{N}$ is arbitrarily large. It follows from Theorem 4.5 that every solution of (5.4) is oscillatory if

$$a + \left(\frac{b}{\eta_1}\right)^{\eta_1}\left(\frac{c}{\eta_2}\right)^{\eta_2} \geq 2. \tag{5.5}$$

Example 5.3. Consider the constant coefficient q-difference equation

$$\begin{aligned}\Delta_q\left(|\Delta_q x(t)|^{\alpha-1}\Delta_q x(t)\right)+a|x(qt)|^{\alpha-1}x(qt)+b|x(qt)|^{\beta_1-1}x(qt)\\ +c|x(qt)|^{\beta_2-1}x(qt)=0,\quad t\geq 1,\end{aligned}\tag{5.6}$$

where $q>1, b, c>0$ and a are real numbers.

Let $u(t)=(q^{b_1}-t)(t-q^{a_1})$, and $a_1=m$ and $b_1=m+2$, $m\in\mathbb{N}$ is arbitrarily large. In view of Theorem 4.8, we see that every solution of (5.6) is oscillatory if

$$a+\left(\frac{b}{\eta_1}\right)^{\eta_1}\left(\frac{c}{\eta_2}\right)^{\eta_2}>0.\tag{5.7}$$

6. Remarks

(1) Literature

Equation (1.1) has been studied by Sun and Wong [34] for the case $\mathbb{T}=\mathbb{R}$ and $\alpha=1$. Our results in Section 4.1 coincide with theirs when $\alpha=1$, and therefore the results can be considered as an extension from $\alpha=1$ to $\alpha>0$. Since the results in [34] are linked to many well-known oscillation criteria in the literature, the interval oscillation criteria we have obtained provide further extensions of these to time scales.

The results in Sections 4.2 and 4.3 are all new for all values of the parameters. Although there are some results for difference equations in the special case $n=1$, there is hardly any interval oscillation criteria for the q-difference equations case.

Moreover, since our main results in Section 4 are valid for arbitrary time scales, similar interval oscillation criteria can be obtained by considering other particular time scales.

(2) Generalization

The results obtained in this paper remain valid for more general equations of the form

$$\left(r(t)\Phi_\alpha(x^\Delta(t))\right)^\Delta+q(t)g(x^\sigma)+\sum_{i=1}^n q_i(t)f_i(x^\sigma)=e(t),\quad t\in[t_0,\infty)_\mathbb{T},\tag{6.1}$$

provided that $g, f_i:\mathbb{R}\to\mathbb{R}$ are continuous and satisfy the growth conditions

$$xg(x)\geq|x|^{\alpha+1},\quad xf_i(x)\geq|x|^{\beta_i+1}\quad\forall x\in\mathbb{R}.\tag{6.2}$$

To see this, we note that if $x(t)$ is eventually positive, then taking into account the intervals where q and q_i are nonnegative, the above inequalities result in

$$\left(r(t)\Phi_\alpha(x^\Delta(t))\right)^\Delta+q(t)\Phi_\alpha(x^\sigma)+\sum_{i=1}^n q_i(t)q_i(x^\sigma)\leq e(t),\quad t\in[t_0,\infty)_\mathbb{T}.\tag{6.3}$$

The arguments afterward follow analogously.

(3) Forms Related to (1.1)

Related to (1.1) are the dynamic equations with mixed delta and nabla derivatives

$$\left(r(t)\Phi_\alpha(x^\Delta)\right)^\nabla + f(t,x) = e(t), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{6.4}$$

$$\left(r(t)\Phi_\alpha(x^\nabla)\right)^\Delta + f(t,x) = e(t), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{6.5}$$

$$\left(r(t)\Phi_\alpha(x^\nabla)\right)^\nabla + f(t,x^\rho) = e(t), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{6.6}$$

where ρ denotes the backward jump operator and

$$f(t,x) = q(t)\Phi_\alpha(x) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x). \tag{6.7}$$

It is not difficult to see that time scale modifications of the previous arguments give rise to completely parallel results for the above dynamic equations. For an illustrative example we provide below the version of Theorem 3.1 for (6.4). The other theorems for (6.4), (6.5), and (6.6) can be easily obtained by employing arguments developed for (1.1) in this paper.

Theorem 6.1. *Suppose that for any given $T \in [t_0,\infty)_{\mathbb{T}}$ there exist subintervals $[a_1,b_1]_{\mathbb{T}}$ and $[a_2,b_2]_{\mathbb{T}}$ of $[T,\infty)_{\mathbb{T}}$ such that*

$$\begin{gathered} q_i(t) \geq 0 \quad \text{for } t \in [a_1,b_1]_{\mathbb{T}} \cup [a_2,b_2]_{\mathbb{T}},\ (i = 1,2,\ldots,n), \\ (-1)^k e(t) \geq 0\ (\not\equiv 0) \quad \text{for } t \in [a_k,b_k]_{\mathbb{T}},\ (k = 1,2). \end{gathered} \tag{6.8}$$

Let $\{\eta_1,\eta_2,\ldots,\eta_n\}$ be an n-tuple satisfying (2.2) *in Lemma 2.1. If there exists a function $u \in \mathcal{B}(a_k,b_k) := \{u \in C^1_{ld}[a,b]_{\mathbb{T}} : u(a) = 0 = u(b),\ u \not\equiv 0\}$, $(k = 1,2)$, such that*

$$\int_{a_k}^{b_k} \left\{ |u(t)|^{\alpha+1} \left[q(t) + \eta |e(t)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(t) \right] - \left|u^\nabla(t)\right|^{\alpha+1} r^\rho(t) \right\} \Delta t \geq 0 \tag{6.9}$$

for $k = 1,2$, where

$$\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \tag{6.10}$$

then (6.4) *is oscillatory.*

(4) An Open Problem

It is of theoretical and practical interest to obtain interval oscillation criteria when there are only sub-half-linear terms in (1.1), that is, when $\beta_i < \alpha$ holds for all $i = 1,2,\ldots,n$. Also, the open problems stated in [34] for the special case $\mathbb{T} = \mathbb{R}$ with $\alpha = 1$ naturally carry over for (1.1).

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