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New Positive Solutions of Nonlinear Elliptic PDEs

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Abstract: We are concerned with positive solutions of two types of nonlinear elliptic boundary value problems (BVPs). We present conditions for existence, uniqueness and multiple positive solutions of a first type of elliptic BVPs. For a second type of elliptic BVPs, we obtain conditions for existence and uniqueness of positive global solutions. We employ mathematical tools including strictly upper (SU) and strictly lower (SL) solutions, iterative sequence method and Amann theorem. We present our research findings in new original theorems. Finally, we summarize and indicate areas of future study and possible applications of the research work.

Keywords: positive (global) solution; (strict) upper and lower solutions; multiplicity of positive solutions; elliptic BVPs

1. Introduction

Nonlinear elliptic boundary value problems (NEBVP) are significantly important type PDEs having applications in different branches of science and engineering including fluid mechanics such as exothermic chemical reactions or auto catalytic reactions, see [1], in physics and chemistry. Some other specific applications of elliptic BVPs may be seen in [2–22].

The interest of such problems come from the thesis [23] (Section 1.5, page 147), where the authors asked open problem concerning multiplicity results. The main question we would like to address in this direction is the existence of more than two solutions, the articles of Chipot and Lovat [1] and Ovono and Rougirel [24], where the authors study classes of nonlocal problems motivated by the fact that they appear in some applied mathematics areas and the diffusion at each point depends on all the values of the solutions in a neighborhood of this point. Moreover, in [8], the authors have mentioned that the importance of such a model lies in the fact that measurements that serve to determine physical constants are not made at a point but represent an average in a neighborhood of a point so that these physical constants depend on local averages. The lack of the existence of the multiple solutions by using bifurcation theory showed that many local branches of solutions exist while, among them, only one is global and has no bifurcation point implying a considerable difficulty to prove the existence of a bifurcation point interior of the ball. The authors in [24] (cf. Theorem 3.1) already pointed out that the existence of a solution to the problem proposed (more exactly, for different kinds of NEBVP involving different conditions) is not guaranteed for the unboundedness of data c. It is natural to ask whether or not we can obtain the existence results of the EBVP or what happens if the data c are unbounded. Up to now, the main scope of these papers consists in the imposing some conditions on the nonlinearity c (c is the data) to prove the existence solutions to the problem (1) in smooth domains in the presence of well-ordered lower and upper solutions. Note that the case where c is unboundedness seems to be new in the literature. In other words, we obtain the existence results under regularity assumptions on c, (see [25–28] for more discussion).

In the present research study, we are interested in existence, uniqueness, multiple positive solutions and existence of weak positive solutions for an NEBVP, (called first type NEBVP), defined as

$$\begin{cases} -\mathcal{A}u = c\,(y,u)\,, & \text{in} \quad \Omega, \\ \mathcal{N}u = h\,(y)\,, & \text{on} \ \ \partial\Omega, \end{cases} \tag{1}$$

and positive global solutions for the second NEBVP defined as

$$\begin{cases} u_t - \Delta u = c\,(y,t,u) & \text{in} \quad \Theta, \\ \mathcal{N}u = h\,(y,t)\,, & \text{on} \ \ \partial\Theta, \\ u\,(y,0) = u_0\,(y)\,, & \text{in} \quad \Omega, \end{cases} \tag{2}$$

in which $\Theta = \Omega \times (0,T]\,,\ \partial\Theta = \partial\Omega \times (0,T]$, Ω is a bounded set $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\Delta u = \mathcal{A}u$ is a second order uniformly elliptic operator and $\mathcal{N}$ is defined as either

$$\mathcal{N}u = u \quad \text{on} \quad \partial\Theta,$$

or

$$\mathcal{N}u = \lambda u_v + \mu u \quad \text{on} \quad \partial\Theta,$$

where u_v is outward derivative in Ω, $\lambda = \lambda(y,t)$, $\mu = \mu\,(y,t)$ are bounded and strictly nonnegative maps on $\partial\Theta$. The initial non-negative smooth map $u_0\,(y)$ satisfies compatibility condition $u_0(y) = 0$ on $\partial\Omega$.

The structure of the article is as follows. In Sections 2 and 3, we prove the existence, uniqueness and multiplicity of positive solutions for the first type and global positive solutions for the second type by employing strictly upper (SU) and strictly lower (SL) solutions, by iterative sequence method for both of them and the Amann theorem for the first type. Section 4 is devoted to the existence of a weak solution to the first type.

2. Multiple Positive Solutions of Nonlinear Elliptic PDEs

Definition 1. *A function $\alpha \in C^2\,(\Omega) \cap C\,(\overline{\Omega})$ is said to be an upper solution (US) of (1) if α satisfies the following inequalities:*

$$\begin{cases} -\mathcal{A}\alpha \geq c\,(y,\alpha) & \text{in} \quad \Omega, \\ \mathcal{N}\alpha \geq h, & \text{on} \ \partial\Omega. \end{cases} \tag{3}$$

Moreover, a function $\beta \in C^2\,(\Omega) \cap C\,(\overline{\Omega})$ is a lower solution (LS) of (1) if for β the conditions

$$\begin{cases} -\mathcal{A}\beta \leq f\,(y,\beta) & \text{in} \quad \Omega, \\ \mathcal{N}\beta \leq h, & \text{on} \ \partial\Omega, \end{cases} \tag{4}$$

hold true.

We say that a map $\underline{u}$ is a strict LS of (1) if,

(i) $\underline{u}$ is an LS of (1), and
(ii) $u > \underline{u}$ for all solutions of (1), such that $u(t) \geq \underline{u}(t)$ holds for all $t \in [0, T]$.

Similarly, a map $\overline{u}$ is an SU solution of (1) if:

(i) $\overline{u}$ is a US of (1) and
(ii) $u < \overline{u}$ for all solutions of (1) with $u(t) \leq \overline{u}(t)$ hold for all $t \in [0, T]$.

Now, we assume that there exist $\alpha,\ \beta \in C^2(\Omega) \cap C(\overline{\Omega})$ to (1), and define

$$(\beta, \alpha) = \left\{u \in C\left(\overline{\Omega}\right) : \beta \leq u \leq \alpha\right\}.$$

(H_1) Let c be monotone nondecreasing in u and $h > 0$ such that

(i) for all $u_i \in \mathbb{R},\ (i = 1, 2)$ and for $y \in \Omega$, c satisfies

$$u_1 < u_2 \Rightarrow c(y, u_1) < c(y, u_2). \tag{5}$$

(ii) for $\alpha,\ \beta \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ with $\beta \leq \alpha$ on $\overline{\Theta}$, and

$$\min \beta \leq u_1 \leq u_2 \leq \max \alpha,$$

suppose that there exists $\sigma > 0$ such that the inequality

$$c(y, u_1) - c(y, u_2) > -\sigma(u_1 - u_2) \text{ holds.} \tag{6}$$

(iii) for every $y \in \Omega$, $y' \in \partial\Omega$, the inequalities $c(y, 0) \geq 0,\ h(y', 0) \geq 0,\ u_0(y) \geq 0$ hold.

We first make some observations on SU and SL solutions. We shall use the SU and SL solutions together with strong maximum principle in the sequel.

Lemma 1. *(Strong maximum principle, see [29]) Let $\mathcal{F}$, $\mathcal{N}$, two elliptic operators, Ω be as in Section 1 and $v \in W^{2,p}(\Omega)$ be given. Then, the following holds.*

(i) (Interior form) Let $y_0 \in \Omega$ and let B_{y_0} be an open ball centered at y_0 and contained Ω. If $\mathcal{F}v \geq 0$ in B_{y_0}, $v(x) \geq v(y_0)$ for all $y \in B_{y_0}$ and $v(y_0) \leq 0$, then $v(y) = v(y_0)$ for all $y \in B_{y_0}$.
(ii) (Boundary form) Let $y_0 \in \partial\Omega$ and let B_{y_0} be an open ball contained in Ω with $y_0 \in \partial B_{y_0}$. If $\mathcal{F}v \geq 0$ in B_{y_0}, $v(y) \geq v(y_0)$ for all $y \in B_{y_0}$ and $v(y_0) \leq 0$, then $\frac{\partial v}{\partial \zeta}(y_0) < 0$ for each ζ satisfying $\langle \zeta \mid \nu \rangle > 0$.
(iii) (Global form) Let $k \geq 0$ be a constant. If $\mathcal{F}v + kv \geq 0$ in Ω and $\mathcal{N}v \geq 0$ on $\partial\Omega$, then either $v = 0$ in Ω or $v(y) > 0$ for all $y \in \Omega \cup \partial\Theta$, and $\frac{\partial v}{\partial \nu}(y) < 0$ for all $y \in \partial\Theta$, where $\mathcal{N}$ is defined as in Section 1.

Lemma 2. *Let (5) hold and u be any solution of (1). Then, every lower (upper, respectively) solution $\underline{u}$ ($\overline{u}$), which is not a solution, (i.e., $u > \underline{u}$ ($u < \overline{u}$)) is an SL (SU) solution of (1).*

Proof. Assume that $\underline{u}$ is an SL solution under hypotheses of Lemma 2. Let us prove this with a contradiction. Let $u, \underline{u}$ be any solutions with $u \geq \underline{u}$. Then,

$$\begin{cases} -\mathcal{A}(u) = c(y, u), & \text{in } \Omega, \\ \mathcal{N}(u) = h(y), & \text{on } \partial\Omega. \end{cases}$$

and

$$\begin{cases} -\mathcal{A}(\underline{u}) \leq c(y, \underline{u}), & \text{in } \Omega, \\ \mathcal{N}(\underline{u}) \leq h(y), & \text{on } \partial\Omega. \end{cases}$$

Putting $v = u - \underline{u}$ implies that $v > 0$. By $u \geq \underline{u}$, and by the definition of the operator $\mathcal{A}$, a subtraction of above equations gives

$$-\mathcal{A}v \leq -\mathcal{A}u + \mathcal{A}\underline{u} \quad \text{in } \Omega$$

$$\leq c\,(y,u) - c\,(y,\underline{u})\,.$$

Thus, by hypothesis of $u = \underline{u}$ and (6), we have

$$-\mathcal{A}v > -\sigma\,(u - \underline{u}) = 0,$$

which is a contradiction that completes the proof.

We can prove that $\overline{u}$ is a strict US under given conditions in a similar manner. □

Lemma 3. *Assume that* (H_1) *holds. Then, by defining* $u, \overline{u}, \underline{u}$ *as above, the inequalities*

$$\underline{u} \leq u \leq \overline{u},$$

hold true.

Now, we are in a position to present some results regarding existence of solutions. To achieve this, starting from suitable maps $u^{(0)} = \alpha$ or $u^{(0)} = \beta$, obtain a sequence $\left\{u^{(n)}\right\}$ from

$$\begin{cases} -\,(\mathcal{A} - \sigma)\,u^{(n)} = c\left(y, u^{(n-1)}\right) + \sigma u^{(n-1)}, & \text{in } \Omega, \\ \mathcal{N}u^{(n)} = h, & \text{on } \partial\Omega. \end{cases}$$

Lemma 4. *Let* α, β, g *be nonnegative, bounded functions and* $\varphi \in C^2\,(\Omega)$ *satisfying*

$$\begin{cases} -\mathcal{A}\varphi + g\varphi \geq 0, & \text{in } \Omega, \\ \mathcal{N}\varphi \geq 0, & \text{on } \partial\Omega. \end{cases}$$

In this case, $\varphi \geq 0$ *in* $\overline{\Omega}$*. Furthermore,* $\varphi > 0$ *in* Ω*, unless* $\varphi = 0$*.*

Theorem 1. *Assume* (H_1) *holds. Let* α, β *be LS and US of* (1) *with* $\beta \leq \alpha$ *and* $c\,(y,u)$ *be a smooth map on* $\min \beta \leq u \leq \max \alpha$*. Then, there are two non-negative solutions* $\overline{u}$ *and* $\underline{u}$ *of problem* (1) *with*

$$\beta \leq \underline{u} \leq \overline{u} \leq \alpha.$$

Proof. Obviously, by the hypothesis (H_1) and $\beta \leq \alpha$, $\beta = 0$ is an LS of (1).

Now, define $\mathcal{Q} : C\left(\overline{\Omega}\right) \mapsto C^2\,(\Omega) \cap C\left(\overline{\Omega}\right)$ as $w = \mathcal{Q}u$, and

$$\begin{cases} -\,(\mathcal{A} - \sigma)\,w = c\,(y,u) + \sigma u, & \text{in } \Omega, \\ \mathcal{N}w = h, & \text{on } \partial\Omega. \end{cases}$$

By Schauder estimates, we deduce $\mathcal{Q}$ is completely continuous and monotonic in the sense of Collatz [30] type, that is; $u_1 \leq u_2$ implies $\mathcal{Q}u_1 < \mathcal{Q}u_2$, provided that u_1 and u_2 restricted to the set $\min \beta \leq u_1,\, u_2 \leq \max \alpha$. In fact, if $u_1 \leq u_2$, then

$$\begin{cases} -\,(\mathcal{A} - \sigma)\,(\mathcal{Q}u_2 - \mathcal{Q}u_1) = c\,(y,u_2) - c\,(y,u_1) + \sigma\,(u_2 - u_1)\,, & \text{in } \Omega, \\ \mathcal{N}\,(\mathcal{Q}u_2 - \mathcal{Q}u_1) = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

Then,

$$\begin{cases} -\mathcal{A}\left(\mathcal{Q}u_2 - \mathcal{Q}u_1\right) \geq -\sigma\left(u_2 - u_1\right) + \sigma\left(u_2 - u_1\right), & \text{in } \Omega, \\ \mathcal{N}\left(\mathcal{Q}u_2 - \mathcal{Q}u_1\right) = 0, & \text{on } \partial\Omega, \end{cases}$$

i.e.,

$$\begin{cases} -\mathcal{A}\left(\mathcal{Q}u_2 - \mathcal{Q}u_1\right) \geq 0, & \text{in } \Omega, \\ \mathcal{N}\left(\mathcal{Q}u_2 - \mathcal{Q}u_1\right) = 0, & \text{on } \partial\Omega, \end{cases}$$

Hence, $\mathcal{Q}u_1 < \mathcal{Q}u_2$, in Ω.
Letting $u^{(0)} = \alpha$ or $u^{(0)} = \beta$, generate $\left\{u^{(n)}\right\} = \left\{\mathcal{Q}u^{(n-1)}\right\}$ as

$$\begin{cases} -\left(\mathcal{A} - \sigma\right)u^{(n)} = c\left(y, u^{(n-1)}\left(y\right)\right) + \sigma u^{(n-1)} \text{ in } \Omega, \\ \mathcal{N}u^{(k)} = h, \quad \text{on } \partial\Omega. \end{cases}$$

When $u^{(0)} = \alpha$, we set $\left\{\overline{u}^{(n)}\right\}$ and $\left\{\underline{u}^{(n)}\right\}$ when $u^{(0)} = \beta$. Then, the sequence $\left\{\overline{u}^{(n)}\right\}$ and $\left\{\underline{u}^{(n)}\right\}$ converges monotonically by the continuity of $\mathcal{Q}$ to $\overline{u}_{\max}$ and $\underline{u}_{\min}$, respectively. Thus, $\overline{u} = \overline{u}_{\max}$ and $\underline{u} = \underline{u}_{\min}$ are two fixed point of $\mathcal{Q}$. The proof is completed. □

Corollary 1. *Let $\{\overline{u}_{\max}\}$ and $\{\underline{u}_{\min}\}$ be two solutions of (1). If w is a solution of (1) satisfying $\beta \leq w \leq \alpha$, the inequalities $\underline{u}_{\min} \leq w \leq \overline{u}_{\max}$ hold.*

Proof. By Theorem 1, we have $w = \mathcal{Q}w$ and $\overline{u}_1 = \mathcal{Q}\alpha$, since $w \leq \alpha$, $\mathcal{Q}w < \mathcal{Q}\alpha$ or $w < \overline{u}_1$.
By induction, $w \leq \overline{u}^{(n)}$ for every n. Thus, $w \leq \overline{u}_{\max}$. Similarly, $w \geq \underline{u}_{\min}$, hence $\underline{u}_{\min} \leq w \leq \overline{u}_{\max}$. □

Theorem 2. *Assume (H_1) holds. (1) has positive local solution $u^+(y)$.*

Proof. Notice that (H_1) implies existence of LS and US. Then, by Theorem 1, there is a local positive solution $u^+(y)$ of (1). □

We adopt the following assumption:
(H_2) Let $u_1,\ u_2 \in (\beta, \alpha)$ with $u_1 \leq u_2$, $c_1\left(y\right) \in \Omega$ be bounded nonnegative maps and the map $c\left(y, u\right)$ satisfies the following inequality

$$c\left(y, u_1\right) - c\left(y, u_2\right) \geq -c_1\left(y\right)\left(u_1 - u_2\right) \text{ in } \Omega.$$

Theorem 3. *Let (5) and (iii) in (H_1), (H_2) hold true. Assume also that $\beta\left(y\right)$, $\alpha\left(y\right)$ are, LS and US of problem (1). Then, problem (1) has unique positive solutions in (β, α).*

Proof. Existence of positive solutions of (1) may be observed by Theorem 1. Let $u_1,\ u_2 \in (\beta, \alpha)$ be two poitive solutions with $u_1 \leq u_2$. Suppose $w = u_1 - u_2$, then $w \leq 0$ and by (H_2), we have

$$\begin{cases} -Aw = c\left(y, u_1\right) - c\left(y, u_2\right) \geq 0, & \text{in } \Omega, \\ Nw = h\left(y\right) - h\left(y\right) = 0, & \text{on } \partial\Omega, \end{cases}$$

Applying Lemma 4 we have $u_1 = u_2$. □

By employing Amann Theorem [31], we show multiple positive solutions. Let α_1, α_2 are two upper solutions and β_1, β_2 are two lower solutions of problem (1).

Theorem 4. *([31]) Assume that E is a Banach space. Assume also that $K \subset E$ is a normal solid cone. Suppose that there are α_1, α_2, β_1, $\beta_2 \in E$ by $\beta_1 < \alpha_1 < \beta_2 < \alpha_2$. The operator $\mathcal{A} : [\beta_1, \alpha_2] \to E$ satisfying*

$$\beta_1 \leq \mathcal{A}\beta_1,\ \mathcal{A}\alpha_1 \leq \alpha_1,\ \beta_2 < \mathcal{A}\beta_2,\ \mathcal{A}\alpha_2 \leq \alpha_2.$$

has at least three fixed points, u_1, u_2, u_3 such that

$$\beta_1 < u_1 < \alpha_1,\ \ \beta_2 < u_2 \leq \alpha_2,\ \ \beta_2 \nleq u_3 \nleq \alpha_1.$$

Theorem 5. *Assume that (H_1) holds. Suppose that β_1, β_2 are LS, and α_1, α_2 are US of (1) such that β_2, α_1 are strict with $\beta_1 < \alpha_1 < \beta_2 < \alpha_2$. In this case, (1) has at least three solutions u_1, u_2, u_3 such that*

$$\beta_1 \leq A\beta_1,\ \mathcal{A}\alpha_1 \leq \alpha_1,\ \beta_2 < \mathcal{A}\beta_2,\ \mathcal{A}\alpha_2 \leq \alpha_2.$$

Proof. We shall show that $\mathcal{A}$ is strongly increasing operator. Equivalently saying, or all u_1, $u_2 \in [\beta_2, \alpha_2]$ with $u_1 < u_2$ $u_1(y) \leq u_2(y)$ and $u_1(y) \neq u_2(y)$. In view of (H_1), we have

$$c(y, u_1) - c(y, u_2) \geq 0 \text{ for all } y \in \Omega.$$

As a result that there exists a neighborhood $\Omega' \subset \Omega$ such that $u_1(y) \leq u_2(y)$ for $y \in \Omega'$ since $u_1(y) \neq u_2(y)$. Hence, by (H_1), we have for all $y \in \Omega$

$$c(y, u_1) - c(y, u_2) \geq 0,\ y \in \Omega'. \tag{8}$$

By (8), we have

$$\begin{aligned} -(\mathcal{A} - \sigma)(\mathcal{Q}u_2 - \mathcal{Q}u_1) &= c(y, u_1) - c(y, u_2) + \sigma(u_2 - u_1) \\ &\geq -\sigma(u_2 - u_1) + \sigma(u_2 - u_1) \geq 0 \text{ for all } y \in \Omega. \end{aligned}$$

Therefore, $\mathcal{Q}u_2 < \mathcal{Q}u_1$ in Ω by strong maximum principle, and we conclude that $\mathcal{Q}$ is a strongly increasing operator.

Now, we prove $\beta_1 \leq \mathcal{Q}\beta_1$. Consider the following problem:

$$\begin{cases} -(\mathcal{A} - \sigma)(\mathcal{Q}\beta_1 - \beta_1) = c(y, \beta_1) + \sigma\beta_1, & in\ \Omega, \\ \mathcal{N}(\mathcal{Q}\beta_1 - \beta_1) = h, & on\ \partial\Omega. \end{cases}$$

In the view of β_1, an LS of (1), we have

$$\begin{aligned} -(\mathcal{A} - \sigma)(\mathcal{Q}\beta_1 - \beta_1) &= -(\mathcal{A} - \sigma)\mathcal{Q}\beta_1 + (\mathcal{A} - \sigma)\beta_1 \\ &= -(\mathcal{A} - \sigma)\mathcal{Q}\beta_1 + \mathcal{A}\beta_1 - \sigma\beta_1 \\ &= -(\mathcal{A} - \sigma)\mathcal{Q}\beta_1 - \sigma\beta_1 + \mathcal{A}\beta_1 \\ &\geq c(y, \beta_1) - c(y, \beta_1) + \sigma\beta_1 - \sigma\beta_1 \geq 0. \end{aligned}$$

Thus, $-(\mathcal{A} - \sigma)(\mathcal{Q}\beta_1 - \beta_1) \geq 0$ and by strong maximum principle, we get $\mathcal{Q}\beta_1 \geq \beta_1$.

In view of β_1, an LS of (1), we have

$$\begin{aligned} \mathcal{N}(\mathcal{Q}\beta_1 - \beta_1) &= \mathcal{N}(\mathcal{Q}\beta_1) - \mathcal{N}\beta_1 \\ &\geq \mathcal{B}(\mathcal{Q}\beta_1) - h \\ &= h - h = 0, \end{aligned}$$

hence $\mathcal{N}(\mathcal{Q}\beta_1 - \beta_1) \geq 0$, that is by strong maximum principle, we conclude $\mathcal{Q}\beta_1 \geq \beta_1$. Then, $\mathcal{Q}\beta_1 \geq \beta_1$.

Similarly, we have $\mathcal{Q}\beta_2 \geq \beta_2$.

We know that $\mathcal{Q}\beta_1 \neq \beta_1$. Since β_2 is an LS of (1) , it is strict solution of (1). Thus, $\beta_2 < T\beta_2$.

According to the same way, we can get

$$\mathcal{A}\alpha_1 \leq \alpha_1, \ \ \mathcal{A}\alpha_2 \leq \alpha_2.$$

Thanks to the Theorem 4, $\mathcal{Q}$ has at least three fixed points u_1, u_2, u_3 with

$$\beta_1 < u_1 < \alpha_1, \ \ \beta_2 < u_2 \leq \alpha_2, \ \ \beta_2 \nleq u_3 \nleq \alpha_1.$$

□

Corollary 2. *Assume that* (H_1) *holds. Let* β_1, β_2 *be LSs and* α_1 *be strict US of* (1) *such that* $\beta_1 < \alpha < \beta_2$. *Then,* (1) *has at least three solutions* u_1, u_2, u_3 *such that*

$$\beta_1 \leq \mathcal{A}\beta_1, \mathcal{A}\alpha_1 \leq \alpha_1, \beta_2 < \mathcal{A}\beta_2, \mathcal{A}\alpha_2 \leq \alpha_2.$$

Proof. We shall apply Theorem 5. That is, assume that there are two upper solutions, α_1, α_2 satisfying:

$$\beta_1(y) \leq \alpha_1(y) < \alpha_2(y) < \beta_2(y) \ \text{ for } y \in \Omega \cup \partial\Theta,$$

Let α_1 and α_2 be US such that $\alpha_1(y) < \alpha_2(y)$, for $y \in \Omega$ and

$$\frac{\partial \alpha_1}{\partial \nu}(y) > \frac{\partial \alpha_2}{\partial \nu}(y), \text{ for } y \in \partial\Theta.$$

Hence, we only verify that

$$\alpha_2(y) < \beta_2(y) \ \text{ for } y \in \Omega.$$

Notice that

$$\beta_2(y) - \alpha_1(y) = 0, \ \text{ for } y \in \partial\Theta,$$

$$\beta_2(y) - \alpha_1(y) > 0, \ \text{ for } y \in \Omega \cup \partial\Theta.$$

Provided that

$$\frac{\partial \beta_2}{\partial \nu}(y) - \frac{\partial \alpha_1}{\partial \nu}(y) < 0, \text{ for } y \in \partial\Theta.$$

Then, by strong maximum principle (Lemma 1), we have

$$\beta_2(y) - \alpha_1(y) > 0, \ \text{ for } y \in \Omega,$$

Suppose there is a $y_0 \in \partial\Theta$ with

$$\frac{\partial \beta_2}{\partial \nu}(y_0) - \frac{\partial \alpha}{\partial \nu}(y_0) = 0.$$

Let $w = \beta_2 - \alpha_1$. Since $\frac{\partial w}{\partial \nu}(y_0) > 0$, we find an open ball B_{y_0} such that $\frac{\partial w}{\partial \nu}(y) > 0$, for all $y \in B_{y_0} \cap \overline{\Omega}$. Since $w(y) = 0$ for $y \in \partial\Theta$, $w(y) < 0$ for $y \in B_{y_0} \cap \Omega$, that implies:

$$0 < \beta_2(y) - \alpha_1(y) < 0, \ \text{ for } y \in B_{y_0} \cap \Omega,$$

which is a contradiction. □

3. Positive Global Solutions for Second Problem

We are interested in existence of global solutions of (2). Suppose that $\mathcal{N}$ is defined either as

$$\begin{cases} \mathcal{N}u\,(y,t) = u\,(y,t)\,, & \text{on } \partial\Theta, \\ & \text{or} \\ \mathcal{N}u\,(y,t) = \lambda u_v\,(y,t) + \mu u\,(y,t)\,, & \text{on } \partial\Theta, \\ u\,(y,0) = u_0\,(y) & \text{in } \Omega. \end{cases}$$

where the initial non-negative smooth map $u_0(y)$ satisfies compatibility condition $u_0\,(y) = 0$ on $\partial\Omega$.

Recall that the operator $\mathcal{F}$ is defined as

$$\mathcal{F}u = u_t - \mathcal{A}u, \text{ where } \Delta u = \mathcal{A}u.$$

As a matter of fact, we have that:

Definition 2. *$\alpha\,(y,t) \in C\left(\overline{Q}\right) \cap C^{1,2}\,(\Theta)$ is a US of (2) provided that*

$$\begin{cases} \mathcal{F}\alpha \geq c\,(y,t,\alpha)\,, & \text{in } \Theta, \\ \mathcal{N}\alpha \geq h\,(y,t)\,, & \text{on } \partial\Theta, \\ u\,(y,0) \geq u_0\,(y)\,, & \text{in } \Omega. \end{cases} \tag{9}$$

Similarly, $\beta\,(y,t) \in C\left(\overline{Q}\right) \cap C^{1,2}\,(\Theta)$ is an LS by changing direction of inequalities in (9), we set $u \in C\left(\overline{\Theta}\right)$ with $\beta \leq u \leq \alpha$ in $\overline{\Theta}$.

It is obvious that the upper and lower solutions of (2) are given by $\alpha\,(y,t)\,,\ \beta\,(y,t)$. Let σ be at (H_1) with

$$c\,(y,u_1) - c\,(y,u_2) + \sigma\,(u_1 - u_2) > 0,$$

on

$$\min \beta\,(y,t) \leq u_i \leq \max \alpha\,(y,t)\,,\ \ i = 1,2.$$

We define $\overline{u}^{(1)}$ as

$$\begin{cases} \mathcal{F}\overline{u}^{(1)} + \sigma\overline{u}^{(1)} = c\,(y,t,\alpha) + \sigma\alpha & \text{in } \Theta, \\ \mathcal{N}\overline{u}^{(1)} = h, & \text{on } \partial\Theta, \\ u^{(1)}\,(y,0) = u_0\,(y)\,, & \text{in } \Omega. \end{cases}$$

Then, by the maximum principle for a parabolic equation, we have

$$\overline{u}^{(1)}\,(y,t) < \alpha\,(y,t)\,, \text{ in } \Omega.$$

Defining $\psi\ :\ \alpha\,(y,t) \mapsto \overline{u}^{(1)}\,(y,t)\,$, we have $\overline{u}^{(1)} = \psi\alpha$ is a monotone operator with type of Collatz [30]. Letting $\underline{u}^{(1)} = \psi\beta$, we get

$$\left\{\overline{u}^{(n)}\right\},\ \left\{\underline{u}^{(n)}\right\},$$

with

$$\overline{u}^{(n)} = \psi\overline{u}^{(n-1)},$$

and

$$\underline{u}^{(n)} = \psi\underline{u}^{(n-1)},$$

in which

$$\overline{u}^{(1)} = \psi\alpha, \ \underline{u}^{(1)} = \psi\beta.$$

Theorem 6. *Suppose that conditions of* (H_1) *hold. Assume also that* $\alpha(y,t) \in \overline{\Theta}$, $\beta(y,t) \in \overline{\Theta}$ *are a US and LS of* (2). *If there is a* σ *such that*

$$c(y,u_1) - c(y,u_2) + \sigma(u_1 - u_2) > 0,$$

where

$$\min_{\Omega} \beta \leq u_i \leq \max_{\Omega} \alpha, \ i = 1,2,$$

there exists a unique strong solution u *of* (2) *with*

$$\lim_{n\to\infty} \overline{u}^{(n)} = \psi u = u = \lim_{n\to\infty} \underline{u}^{(n)},$$

where $\left\{\overline{u}^{(n)}\right\}$ *is decreasing and* $\left\{\underline{u}^{(n)}\right\}$ *is increasing sequences. We address the situation where* h *is time independent next.*

Corollary 3. *Suppose that conditions of* (H_1) *hold. If* $u(y) \in (\underline{u}(y), \overline{u}(y))$ *is a solution of*

$$\begin{cases} -\mathcal{A}u = c(y, u(y)), & in \quad \Omega, \\ \mathcal{N}u = h, & on \quad \partial\Omega, \end{cases}$$

where $\overline{u}(y)$ *and* $\underline{u}(y)$ *are an upper and an LS, respectively, there is a global regular solution* $u(y,t) \in (\underline{u}(y), \overline{u}(y))$, *for all* $t > 0$.

Now, we introduce two identities:

$$\begin{cases} -\mathcal{A}u = c(y,u), & \text{on} \quad \Theta, \\ u = 0, & \text{on} \ \partial\Omega, \end{cases} \tag{10}$$

and

$$\begin{cases} \mathcal{F}u = c(y,t,u), & \text{in} \quad \Theta, \\ u = 0, & \text{on} \quad \partial\Theta, \\ u(y,0) = \overline{u}(y), & \text{in} \quad \Omega. \end{cases} \tag{11}$$

Theorem 7. *Suppose that conditions of* (H_1) *hold and also* $\overline{u}(y)$ *is a US of* (10). *If* $u(y,t)$ *is a solution of* (11), $u_t \leq 0$.

Proof. Assume $\left\{u^(n)(y,t)\right\} \subset \Theta$ is a sequence of maps by $u^{(0)}(y,t) = \overline{u}(y)$, and for $n \geq 1$,

$$\begin{cases} \mathcal{F}u^{(n)} + \sigma u^{(n)} = c\left(y, u^{(n-1)}\right) + \sigma u^{(n-1)}, & \text{in} \ \ \Theta, \\ u^{(n)} = 0, & \text{on} \ \partial\Theta, \\ u^{(n)}(y,0) = \overline{u}(y), & \text{in} \ \ \Omega. \end{cases} \tag{12}$$

In this case,

$$\overline{u}(y) \geq u^{(1)}(y,t) \geq \ldots \geq u^{(n-1)}(y,t) \geq u^{(n)}(y,t) \geq \ldots. \tag{13}$$

Recall that

$$\begin{cases} \mathcal{F}\left(u^{(1)}-\overline{u}\right)+\sigma\left(u^{(1)}-\overline{u}\right)=-\left[c\left(y,\overline{u}\right)-\mathcal{A}\overline{u}\right]\geq 0, \\ \mathcal{N}\left(u^{(1)}-\overline{u}\right)=h\left(y,t\right)-\mathcal{N}\overline{u}\leq 0. \end{cases}$$

We conclude by strong maximum principle that $\overline{u}\geq u^{(1)}$.
By induction and from (13) for $n\in\mathbb{N}$, we deduce the existence

$$u^{(n)}\rightarrow u^{*}\ as\ n\rightarrow\infty.$$

Thus, $u^{*}(y,t)$ is a solution of

$$\begin{cases} \mathcal{F}u^{*}=c\left(x,u^{*}\right), & \text{in} \quad \Theta, \\ u^{*}=0, & \text{on } \partial\Theta, \\ u^{*}\left(y,0\right)=\overline{u}\left(y\right), & \text{in } \Omega, \end{cases}$$

Hence, $u^{*}\left(y,t\right)=u\left(y,t\right)\in\Theta$, via uniqueness condition. Differentiating (12) with respect to t, we have

$$\begin{cases} \mathcal{F}\left(u^{(n)}\right)_{t}+\sigma\left(u^{(n)}\right)_{t}=c_{U}\left(y,U\right)U_{t}, & \text{in} \quad \Theta, \\ \left(u^{(n)}\right)_{t}=0, & \text{on} \quad \partial\Theta, \end{cases}$$

in which $U=u^{(n-1)}$.
Since $c_{U}(y,U)U_{t}$ is located in Θ, it is bounded.
We set

$$w_{n}=\frac{u^{(n)}\left(y,\delta\right)-u^{(n)}\left(y,0\right)}{\delta},\ y\in\Omega.\ \delta>0,$$

As a result by (12) and (13), we get $w_{n}\leq 0$. Hence, $\left(u^{(n)}\left(y,0\right)\right)_{t}\leq 0$, $y\in\Omega$. Furthermore, we have $\left(u^{(n)}\right)_{t}\leq 0$.
We can apply the same proof of Theorem 1 to get

$$u^{(n)}\mapsto u\in C^{1+\eta}\ \ \text{on}\ \ t,\ \ \text{for}\ \ 0<\eta<1.$$

Thus, $u_{t}\left(y,t\right)\leq 0$ in Θ. Herewith, the proof is complete. □

Let us assume $c\left(y,t,u\right)$ is a C^{1}-mapping for u and satisfies the inequalities:

$$\begin{cases} c\left(y,t,c_{1}\right)\geq 0,\quad c\left(y,t,c_{2}\right)\leq 0, & \text{in} \quad \Omega\cap\mathbb{R}^{+}, \\ c_{1}\mu\left(y,t\right)\leq h\left(y,t\right)\leq c_{2}\mu\left(y,t\right), & \text{on } \partial\Omega\cap\mathbb{R}^{+}, \end{cases} \tag{14}$$

where $c_{1}>0$, $c_{2}>0$ are constants with $c_{1}<c_{2}$.

Theorem 8. *Assume that (14) holds. If there exists constants $c_{1}>0$, $c_{2}>0$ with $c_{1}<c_{2}$. Then, for all $u\in[c_{1},c_{2}]$, (2) has a unique positive and bounded global solution.*

Proof. Let $\alpha=c_{2}$, $\beta=c_{1}$; then, by (14), we get

$$\begin{cases} \mathcal{F}\alpha=0\geq c\left(y,t,c_{2}\right)=c\left(y,t,\alpha\right), & \text{in} \quad \Omega\times\mathbb{R}^{+}, \\ \mathcal{N}\alpha=\lambda\alpha_{v}+\mu\alpha=c_{2}\mu\left(y,t\right)\geq h\left(y,t\right), & \text{on } \partial\Omega\times\mathbb{R}^{+}, \\ \alpha=c_{2}, & \text{in} \quad \Omega. \end{cases}$$

This allows to conclude that $\alpha = c_2$, is a US. Similarly, $\beta = c_1$ is an LS. Consequently, thanks to the Theorem 6, we conclude the result. □

We are now ready to prove the uniqueness result of global positive solution. To this purpose, we assume that:

(a) For every $y \in \Omega$, $y' \in \partial\Omega$: $c\,(y,t) \geq 0$, $h\,(y',t) \geq 0$ and $u_0\,(y) \geq 0$.

Theorem 9. *Suppose that* (a), (5) *and* (6) *hold. If there exists a mapping* M *with*

$$c_u\,(y,t,u)\,u \leq M\,(y,t)\,u, \text{ for every } T < \infty, \; u \geq 0, \text{ in } \Theta, \tag{15}$$

then, (2) *has a unique global positive solution.*

Proof. Using the mean-value theorem, (5) and (6), we have

$$\begin{aligned} c\,(y,t,u) &= c_u\,(y,t,\xi)\,u + c\,(y,t,0)\,, \\ &\geq c_u\,(y,t,\xi)\,u, \text{ in } \Theta, \end{aligned} \tag{16}$$

in which $\xi = \xi\,(y,t)$ is intermediate value between u and 0.

By Lemma 3 and $\mathcal{F}$, we write

$$c_3\,(y,t) = -c_u\,(y,t,\xi)\,.$$

Hence, $u = 0$, *or* $0 < u$ in Θ. u is positive because, if it is not, $u = 0$ only if all of the maps in (a) is equal to 0. This implies that $u = \beta$ is an LS of (2).

Let w be a solution of:

$$\begin{cases} \mathcal{F}w = Mw + c\,(y,t,0) & \text{in } \Theta, \\ \mathcal{N}w = h\,(y,t) & \text{on } \partial\Theta, \\ w\,(y,0) = u_0\,(y) & \text{in } \Omega, \end{cases}$$

Therefore, w is a upper positive solution of (2). As a matter of fact, for $\alpha = w$ and again applying the mean-value theorem,

$$c\,(y,t,\alpha) = c_\alpha\,(y,t,\xi)\,(\alpha) + c\,(y,t,0)\,,$$

where $\xi = \xi\,(y,t)$ is located between α and 0.

Combining with (a), (5), (6) and (15), we have

$$c\,(y,t,\alpha) \leq M\,(y,t)\,\alpha + c\,(y,t,0)\,, \text{ in } \Theta.$$

Thus,

$$\begin{cases} \mathcal{F}\alpha = M\alpha + c\,(y,t,0) \geq c\,(y,t,\alpha)\,, & \text{in } \Theta, \\ \mathcal{N}w = h\,(y,t) \geq 0, & \text{on } \partial\Theta, \\ w\,(y,0) = u_0\,(y) \geq 0, & \text{in } \Omega, \end{cases}$$

which implies that α is an upper positive solution of (2). Hence, by Theorem 8, we deduce the unique positive global solution of (2). □

4. Weak Solutions for the First Problem

Now, our main result shows the existence of a weak solution to problem (1) with Dirichlet boundary condition ($u = 0$ i.e., $h = 0$ and $\mathcal{N}u = u$) under a unboundedness on c. Before doing this, we introduce the following notion of weak solution to (1). For that, we need Lemmas 5–7 and the assumptions (H_3) and (H_4) (see later). Note that the notion of weak solution to (1) is essentially the same as in Definition 1, the only difference is that we now require that u belong to $H_0^1\,(\Omega)$.

Definition 3. *Let $u \in H_0^1(\Omega)$, u is said to be a weak solution of (1) if it satisfies*

$$\int_{\Omega} \nabla u \nabla \phi dx = \int_{\Omega} c(u)\phi dx \quad in \quad \Omega.$$

A nonnegative function $\underline{u}$, $\overline{u} \in H_0^1(\Omega)$ is called a weak lower solution (WLS) and weak upper solution (WUS) of (1) if they satisfy

$$\int_{\Omega} \nabla \underline{u} \nabla \phi dx \leq \int_{\Omega} c(\underline{u})\phi dx \quad in \quad \Omega,$$

and

$$\int_{\Omega} \nabla \overline{u} \nabla \phi dx \geq \int_{\Omega} c(\overline{u})\phi dx \quad in \quad \Omega,$$

for all $\phi \in H_0^1(\Omega)$.

Lemma 5. *([26]) Let v solve $\Delta v = c$ in Ω. If $c \in C(\Omega)$, then $v \in C^{1,\eta}(\Omega)$ for any $\eta \in (0,1)$, thus in particular, v is continuous in Ω.*

Lemma 6. *([32]) For each $c \in L^2(\Omega)$. Then, there exists a unique solution $v \in H_0^1(\Omega)$ to problem (1).*

Lemma 7. *([33]) Assume that u and v are two non-negative functions such that*

$$\begin{cases} -\Delta u \geq -\Delta v, & in \quad \Omega, \\ u = v = 0, & in \; \partial\Omega. \end{cases}$$

Then, $u \geq v$, a.e., in Ω.

(H_3) $c \in C^1((0,+\infty))$ is increasing function such that

$$\lim_{u \to +\infty} c(u) = +\infty.$$

(H_4) Moreover, $c \in C^1((0,+\infty))$ satisfies

$$\lim_{u \to +\infty} \frac{c(u)}{u} = 0.$$

Theorem 10. *Let (H_3) and (H_4) hold. Then, problem (1) has a positive weak solution.*

Proof. Let σ be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and ϕ_1 be the corresponding positive eigenfunction with $\|\phi_1\| = 1$.

Let $\delta > 0$ be such that $|\nabla\phi_1|^2 - \sigma\phi_1^2 > 0$ on

$$\overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}.$$

Let us define

$$\underline{u} = \frac{1}{2}\phi_1^2.$$

We shall verify that $\underline{u}$ is a weak LS of (1). Indeed, let $\phi \in H_0^1(\Omega)$ with $\phi > 0$ in Ω. A simple calculation shows that

$$\int_{\overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx = \int_{\overline{\Omega}_\delta} \phi_1 \nabla \phi_1 \nabla \phi dx$$

$$= \int_{\overline{\Omega}_\delta} \nabla \phi_1 \nabla \left(\phi_1 \cdot \phi \right) dx - \int_{\overline{\Omega}_\delta} \left| \nabla \phi \right|^2 \phi dx$$
$$= \int_{\overline{\Omega}_\delta} \left(\sigma \phi_1^2 - \left| \nabla \phi \right|^2 \right) \phi dx.$$

On $\overline{\Omega}_\delta$, we have $\left| \nabla \phi_1 \right|^2 - \sigma \phi_1^2 > 0$, then $\sigma \phi_1^2 - \left| \nabla \phi_1 \right|^2 < 0$. Hence,

$$\int_{\overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx < 0.$$

By (H_3), we get $c\left(\underline{u}\right) > 0$ (large enough).
Then,

$$\int_{\overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx \leq \int_{\overline{\Omega}_\delta} c\left(\underline{u}\right) \phi dx. \tag{17}$$

Next, on $\Omega \setminus \overline{\Omega}_\delta$, we have $\phi_1 \geq d$ for some $d > 0$. By (H_3) and by the definition of $\underline{u}$, it follows that:

$$\int_{\Omega \setminus \overline{\Omega}_\delta} c\left(\underline{u}\right) \phi dx \geq \int_{\Omega \setminus \overline{\Omega}_\delta} \sigma \phi dx$$
$$\geq \int_{\Omega \setminus \overline{\Omega}_\delta} \left(\sigma \phi_1^2 - \left| \nabla \phi \right|^2 \right) \phi dx$$
$$= \int_{\overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx, \tag{18}$$

From (17) and (18), we deduce that

$$\int_{\Omega} \nabla \underline{u} \nabla \phi dx \leq \int_{\Omega} c\left(\underline{u}\right) \phi dx,$$

for any $\phi \in H_0^1\left(\Omega\right)$. That is, $\underline{u}$ is a weak LS of problem (1).
Next, we shall construct a WUS of (1). Let e be the solution of the following problem

$$\begin{cases} \Delta e = 1 & \text{in } \Omega, \\ e = 0, & \text{on } \partial\Omega. \end{cases} \tag{19}$$

Let $\overline{u} = Ce$, where C is a positive real number which will be chosen later. We will verify that $\overline{u}$ is a weak US (1). Let $\phi \in H_0^1\left(\Omega\right)$ with $\phi > 0$ in Ω. Then, from (19), we get

$$\int_{\Omega} \nabla \overline{u} \nabla \phi dx = C \int_{\Omega} \nabla e \nabla \phi dx$$
$$= C \int_{\Omega} \phi dx.$$

By (H_3), we can choose C large enough so that

$$C \geq c\left(C \left\| e \right\|_\infty\right).$$

Therefore,

$$\begin{aligned}\int_{\Omega} \nabla \overline{u} \nabla \phi dx &\geq c\left(C\|e\|_{\infty}\right) \int_{\Omega} \phi dx \\ &\geq \int_{\Omega} c\left(C\|e\|_{\infty}\right) \phi dx \\ &\geq \int_{\Omega} c(\overline{u}) \phi dx.\end{aligned}$$

that is, $\overline{u}$ a weak US of (1) for a large enough C. In order to obtain a weak solution of (1) , we define the sequence

$$\{u_n\} \subset E = H_0^1(\Omega) \cap C(\Omega),$$

as $u_0 = \overline{u} \subset E = H_0^1(\Omega) \cap C(\Omega)$, and u_n is the unique solution of the problem

$$\begin{cases} -\Delta u_n = c(u_{n-1}), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{20}$$

If $u_{n-1} \in E$ is given, the right-hand-side of (20) is independent of u_n since $c(u_0) \in C(\Omega) \subset L^2(\Omega)$ in x and from Lemma 6. Then, (20) with $n = 1$ has unique solution $u_1 \in H_0^1(\Omega)$.

We deduce from Lemma 5 that $u_1 \in C(\Omega)$. Consequently, we conclude that $u_1 \in E$. In the same way, we construct the following elements $u_n \in E$ of our sequence. From (20), and by the fact that u_0 is a weak US of (1), we have

$$\begin{cases} -\Delta u_0 \geq c(u_0) = -\Delta u_1, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases}$$

from which and Lemma 7, we have $u_0 \geq u_1$.

Since $u_0 \geq \underline{u}$ and by the monotonicity of c, we have

$$-\Delta u_1 = c(u_0) \geq c(\underline{u}) \geq -\Delta \underline{u}.$$

from which and Lemma 7, we deduce that $u_1 \geq \underline{u}$.

From (20), and by the monotonicity of c, $u_1 \geq \underline{u}$ and $u_0 \geq u_1$, for u_2, we write

$$-\Delta u_1 = c(u_0) \geq c(u_1) = -\Delta u_2,$$

and

$$-\Delta u_2 = c(u_1) \geq c(\underline{u}) \geq -\Delta \underline{u}.$$

Then, thanks to the Lemma 7, we get $u_1 \geq u_2$ and $u_2 \geq \underline{u}$. Repeating this argument, we get a bounded monotonic sequence $\{u_n\}$ satisfying

$$\overline{u} = u_0 \geq u_1 \geq \cdots \geq u_n \geq \underline{u} > 0.$$

Thanks to the continuity of the function c and by the definition of the sequences $\{u_n\}$, there exists a constant $\mathcal{R}$, which is independent from n such that

$$|c(u_{n-1})| < \mathcal{R}. \tag{21}$$

Using (21), multiplying the first equation of (20) by u_n, integrating and using the Hölder inequality and Sobolev's embedding, we can show that

$$\int_{\Omega} |\nabla u_n|^2 \, dx \leq \int_{\Omega} c\,(u_n)\, u_n dx$$
$$\leq \mathcal{R} \int_{\Omega} |u_n| \, dx$$
$$\leq \mathcal{R} \, \|u_n\|_{H_0^1(\Omega)} \, .$$

Then,

$$\|u_n\|_{H_0^1(\Omega)} \leq \mathcal{R}, \quad for.\ all\ n. \tag{22}$$

where $\mathcal{R}'$ is a constant and independent from n. By (22), we infer that $\{u_n\}$ has a subsequence which weakly converges in $H_0^1(\Omega, \mathbb{R})$ to u with $u \geq \underline{u} > 0$. Now, letting $n \to +\infty$, we deduce that u is a positive weak solution of (1). Hereby, the proof is completed. □

5. Conclusions and Outlook

There have been two main objectives in this paper. As explained in introduction part, the first one is from the lack of existence of multiple solutions by using the bifurcation theory. Second one is concerning multiplicity of more than two solutions. We presented new original theorems regarding the existence of positive solutions of nonlinear elliptic BVPs (Theorem 10). Extension of obtained results to fractional-stochastic PDEs will be investigated in a future research work.

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Abbreviations

The following abbreviations are used in this manuscript:

BVP	Boundary Value Problems
NEBVP	Nonlinear Elliptic Boundary Value Problems
PDEs	Partial Differential Equations
US	Upper Solution
LS	Lower Solution
SU	Strictly Upper
SL	Strictly Lower
WLS	Weak lower Solution
WUS	Weak upper Solution

References

1. Chipot, M.; Lovat, B. On the asymptotic behaviour of some nonlocal problems. *Positivity* **1999**, *3*, 65–81. [CrossRef]
2. Ako, K. On the Dirichlet problem for quasi-linear elliptic differential equations of second order. *J. Math. Soc. Jpn.* **1961**, *13*, 45–62. [CrossRef]
3. Amann, H.; Laestch, T. On the existence of positive solutions of nonlinear elliptic BVPs. *Indiana Univ. Math. J.* **1971**, *13*, 125–146. [CrossRef]
4. Afrouzi, G.A.; Naghizadeh, Z.; Mahdavi, S. Monotone methods in nonlinear elliptic BVP. *Int. J. Nonlinear Sci.* **2009**, *7*, 283–289.
5. Bouteraa, N.; Benaicha, S. Existence of solutions for third-order three-point BVP. *Mathematica* **2018**, *60*, 12–22.

6. Bouteraa, N.; Benaicha, S. Existence and multiplicity of positive radial solutions to the Dirichlet problem for nonlinear elliptic equations on annular domains. *Stud. Univ. Babes-Bolyai Math.* **2020**, *65*, 109–125. [CrossRef]
7. Bouteraa, N.; Benaicha, S.; Djourdem, H. On the existence and multiplicity of positive solutions for nonlinear elliptic equation on bounded annular domains via fixed point index. *Maltepe J. Math.* **2019**, *I*, 30–47.
8. Cano-Casanova, C. Linear elliptic and parabolic PDEs with nonlinear mixed boundary conditions and spacial heterogeneties. *Electron. J. Differ. Equ.* **2018**, *166*, 1–27.
9. Keller, H.K. Elliptic BVPs suggested by nonlinear diffusion processes. *Arch. Rat. Mech. Anal.* **1969**, *5*, 363–381. [CrossRef]
10. Ma, R.; Chen, R.; Lu, Y. Method of Lower and Upper Solutions for Elliptic Systems with Nonlinear Boundary Condition and Its Applications. *J. Appl. Math.* **2014**, *2014*, 705298. [CrossRef]
11. Nagumo, M. On principally linear elliptic differential equations of second order. *Osaka Math. J.* **1954**, *6*, 207–229.
12. Puel, J.P. Existence comportement a l'infini et stabilite dans certaines problemes quasilineares elliptiques et paraboliques d'ordre 2. *Ann. Scuola Norm. Sup. Pisa* **1977**, *IV 3*, 89–119.
13. Pao, C.V. Asymptotic behavior and nonexistence of global Solutions for a class of nonlinear BVPs of Parabolic Type. *J. Math. Anal. Appl.* **1978**, *65*, 616–637. [CrossRef]
14. Pao, C.V.; Ruan, W.H. Positive solutions of quasilinear parabolic systems with Dirichlet boundary condition. *J. Math. Anal. Appl.* **2007**, *333*, 472–499. [CrossRef]
15. Sattinger, D.H. Monotone methods in nonlinear elliptic and parabolic BVPs. *Ind. J. Math.* **1972**, *211*, 979–1000. [CrossRef]
16. Kropat, E.; Pickl, S.; Rößler, A.; Weber, G.-W. A new algorithm from semi-infinite optimization for a problem of time-minimal control. *J. Comput. Technol.* **2000**, *5*, 67–81.
17. Yilmaz, F.; Öz, H.; Weber, G.-W. Simulation of Stochastic Optimal Control Problems with Symplectic Partitioned Runge-Kutta Scheme. *Dyn. Contin. Discret. Impuls. Syst. Ser. B* **2015**, *22*, 425–440.
18. Öz, H.; Yilmaz, F.; Weber, G.-W. A discrete optimality system for an optimal harvesting problem. *Comput. Manag. Sci.* **2017**, *14*, 519–533.
19. Öz Bakan, H.; Yilmaz, F.; Weber, G.-W. Minimal Truncation Error Constants for Symplectic Partitioned Runge-Kutta Method for Stochastic Optimal Control Problems. *J. Comput. Appl. Math. (JCAM)* **2018**, *331*, 196–207. [CrossRef]
20. Yilmaz, F.; Öz, H.; Weber, G.-W. Calculus and Digitalization in Finance: Change of Time Method and Stochastic Taylor Expansion with Computation of Expectation. In *Modeling, Optimization, Dynamics and Bioeconomy I*; Springer: Berlin, Germany, 2014; Volume 73, pp. 739–753, Chapter 40.
21. Kim, M.-K. Optimal Control and Operation Strategy for Wind Turbines Contributing to Grid Primary Frequency Regulation. *Appl. Sci.* **2017**, *7*, 927. [CrossRef]
22. Yang, D.; Zhao, K.; Tian, H.; Liu, Y. Decision Optimization for Power Grid Operating Conditions with High- and Low-Voltage Parallel Loops. *Appl. Sci.* **2017**, *7*, 487. [CrossRef]
23. Sánchez, A.J.F. Existence and Multiplicity of Solutions for Elliptic Problems with Critical Growth in the Gradient. 2019; pp. 1–167. Available online: https://tel.archives-ouvert.fr/tel-02299049 (accessed on 7 July 2020).
24. Ovono, A.A.; Rougirel, A. Elliptic equations with diffusion parameterized by the range of nonlocal interactions. *Ann. Mat. Pura Appl.* **2010**, *1*, 163–183. [CrossRef]
25. Alves, C.O.; Covei, D.P. Existence of solution for a class of nonlocal elliptic problem via sub-supersolution method. *Nonlinear Anal. Real Word Appl.* **2015**, *23*, 1–8. [CrossRef]
26. Diaz, J.I.; Goomez-Castro, D. An application of shape differentiation to the effectiveness of steady state reaction-diffusion problem arising in chemical engineering. *Electron. J. Differ. Equ.* **2015**, *2*, 31–45.
27. Yan, B.; Ren, Q. Existence, uniqueness and multiplicity of positive solutions for some nonlocal singular elliptic problems. *Electron. J. Differ. Equ.* **2017**, *138*, 1–21.
28. Yan, B.; O'Regan, D.; Agarwal, R.P. The existence of positive solutions for Kirchhoff-type problems via the sub-supersolution method. *An. St. Univ. Ovidius Constanta* **2018**, *26*, 5–41. [CrossRef]
29. Troianiello, G.M. *Elliptic Differential Equations and Obstacle Problems*; Plenum: New York, NY, USA, 1987.
30. Collatz, L. *Functional Analysis and Numerical Mathematics*; Academic Press Inc.: New York, NY, USA; London, UK, 1966.

31. Amann, H. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **1976**, *18*, 620–709. [CrossRef]
32. Alves, C.O.; Correa, F.G.S.A. On existence of solutions for a class of problem involving a nonlinear operator. *Comm. Appl. Nonlinear Anal.* **2001**, *8*, 43–56.
33. Azouz, N.; Bensedik, A. Existence result for an elliptic equation of Kirchhoff type with changing sign data. *Funkcial. Ekvac.* **2012**, *55*, 55–66. [CrossRef]