

Research Article

Relativistic Two-Dimensional Harmonic Oscillator Plus Cornell Potentials in External Magnetic and AB Fields

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The Klein-Gordon (KG) equation for the two-dimensional scalar-vector harmonic oscillator plus Cornell potentials in the presence of external magnetic and Aharonov-Bohm (AB) flux fields is solved using the wave function ansatz method. The exact energy eigenvalues and the wave functions are obtained in terms of potential parameters, magnetic field strength, AB flux field, and magnetic quantum number. The results obtained by using different Larmor frequencies are compared with the results in the absence of both magnetic field ($\omega_L = 0$) and AB flux field ($\xi = 0$) cases. Effect of external fields on the nonrelativistic energy eigenvalues and wave function solutions is also precisely presented. Some special cases like harmonic oscillator and Coulombic fields are also studied.

1. Introduction

The exact solution of Schrödinger equation (SE) and the relativistic wave equations for some physical potentials are very important in many fields of physics and chemistry since they contain all the necessary information for the quantum system under investigation. The hydrogen atom and the harmonic oscillator are usually given in textbooks as two of several exactly solvable problems in both classical and quantum physics [1]. The exact l -state solutions of the SE are possible only for a few potentials, and hence approximation methods are used to obtain their solutions. According to the Schrödinger formulation of quantum mechanics, a total wave function provides implicitly all relevant information about the behaviour of a physical system. Hence, if it is exactly solvable for a given potential, the wave function can describe such a system completely. Until now, many efforts have been made to solve the stationary SE with anharmonic potentials in three dimensions (3D) and two dimensions (2D) [2–7] with many applications to molecular and chemical physics. However, the study of SE with some of these potentials in arbitrary dimension D is also solved in the following (cf.

[8] and the references therein). The study of bound states is fundamental in the understanding of molecular spectrum of a diatomic molecule and provides us with insight into the physical problem under consideration in quantum mechanics [9].

Recently, some authors have studied the bound states of the l -wave equations with some typical potentials in the presence of an equal scalar potential $S(r)$ and a vector potential $V(r)$. These potentials include the harmonic oscillator potential [10, 11], ring-shaped Kratzer-type potential [12], pseudo-harmonic oscillator potential [13], double ring-shaped harmonic oscillator potential [14], and ring-shaped pseudo-harmonic oscillator potential [15–17].

It is well-known that the non-relativistic quantum mechanics is an approximate theory of the relativistic one. However, when a particle moves in a strong potential field, the relativistic effects must be considered which give the corrections for non-relativistic quantum mechanics [18]. Therefore, the motion of spin-0 and spin-1/2 particles satisfies the KG and the Dirac equations, respectively. In particular, solutions to relativistic equations play an important role in many aspects of modern physics. For instance, the Dirac

equation has been used to explain the antinucleon bound in a nucleus [19], deformed nuclei [20], and super deformation [21] and to establish an effective nuclear shell model scheme [22–24], while the KG equation has been used in describing a wide variety of phenomena, which include classical wave systems, such as the displacement of a string attached to an elastic bed [25] and quantum system based on scalar field theories [26].

Recently, the Schrödinger equation is solved exactly for its bound states (energy spectrum and wave functions) [27–29] to study the spectral properties in a 2D charged particle (electron or hole) confined by a harmonic oscillator in the presence of external strong uniform magnetic field \vec{B} along the z direction and Aharonov-Bohm (AB) flux field created by a solenoid. The energy levels and the wave functions of an electron confined by 2D harmonic and pseudo-harmonic oscillators have been studied in presence of external fields [30, 31] using the Nikiforov-Uvarov (NU) method. Overmore, it is natural to study the relativistic effects of the external magnetic and AB flux fields on the KG equation for pseudo-harmonic oscillator potential, especially for a strong coupling by means of the NU method [32]. The 2D solution of Schrödinger equation for the Kratzer potential with and without the presence of a constant magnetic field is studied within the framework of the asymptotic iteration method [33]. The energy eigenvalues are obtained analytically (numerically) for the absence (presence) of uniform magnetic field case. These results have been obtained by using different Larmor frequencies ($\omega_L \neq 0$) and potential parameters are compared with the results in the absence of magnetic field case ($\omega_L = 0$). Overmore, the spectral properties of the 2D Schrödinger equation for the pseudo-harmonic-Coulomb-linear potential are studied using the analytical iteration method [34]. Under spin symmetry, the energy states and wave functions of the Dirac equation for the Killingbeck (harmonic oscillator plus Cornell) potential have been carried out using the wave function ansatz method [35]. The spin and pseudospin symmetry in Dirac equation have been studied in the Killingbeck potential within the context of quasi-exact solution [36]. Further, the energy eigenvalues and normalized eigenfunctions of the radial Schrödinger equation in N -dimensional Hilbert space for the quark-antiquark interaction Killingbeck potential have been obtained using the power series technique via a suitable choice of ansatz to the wave function [37, 38].

Very recently, we studied the scalar charged particle exposed to relativistic scalar-vector Killingbeck potentials in presence of magnetic and Aharonov-Bohm flux fields and obtained its energy eigenvalues and wave functions using the analytical exact iteration method [39]. Therefore, the behavior of a spinless relativistic particle moving under the Killingbeck potential in a static magnetic and AB flux fields has not been investigated yet, and in this work, we aim to solve the KG equation in 2D for equal mixture of scalar and vector Killingbeck potentials with and without constant magnetic and AB flux fields for the first time. We present the exact energy eigenvalues and wave functions of the Killingbeck potential for any n and m quantum numbers in a constant magnetic field with the different Larmor frequencies using

the wave function ansatz method [36]. The non-relativistic energy eigenvalues and wave functions of our solution are presented by making an appropriate mapping of parameters. Overmore, special cases of KG for equal scalar-vector Killingbeck potentials are also presented in the presence ($\omega_L \neq 0$, $\xi \neq 0$) and absence ($\omega_L = 0$, $\xi = 0$) of uniform fields.

The structure of this paper is as follows. We study effect of external uniform magnetic and AB flux fields on a relativistic spinless particle (antiparticle) under equal mixture of scalar and vector Killingbeck potentials in Section 2. We discuss some special cases in Section 3. Finally, we give our concluding remarks in Section 4.

2. The Klein-Gordon Atom

The Klein-Gordon atom for the spinless particle with mass m_e and charge $-e$ moving in external electromagnetic field and AB flux field given by potentials $V(r)$, $S(r)$ and \vec{A} reads [40, 41]

$$\left[c^2 \left(\vec{p} + \frac{e\vec{A}}{c} \right)^2 - (E - V(r))^2 + (m_e c^2 + S(r))^2 \right] \times \psi(r, \phi) = 0. \quad (1)$$

The scalar and vector potentials are chosen in the following form [27–30]:

$$V_K(r) = \lambda r^2 + \sigma r - \frac{\kappa}{r}, \quad \vec{A} = \frac{1}{2} \vec{B} \times \vec{r} + \frac{\Phi_{AB}}{2\pi r} \hat{\phi}, \quad (2)$$

where $V_K(r)$ is the Killingbeck potential, that is, harmonic oscillator potential plus Cornell potential [35–38], which is extensively used in particle physics [42, 43]. Moreover, the vector potential in the symmetric gauge is defined by $\vec{A} = \vec{A}_1 + \vec{A}_2$ such that $\vec{\nabla} \times \vec{A}_1 = \vec{B}$ and $\vec{\nabla} \times \vec{A}_2 = 0$, where the applied magnetic field $\vec{B} = (0, 0, B)$ is perpendicular to the plane of transversal motion of the particle and \vec{A}_2 describes the additional AB flux field Φ_{AB} created by a solenoid in cylindrical coordinates [32]. The wave function in (1) is defined by

$$\psi(r, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \frac{R(r)}{\sqrt{r}}, \quad m = 0, \pm 1, \pm 2, \dots, \quad (3)$$

where m is the eigenvalue of angular momentum. The relationship between the attractive scalar and repulsive vector potentials is given by $S(r) = \beta V(r)$, where $-1 \leq \beta \leq 1$ is arbitrary constant and hence the KG equation could be reduced to a Schrödinger-type second-order differential equation as follows:

$$\left[c^2 \left(\vec{p} + \frac{e\vec{A}}{c} \right)^2 + 2(EV(r) + m_e c^2 S(r)) + S^2(r) - V^2(r) + m_e^2 c^4 - E^2 \right] \psi(r, \phi) = 0. \quad (4)$$

Now, we will treat the bound-state solutions of the two cases in (4) as follows.

2.1. *The Positive Energy Solution.* The positive energy states require that $S(r) = V(r)$ (i.e., $\beta = 1$ case) which, in the non-relativistic limit, corresponds to the solution of the wave equation:

$$\left\{ \frac{1}{2\mu} \left[-i\hbar \vec{\nabla} + \frac{e}{c} \left(\frac{Br}{2} + \frac{\Phi_{AB}}{2\pi r} \right) \hat{\phi} \right]^2 + 2V_K(r) - E \right\} \times \psi(r, \phi) = 0, \quad (5)$$

where $\psi(r, \phi)$ stands for non-relativistic wave function. Thus, the choice $S(r) = V(r)$ produces a non-relativistic limit with a potential function $2V(r)$ and not $V(r)$. Accordingly, it would be natural to scale the potential term in (4) and (5) so that in the non-relativistic limit the interaction potential becomes $V(r)$ not $2V(r)$. Thus, we need to recast (4) for the $S(r) = V(r)$ as [32, 39, 40, 44]

$$\left[c^2 \left(-i\hbar \vec{\nabla} + \frac{e}{c} \vec{A} \right)^2 + (E + m_e c^2) V(r) \right] \psi(r, \phi) = (E^2 - m_e^2 c^4) \psi(r, \phi), \quad (6)$$

where

$$\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}, \quad (7)$$

and in order to simplify (6), we introduce new parameters $\alpha_1 = E + m_e c^2$ and $\alpha_2 = E - m_e c^2$ so that it can be reduced to the following form:

$$\left\{ c^2 \left[-i\hbar \vec{\nabla} + \frac{e}{c} \left(\frac{Br}{2} + \frac{\Phi_{AB}}{2\pi r} \right) \hat{\phi} \right]^2 - \alpha_1 (\alpha_2 - V_K(r)) \right\} \times \psi(r, \phi) = 0. \quad (8)$$

By inserting (2) and (3) into (8), we obtain

$$\frac{d^2 R(r)}{dr^2} + \frac{\alpha_1}{\hbar^2 c^2} [\alpha_2 - U_{\text{eff}}(r, \omega_L, \xi)] R(r) = 0, \quad (9a)$$

$$U_{\text{eff}}(r, \omega_L, \xi) = V_K(r) + \frac{m_e^2 c^2 \omega_L^2}{\alpha_1} r^2 + \frac{\hbar^2 c^2 (m'^2 - 1/4)}{\alpha_1 r^2} + \frac{2\hbar \omega_L m_e c^2}{\alpha_1} m', \quad (9b)$$

$$\omega_L = \frac{\Omega}{2}, \quad \Omega = \frac{|e|B}{m_e c}, \quad m' = m + \xi, \quad (9c)$$

$$m' = 1, 2, \dots, \quad \xi = \frac{\Phi_{AB}}{\Phi_0},$$

where the effective potentials depending on the magnitudes of two fields strength with ω_L and m' are the Larmor frequency and a new eigenvalue of angular momentum (magnetic quantum number), respectively. It is worthy to mention that the frequency Ω is called the cyclotron frequency [45]. This is the frequency of rotation corresponding to the classical

motion of a charged particle in a uniform magnetic field and $\Omega/2$ is the Larmor frequency in units of Hz [45]. Moreover, we take ξ as integer with the flux quantum $\Phi_0 = hc/e$. Here, $V_K(r)$ is a pure Killingbeck potential (2), the second term is the harmonic oscillator-type potential, and the other terms are the rotational potential creating the rotational energy levels. Equations (9a), (9b), and (9c) can be alternatively expressed as

$$\frac{d^2 R(r)}{dr^2} + \left(\frac{C_1}{r^2} + \frac{C_2}{r} - C_3 - C_4 r - C_5 r^2 \right) R(r) = 0, \quad (10a)$$

$$C_1 = -\left(m'^2 - \frac{1}{4} \right), \quad C_2 = \frac{\alpha_1 \kappa}{\hbar^2 c^2}, \quad (10b)$$

$$C_3 = \frac{2m_e \omega_L m'}{\hbar} - \frac{\alpha_1 \alpha_2}{\hbar^2 c^2},$$

$$C_4 = \frac{\alpha_1 \sigma}{\hbar^2 c^2}, \quad C_5 = \frac{\alpha_1 \lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar} \right)^2, \quad (10c)$$

with the asymptotic behaviors $R(0) = 0$ and $R(\infty) \rightarrow 0$. It is interesting to look at the results obtained from (10a) for a special case $C_2 = C_4 = 0$; that is, we have

$$\frac{d^2 R(r)}{dr^2} + \left(\frac{C_1}{r^2} - C_5 r^2 \right) R(r) = C_3 R(r), \quad (11)$$

which corresponds to the differential equation of a harmonic oscillator with a centrifugal term. It is a form similar to the one given by (7) in [46] with the equivalence $C_1 \rightarrow -l'(l'+1)$, $C_5 \rightarrow \alpha^2$ and $C_3 \rightarrow -\lambda_n$ (α, l , and λ are the parameters used in [46]). So by putting the parameter values

$$l' = m' - \frac{1}{2}, \quad \alpha = \sqrt{\frac{(E + m_e c^2) \lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar} \right)^2}, \quad (12)$$

$$\lambda_n = \frac{E^2 - m_e^2 c^4}{\hbar^2 c^2} - \frac{2m_e \omega_L m'}{\hbar}$$

into (14) of [45], we obtain the energy equation as

$$E^2 - m_e^2 c^4 = 2m_e c^2 \hbar \omega_L m' + 2\hbar c^2 (2n + m' + 1) \times \sqrt{\frac{(E + m_e c^2) \lambda}{c^2} + (m_e \omega_L)^2}. \quad (13)$$

Now, we find a solution to (10a) by making the following choice of the wave function [37, 38, 47, 48]:

$$R_{nm}(r) = \exp\left(\frac{1}{2}pr^2 + qr\right) \sum_{n=0}^{\infty} a_n r^{n+\delta}, \quad (14)$$

where p and q are parameters whose values are to be determined in terms of the potential parameters λ , σ , and κ . Substituting (14) into (10a), we obtain the series [36]

$$\sum_{n=0}^{\infty} a_n S_n r^{n+\delta-2} + \sum_{n=1}^{\infty} a_{n-1} T_{n-1} r^{n+\delta-2} + \sum_{n=2}^{\infty} a_{n-2} W_{n-2} r^{n+\delta-2} = 0, \quad (15)$$

$$p^2 = C_5, \quad 2pq = C_4,$$

with

$$\begin{aligned} S_n &= (n + \delta)(n + \delta - 1) + C_1, \\ T_{n-1} &= 2q(n + \delta - 1) + C_2, \\ W_{n-2} &= q^2 + 2p\left(n + \delta - \frac{3}{2}\right) - C_3, \end{aligned} \quad (16)$$

where the two parameters p and q are determined in order for the radial wave functions $R_{nm}(r)$ to be finite everywhere and vanish at $r = 0$ and as $r \rightarrow \infty$. To obtain the recurrence relation which can connect various expansion coefficients a_n , we make identical powers of r in (15), that is, equate the coefficients of $r^{n+\delta-2}$ to zero. Thus, the relations become

$$\begin{aligned} a_0 [\delta(\delta - 1) + C_1] &= 0 \implies \delta^2 + C_1 = \delta, \\ a_0 \neq 0 &\implies \delta = \pm m' + \frac{1}{2}, \\ a_1 &= -\frac{(2q\delta + C_2)}{2\delta} a_0, \\ a_2 &= -\frac{[p(2\delta + 1) + q^2 - C_3] a_0 + [2q(\delta + 1) + C_2] a_1}{2(2\delta + 1)}, \\ a_3 &= -\left(\frac{(2pq - C_4) a_0 + [p(2\delta + 3) + q^2 - C_3] a_1}{6(\delta + 1)} \right. \\ &\quad \left. + \frac{[2q(\delta + 2) + C_2] a_2}{6(\delta + 1)} \right), \\ &\quad \vdots \\ a_n &= -\left(\frac{(p^2 - C_5) a_{n-4} + (2pq - C_4) a_{n-3}}{n(2\delta + n - 1)} \right. \\ &\quad \left. + \frac{[p(2\delta + 2n - 3) + q^2 - C_3] a_{n-2}}{n(2\delta + n - 1)} \right. \\ &\quad \left. + \frac{[2q(\delta + n - 1) + C_2] a_{n-1}}{n(2\delta + n - 1)} \right), \end{aligned} \quad (17)$$

where $n = 0, 1, 2, \dots$, with $a_0 \neq 0$. Here, the positive sign of parameter $\delta = m' + 1/2$ has been selected. The power series for large values of δ or m' is convergent. For convenience, we take the ratio of two successive terms, that is, a_{n+1}/a_n , which becomes

$$\begin{aligned} \frac{a_1}{a_0} &= -\left(q + \frac{C_2}{2\delta}\right) \longrightarrow -q \quad \text{when } \delta \longrightarrow \infty \\ \frac{a_2}{a_0} &= \frac{q^2(\delta + C_2/2q)[\delta + C_2/2q + 1]}{\delta(2\delta + 1)} \\ &\quad - \frac{[p(2\delta + 1) + q^2 - C_3]}{2(2\delta + 1)} \longrightarrow \frac{q^2 - p}{2} \\ &\quad \text{when } \delta \longrightarrow \infty. \end{aligned} \quad (18)$$

It is apparent from the above relations that the power series converges to zero when $\delta \rightarrow \infty$. Hence, the series must be truncated (bounded) for $n = n_{\max}$. At this value of n , we obtain the following equations:

$$p = \pm \sqrt{\frac{(E + m_e c^2)\lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar}\right)^2} \quad (19a)$$

$$\implies p = -\sqrt{\frac{(E + m_e c^2)\lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar}\right)^2}, \quad p \neq 0,$$

$$2pq = C_4$$

$$\implies q = -\frac{(E + m_e c^2)\sigma}{2\hbar^2 c^2 \sqrt{((E + m_e c^2)\lambda/\hbar^2 c^2) + (m_e \omega_L/\hbar)^2}}, \quad (19b)$$

$$\frac{(E + m_e c^2)\kappa}{\hbar^2 c^2} = -2q\left(n + m' - \frac{1}{2}\right), \quad (19c)$$

$$\begin{aligned} \frac{(E + m_e c^2)(E - m_e c^2)}{\hbar^2 c^2} &= \frac{2m_e \omega_L m'}{\hbar} + 2\sqrt{\frac{(E + m_e c^2)\lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar}\right)^2} \\ &\quad \times (m' + n - 1) - q^2, \end{aligned} \quad (19d)$$

where the negative sign of the coefficient p has been chosen in (19a). It is worth noting that the potential parameter λ must be positive when $\omega_L = 0$. Hence, (19c) gives a restriction on the potential parameters λ , σ , and κ as follows:

$$\sigma = \frac{\kappa}{(n + m' - 1/2)} \sqrt{\frac{(E + m_e c^2)\lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar}\right)^2}, \quad (20)$$

and it follows that (19d) gives the energy formula as

$$\begin{aligned} E^2 - m_e^2 c^4 &= 2m_e c^2 \hbar \omega_L m' + 2\hbar^2 c^2 (n + m' - 1) \\ &\quad \times \sqrt{\frac{(E + m_e c^2)\lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar}\right)^2} \\ &\quad - \frac{(E + m_e c^2)^2 \kappa^2}{4\hbar^2 c^2 (n + m' - 1/2)^2}, \end{aligned} \quad (21)$$

where $n = 0, 1, 2, \dots$. We may find a solution to the above transcendental equation as $E = E_{\text{KG}}^{(+)} = E_{nm'}(\omega_L, \xi)$. Overmore, the wave function (14) with the help of (17), (19a), and (19b) becomes

$$\psi_{n,m}^{(+)}(r, \phi) = C_{n,m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{m+\xi} e^{-(1/2)\sqrt{(E+m_e c^2)\lambda/\hbar^2 c^2 + (m_e \omega_L/\hbar)^2} [r^2 + ((E+m_e c^2)\sigma/(E+m_e c^2)d_2 + \hbar^2 c^2 (m_e \omega_L/\hbar)^2)]r} \sum_{n=0}^{n_{\max}} a_n r^n, \quad (22a)$$

$$a_1 = -a_0 \left[q + \frac{(E + m_e c^2) \kappa}{2\hbar^2 c^2 (m' + 1/2)} \right], \quad (22b)$$

$$a_2 = -\frac{1}{2} a_0 \left[p - q^2 - \frac{m_e \omega_L m'}{\hbar (m' + 1)} - \frac{(E + m_e c^2) \kappa}{\hbar^2 c^2 (m' + 1/2)} \left(q + \frac{(E + m_e c^2) \kappa}{4\hbar^2 c^2 (m' + 1)} \right) \right], \quad (22c)$$

where $C_{n,m}$ is the normalization constant.

In the non-relativistic limit, when $E + m_e c^2 \rightarrow 2\mu$, $E + m_e c^2 \rightarrow E_{nm'}$ and $c = 1$, (21) and (22a) give the energy formula

$$E_{nm'}(\omega_L, \xi) = \hbar \omega_L m' + \hbar (n + m' - 1) \times \sqrt{\frac{2\lambda}{\mu} + \omega_L^2} - \frac{\mu \kappa^2}{2\hbar^2 (n + m' - 1/2)^2}, \quad (23)$$

and wave function

$$\psi_{n,m}^{(+)}(r, \phi) = C_{n,m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{m+\xi} \times e^{-(\mu/2\hbar)\sqrt{2\lambda/\mu + \omega_L^2} [r^2 + (2\sigma/\mu(2d_2/\mu + \omega_L^2))]r} \sum_{n=0}^{n_{\max}} a_n r^n, \quad (24)$$

respectively, for the Schrödinger-Killingbeck system with the following restriction on the potential parameters λ , σ , and κ given by

$$\sigma = \frac{\mu \kappa}{\hbar (n + m' - 1/2)} \sqrt{\frac{2\lambda}{\mu} + \omega_L^2}. \quad (25)$$

Notice that the present model has been solved in 2D space with an external uniform magnetic field since it is perpendicular to the plane, where the vector and scalar Cornell potentials have the dimensions of $r = \rho$ and ϕ ; that is, $V = S = V(\rho, \phi)$ [32]. However, without the magnetic field, the model can be solved in any desired dimensional space by considering the change $m \rightarrow l + (D - 2)/2$ [49, 50], where D refers to spatial dimension and also the existence of interdimensional degeneracy.

As shown in Figure 1(a) and (9b), the effective potential function changes in shape when the magnetic field strength

increases, say $\omega_L = 8$ and in absence of AB flux field. The energy levels are raised when the strength of the magnetic field increases and in absence of AB flux field $\xi = 0$. We see that the effective potential changes gradually from the pure pseudo-harmonic oscillator potential, when $\omega_L = 0$, to a pure harmonic oscillator type behavior in short potential range when $\omega_L = 8$. In Figure 1(b), the effective potential (9b) which is pseudo-harmonic oscillator when $\omega_L = 0$ becomes sensitive to the increasing AB flux field $\xi = 8$ in the short range region; that is, $0 < r < 4$ a.u.

2.2. The Bound States for Negative Energy. In this case of $S(r) = -V(r)$, the first inspection of (4) shows that the following changes $V(r) \rightarrow -V(r)$, (i.e., $\lambda \rightarrow -\lambda$, $\sigma \rightarrow -\sigma$, $\kappa \rightarrow -\kappa$), $\psi_{n,m}^{(+)}(r, \phi) \rightarrow \psi_{n,m}^{(-)}(r, \phi)$ and $E \rightarrow -E$ give the negative energy solution for antiparticles as

$$E^2 - m_e^2 c^4 - 2m_e c^2 \hbar \omega_L m' = 2\hbar^2 c^2 (n + m' - 1) \sqrt{\frac{(E - m_e c^2) \lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar}\right)^2} - \frac{(E - m_e c^2)^2 \kappa^2}{4\hbar^2 c^2 (n + m' - 1/2)^2}, \quad (26)$$

with restriction on potential parameters

$$\sigma = \frac{\kappa}{(n + m' - 1/2)} \sqrt{\frac{(E - m_e c^2) \lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar}\right)^2}. \quad (27)$$

We may find solution to the above transcendental equation as $E = E_{KG}^{(-)} = E_{nm'}(\omega_L, \xi)$. Overmore, the wave function for antiparticle reads

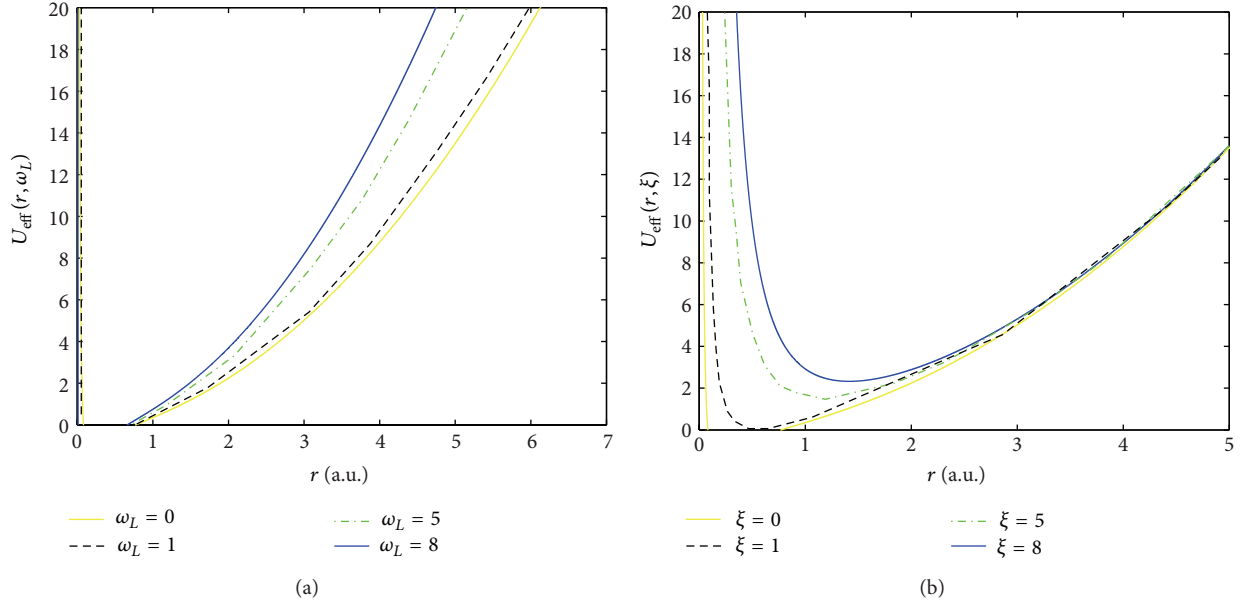


FIGURE 1: The KG effective potential function for (a) $\omega_L = 0, 1, 5, 8$ and with $\xi = 0$. (b) $\xi = 0, 1, 5, 8$ and with $\omega_L = 0$ (colour online). Here, $m_e = \hbar = c = 1$.

$$\psi_{n,m}^{(-)}(r, \phi) = D_{n,m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{m+\xi} e^{-(1/2)\sqrt{(E-m_e c^2)\lambda/\hbar^2 c^2 + (m_e \omega_L/\hbar)^2} [r^2 + ((E-m_e c^2)\sigma)/((E-m_e c^2)d_2 + \hbar^2 c^2 (m_e \omega_L/\hbar)^2)] r} \sum_{n=0}^{n_{\max}} a_n r^n, \quad (28)$$

$$a_1 = -a_0 \left[q + \frac{(E - m_e c^2) \kappa}{2\hbar^2 c^2 (m' + 1/2)} \right], \quad (29)$$

$$a_2 = -\frac{1}{2} a_0 \left[p - q^2 - \frac{m_e \omega_L m'}{\hbar (m' + 1)} - \frac{(E - m_e c^2) \kappa}{\hbar^2 c^2 (m' + 1/2)} \left(q + \frac{(E - m_e c^2) \kappa}{4\hbar^2 c^2 (m' + 1)} \right) \right], \quad (30)$$

where $D_{n,m}$ is the normalization constant.

3. Discussions

In this section, we discuss some special cases of interest from our general solution.

(i) If we set $\sigma = \kappa = 0$, the effective Killingbeck potential turns to effective harmonic oscillator potential in relativistic case as

$$U_{\text{eff}}(r, \omega_L, \xi) = \lambda r^2 + \frac{m_e^2 c^2 \omega_L^2}{(E + m_e c^2)} r^2 + \frac{\hbar^2 c^2}{(E + m_e c^2)} \frac{(m'^2 - 1/4)}{r^2} + \frac{2\hbar \omega_L m_e c^2 m'}{(E + m_e c^2)}, \quad \lambda = \frac{1}{2} \mu \omega^2. \quad (31)$$

The bound state solutions (with the change $n = 2(n_r + 1)$) are [32, 36]

$$E^2 - m_e^2 c^4 = 2m_e c^2 \hbar \omega_L m' + 4\hbar c^2 \left(n_r + \frac{m' + 1}{2} \right) \times \sqrt{\frac{(E + m_e c^2) \lambda}{c^2} + (m_e \omega_L)^2}, \quad (32)$$

$$\psi_{n_r, m}^{(+)}(r, \phi) = A_{n_r, m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{m+\xi} \times e^{-(1/2)\sqrt{(E+m_e c^2)\lambda/\hbar^2 c^2 + (m_e \omega_L/\hbar)^2} r^2} \sum_{n=0}^{n_{\max}} a_n r^{2(n_r+1)}, \quad (33a)$$

$$a_1 = 0, \quad a_2 = \frac{1}{2} a_0 \left[\sqrt{\frac{(E + m_e c^2) \lambda}{\hbar^2 c^2} + \left(\frac{m_e \omega_L}{\hbar} \right)^2} + \frac{m_e \omega_L m'}{\hbar (m' + 1)} \right], \quad (33b)$$

where $A_{n_r, m}$ is the normalization constant. On the other hand, the nonrelativistic effective harmonic oscillator potential

$$U_{\text{eff}}(r, \omega_L, \xi) = \lambda r^2 + \frac{1}{2} \omega_L^2 r^2 + \frac{(m'^2 - 1/4) \hbar^2}{2m_e r^2} + \hbar \omega_L m' \quad (34)$$

has bound states as [32]

$$E_{mm'}(\omega_L, \xi) = \hbar \omega_L m' + 2\hbar \Omega' \left(n_r + \frac{|m'| + 1}{2} \right), \quad \Omega' = \sqrt{\omega_L^2 + \omega^2}, \quad (35)$$

$$\begin{aligned} \psi_{n_r, m}^{(+)}(r, \phi) &= C_{n_r, m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{m+\xi} e^{-(\mu/2\hbar)\Omega' r^2} \\ &\times \sum_{n=0}^{n_{\max}} a_n r^{2(n_r+1)}, \quad n_r = 0, 1, 2, \dots, \\ &= \frac{1}{\sqrt{2\pi}} e^{im\phi} \sqrt{\frac{2b^{m'+1} n!}{(n+m')!}} r^{m'} e^{-br^2/2} L_n^{(m')} (br^2), \\ & \quad b = \frac{\mu}{\hbar} \Omega', \end{aligned} \quad (36a)$$

$$a_1 = 0, \quad a_2 = \frac{\mu}{2\hbar} \Omega' a_0, \quad a_0 \neq 0. \quad (36b)$$

(ii) Setting $\lambda = \sigma = 0$ in (21), the killingbeck potential turns to the Coulomb potential case (i.e., $V(r) = -A/r$ and $n = n_r + 1$). Thus, we obtain the relativistic energy formula

$$E^2 - m_e^2 c^4 = 2m_e c^2 (n_r + 2m') \hbar \omega_L - \frac{(E + m_e c^2)^2 \kappa^2}{4\hbar^2 c^2 (n_r + m' + 1/2)^2}, \quad \kappa = Z e^2, \quad (37)$$

and the wave function

$$\psi_{n_r, m}^{(+)}(r, \phi) = C_{n_r, m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{m+\xi} e^{-(m_e \omega_L)/2\hbar r^2} \times \sum_{n=0}^{n_{\max}} a_n r^{n_r+1}, \quad (38a)$$

$$a_1 = -\frac{(E + m_e c^2) \kappa}{2\hbar^2 c^2 (m' + 1/2)} a_0, \quad (38b)$$

$$a_2 = -\frac{1}{2} \left[\frac{m_e \omega_L}{\hbar} - \frac{m_e \omega_L m'}{\hbar (m' + 1)} - \frac{(E + m_e c^2)^2 \kappa^2}{4\hbar^4 c^4 (m' + 1)(m' + 1/2)} \right] a_0. \quad (38c)$$

Overmore, in the nonrelativistic limit, we have

$$E = \hbar \omega_L m' + \hbar (n_r + m') \omega_L - \frac{m_e Z^2 e^4}{2\hbar^2 (n_r + m' + 1/2)^2}, \quad (39)$$

and the wave function becomes

$$\psi_{n_r, m}^{(+)}(r, \phi) = C_{n_r, m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{m+\xi} e^{-(\mu \omega_L / 2\hbar) r^2} \sum_{n=0}^{n_{\max}} a_n r^{n_r+1}, \quad (40a)$$

$$a_1 = -\frac{m_e A}{\hbar^2 (m' + 1/2)} a_0, \quad (40b)$$

$$a_2 = -\frac{1}{2} \left[\frac{\mu \omega_L}{\hbar} - \frac{\mu \omega_L m'}{\hbar (m' + 1)} - \frac{\mu^2 Z^2 e^4}{\hbar^4 (m' + 1)(m' + 1/2)} \right] a_0. \quad (40c)$$

Notice that in the absence of external fields $\omega_L = \xi = 0$, the problem can be solved in any dimension. Thus, applying the transformation $|m| \rightarrow l + 1/2$, our previous results are reduced to the well-known three-dimensional bound state solutions for the harmonic oscillator and the hydrogen-like atoms in the Coulomb fields [40]:

$$E^\pm = \pm m_e c^2 \left[1 + \frac{2\hbar c}{m_e c^2} \left(2n_r + l + \frac{3}{2} \right) \sqrt{\frac{(E + m_e c^2) \lambda}{m_e^2 c^4}} \right]^{1/2},$$

$$E^\pm = \pm m_e c^2 \left[1 - \frac{Z^2 e^4}{4\hbar^2 c^2 (n_r + l + 1)^2} \left(1 + \frac{E}{m_e c^2} \right)^2 \right]^{1/2}, \quad (41)$$

respectively. According to (41), the energy spectrum can be found for the scalar particle and antiparticle.

In Figure 2(a), we plot the effective potential for the case of low vibrational ($n = 0, 1, 2, 3$) and rotational ($m = 1$) levels for various Larmor frequencies $\omega_L = 0, 1, 5, 8$ and $\xi = 0$ case. As shown in Figure 2(a) and (31), the effective potential function changes in shape as well as the bound state energy eigenvalues increase when $\omega_L = 8$. It is shown that the energy levels are raised when the strength of the magnetic field increases and in the absence of AB flux field. It is also obvious that the effective potential changes gradually from the pure pseudo-harmonic oscillator potential, which is a nonmagnetic ($\omega_L = 0$) and AB flux ($\xi = 0$) fields case, to a pure harmonic oscillator type behavior in short potential range when the strength of the applied magnetic field is increased to $\omega_L = 8$. If we consider a strong magnetic field case $\omega_L = 8$ which has the shape of pure harmonic oscillator potential function, the energy difference between adjacent energy levels is nearly equal which is a known characteristic of the pure harmonic oscillator potential. In Figure 2(b), we plot the effective potential (34) in the absence of the magnetic field and in the presence of AB flux field in the short range region.

In Tables 1 and 2, we show the effect of magnetic field and AB flux field, respectively, on the low vibrational n

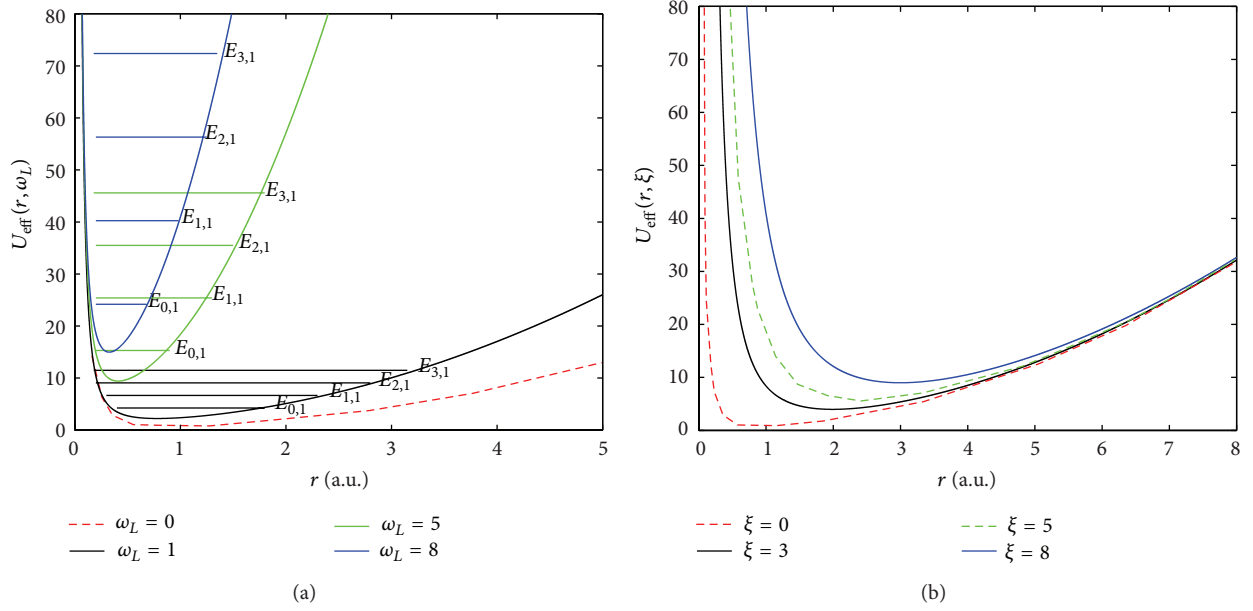


FIGURE 2: The Schrödinger effective potential function and corresponding bound state energy levels (E_{nm}) in low vibrational ($n = 0, 1, 2, 3$) and rotational ($m = 1$) levels for (a) $\omega_L = 0, 1, 5, 8$ with $\xi = 0$. (b) $\xi = 0, 1, 5, 8$ and with $\omega_L = 0$ (color online). Here, $\mu = \hbar = 1$.

and rotational m relativistic energy states of the harmonic oscillator potential. As shown in Table 1, when the magnetic field is not applied and without AB flux field ($\omega_L = 0$, $\xi = 0$), the spacing between the energy levels of the effective potential is narrow and decreases with increasing n . But when the magnetic field strength increases, the energy levels of the effective potential increase and the spacings between states also increase. In Table 2, when the AB flux field is applied and without magnetic field, the energy states become degenerate for various values of n and m and for various AB flux field strength values. In Tables 3 and 4, we show the effect of magnetic field and AB flux field, respectively, on the low vibrational n and rotational m nonrelativistic energy states of the harmonic oscillator potential. As shown in Table 3, when the magnetic field is not applied and without AB flux field ($\omega_L = 0$, $\xi = 0$), the energy states are equally spaced (the pure harmonic oscillator case). But when the magnetic field strength is raised, the energy levels of the effective potential increase and the spacings between states also increase. In Table 4, when the AB flux field is applied and without magnetic field, the energy states become degenerate and equally spaced for various values of n and m and for various AB flux field strength values.

4. Concluding Remarks

To sum up, in this paper, we have studied the solution of two-dimensional KG and Schrödinger equations with the Killingbeck potential for low vibrational and rotational energy levels without and with a constant magnetic field having arbitrary Larmor frequency and AB flux field. We have applied the wave function ansatz method for $\omega_L \neq 0$

(with magnetic field) and $\xi \neq 0$ (with AB flux field) to obtain analytical expressions, in closed form, for bound state energies and wave functions of the spinless relativistic particle subject to a Killingbeck interaction expressed in terms of external uniform magnetic and AB flux fields in any vibrational n and rotational m states. The above results show that the problems of relativistic quantum mechanics can be also solved exactly as in the non-relativistic ones. We considered the solution of both positive (particle) and negative (antiparticle) KG energy states. It is noticed that the solution with equal mixture of scalar-vector potentials can be easily reduced into the well-known Schrödinger solution for a particle with an interaction potential field and a free field, respectively. We have also studied the bound-state solutions for some special cases including the non-relativistic limits (Schrödinger equation for harmonic oscillator and Coulomb potentials under external magnetic and AB flux fields) and also the KG equation for harmonic oscillator and Coulomb interactions. The results show that the splitting is not constant and is mainly dependent on the strength of the external magnetic field and AB flux field. In order to show the effect of constant magnetic and AB flux fields on the vibrational and rotational energy levels of the harmonic oscillator, we plot the effective potential and corresponding energy levels with increasing Larmor frequency and flux field for special potential parameters. We have seen that the effective potential function and corresponding energy levels are raised in energy when magnetic and AB flux field strengths increase. The effective potential function behavior gradually changes from the pure pseudo-harmonic oscillator to a pure harmonic oscillator shape in short potential range as the magnetic and AB flux fields strengths increase.

TABLE 1: The KG energy eigenvalues (E_{nm} in atomic units) of the harmonic oscillator potential for various ω_L values and $\xi = 0$. Here, $\hbar = c = m_e = k = 1$.

m	n	E_{nm}								
		$\omega_L = 0$	$\omega_L = 1$	$\omega_L = 2$	$\omega_L = 3$	$\omega_L = 4$	$\omega_L = 5$	$\omega_L = 6$	$\omega_L = 7$	$\omega_L = 8$
0	0	1.83929	2.04353	2.40325	2.75615	3.08137	3.38066	3.65815	3.91752	4.16166
	1	3.09625	3.4289	4.03986	4.65113	5.21808	5.74094	6.22602	6.67941	7.10609
	2	4.12383	4.51488	5.26359	6.03138	6.75146	7.4193	8.04087	8.62301	9.17158
	3	5.03104	5.45806	6.30283	7.18744	8.02542	8.80667	9.53605	10.2205	10.8663
1	0	2.50976	3.16597	3.89307	4.55265	5.14478	5.68284	6.17802	6.63858	7.0706
	1	3.62919	4.27262	5.12444	5.93706	6.68087	7.36315	7.99437	8.58337	9.13707
	2	4.58916	5.22885	6.16824	7.09549	7.95634	8.7516	9.49036	10.1815	10.8323
	3	5.45354	6.09216	7.09891	8.11893	9.0768	9.96689	10.7966	11.5745	12.3081

TABLE 2: The KG energy eigenvalues (E_{nm} in atomic units) of the harmonic oscillator potential for various ξ values and $\omega_L = 0$.

m	n	E_{nm}								
		$\xi = 0$	$\xi = 1$	$\xi = 2$	$\xi = 3$	$\xi = 4$	$\xi = 5$	$\xi = 6$	$\xi = 7$	$\xi = 8$
0	0	1.83929	2.50976	3.09625	3.62919	4.12383	4.58916	5.03104	5.45354	5.85966
	1	3.09625	3.62919	4.12383	4.58916	5.03104	5.45354	5.85966	6.25166	6.6313
	2	4.12383	4.58916	5.03104	5.45354	5.85966	6.25166	6.63137.0	7.0	7.35892
	3	5.03104	5.45354	5.85966	6.25166	6.6313	7.0	7.35892	7.70901	8.05108
1	0	2.50976	3.09625	3.62919	4.12383	4.58916	5.03104	5.45354	5.85966	6.25166
	1	3.62919	4.12383	4.58916	5.03104	5.45354	5.85966	6.25166	6.6313	7.0
	2	4.58916	5.03104	5.45354	5.85966	6.25166	6.6313	7.0	7.35892	7.70901
	3	5.45354	5.85966	6.25166	6.6313	7.0	7.35892	7.70901	8.05108	8.38582

TABLE 3: The nonrelativistic energy eigenvalues (E_{nm} in atomic units) of the harmonic oscillator potential for various ω_L values and $\xi = 0$.

m	n	E_{nm}								
		$\omega_L = 0$	$\omega_L = 1$	$\omega_L = 2$	$\omega_L = 3$	$\omega_L = 4$	$\omega_L = 5$	$\omega_L = 6$	$\omega_L = 7$	$\omega_L = 8$
0	0	1.0	1.41421	2.23607	3.16228	4.12311	5.09902	6.08276	7.07107	8.06226
	1	3.0	4.24264	6.7082	9.48683	12.3693	15.2971	18.2483	21.2132	24.1868
	2	5.0	7.07107	11.1803	15.8114	20.6155	25.4951	30.4138	35.3553	40.3113
	3	7.0	9.89949	15.6525	22.1359	28.8617	35.6931	42.5793	49.4972	56.4358
1	0	2.0	3.82843	6.47214	9.32456	12.2462	15.198	18.1655	21.1421	24.1245
	1	4.0	6.65685	10.9443	15.6491	20.4924	25.3961	30.3311	35.2843	40.249
	2	6.0	9.48528	15.4164	21.9737	28.7386	35.5941	42.4966	49.4264	56.3735
	3	8.0	12.3137	19.8885	28.2982	36.9848	45.7922	54.6621	63.5685	72.4981

TABLE 4: The nonrelativistic energy eigenvalues (E_{nm} in atomic units) of the harmonic oscillator potential for various ξ values and $\omega_L = 0$.

m	n	E_{nm}								
		$\xi = 0$	$\xi = 1$	$\xi = 2$	$\xi = 3$	$\xi = 4$	$\xi = 5$	$\xi = 6$	$\xi = 7$	$\xi = 8$
0	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
	1	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0
	2	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0
	3	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0
1	0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	1	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
	2	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0
	3	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0

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