# BRIDGING THE GAP BETWEEN
# VARIATIONAL HOMOGENIZATION RESULTS AND
# TWO-SCALE ASYMPTOTIC AVERAGING TECHNIQUES
# ON PERIODIC NETWORK STRUCTURES

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Abstract. In modern material sciences and multi-scale physics homogenization approaches provide a global characterization of physical systems that depend on the topology of the underlying microgeometry. Purely formal approaches such as averaging techniques can be applied for an identification of the averaged system. For models in variational form, two-scale convergence for network functions can be used to derive the homogenized model. The sequence of solutions of the variational microcsopic models and the corresponding sequence of tangential gradients converge toward limit functions that are characterized by the solution of the variational macroscopic model. Here, a further extension of this result is proved. The variational macroscopic model can be equivalently represented by a homogenized model on the superior domain and a certain number of reference cell problems. In this way, the results obtained by averaging strategies are supported by notions of convergence for network functions on varying domains.

1. **Introduction.** *Micro-architectured devices* and *networked materials* play an important role in modern *material sciences* and *multi-scale physics*. In many applications, the physical process on the extremely large network is governed by the effects of the inherent periodic microstructure. In other words, the microgeometry significantly affects the overall properties of the physical device under consideration. Global characteristics like the elasticity or stability of a material are controlled by the network's topology. Typically, the microstructure takes the shape of a grid or

a periodic curvilinear network that is formed from a cellular structure. A precise mathematical modelling of such systems has to take both the microscale and the macroscale into account. Approaches from *homogenization theory* can be applied to reveal the hidden influence of the *microscale* (or *local scale*) on the properties of the *macroscale* (or *global scale*). As result, so-called *homogenized models* are derived that describe the behaviour of the system or device on the global scale.

Our studies on homogenization principles for differential equations on large periodic networks are motivated by our research in the field of groundwater contamination and groundmotion prediction [26]. Here, large capillary networks in the soil are considered in order to identify the distribution of certain (toxic) substances. A numerical solution of these very large systems of network differential equations cannot be obtained in a reasonable time. For this reason, we applied homogenization theory in order to receive an approximate solution on the scale of the physical region. Various theoretical aspects and applications of homogenization theory have been discussed in the literature (for further reading we refer for example to [3, 7, 8, 12, 13, 27, 29]). Nevertheless, only a relatively small number of publications has set a particular focus on *network structures*. Most publications in this field are concerned with *thin domains*, *fattened graphs* or *reticulated structures* [7].

### Homogenization problems on periodic networks

Only a few authors addressed homogenization problems on one-dimensional periodic manifolds so far. For example, in [30] a model for a class of two-dimensional *resistive networks* has been discussed by Vogelius. The viscoelasticity and elastic response of *rubbery polymers* on a micro-mechanically based network model is presented in [11, 24, 25]. Here, approaches from homogenization theory are applied in order to derive a characterization of the micro-to-macro transition and the global response of the *polymer network*. *Microelectromechanical systems* composed of *recurrent cellular electronic circuits* have been presented by Lenczner et al. in [22, 23].

### Challenging topologies

Homogenization problems on periodic graphs are very challenging both from the theoretical and the numerical perspective. On the theoretical side, the following difficulties can arise:

1. *notions of convergence* typically used in homogenization theory are not directly applicable to microscopic models on networks

2. *extension operators* that extend functions from the network to the superior domain (i.e., the global scale) are not known

3. *periodic networks* are represented by singularly perturbed one-dimensional manifolds.

In addition, the length of each edge tends to zero during the homogenization process. For this reason, the question about the limit behavior of the *tangential derivatives* along the vanishing edges arises.

Beside these analytical questions further problems have to be addressed on the numerical side:

1. a large number of edges and a huge number of vertices at the network's junctions have to be considered in the numerical discretization scheme

2. the periodic microstructure results in *highly oscillating coefficients*

3. *transmission conditions* have to be imposed at the inner nodes, that often destroy the (symmetric) structure of the system's matrix of numerical schemes

4. *boundary layers* occur at the end of *each* edge in case of transport-dominated models [9].

In fact, a numerical solution cannot be obtained in a reasonable time for the *microscopic model* on the $\varepsilon$-periodic networks at the scale of real applications (see [10, 28] for further numerical aspects).

## Homogenization on networks

Homogenization theory can be applied to microscopic models on periodic networks. Until now, only a few authors have studied homogenization problems on singularly perturbed one-dimensional periodic manifolds. Mazja et al. presented an *averaging technique* for *self-adjoint* second-order operators on a periodic graph. In [14, 16, 17], this technique has been further extended to *non self-adjoint operators* and *diffusion-advection-reaction models*. In particular, the case of a vanishing diffusion part and *transport-dominated systems* is discussed.

Lenczner et al. introduced a notion of two-scale convergence for network functions and applied it to homogenization problems in the fields of *mechatronics* and *electrical networks* [20]. This type of two-scale convergence will also be used in this paper. In this study, we set a particular focus on two different approaches for the homogenization of network differential equations that have been discussed in the literature.

The first approach is based on a *two-scale averaging technique*. Here, the microscopic model is given by a system of second order differential equations on the branches of the network. These equations describe a *singularly perturbed diffusion-advection-reaction process*. The solution of the microscopic model is represented by an *asymptotic expansion*. For a vanishing length of periodicity $\varepsilon > 0$ this *averaging strategy* finally leads to the *homogenized model* on the superior domain $\Omega$. The *homogenized coefficients* provide a characterization of the effective behavior of the system on a global scale. In particular, the homogenized coefficients directly depend on the network's topology. *Error results* compare the solution of the microscopic model and its first order approximation guarantee the quality of the approximation. The derivation of the averaged model in case of singularly perturbed diffusion-advection-reaction processes has been discussed in [16]. A numerical example on the averaging of such problems for large three-dimensional curvilinear networks with a periodic microstructure is presented in [18]. In addition, aspects of *topology optimization* of the underlying periodic microstructure are investigate in [19].

Nevertheless, the averaging strategy is a pure *formal derivation* of a homogenized model, because no particular *notion of convergence* has been applied. In this sense, the averaging strategy can be considered as a way to provide a picture of the homogenized model. However, an appropriate homogenization technique should address the convergence of the sequence of solutions of the microscopic model as well as the sequence of the corresponding tangential gradients. In particular, this technique should also clarify how the corresponding limits are related to the homogenized model. Such an approach has been presented in [18] for *microscopic models in variational form*. Here, a *notion of convergence for network functions* introduced

by Lenczner et al. has been applied (see [20]). This notion of convergence is based on a so-called *two-scale transform*. Basically, network functions on the varying $\varepsilon$-periodic domains $\mathcal{N}_\varepsilon^\Omega$ are transformed into functions on the fixed domain $\Omega \times \mathcal{Y}$, consisting of the superior domain $\Omega$ and the reference graph $\mathcal{Y}$. Then, traditional notions of convergence like weak convergence or strong convergence can be applied. In [18], the two-scale limit of the sequence of solutions of the variational microscopic models as well as the two-scale limit of the sequence of the corresponding tangential gradients have been identified. Both limits can be represented by the solution of the *homogenized macroscopic model in variational form*. We note that these results have been further extended to the corresponding optimal control problems periodic networks. These studies are based on the concepts of $\Gamma$-convergence and $S$-homogenization [15].

### Scope of the paper

In this paper, we further extend our approach on the homogenization of microscopic network models in variational form and derive a new representation of the homogenized model. As a model example, we consider *diffusion-advection-reation systems* on varying networks. In [18], variational microscopic models have been introduced and the corresponding homogenized variational model has been derived by a two-scale analysis. The homogenized variational model is defined on the domain $\Omega \times \mathcal{Y}$, i.e., on the superior domain $\Omega$ and the reference graph $\mathcal{Y}$. Averaging strategies and the method of two-scale asymptotic expansions lead to another type of homogenized model in terms of a second order differential equation on the superior domain $\Omega$. Here, the influence of the mircogeometry is completely included in the homogenized coefficients.

Since the method of two-scale asymptotic expansions is a purely formal approach, we aim to support this result by a mathematical rigoros proof based on the notion of two-scale convergence for network functions on varying domains. We identify a *homogenized model* on the superior domain $\Omega$ that is given by a second order differential equation. Indeed, we can show that the homogenized model in variational form derived in [18] is equivalent to a particular *homogenized model* on the global domain $\Omega$. Here, a *scale-separating ansatz* regarding the solution of the homogenized model and its gradient is applied and the corresponding factors for the microgeometry turn out to be the solution of additional cell problems on the reference graph $\mathcal{Y}$.

### Outline

The paper is organized as follows: In *Section 2*, some basic facts about periodic networks are introduced. *Section 3* discusses some function spaces on scalable networks with a periodic microgeometry. Then, in *Section 4*, the two-scale transform and the two-scale convergence for network functions are introduced. Two different representations of the microscopic model on periodic networks are presented in *Section 5*. The first one, is based on a system of differential equations of second order on the branches of the periodic network. In [16, 17], an averaging technique based on two-scale asymptotic expansions has been applied to this microscopic model for the deriviation of the macroscopic model in terms of a second order partial differential equation on the full domain $\Omega$. The second microscopic model is given in variational form. A two-scale convergence for network functions has been used in [18] in order to derive a macroscopic variational model. Then, in *Section 6*, the homogenization result is presented and the corresponding macroscopic variational model is derived.

In *Section 7* we derive a new representation of the homogenized model. We show that the macroscopic variational model can be represented in an equivalent way by a combination of the so-called *homogenized model* - a partial differential equation on $\Omega$ - and $d+1$ *reference cell problems* on the reference graph. We conclude with a discussion of potential developments and future works.
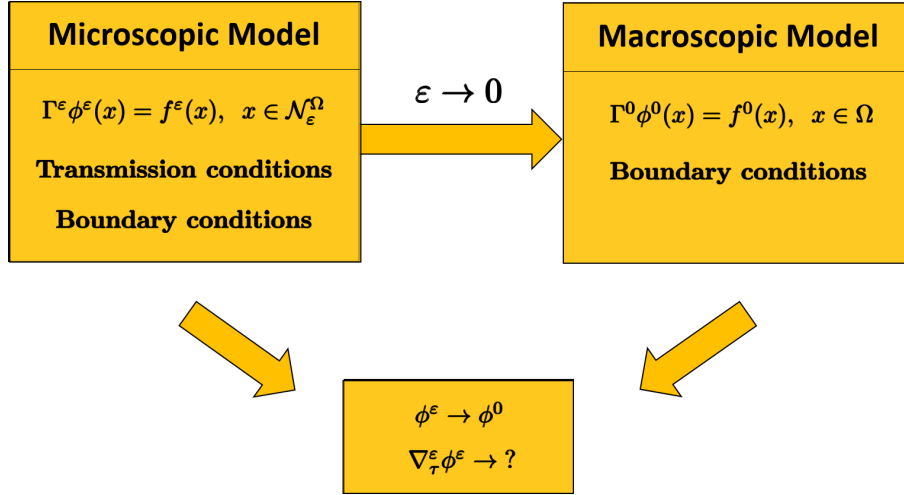


FIGURE 1. *The homogenization process*: The sequence of solutions of the microscopic model as well as the corresponding sequence of tangential gradients are weakly two-scale convergent. The limits of these sequences can be represented by the solution of the homogenized model.

2. **Periodic Networks.** The physical system under consideration is defined on the edges of a very large periodic network in $\mathbb{R}^d$, where $d \in \mathbb{N}$. For this reason, we introduce the infinite and $\varepsilon$-periodic network $\mathcal{N}_\varepsilon$ in $\mathbb{R}^d$, where $\varepsilon > 0$ denotes the *length of periodicity*. The $\varepsilon$-periodic network $\mathcal{N}_\varepsilon$ can be identified with a *singularly perturbed one-dimensional manifold* that represents the *branches* of the graph. The connections of two adjacent branches of $\mathcal{N}_\varepsilon$ (i.e., the *nodes*) are the *singular perturbations* of the corresponding one-dimensional manifold.

**The restricted network $\mathcal{N}_\varepsilon^\Omega$**

In our study, we assume that the physical device is contained in a *polyhedral domain* $\Omega \subset \mathbb{R}^d$, where $\partial\Omega$ is the outer boundary of the system. The microscopic model is defined on the $\varepsilon$-periodic network $\mathcal{N}_\varepsilon^\Omega$, where $\varepsilon > 0$. Here, $\mathcal{N}_\varepsilon^\Omega$ is the restriction of the infinite $\varepsilon$-periodic network $\mathcal{N}_\varepsilon$ to the domain $\Omega$, i.e., $\mathcal{N}_\varepsilon^\Omega := \mathcal{N}_\varepsilon \cap \Omega$. The superior domain $\Omega$ and the $\varepsilon$-network $\mathcal{N}_\varepsilon^\Omega$ on the microscopic level fulfill the following assumption.

**Assumption 2.1 (Length scales).**

**(P)** *The length of periodicity $\varepsilon > 0$ of the network $\mathcal{N}_\varepsilon^\Omega$ is considered as "very small" when compared to the diameter of the polyhedral domain $\Omega \subset \mathbb{R}^d$ (i.e., $\operatorname{diam}(\Omega) \gg \varepsilon > 0$).*

**Nodes and branches**

With $\mathcal{V}_\varepsilon$ and $\mathcal{V}_\varepsilon^\Omega$ we denote the set of nodes of the networks $\mathcal{N}_\varepsilon$ and $\mathcal{N}_\varepsilon^\Omega$, respectively. The set of nodes of the restricted network $\mathcal{N}_\varepsilon^\Omega$ consists of the *boundary nodes* (or *outer nodes*) $\partial^B(\mathcal{N}_\varepsilon^\Omega) := \mathcal{N}_\varepsilon \cap \partial\Omega$ and the *ramification nodes* (or *inner nodes*) $\partial^R(\mathcal{N}_\varepsilon^\Omega) := \mathcal{V}_\varepsilon \cap \Omega$. That means, $\mathcal{V}_\varepsilon^\Omega = \partial^R(\mathcal{N}_\varepsilon^\Omega) \,\dot\cup\, \partial^B(\mathcal{N}_\varepsilon^\Omega)$. The *index set of edges* $\mathcal{J}_\varepsilon^\Omega$ of the restricted network $\mathcal{N}_\varepsilon^\Omega$ is given by $\mathcal{J}_\varepsilon^\Omega$. For each index $j \in \mathcal{J}_\varepsilon^\Omega$ the branch $B_j^\varepsilon \subset \mathbb{R}^d$ is parameterized in terms of its arc length $\mathcal{L}_j^\varepsilon$ with respect to the interval $\mathcal{I}_j^\varepsilon := (0, \mathcal{L}_j^\varepsilon)$. With $\mathcal{E}^-(B_j^\varepsilon)$ and $\mathcal{E}^+(B_j^\varepsilon)$ we denote the tips of the directed branch $B_j^\varepsilon$. We further assume that no node of $\mathcal{N}_\varepsilon$ is located on the boundary of $\Omega$.

**Assumption 2.2 (Nodes of $\mathbf{\mathcal{N}_\varepsilon}$).**

**(B)** *The set of nodes of the infinite $\varepsilon$-periodic network $\mathcal{N}_\varepsilon$ does not intersect with the boundary of $\Omega$, i.e., $\mathcal{V}_\varepsilon \cap \partial\Omega = \emptyset$.*

**Unit cell and reference graph**

With $\square = [0,1)^d$ we denote the *unit cell* (or *model cell*). It contains the *reference graph* $\mathcal{Y} := \mathcal{N}_1 \cap \overline{\square}$. The networks $\mathcal{N}_\varepsilon$ and $\mathcal{N}_\varepsilon^\Omega$ are composed of recurrent elements that are obtained from the model cell and the reference graph by copying and scaling with factor $\varepsilon$. The index set of nodes of the reference graph $\mathcal{Y}$ is given by $\mathcal{V}(\mathcal{Y})$. It consists of the *boundary nodes* (or *outer nodes*) $\partial^B(\mathcal{Y}) := \mathcal{N}_1 \cap \partial\square$ and *ramification nodes* (or *inner nodes*) $\partial^R(\mathcal{Y}) := \mathcal{V}_1 \cap \operatorname{int}\square$ (i.e., $\mathcal{V}(\mathcal{Y}) = \partial^R(\mathcal{Y}) \,\dot\cup\, \partial^B(\mathcal{Y})$). The *index set of edges* of the reference graph is denoted $\mathcal{J}^\mathcal{Y}$. Each branch $j \in \mathcal{J}^\mathcal{Y}$ is parameterized in terms of its arc length $\mathcal{L}_j^\mathcal{Y}$ with respect to the interval $\mathcal{I}_j^\mathcal{Y} := (0, \mathcal{L}_j^\mathcal{Y})$.

The following assumption reflects the geometry of the reference graph.

**Assumption 2.3 (Geometry).**

**(G1)** *Each branch of the reference graph $\mathcal{Y}$ crosses the unit cell $\square$ completely and connects two opposite boundary nodes $\tau^-, \tau^+ \in \partial^B(\mathcal{Y})$.*

**(G2)** *A (directed) branch can intersect the boundary of the unit cell $\square$ only with its nodes.*

3. **Function Spaces on Periodic Networks.** We provide some function spaces on periodic networks in the $d$-dimensional space (typically, $d \in \{1, 2, 3\}$) and on the set product $\Omega \times \mathcal{Y}$ of the superior domain and the reference graph that represents the microscopic level. In particular, we will refer to the *tangential derivative* $\nabla_\tau^\varepsilon u^\varepsilon$ of a function $u^\varepsilon$ on the network $\mathcal{N}_\varepsilon^\Omega$ and the *tangential derivative* $\nabla_\tau^\mathcal{Y} u$ of a function $u$ on the reference graph $\mathcal{Y}$.

**Function spaces on the network $\mathbf{\mathcal{N}_\varepsilon^\Omega}$**

Firstly, we define some function spaces on the restricted network $\mathcal{N}_\varepsilon^\Omega$.

**Definition 3.1.** Let $1 \le p < \infty$. On the $\varepsilon$-periodic network $\mathcal{N}_\varepsilon^\Omega$ we introduce the function space $L^p\big(\mathcal{N}_\varepsilon^\Omega\big)$ that is equipped with the norm

$$\|\phi^\varepsilon\|_{L^p(\mathcal{N}_\varepsilon^\Omega)}^p := \int_{\mathcal{N}_\varepsilon^\Omega} |\phi^\varepsilon(x)|^p \, dx.$$

The usual $L^2$-inner product for network functions on $\mathcal{N}_\varepsilon^\Omega$ is introduced in the following definition.

**Definition 3.2.** Let $u, v \in L^2(\mathcal{N}_\varepsilon^\Omega)$. The $L^2$-inner product is denoted by

$$\langle u, v \rangle_{\mathcal{N}_\varepsilon^\Omega} := \int_{\mathcal{N}_\varepsilon^\Omega} u(x) \cdot v(x) \, dx.$$

In addition, some Sobolev-spaces on the restricted network $\mathcal{N}_\varepsilon^\Omega$ are required.

**Definition 3.3.** On the $\varepsilon$-periodic network $\mathcal{N}_\varepsilon^\Omega$ we introduce the Hilbert spaces

$$\widetilde{\mathcal{H}^1}\big(\mathcal{N}_\varepsilon^\Omega\big) := H^1\big(\mathcal{N}_\varepsilon^\Omega\big)$$

and

$$\mathcal{H}^1\big(\mathcal{N}_\varepsilon^\Omega\big) := \left\{ \phi^\varepsilon \in \widetilde{\mathcal{H}^1}\big(\mathcal{N}_\varepsilon^\Omega\big) \,\middle|\, \phi^\varepsilon \text{ is continuous at } x \in \mathcal{V}_\varepsilon^\Omega \right\},$$

where the network functions are continuous at the ramification nodes. If a network function has to satisfy *Dirichlet boundary conditions* we refer to the following Hilbert space:

$$\mathcal{H}_0^1\big(\mathcal{N}_\varepsilon^\Omega\big) := \left\{ \phi^\varepsilon \in \mathcal{H}^1\big(\mathcal{N}_\varepsilon^\Omega\big) \,\middle|\, \phi^\varepsilon(x) = 0 \text{ for each } x \in \partial^B(\mathcal{N}_\varepsilon^\Omega) \right\}.$$

These function spaces can be equipped with the equivalent norms

$$\big\|\phi^\varepsilon\big\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)} := \left\{ \int_{\mathcal{N}_\varepsilon^\Omega} \big[\nabla_\tau^\varepsilon \phi^\varepsilon(x)\big]^2 + [\phi^\varepsilon(x)]^2 \, dx \right\}^{\frac{1}{2}},$$

$$\big\|\phi^\varepsilon\big\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} := \left\{ \int_{\mathcal{N}_\varepsilon^\Omega} \big[\nabla_\tau^\varepsilon \phi^\varepsilon(x)\big]^2 \, dx \right\}^{\frac{1}{2}}.$$

Later on we will use the following normed space.

**Definition 3.4.** The solution of the variational microscopic model with respect to Dirichlet boundary conditions will be an element of the function space

$$\mathcal{H}_{0,*}^1 := \big(\mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega), \|\cdot\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)}\big).$$

**Remark 1.** In some situations we will also need a scaled norm on $\mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)$:

$$\big\|u^\varepsilon\big\|_{\mathcal{H}_{\varepsilon,*}^1(\mathcal{N}_\varepsilon^\Omega)} := \varepsilon^{\frac{d-1}{2}} \cdot \big\|u^\varepsilon\big\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)}.$$

**Periodic functions**

The periodic structure of the network and the small length of periodicity lead to highly oscillating coefficients. Because of the scalability of the networks, they are defined on the network $\mathcal{N}_1$.

**Definition 3.5.** On the infinite network $\mathcal{N}_1$ we define the spaces of periodic functions

$$L_{per}^2 := \{f \in L^2(\mathcal{N}_1) \mid f \text{ is } \mathcal{Y}\text{-periodic}\},$$
$$\mathcal{H}_{per}^1 := \{f \in \mathcal{H}^1(\mathcal{N}_1) \mid f \text{ is } \mathcal{Y}\text{-periodic}\}.$$

The $\varepsilon$-periodic $\mathcal{N}_\varepsilon$ is obtained by scaling from $\mathcal{N}_1$. For this reason, the corresponding coefficients on the edges of $\mathcal{N}_\varepsilon$ are given by $u^\varepsilon(x) := u(\varepsilon^{-1}x)$ for $x \in \mathcal{N}_\varepsilon$.

### Functions on $\Omega \times \mathcal{Y}$

The two-scale transform applied in this paper connects functions on the scalable network $\mathcal{N}_\varepsilon^\Omega$ with functions on the fixed domain $\Omega \times \mathcal{Y}$.

**Definition 3.6.** Let $1 \le p < \infty$. The function space $L^p(\Omega \times \mathcal{Y})$ is equipped with the norm

$$\|\phi\|_{L^p(\Omega \times \mathcal{Y})}^p := \int\limits_\Omega \int\limits_\mathcal{Y} |\phi(z,y)|^p \, dy \, dz.$$

In particular, the following function spaces are required for the notion of the *two-scale transform*.

**Definition 3.7.** Let $\Omega \subset \mathbb{R}^d$ be the superior domain and $\mathcal{Y}$ denotes the reference graph. We set

$$\mathcal{H}_{0,\tau^\mathcal{Y}}^1(\Omega, \mathcal{Y}) := \{\phi \in L^2(\Omega \times \mathcal{Y}) \,|\, \nabla_z \phi(z,y) * \tau^\mathcal{Y}(y) \in L^2(\Omega) \text{ for each } y \in \mathcal{Y},$$
$$\nabla_\tau^\mathcal{Y} \phi(z,y) = 0 \text{ for each } (z,y) \in \Omega \times \mathcal{Y}\}$$

and

$$\mathcal{H}_{0,\tau^\mathcal{Y}}^2(\Omega, \mathcal{Y}) := \{\phi \in L^2(\Omega \times \mathcal{Y}) \,|\, \nabla_z \phi(z,y) * \tau^\mathcal{Y}(y) \in H^1(\Omega) \text{ for each } y \in \mathcal{Y},$$
$$\nabla_\tau^\mathcal{Y} \phi(z,y) = 0 \text{ for each } (z,y) \in \Omega \times \mathcal{Y}\}.$$

Here, $*$ denotes the dot product in $\mathbb{R}^n$.

4. **Two-Scale Transform and Two-Scale Convergence.** In this section, we briefly review the notions of *two-scale transform* and *two-scale convergence* for network functions introduced by Lenczner et al. [20]. Functions defined on varying networks are transformed into functions on fixed domains. Then, traditional notions of convergence such as *weak convergence* and *strong convergence* can be applied.

4.1. **Feasible networks.** The definition of the two-scale transform is based on disjoint coverings by cells of the domain $\Omega$ and the networks $\mathcal{N}_\varepsilon$ and $\mathcal{N}_\varepsilon^\Omega$.

### Covering of $\mathbb{R}^d$ with $\varepsilon$-cells

We define $\square_\varepsilon := [0, \varepsilon)^d$ for $\varepsilon > 0$ and $c_i^\varepsilon := \varepsilon i \in \mathbb{R}^d$ for the *multi-index $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$*. The $\boldsymbol{\varepsilon}$-**cell in** $\mathbb{R}^{\boldsymbol{d}}$ **with corner point** $\boldsymbol{c_i^\varepsilon}$ is given by $\mathcal{C}_i^\varepsilon := \varepsilon(i + \square) = c_i^\varepsilon + \square_\varepsilon$. The union of these cells provides a disjoint covering of $\mathbb{R}^d$.

### Covering of $\mathcal{N}_\varepsilon$

For $\varepsilon > 0$ we set $\mathcal{Y}_\varepsilon := \varepsilon \mathcal{Y}$ and $\mathcal{Y}_i^\varepsilon := \varepsilon(i + \mathcal{Y}) = c_i^\varepsilon + \mathcal{Y}_\varepsilon$ is the part of the network $\mathcal{N}_\varepsilon$ that is contained in the $\varepsilon$-cell $\mathcal{C}_i^\varepsilon$. The network $\mathcal{N}_\varepsilon$ can be represented as a disjoint union of all these sets

### Covering of $\Omega$ with $\varepsilon$-cells

For a covering of $\Omega$ we used the recurrent segments of the network $\mathcal{N}_\varepsilon^\Omega$ in the domain $\Omega$. For each length of periodicity $\varepsilon > 0$, the set of multi-indices $\mathcal{I}_\varepsilon^\Omega := \{i \in \mathbb{R}^d \,|\, \mathcal{C}_i^\varepsilon \cap \Omega \ne \emptyset\}$ is called the *index set of all $\varepsilon$-cells in $\Omega$*. The $\boldsymbol{\varepsilon}$-**cell in** $\boldsymbol{\Omega}$ is

given by $\mathcal{C}_i^{\Omega,\varepsilon} := \mathcal{C}_i^{\varepsilon} \cap \Omega$ for each $i \in \mathcal{I}_{\varepsilon}^{\Omega}$. The set of **feasible lengths of periodicity** is introduced by

$$E := \left\{ \varepsilon \in (0,1] \,\middle|\, \Omega = \dot{\bigcup_{i \in \mathcal{I}_{\varepsilon}^{\Omega}}} \mathcal{C}_i^{\Omega,\varepsilon} \right\}.$$

The domain $\Omega$ is covered by pairwise disjoint $\varepsilon$-cells for each *feasible length of periodicity* $\varepsilon \in E$.

## Covering of $\mathcal{N}_{\varepsilon}^{\Omega}$

For each $\varepsilon \in E$ the set $\mathcal{Y}_i^{\Omega,\varepsilon} := \mathcal{Y}_i^{\varepsilon} \cap \Omega$ is defined for each $i \in \mathcal{I}_{\varepsilon}^{\Omega}$. Then, we obtain the covering

$$\mathcal{N}_{\varepsilon}^{\Omega} = \bigcup_{i \in \mathcal{I}_{\varepsilon}^{\Omega}} \mathcal{Y}_i^{\Omega,\varepsilon}. \tag{1}$$

**4.2. The two-scale transform.** In [18], the homogenization of the microscopic model in variational form is based on the notions of weak two-scale convergence and strong two-scale convergence for network functions introduced by Lenczner et al. [20]. The definition of this type of convergence depends on the *inverse two-scale transform* that transforms a function from the fixed domain $\Omega \times \mathcal{Y}$ to the varying network $\mathcal{N}_{\varepsilon}^{\Omega}$. This *surjective* mapping is applied to assign a *two-scale transform* $\widehat{u^{\varepsilon}} \in L^2(\Omega \times \mathcal{Y})$ to each function $u^{\varepsilon} \in L^2(\mathcal{N}_{\varepsilon}^{\Omega})$.

### Two scale transform and inverse two-scale transform

Let $(z,y) \in \Omega \times \mathcal{Y}$ and consider the feasible length of periodicity $\varepsilon \in E$. Because of Equation 1, the domain $\Omega$ is covered by pairwise disjoint $\varepsilon$-cells. That means, for each $z \in \Omega$ there *exists a unique* cell index $i \in \mathcal{I}_{\varepsilon}^{\Omega}$ with $z \in \mathcal{C}_i^{\Omega,\varepsilon}$. With $x(z,y) := \varepsilon(i+y) = c_i^{\varepsilon} + \varepsilon y \in \mathcal{Y}_i^{\varepsilon}$ we select a point on $\mathcal{N}_{\varepsilon}^{\Omega}$.

Assume that $\widehat{y} \in \mathcal{Y}$ is fixed. Then, the function $x(z,\widehat{y})$ takes the same value for all $z \in \mathcal{C}_i^{\Omega,\varepsilon}$. In other words, $x(z,\widehat{y})$ depends only on the $\varepsilon$-cell $\mathcal{C}_i^{\Omega,\varepsilon}$ with $z \in \mathcal{C}_i^{\Omega,\varepsilon}$, but not on the position of $z$ within the $\varepsilon$-cell $\mathcal{C}_i^{\Omega,\varepsilon}$. That means,

$$z_1, z_2 \in \mathcal{C}_i^{\Omega,\varepsilon} \text{ for an } i \in \mathcal{I}_{\varepsilon}^{\Omega}, \, \widehat{y} \in \mathcal{Y} \quad \Rightarrow \quad x(z_1,\widehat{y}) = x(z_2,\widehat{y}). \tag{2}$$

Instead of $z \in \mathcal{C}_i^{\Omega,\varepsilon}$, each other element of the $\varepsilon$-cell $\mathcal{C}_i^{\Omega,\varepsilon}$ can be used as an argument of $x(\cdot,\widehat{y})$ and the value of $x$ does not change.

For example, we can choose the corner point $c_i^{\varepsilon} = \varepsilon i$ of the $\varepsilon$-cell $\mathcal{C}_i^{\Omega,\varepsilon}$. Since $x(c_i^{\varepsilon},y) = x(z,y)$ for all $z \in \mathcal{C}_i^{\Omega,\varepsilon}$) and each $y \in \mathcal{Y}$, we can consider this point as a representative of the $\varepsilon$-cell $\mathcal{C}_i^{\Omega,\varepsilon}$. In addition,

$$\overline{\mathcal{Y}_i^{\Omega,\varepsilon}} = \left\{ x(c_i^{\varepsilon},y) \,\middle|\, y \in \mathcal{Y} \right\} = \left\{ x(z,y) \,\middle|\, z \in \mathcal{C}_i^{\Omega,\varepsilon}, \, y \in \mathcal{Y} \right\}$$

for each $i \in \mathcal{I}_{\varepsilon}^{\Omega}$ and because of Equation (1) we get

$$\mathcal{N}_{\varepsilon}^{\Omega} = \bigcup_{i \in \mathcal{I}_{\varepsilon}^{\Omega}} \left\{ x(c_i^{\varepsilon},y) \,\middle|\, y \in \mathcal{Y} \right\} = \bigcup_{i \in \mathcal{I}_{\varepsilon}^{\Omega}} \left\{ x(z,y) \,\middle|\, z \in \mathcal{C}_i^{\Omega,\varepsilon}, \, y \in \mathcal{Y} \right\}.$$

The function $x : \Omega \times \mathcal{Y} \to \mathcal{N}_{\varepsilon}^{\Omega}$ is *surjective*, but *not injective*. For each $s \in \{1,\ldots,d\}$ we introduce the vector

$$\mathbf{1}_s := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^d.$$
$$\uparrow$$
$$\text{position } s$$

For each $i, j \in \mathcal{I}_\varepsilon^\Omega$ with $\mathcal{C}_j^{\Omega,\varepsilon} = \mathcal{C}_i^{\Omega,\varepsilon} + \mathbf{1}_s$ and $v^-, v^+ \in \partial \mathcal{V}_\mathcal{Y}$ with $v^+ = v^- + \mathbf{1}_s$ we get $x(c_i^\varepsilon, v^+) = x(c_j^\varepsilon, v^-)$. That means, the function $x(\cdot, \cdot)$ is not injective (see Figure 2).
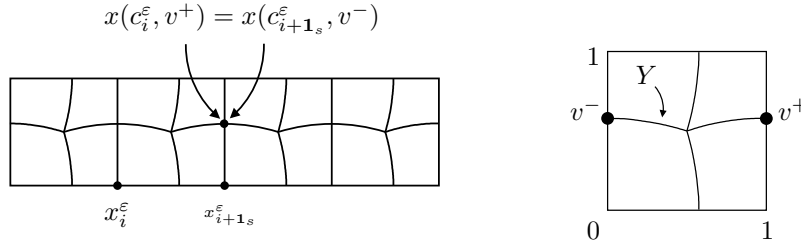


FIGURE 2. *Two-scale transform*: The function $x : \Omega \times \mathcal{Y} \to \mathcal{N}_\varepsilon^\Omega$ is surjective, but not injective.

With the results obtained so far, we can define the *inverse two-scale transform*. The set $\mathcal{C}_i^{\Omega,\varepsilon} \times \{y\}$ is assigned to each point $x \in \mathcal{N}_\varepsilon^\Omega$, where $x(c_i^\varepsilon, y) = x(z, y)$ for all $z \in \mathcal{C}_i^{\Omega,\varepsilon}$. In this way, we can define the **two-scale transform** $\mathcal{T}_{TS}$ on $\mathcal{N}_\varepsilon^\Omega$ with $\mathcal{T}_{TS}(x) := \mathcal{C}_i^{\Omega,\varepsilon} \times \{y\}$. In addition, each element $(z, y) \in \mathcal{C}_i^{\Omega,\varepsilon} \times \{y\}$ leads to the point $x \in \overline{\mathcal{Y}_i^{\Omega,\varepsilon}} \left( \subset \mathcal{N}_\varepsilon^\Omega \right)$.

Now, for each index $i \in \mathcal{I}_\varepsilon^\Omega$ we introduce the function

$$\left( \mathcal{T}_{TS}^i \right)^{-1} : \mathcal{C}_i^{\Omega,\varepsilon} \times \mathcal{Y} \to \overline{\mathcal{Y}_i^\varepsilon}$$
$$(z, y) \mapsto \left( \mathcal{T}_{ZS}^i \right)^{-1}(z, y) := x(z, y) = c_i^\varepsilon + \varepsilon y.$$

By extending this function to $\mathcal{N}_\varepsilon^\Omega$ and $\Omega \times \mathcal{Y}$ we obtain

$$\left( \mathcal{T}_{TS} \right)^{-1} : \Omega \times \mathcal{Y} \to \mathcal{N}_\varepsilon^\Omega$$
$$(z, y) \mapsto \left( \mathcal{T}_{TS} \right)^{-1}(z, y) := \left( \mathcal{T}_{ZS}^i \right)^{-1}(z, y) \qquad \text{for all } z \in \mathcal{C}_i^{\Omega,\varepsilon}.$$

The surjective function $\left( \mathcal{T}_{TS} \right)^{-1}$ is called the **inverse two-scale transform**.

In the next step, we define the *two-scale transform* of a function on a network in the sense of Lenczner et al. [20]. The two-scale transform assigns a measurable function on $\Omega \times \mathcal{Y}$ to a measurable function on $\mathcal{N}_\varepsilon^\Omega$.

**Definition 4.1.** Let $u^\varepsilon \in L^1(\mathcal{N}_\varepsilon^\Omega)$. The function $\widehat{u^\varepsilon} : \Omega \times \mathcal{Y} \to \mathbb{R}$ defined by $\widehat{u^\varepsilon}(z, y) := u^\varepsilon(x)$, where $x = \left( \mathcal{T}_{TS} \right)^{-1}(z, y)$, is called the **two-scale transform** of $u^\varepsilon$.

**Remark 2.** Traditional homogenization approaches for *thin domains* often rely on the application of so-called *extension operators* These operators extend transform a function from the thin domain to the full superior domain while certain criteria on

the boundedness of the function or its gradient are fulfilled. Such extension operators are not known for network functions on singularly perturbed one-dimensional manifolds.

In [2], *Arbogast/Douglas/Hornung* discussed the so-called *dilatation operator* ”∼“ in context with the flow problems through a thin network of channels in the soil of a petroleum reservoir. Here, the dilatation-operator extends a function $u^\varepsilon \in L^2(\Omega_\varepsilon)$ with domain $\Omega_\varepsilon \subset \Omega$ into a function $\widetilde{u^\varepsilon} \in L^2(\Omega \times \mathcal{Y})$. In [20, 23] *Lenzner et al.* made this technique applicable to network functions und introduced the two-scale transform of a function $u^\varepsilon \in L^1(\mathcal{N}_\varepsilon^\Omega)$.
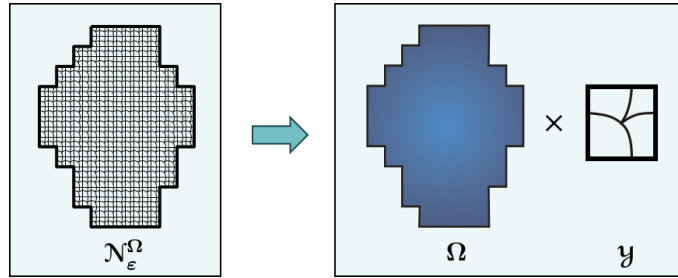


FIGURE 3.  *Two-scale transform*: Mapping from $\mathcal{N}_\varepsilon^\Omega$ to the product $\Omega \times \mathcal{Y}$.

4.3. **The two-scale convergence.** After the preparations of the previous section, the two-scale convergence of a sequence of network functions in the sense of Lenczner et al. can be introduced (see [20, 23]).

**Definition 4.2.** Let $\left\{ u^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega) \right\}_{\varepsilon \in E}$ be a sequence of network functions and let $u_0 \in L^2(\Omega \times \mathcal{Y})$.

**1.** The sequence $\left\{ u^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega) \right\}_{\varepsilon \in E}$ is **weakly two-scale convergent** to $u_0$
 $[u^\varepsilon \xrightarrow{2} u_0]$, if $\left\{ \widehat{u^\varepsilon} \in L^2(\Omega \times \mathcal{Y}) \right\}_{\varepsilon \in E}$ is weakly convergent in $L^2(\Omega \times \mathcal{Y})$ to $u_0$.

**2.** The sequence $\left\{ u^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega) \right\}_{\varepsilon \in E}$ is **strong two-scale convergent** to $u_0$
 $[u^\varepsilon \xrightarrow{2} u_0]$, if $\left\{ \widehat{u^\varepsilon} \in L^2(\Omega \times \mathcal{Y}) \right\}_{\varepsilon \in E}$ strongly converges in $L^2(\Omega \times \mathcal{Y})$ to $u_0$.

The strong two-scale convergence in $L^2(\Omega \times \mathcal{Y})$ implies the weak two-scale convergence.

**Corollary 1.** *If the sequence $\left\{ u^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega) \right\}_{\varepsilon \in E}$ is strongly two-scale convergent to $u_0 \in L^2(\Omega \times \mathcal{Y})$, then it is also weakly two-scale convergent to $u_0$.*

The two-scale convergence of a sequence of network functions and the corresponding sequence of tangential gradients is addressed the following theorem[1].

**Theorem 4.3** (**Two-scale convergence**)**.**
*Let the assumptions $(P)$, $(B)$, $(G1)$, $(G2)$ be fulfilled. Let $\left\{ u^\varepsilon \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega) \right\}_{\varepsilon \in E}$ be sequence of network functions such that the sequence $\left\{ \|u^\varepsilon\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)} \in \mathbb{R} \right\}_{\varepsilon \in E}$ is bounded, then there exists a subsequence $\left\{ u^{\varepsilon_k} \in \mathcal{H}_0^1(\mathcal{N}_{\varepsilon_k}^\Omega) \right\}_{\varepsilon_k \in E_k}$, such that*

---

[1]The theorem on the two-scale convergence for network functions has been stated in [20, 23] in context with electrical networks. We also refer to [14, 18] for the corresponding theorem with respect to the variational problems discussed in this paper.

(i)   $\left\{u^{\varepsilon_k} \in L^2\big(\mathcal{N}_{\varepsilon_k}^{\Omega}\big)\right\}_{\varepsilon_k \in E_k}$ is weakly two-scale convergent to $u_0 \in \mathcal{H}_{0,\tau^{\mathcal{Y}}}^1(\Omega, \mathcal{Y})$.

(ii)   $\left\{\nabla_\tau^{\varepsilon_k} u^{\varepsilon_k} \in L^2\big(\mathcal{N}_{\varepsilon_k}^{\Omega}\big)\right\}_{\varepsilon_k \in E_k}$ is weakly two-scale convergent to

$$\nabla_z u_0 * \tau^{\mathcal{Y}} + \nabla_\tau^{\mathcal{Y}} u_1,$$

where $u_1 \in L^2\big(\Omega; \mathcal{H}_{\mathcal{Y}}^1\big)$.

## 5. Microscopic Models.

In this section, we introduce two representations of the *microscopic model* on the $\varepsilon$-periodic network $\mathcal{N}_\varepsilon^\Omega$. Firstly, we consider a microscopic model where a diffusion-advection-reaction process in terms of a second order differential equation is given on each branch of the network. This model has been discussed in [16, 18] and the corresponding macroscopic model - a second order differential equation on the superior domain $\Omega$ - has been derived. Secondly, the microscopic model can be represented in variational form. In [18], a notion of two-scale convergence for network functions has been applied and a macroscopic model in variation form is derived. In Section 7, we show that a result similar to the averaging approach can be obtained and that there exists a homogenized model on the fixed domain $\Omega \times \mathcal{Y}$.

5.1. **The Microscopic Model as a System of Differential Equations.** In [16], the microscopic model on the on the $\varepsilon$-periodic network $\mathcal{N}_\varepsilon^\Omega$ is given by a system of second order differential equations on the branches of the network. The network differential equations describe a *diffusion-advection-reaction process* on each edge. *Transition conditions* in terms of *continuity conditions* and *Kirchhoff laws* have to fulfilled at the inner nodes. In addition, homogeneous Dirichlet conditions are imposed at the outer nodes of the network. This leads us to the following *microscopic problem on the $\varepsilon$-periodic network $\mathcal{N}_\varepsilon^\Omega$*:

$$\left.\begin{array}{l} \text{Find } u^\varepsilon \in \mathcal{H}^2\big(\mathcal{N}_\varepsilon^\Omega\big), \text{ such that} \\[4pt] \qquad \Gamma^\varepsilon u^\varepsilon(x) = f^\varepsilon(x), \; x \in \mathcal{N}_\varepsilon^\Omega \\[4pt] \qquad \quad u^\varepsilon(\tau) = 0, \qquad \tau \in \partial^B(\mathcal{N}_\varepsilon^\Omega) \\[4pt] \quad \mathcal{K}_\tau^{\Omega,\varepsilon}\Big(\delta a^\varepsilon \nabla_\tau^{\mathcal{Y}} u^\varepsilon\Big) = 0, \qquad \tau \in \partial^R(\mathcal{N}_\varepsilon^\Omega) \end{array}\right\} \quad (\mathrm{MP}_{\boldsymbol{\varepsilon}})$$

where

$$\Gamma^\varepsilon u^\varepsilon(x) := - \nabla_\tau^\varepsilon\Big(\delta a^\varepsilon(x) \cdot \nabla_\tau^\varepsilon u^\varepsilon(x)\Big) + \nabla_\tau^\varepsilon\Big(b(x) \cdot u^\varepsilon(x)\Big)$$

$$+ c^\varepsilon(x) \cdot \nabla_\tau^\varepsilon u^\varepsilon(x) + d(x) \cdot u^\varepsilon(x)$$

for each $x \in \mathcal{N}_\varepsilon^\Omega$. Here, we have used the following notation: If $\phi : \mathcal{N}_1 \to \mathbb{R}$ is a (periodic) function on the infinite network $\mathcal{N}_1$, then we define the function $\phi^\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}$ with $\phi^\varepsilon(x) := \phi(\varepsilon^{-1}x)$. The *singular perturbation parameter* $\delta > 0$ controls the strength of the diffusion part in $(\mathrm{MP}_\varepsilon)$. For $\delta \to 0$, the diffusion part vanishes and a *transport-dominated* microscopic model arises.

**Kirchhoff functionals**

For a given function $Q^\varepsilon$ on the network $\mathcal{N}_\varepsilon^\Omega$, the *Kirchhoff functional*

$$\mathcal{K}_\tau^{\Omega,\varepsilon}(Q^\varepsilon) = \sum_{j \in J_\varepsilon^{\Omega,+}(\tau)} \lim_{l_j \to \mathcal{L}_j^\varepsilon} Q_j^\varepsilon(l_j) - \sum_{j \in J_\varepsilon^{\Omega,-}(\tau)} \lim_{l_j \to 0} Q_j^\varepsilon(l_j)$$

is defined at each inner node $\tau \in \partial^R(\mathcal{N}_\varepsilon^\Omega)$. The index set $J_\varepsilon^{\Omega,+}(\tau) \subset \mathcal{J}_\varepsilon^\Omega$ consists of all edges that terminate in $\tau$. The index set $J_\varepsilon^{\Omega,-}(\tau) \subset \mathcal{J}_\varepsilon^\Omega$ comprises all edges which leave the node $\tau$. Similarly, a Kirchhoff functional $\mathcal{K}_\tau$ can be introduced at the inner nodes $\tau \in \mathcal{V}_\varepsilon$ of the infinite network $\mathcal{N}_\varepsilon$ and the inner nodes $\tau \in \partial^R(Y)$ of the reference graph, respectively.

**Coefficients**

The coefficients $a$, $b$, $c$, $d$ and the right hand side $f$ of the microscopic model $(\mathrm{MP}_\varepsilon)$ have to satisfy the following conditions:

---

(C1)  The coefficients $a$, $b$, $c$, $d$ and the function $f$ are $\mathcal{Y}$-periodic and they fulfill the following conditions:
$$a \in \mathcal{H}_{per}^1(\mathcal{N}_1), \quad a_{max} \geq a \geq a_0 > 0,$$
$$b \in \mathcal{H}_{per}^1(\mathcal{N}_1), \quad b_{max} \geq b \geq 0,$$
$$c \in \mathcal{H}_{per}^1(\mathcal{N}_1), \quad c_{max} \geq c \geq 0,$$
$$d \in \mathcal{H}_{per}^1(\mathcal{N}_1), \quad d_{max} \geq d \geq 0,$$
$$f \in L_{per}^2(\mathcal{N}_1).$$

(C2)  For each node $\tau \in \mathcal{V}_1$ we have
$$\mathcal{K}_\tau(b) = \mathcal{K}_\tau(c) = 0.$$

(C3)  For each $y \in \mathcal{Y}$ we have
$$\frac{1}{2} \cdot \nabla_\tau^\mathcal{Y}(b(y) - c(y)) \geq 0.$$

---

We note that the microscopic model can be represented in equivalent way by the following model.

---

Find $u^\varepsilon \in \mathcal{K}_0^2(\mathcal{N}_\varepsilon^\Omega)$, such that
$$\Gamma^\varepsilon u^\varepsilon(x) = f^\varepsilon(x), \ x \in \mathcal{N}_\varepsilon^\Omega,$$

---

where
$$\mathcal{K}_0^2(\mathcal{N}_\varepsilon^\Omega) := \left\{ u^\varepsilon \in \mathcal{H}_0^2(\mathcal{N}_\varepsilon^\Omega) \,\middle|\, K_\tau^{\Omega,\varepsilon}(\delta a^\varepsilon \nabla_\tau^\varepsilon u^\varepsilon) = 0 \ \left(\forall \tau \in \partial^R(\mathcal{N}_\varepsilon^\Omega)\right) \right\},$$

$$\mathcal{H}_0^2(\mathcal{N}_\varepsilon^\Omega) := \left\{ u^\varepsilon \in \mathcal{H}^2(\mathcal{N}_\varepsilon^\Omega) \,\middle|\, u^\varepsilon(\tau) = 0 \ \left(\forall \tau \in \partial^B(\mathcal{N}_\varepsilon^\Omega)\right) \right\}.$$

**5.2. The Variational Form of the Microscopic Model.** In this section, we introduce a *variational model of the physical system* on the branches of the $\varepsilon$-periodic network $\mathcal{N}_\varepsilon^\Omega$. This model is discussed in [18], where a homogenized model in variational form is received with a notion of two-scale convergence for network functions on varying domains.

**Definition 5.1 (Variational microscopic model).**
Let the assumptions (P), (B), (G1), (G2) and the conditions (C1), (C2), (C3) be fulfilled. Let $f \in \left[\mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)\right]^{-1}$. The *variational microscopic model* is defined by

---

Find $u^\varepsilon \in \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$, such that
$$\mathcal{A}^\varepsilon(u^\varepsilon, v^\varepsilon) = \langle f^\varepsilon, v^\varepsilon \rangle_{\mathcal{N}_\varepsilon^\Omega} \quad \left(\forall v^\varepsilon \in \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)\right),$$
$\left.\right\}$ $(\mathrm{VMP}_\varepsilon)$

---

where the *bilinear form* $\mathcal{A}^\varepsilon : \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega) \times \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega) \to \mathbb{R}$ is given by

$$\mathcal{A}^\varepsilon(u^\varepsilon, v^\varepsilon) := \int_{\mathcal{N}_\varepsilon^\Omega} \delta a^\varepsilon(x)\nabla_\tau^\varepsilon u^\varepsilon(x)\nabla_\tau^\varepsilon v^\varepsilon(x) - b^\varepsilon(x)u^\varepsilon(x)\nabla_\tau^\varepsilon v^\varepsilon(x)$$
$$+ c^\varepsilon(x)\nabla_\tau^\varepsilon u^\varepsilon(x)v^\varepsilon(x) + d^\varepsilon(x)u^\varepsilon(x)v^\varepsilon(x)\, dx.$$

Since the bilinear form is $\mathcal{A}^\varepsilon$ bounded and coercive, the *Lax-Milgram-theorem* leads to the following result.

**Theorem 5.2.** *(Existence and uniqueness)*
*The variational problem* $(VMP)_\varepsilon$ *has a unique solution in* $\mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$.

In addition, we can state an *a priori estimation* of the solution of the variational formulation of the microscopic model.

**Theorem 5.3.** *(A priori estimations)*
*The solution* $u^\varepsilon \in \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$ *of the variational problem* $(VMP)_\varepsilon$ *satisfies*

$$\left\|u^\varepsilon\right\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \leq \frac{\sqrt{C}}{\delta a_0}\left\|f^\varepsilon\right\|_{L^2(\mathcal{N}_\varepsilon^\Omega)},$$

*where* $C \in \mathcal{O}(1)$.

## 6. The Macroscopic Model.

6.1. **Averaging Technique.** In [14, 16, 17], an averaging technique is applied to derive a macroscopic model. The solution $u^\varepsilon$ of the microscopic model $(MP_\varepsilon)$ is represented in terms of the *two-scale asymptotic expansion*

$$u^\varepsilon(x) = u_0(x) + \sum_{k=1}^\infty \varepsilon^k \cdot u_k\left(x, \frac{x}{\varepsilon}\right), \tag{A}$$

with $u_k : \overline{\Omega} \times \mathcal{N}_1 \to \mathbb{R}$, which is periodic with respect to the second variable. For the limit analysis $\varepsilon \to 0$ a first order asymptotic expansion

$$u^\varepsilon(x) = u_0(x) + u_1\left(x, \frac{x}{\varepsilon}\right)$$

is sufficient. The function $u_1$ is represented in the form

$$u_1 = \mathcal{S} \circ \nabla u_0 - \mathcal{T} \cdot u_0,$$

where

$$\mathcal{S} : \overline{\Omega} \times \mathcal{N}_1 \to \mathbb{R}^d \text{ and } \mathcal{T} : \overline{\Omega} \times \mathcal{N}_1 \to \mathbb{R}$$

are $Y$-periodic and the vector function $\mathcal{S}$ is composed of functions $\mathcal{S}^{(k)}$ with $k \in \{1, 2, \ldots, d\}$. The functions $\mathcal{S}^{(k)}$ are the solutions of the system

$$\left. \begin{aligned} &-\frac{\partial}{\partial \sigma_j}\left(\delta \cdot a_j(\sigma_j) \cdot \frac{\partial}{\partial \sigma_j}\mathcal{S}_j^{(k)}(\sigma_j)\right) \\ &= \frac{\partial}{\partial \sigma_j}\left(\delta \cdot a_j(\sigma_j) \cdot \gamma_j^{(k)}(\sigma_j)\right), \ j \in \mathcal{J}_{\mathcal{Y}}, \\ &\mathcal{K}_1^{(\tau)}(\mathcal{S}^{(k)}) = -\mathcal{K}_0^{(\tau)}((x)_k), \ \tau \in \partial^R(\mathcal{Y}), \\ &\mathcal{S}^{(k)} \text{ is } \mathcal{Y} - \text{periodic}, \end{aligned} \right\} (\mathcal{S}_k)$$

where $(x)_k$ denotes the $k$-th component of the vector $x$ and $\gamma_j$ is a vector where the $i$th component is given by the cosine of the angle between the tangential direction at the branch $j$ and the $i$th coordinate axis. The function $\mathcal{T}$ is the solution of the system

$$\left.\begin{array}{l} -\dfrac{\partial}{\partial \sigma_j}\left(\delta \cdot a_j(\sigma_j) \cdot \dfrac{\partial}{\partial \sigma_j}\mathcal{T}_j(\sigma_j)\right) \\[2mm] \quad = \dfrac{\partial}{\partial \sigma_j}b_j(\sigma_j),\ j \in \mathfrak{J}_{\mathcal{Y}}, \\[3mm] \mathcal{K}_1^{(\tau)}(\mathcal{T}) = 0,\ \tau \in \partial^R(\mathcal{Y}), \\[2mm] \mathcal{T}\ is\ \mathcal{Y} - periodic. \end{array}\right\} (\mathcal{T})$$

The function $u_0$ is the solution of the *homogenized equation*

$$\mathcal{L}^0 u_0(x) := -\sum_{s,k=1}^{d}\frac{\partial}{\partial x_s}\left(\widehat{a}_{sk}\cdot\frac{\partial}{\partial x_k}u_0(x)\right) + \sum_{k=1}^{d}\frac{\partial}{\partial x_k}\left(\widehat{b}_k\cdot u_0(x)\right)$$

$$+ \sum_{k=1}^{d}\widehat{c}_k\cdot\frac{\partial}{\partial x_k}u_0(x) + \widehat{d}\cdot u_0(x) = \widehat{f}.$$

When we add homogeneous Dirichlet boundary conditions, we obtain the *macroscopic model*:

$$\left.\begin{array}{l} \mathcal{L}^0 u_0(x) = \widehat{f}(x),\ x \in \Omega, \\[1mm] u_0(x) = 0\quad,\ x \in \partial\Omega. \end{array}\right\} (\text{MP}_0)$$

The corresponding *homogenized coefficients* are given by

$$\widehat{a}_{sk} = \sum_{j\in\mathfrak{J}_{\mathcal{Y}}}\int_{\mathfrak{J}_j^{\mathcal{Y}}}\delta\cdot a_j(\sigma_j)\cdot\gamma_j^{(s)}(\sigma_j)\cdot\gamma_j^{(k)}(\sigma_j)$$

$$+\ \delta\cdot a_j(\sigma_j)\cdot\gamma_j^{(s)}(\sigma_j)\cdot\frac{\partial}{\partial\sigma_j}\mathcal{S}_j^{(k)}(\sigma_j)\,d\sigma_j,$$

$$\widehat{b}_k(x) = \sum_{j\in\mathfrak{J}_{\mathcal{Y}}}\int_{\mathfrak{J}_j^{\mathcal{Y}}}b_j(x,\sigma_j)\cdot\gamma_j^{(k)}(\sigma_j)$$

$$+\ \delta\cdot a_j(\sigma_j)\cdot\gamma_j^{(k)}(\sigma_j)\cdot\frac{\partial}{\partial\sigma_j}\mathcal{T}_j(\sigma_j)\,d\sigma_j,$$

$$\widehat{c}_k = \sum_{j\in\mathfrak{J}_{\mathcal{Y}}}\int_{\mathfrak{J}_j^{\mathcal{Y}}}c_j(\sigma_j)\cdot\gamma_j^{(k)}(\sigma_j) + c_j(\sigma_j)\cdot\frac{\partial}{\partial\sigma_j}\mathcal{S}_j^{(k)}(\sigma_j)$$

$$+\ \frac{\partial}{\partial\sigma_j}\left(b_j(\sigma_j)\cdot\mathcal{S}_j^{(k)}(\sigma_j)\right)d\sigma_j,$$

$$\widehat{d}(x) = \sum_{j\in\mathfrak{J}_{\mathcal{Y}}}\int_{\mathfrak{J}_j^{\mathcal{Y}}}d_j(\sigma_j) - c_j(x,\sigma_j)\cdot\frac{\partial}{\partial\sigma_j}\mathcal{T}_j(\sigma_j)$$

$$-\frac{\partial}{\partial\sigma_j}\left(b_j(\sigma_j)\cdot\mathcal{T}_j(\sigma_j)\right)d\sigma_j,$$

$$\widehat{f}(x)=\sum_{j\in\partial y}\int_{\mathcal{I}_j^{y}}f_j(\sigma_j)\,d\sigma_j.$$

The quality of the approximation is guaranteed by error results. The residuum

$$\mathfrak{R}^\varepsilon:=u^\varepsilon-u_0-\varepsilon u_1$$

compares the exact solution, $u^\varepsilon$, of the microscopic model on the network and the first order approximation. It can be shown that

$$\|\mathfrak{R}^\varepsilon\|_{H^1(\mathcal{N}_\varepsilon^\Omega)}\leq C\cdot\varepsilon^{\frac{1}{2}}.$$

This result provides a full justification of the two-scale asymptotic approach. For $\varepsilon\to 0$, the difference between the exact solution of the microscopic model and the first order approximation tends to zero. In other words, the macroscopic model provides a good approximation to the microscopic model on the periodic network with a small length of periodicity that can be easily solved by standard PDE-solvers in a few seconds.

6.2. **Two-scale Convergence.** The second approach considered here is based on the notion of two-scale convergence. In [18], the variational microscopic model (VMP$_\varepsilon$) is addressed and the *macroscopic variational problem* (or *variational two-scale homogenized problem*) on $\Omega\times\mathcal{Y}$ is derived. This result is based on the following theorem.

**Theorem 6.1. (Two-scale convergence)**
*Let $\left\{u^\varepsilon\in\mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)\right\}_{\varepsilon\in E}$ be the sequence of solutions of the variational microscopic model (VMP)$_\varepsilon$. Then there exists a subsequence $\left\{u^{\varepsilon_k}\in\mathcal{H}^1(\mathcal{N}_{\varepsilon_k}^\Omega)\right\}_{\varepsilon_k\in E_k}$, such that*

*(i)  $\left\{u^{\varepsilon_k}\in L^2(\mathcal{N}_{\varepsilon_k}^\Omega)\right\}_{\varepsilon_k\in E_k}$ is weakly two-scale convergent to $u_0\in\mathcal{H}_{\tau\mathcal{Y}}^1(\Omega,\mathcal{Y})$.*

*(ii) $\left\{\nabla_\tau^{\varepsilon_k}u^{\varepsilon_k}\in L^2(\mathcal{N}_{\varepsilon_k}^\Omega)\right\}_{\varepsilon_k\in E_k}$ is weakly two-scale convergent to*

$$\nabla_z u_0*\tau^{\mathcal{Y}}(y)+\nabla_\tau^{\mathcal{Y}}u_1,$$

*where $u_1\in L^2(\Omega;\mathcal{H}_\mathcal{Y}^1)$.*

We assume that the following conditions are fulfilled.

**Assumption 6.2 (Coefficients).** *The coefficients and the right hand side of the variational microscopic model satisfy the following conditions:*

$$\boxed{(K)\quad a^\varepsilon\xrightarrow{2}a^0,\ \ b^\varepsilon\xrightarrow{2}b^0,\ \ c^\varepsilon\xrightarrow{2}c^0,\ \ d^\varepsilon\xrightarrow{2}d^0,\ \ f^\varepsilon\xrightarrow{2}f^0.}$$

**Remark 3.** Each strongly two-scale convergent sequence is also weakly two-scale convergent by Theorem 1. For this reason, it is sufficient to assume that $f^\varepsilon\xrightarrow{2}f^0$.

Now, we can state the *main homogenization result*.

**Theorem 6.3. (Homogenization result[2])**

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[2]For further details see [18].

*Let the assumptions (C1), (C2), (C3), (G1), (G2), (P), (B), and (K) be fulfilled. Let $\left\{u^\varepsilon \in \mathcal{H}_0^1\big(\mathcal{N}_\varepsilon^\Omega\big)\right\}_{\varepsilon \in E}$ be the sequence of solutions of the variational problems ($VPM_\varepsilon$). Then, there exists a subsequence $\left\{u^{\varepsilon_k} \in \mathcal{H}_0^1(\mathcal{N}_{\varepsilon_k}^\Omega)\right\}_{\varepsilon_k \in E_k}$ such that*

$$u^{\varepsilon_k} \overset{2}{\rightharpoonup} u_0,$$

$$\nabla_\tau^{\varepsilon_k} u^{\varepsilon_k} \overset{2}{\rightharpoonup} \nabla_z u_0 * \tau^{\mathcal{Y}} + \nabla_\tau^{\mathcal{Y}} u_1.$$

*Here, $(u_0, u_1)$ is the unique solution of the **variational two-scale homogenized problem**:*

---

*Find $(u_0, u_1) \in \mathcal{H}_{0,\tau^{\mathcal{Y}}}^1(\Omega, \mathcal{Y}) \times L^2(\Omega; \mathcal{H}_{\mathcal{Y}}^1)$, such that*

$$\int_\Omega \int_{\mathcal{Y}} \delta \cdot a^0(z, y) \cdot \big(\nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) + \nabla_\tau^{\mathcal{Y}} u_1(z, y)\big)$$

$$\cdot \big(\nabla_z \psi^0(z) * \tau^{\mathcal{Y}}(y) + \nabla_\tau^{\mathcal{Y}} \psi^1(z, y)\big)$$

$$- b^0(z, y) \cdot u_0(z) \cdot \big(\nabla_z \psi^0(z) * \tau^{\mathcal{Y}}(y) + \nabla_\tau^{\mathcal{Y}} \psi^1(z, y)\big)$$

$$+ c^0(z, y) \cdot \big(\nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) + \nabla_\tau^{\mathcal{Y}} u_1(z, y)\big) \cdot \psi^0(z)$$

$$+ d^0(z, y) \cdot u_0(z) \cdot \psi^0(z)\, dy\, dz$$

$$= \int_\Omega \int_{\mathcal{Y}} f^0(z, y) \cdot \psi^0(z)\, dy\, dz$$

*for all $(\psi^0, \psi^1) \in \mathcal{H}_{0,\tau^{\mathcal{Y}}}^1(\Omega, \mathcal{Y}) \times L^2(\Omega; \mathcal{H}_{\mathcal{Y}}^1)$.*          (VHP$_\mathbf{0}$)

---

**7. The Homogenized Model.** From a theoretical point of view the considerations on the homogenization of the microscopic model can be finished here. However, for numerical considerations we want to find another representation of the homogenized model (HP$_0$) that does not depend on the fast variable $y$ and the microgeometry. In the following we demonstrate that the two-scale homogenized problem can be equivalently represented by a combination of the *homogenized problem* and $d + 1$ *reference cell problems*.

**Assumption 7.1 (Regularity).**
*We assume that the unique solution $(u_0, u_1) \in \mathcal{H}_{0,\tau^{\mathcal{Y}}}^1(\Omega, \mathcal{Y}) \times L^2(\Omega; \mathcal{H}_{\mathcal{Y}}^1)$ of the **variational two-scale homogenized problem** has a higher regularity such that $u_0 \in \mathcal{H}_{0,\tau^{\mathcal{Y}}}^2(\Omega, \mathcal{Y})$ and $u_1 \in L^2(\Omega; \mathcal{H}_{\mathcal{Y}}^2)$.*

If we choose $\psi^0 \equiv 0$ and then $\psi^1 \equiv 0$ in the equation of the variational two-scale homogenized problem, we can see that it is equivalent to the so-called **two-scale homogenized problem**.

**Theorem 7.2.** *Let the regularity assumption (7.1) be fulfilled. Then, the solution $(u_0, u_1)$ coincides with the solution of the **two-scale homogenized problem**:*

*Find* $(u_0, u_1) \in \mathcal{H}^2_{0,\tau^{\mathcal{Y}}}(\Omega, \mathcal{Y}) \times L^2(\Omega; \mathcal{H}^2_{\mathcal{Y}})$, *such that*

$$- \nabla^{\mathcal{Y}}_\tau \Big( \delta a^0(z,y) \cdot \nabla^{\mathcal{Y}}_\tau u_1(z,y) \Big)$$

$$= \nabla^{\mathcal{Y}}_\tau \Big( \delta a^0(z,y) \cdot \nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) - b^0(z,y) \cdot u_0(z) \Big) \text{ on } \Omega \times \mathcal{Y},$$

$$- \sum_{s=1}^d \frac{\partial}{\partial z_s} \left( \int_{\mathcal{Y}} \delta a^0(z,y) \cdot \big(\tau^{\mathcal{Y}}(y)\big)_s \cdot \big(\nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) + \nabla^{\mathcal{Y}}_\tau u_1(z,y)\big) \right.$$

$$\left. - b^0(z,y) \cdot \big(\tau^{\mathcal{Y}}(y)\big)_s \cdot u_0(z) \, dy \right)$$

$$+ \int_{\mathcal{Y}} c^0(z,y) \cdot \big(\nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) + \nabla^{\mathcal{Y}}_\tau u_1(z,y)\big) + d^0(z,y) \cdot u_0(z) \, dy$$

$$= \int_{\mathcal{Y}} f^0(z,y) \, dy \text{ on } \Omega,$$

$u_0 = 0$ *on* $\partial\Omega$,

$u_1(z, \cdot)$ *is* $\mathcal{Y}$-*periodic.*                                                    **(HP$_0$)**

We now assume that $z \in \Omega$, i.e., $z$ is a parameter. In the sequel, several *reference cell problems* of the following type are considered:

**Definition 7.3.** The *abstract reference cell problem* is given by

$$\boxed{\begin{aligned} &\text{Find } u \in \mathcal{H}^2_{\mathcal{Y}}, \text{ such that} \\ &\quad A_{\mathcal{Y}} u(y) = F(y) \qquad \text{for each } y \in \mathcal{Y}. \end{aligned}}$$                                              (3)

Here, we use

$$A_{\mathcal{Y}} := -\nabla^{\mathcal{Y}}_\tau \Big( \delta a^0(z,y) \cdot \nabla^{\mathcal{Y}}_\tau \Big)$$

and the right hand side

$$F(y) := \nabla^{\mathcal{Y}}_\tau g(y)$$

that depends on a function $g \in \mathcal{H}^1_{\mathcal{Y}}$.

We now turn to the solvability of the abstract reference cell problem (3). Each solution of problem (3) is known up to an additive constant because

$$\nabla^{\mathcal{Y}}_\tau u = \nabla^{\mathcal{Y}}_\tau (u + c)$$

for each $u \in \mathcal{H}^1_{\mathcal{Y}}$ and every $c \in \mathbb{R}$. For this reason, the quotient space

$$\mathcal{W}^1_{per}(\mathcal{Y}) := \mathcal{H}^1_{\mathcal{Y}}/\mathbb{R},$$

is introduced, i.e., the space of all possible equivalence classes of $\mathcal{H}^1_{\mathcal{Y}}$ with respect to the equivalence relation

$$u \sim v \quad :\Leftrightarrow \quad u - v = \text{constant} \qquad \big(\forall u, v \in \mathcal{H}^1_{\mathcal{Y}}\big).$$

We replace the abstract reference cell problem (3) by

$$\boxed{\begin{aligned} &\text{Find } \dot{u} \in \mathcal{W}^1_{per}(\mathcal{Y}), \text{ such that} \\ &\quad \dot{A}_\mathcal{Y} \dot{u} = F \qquad \text{on } \mathcal{Y} \end{aligned}} \tag{4}$$

with

$$\dot{A}_\mathcal{Y} \dot{u} := A_\mathcal{Y} u = -\nabla^\mathcal{Y}_\tau \big(\delta a^0(z,y) \cdot \nabla^\mathcal{Y}_\tau u\big)$$

for each $\dot{u} \in \mathcal{W}^1_{per}(\mathcal{Y})$ and $u \in \dot{u}$, where $\dot{u}$ is the equivalence class of $u$.

Next, we introduce the abstract reference cell problem in variational form on the reference graph.

**Definition 7.4.** The abstract reference cell problem in variational form is defined by

$$\boxed{\begin{aligned} &\text{Find } \dot{u} \in \mathcal{W}^1_{per}(v), \text{ such that} \\ &\quad \dot{\mathcal{A}}_\mathcal{Y}(\dot{u}, \dot{v}) = \dot{\mathcal{F}}(\dot{v}) \qquad \text{for each } \dot{v} \in \mathcal{W}^1_{per}(\mathcal{Y}). \end{aligned}} \tag{5}$$

Here, the bilinear form $\mathcal{A}_Y : \mathcal{H}^1_\mathcal{Y} \times \mathcal{H}^1_\mathcal{Y} \to \mathbb{R}$ is defined by

$$\mathcal{A}_\mathcal{Y}(u,v) := \int_\mathcal{Y} \delta a^0(z,y) \cdot \nabla^\mathcal{Y}_\tau u(y) \cdot \nabla^\mathcal{Y}_\tau v(y) \, dy$$

and the bilinear form $\dot{\mathcal{A}}_\mathcal{Y} : \mathcal{W}^1_{per}(\mathcal{Y}) \times \mathcal{W}^1_{per}(\mathcal{Y}) \to \mathbb{R}$ is given by

$$\dot{\mathcal{A}}_\mathcal{Y}(\dot{u}, \dot{v}) := \mathcal{A}_\mathcal{Y}(u,v)$$

for each $\dot{u} \in \mathcal{W}^1_{per}(\mathcal{Y})$, $u \in \dot{u}$, $\dot{v} \in \mathcal{W}^1_{per}(\mathcal{Y})$ and each $v \in \dot{v}$. In addition, the function $\dot{\mathcal{F}} : \mathcal{W}^1_{per}(\mathcal{Y}) \to \mathbb{R}$ is defined by the linear form $\mathcal{F} : \mathcal{H}^1_\mathcal{Y} \to \mathbb{R}$ with

$$\mathcal{F}(v) := \int_\mathcal{Y} F(y) \cdot v(y) \, dy = \int_\mathcal{Y} \nabla^\mathcal{Y}_\tau g(y) \cdot v(y) \, dy = -\int_\mathcal{Y} g(y) \cdot \nabla^\mathcal{Y}_\tau v(y) \, dy,$$

where

$$\dot{\mathcal{F}}(\dot{v}) := \mathcal{F}(v)$$

for each $\dot{v} \in \mathcal{W}^1_{per}(\mathcal{Y})$ and every $v \in \dot{v}$. In particular, we get

$$\mathcal{F}(v_1) = \mathcal{F}(v_2)$$

for all $v_1, v_2 \in \dot{v}$ with $\dot{v} \in \mathcal{W}^1_{per}(\mathcal{Y})$, i.e., $\dot{\mathcal{F}} \in (\mathcal{W}^1_{per}(\mathcal{Y}))'$.

We now apply the *Lax-Milgram theorem* in order to prove that the variational problem (5) has a unique solution. On $\mathcal{W}^1_{per}(\mathcal{Y})$ we define the norm

$$\|\dot{v}\|_{\mathcal{W}^1_{per}(\mathcal{Y})} := \|\nabla^\mathcal{Y}_\tau v\|_{L^2(\mathcal{Y})} = \left\{ \int_\mathcal{Y} \left[\nabla^\mathcal{Y}_\tau v(y)\right]^2 dy \right\}^{\frac{1}{2}}$$

for each $\dot{v} \in \mathcal{W}^1_{per}(\mathcal{Y})$ and every $v \in \dot{v}$. If $\dot{u}, \dot{v} \in \mathcal{W}^1_{per}(\mathcal{Y})$ and $u \in \dot{u}$, $v \in \dot{v}$, then we obtain

$$\left|\dot{\mathcal{A}}_\mathcal{Y}(\dot{u}, \dot{v})\right| = \left|\mathcal{A}_\mathcal{Y}(u,v)\right| = \left| \int_\mathcal{Y} \delta a^0(z,y) \cdot \nabla^\mathcal{Y}_\tau u(y) \cdot \nabla^\mathcal{Y}_\tau v(y) \, dy \right|$$

$$\leq \delta a_{max} \left| \int_\mathcal{Y} \nabla^\mathcal{Y}_\tau u(y) \cdot \nabla^\mathcal{Y}_\tau v(y) \, dy \right| \leq \delta a_{max} \|\nabla^\mathcal{Y}_\tau u\|_{L^2(\mathcal{Y})} \cdot \|\nabla^\mathcal{Y}_\tau v\|_{L^2(\mathcal{Y})}$$

$$= \delta a_{max} \|\dot{u}\|_{\mathcal{W}^1_{per}(\mathcal{Y})} \cdot \|\dot{v}\|_{\mathcal{W}^1_{per}(\mathcal{Y})}$$

and $\dot{\mathcal{A}}_y$ is bounded on $\mathcal{W}^1_{per}(\mathcal{Y})$. In addition, we have

$$\dot{\mathcal{A}}_y(\dot{u}, \dot{u}) = \mathcal{A}_y(u, u) = \int_{\mathcal{Y}} \delta a^0(z, y) \cdot \left[\nabla^{\mathcal{Y}}_\tau u(y)\right]^2 dy$$

$$\geq \delta a_0 \cdot \int_{\mathcal{Y}} \left[\nabla^{\mathcal{Y}}_\tau u(y)\right]^2 dy = \delta a_0 \cdot \|\dot{u}\|^2_{\mathcal{W}^1_{per}(\mathcal{Y})}$$

for each $\dot{u} \in \mathcal{W}^1_{per}(\mathcal{Y})$ and $u \in \dot{u}$. It follows that $\dot{\mathcal{A}}_Y$ is $\mathcal{W}^1_{per}(\mathcal{Y})$-coercive. Now, we obtain the following theorem.

**Theorem 7.5.** *The variational problem (4) and problem (4) have a unique solution* $\dot{u} \in \mathcal{W}^1_{per}(\mathcal{Y})$.

Each element of $\mathcal{W}^1_{per}(\mathcal{Y})$ represents a class of $\mathcal{H}^1_{\mathcal{Y}}$-functions that can be considered as equivalent with respect to the relation $\sim$. Because of Theorem 7.5, the reference cell problem (3) has a unique solution in $\mathcal{H}^1_{\mathcal{Y}}$. This solution is unique up to an additive constant. If we further assume that the *average* of a solution of problem (3) must be equal to zero, the solution of (3) is indeed unique. The average of a function $u \in \mathcal{H}^1_{\mathcal{Y}}$ is defined by

$$\mathcal{M}_{\mathcal{Y}}(u) := \int_{\mathcal{Y}} u(y) \, dy.$$

The unique solution is obtained as the result of the following problem.

$$\boxed{\begin{array}{l} \text{Find } u \in \mathcal{H}^1_{\mathcal{Y}}, \text{ such that} \\ \quad A_{\mathcal{Y}} u(y) = F(y) \qquad \text{for each } y \in \mathcal{Y}, \\ \quad \mathcal{M}_{\mathcal{Y}}(u) = 0. \end{array}} \qquad (6)$$

The corresponding variational problem is given by

$$\boxed{\begin{array}{l} \text{Find } u \in W^{1,0}_{per}(\mathcal{Y}), \text{ such that} \\ \quad \mathcal{A}_{\mathcal{Y}}(u, v) = \mathcal{F}(v) \qquad \text{for each } v \in W^{1,0}_{per}(\mathcal{Y}), \end{array}} \qquad (7)$$

where

$$W^{1,0}_{per}(\mathcal{Y}) := \left\{ u \in \mathcal{H}^1_{\mathcal{Y}} \,\middle|\, \mathcal{M}_{\mathcal{Y}}(u) = 0 \right\}.$$

For problem (6) and problem (7) we finally obtain the following theorem.

**Theorem 7.6.** *The variational problem (6) and problem (7) have a unique solution* $u \in W^{1,0}_{per}(\mathcal{Y})$.

Until now we have considered the solvability of the reference cell problems. In the following, we show that the *two-scale homogenized problem* $(\mathrm{HP}_0)$ can be equivalently represented by the *homogenized problem* and $d + 1$ *reference cell problems*. Here, $z \in \Omega$ is considered as a parameter. The first equation of the two-scale homogenized problem leads to the problem

$$\boxed{\begin{array}{l} \text{Find } u_1 = u_1(z, \cdot) \in \mathcal{H}^1_{\mathcal{Y}}, \text{ such that} \\ \quad A_{\mathcal{Y}} u_1 = \nabla^{\mathcal{Y}}_\tau \left( \delta a^0(z, y) \cdot \nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) - b^0(z, y) \cdot u_0(z) \right) \\ \text{for each } y \in \mathcal{Y}. \end{array}} \qquad (8)$$

This problems is of the type of the reference cell problem (3) with

$$g(y) = \delta a^0(z,y) \cdot \nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) - b^0(z,y) \cdot u_0(z).$$

In the same way as above, we obtain the problem

$$\boxed{\begin{array}{l} \text{Find } \dot{u}_1 = \dot{u}_1(z,\cdot) \in \mathcal{W}^1_{per}(\mathcal{Y}), \text{ such that} \\[6pt] \qquad \dot{A}_{\mathcal{Y}} \dot{u}_1(z,y) = F(y) \qquad \text{for each } y \in \mathcal{Y} \end{array}} \tag{9}$$

with

$$F(y) = \nabla^Y_\tau \Big( \delta a^0(z,y) \cdot \nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) - b^0(z,y) \cdot u_0(z) \Big)$$

and the variational problem

$$\boxed{\begin{array}{l} \text{Find } \dot{u}_1 = \dot{u}_1(z,\cdot) \in \mathcal{W}^1_{per}(\mathcal{Y}), \text{ such that} \\[6pt] \qquad \dot{\mathcal{A}}_Y(\dot{u}_1, \dot{v}) = \dot{\mathcal{F}}(\dot{v}) \qquad \text{for each } \dot{v} \in \mathcal{W}^1_{per}(\mathcal{Y}) \end{array}} \tag{10}$$

with

$$\dot{\mathcal{F}}(\dot{v}) = -\int_{\mathcal{Y}} \Big( \delta a^0(z,y) \cdot \nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) - b^0(z,y) \cdot u_0(z) \Big) \cdot \nabla^{\mathcal{Y}}_\tau v(y) \, dy.$$

Theorem 7.5 shows that there exists a unique solution $\dot{u}_1 = \dot{u}_1(z,\cdot) \in \mathcal{W}^1_{per}(\mathcal{Y})$. In order to find a representation of this solution we use the ansatz

$$\boxed{\dot{u}_1(z,y) = \sum_{s=1}^{d} \dot{\chi}_s(y) \cdot \frac{\partial}{\partial z_s} u_0(z) - \dot{\eta}(y) \cdot u_0(z)} \tag{11}$$

with the unknown variables $\dot{\chi}_s \in \mathcal{W}^1_{per}(\mathcal{Y})$ for $s \in \{1, \ldots, d\}$ and $\dot{\eta} \in \mathcal{W}^1_{per}(\mathcal{Y})$. By inserting the ansatz (11) in problem (9) we obtain

$$-\sum_{s=1}^{d} \nabla^{\mathcal{Y}}_\tau \Big( \delta a^0(z,y) \cdot \nabla^{\mathcal{Y}}_\tau \chi_s(y) \Big) \cdot \frac{\partial}{\partial z_s} u_0(z) + \nabla^{\mathcal{Y}}_\tau \Big( \delta a^0(z,y) \cdot \nabla^{\mathcal{Y}}_\tau \eta(y) \Big) \cdot u_0(z)$$

$$= \sum_{s=1}^{d} \nabla^{\mathcal{Y}}_\tau \Big( \delta a^0(z,y) \cdot \big( \tau^{\mathcal{Y}}(y) \big)_s \Big) \cdot \frac{\partial}{\partial z_s} u_0(z) - \nabla^{\mathcal{Y}}_\tau \big( b^0(z,y) \big) \cdot u_0(z)$$

for each $u \in \dot{u}$. That means, $\dot{\chi}_s$ is the solution of the *reference cell problem*

$$\boxed{\begin{array}{l} \text{Find } \dot{\chi}_s \in \mathcal{W}^1_{per}(\mathcal{Y}), \text{ such that} \\[6pt] \qquad \dot{A}_{\mathcal{Y}} \dot{\chi}_s(y) = \nabla^{\mathcal{Y}}_\tau \Big( \delta a^0(z,y) \cdot \big( \tau^{\mathcal{Y}}(y) \big)_s \Big) \qquad \text{for each } y \in \mathcal{Y} \end{array}} \tag{12}$$

and $\dot{\eta}$ is the solution of the *reference cell problem*

$$\boxed{\begin{array}{l} \text{Find } \dot{\eta} \in \mathcal{W}^1_{per}(\mathcal{Y}), \text{ such that} \\[6pt] \qquad \dot{A}_{\mathcal{Y}} \dot{\eta}(y) = \nabla^{\mathcal{Y}}_\tau \Big( b^0(z,y) \Big) \qquad \text{for each } y \in \mathcal{Y}. \end{array}} \tag{13}$$

System (12) and system (13) have the same structure as problem (4). Theorem 7.5 shows that there exist unique solutions

$$\dot{\chi}_s \in \mathcal{W}^1_{per}(\mathcal{Y}) \qquad \text{and} \qquad \dot{\eta} \in \mathcal{W}^1_{per}(\mathcal{Y}),$$

where $s \in \{1, \ldots, d\}$. That means, $\dot{u}_1 \in \mathcal{W}_{per}^1(\mathcal{Y})$ is defined. In order to provide the unique solution $u_1 \in \mathcal{H}_{\mathcal{Y}}^1$ of problem (8) it is necessary that the average of $u_1$ takes the value zero. That means, $u_1$ is the solution of

$$
\boxed{
\begin{aligned}
&\text{Find } u_1 = u_1(z, \cdot) \in W_{per}^{1,0}(\mathcal{Y}), \text{ such that} \\
&\qquad A_Y u_1(z, y) = \nabla_\tau^{\mathcal{Y}}\Big( \delta a^0(z, y) \cdot \nabla_z u_0(z) * \tau^{\mathcal{Y}} - b^0(z, y) \cdot u_0(z) \Big) \\
&\text{for each } y \in \mathcal{Y}.
\end{aligned}
}
\qquad (14)
$$

Since

$$
\mathcal{M}_{\mathcal{Y}}(u_1) = 0
$$

we have to find an element with average value zero in each of the equivalence classes $\dot{\chi}_s \in \mathcal{W}_{per}^1(\mathcal{Y})$ and $\dot{\eta} \in \mathcal{W}_{per}^1(\mathcal{Y})$ for a unique representation of $u_1$. That means, instead of the solutions of (12) and (13), that are required for a representation of (11), we have to consider the solutions of the **reference cell problems**

$$
\boxed{
\begin{aligned}
&\text{Find } \chi_s \in W_{per}^{1,0}(\mathcal{Y}), \text{ such that} \\
&\qquad A_{\mathcal{Y}} \chi_s(y) = \nabla_\tau^Y \Big( \delta a^0(z, y) \cdot \big( \tau^{\mathcal{Y}}(y) \big)_s \Big) \\
&\text{for each } y \in \mathcal{Y}
\end{aligned}
}
\qquad (\textbf{RCP1}_s)
$$

and

$$
\boxed{
\begin{aligned}
&\text{Find } \eta \in W_{per}^{1,0}(\mathcal{Y}), \text{ such that} \\
&\qquad A_{\mathcal{Y}} \eta(y) = \nabla_\tau^{\mathcal{Y}} \Big( b^0(z, y) \Big) \qquad \text{for each } y \in \mathcal{Y}
\end{aligned}
}
\qquad (\textbf{RCP2}_s)
$$

with the corresponding variational formulations

$$
\boxed{
\begin{aligned}
&\text{Find } \chi_s \in W_{per}^{1,0}(\mathcal{Y}), \text{ such that} \\
&\qquad \mathcal{A}_{\mathcal{Y}}(\chi_s, v) = -\int_{\mathcal{Y}} \delta a^0(z, y) \cdot \big( \tau^{\mathcal{Y}} \big)_s \cdot \nabla_\tau^{\mathcal{Y}} v(y)\, dy \\
&\text{for each } v \in W_{per}^{1,0}(\mathcal{Y})
\end{aligned}
}
\qquad (\textbf{VRCP1}_s)
$$

and

$$
\boxed{
\begin{aligned}
&\text{Find } \chi_s \in W_{per}^{1,0}(\mathcal{Y}), \text{ such that} \\
&\qquad \mathcal{A}_{\mathcal{Y}}(\eta, v) = -\int_{\mathcal{Y}} b^0(z, y) \nabla_\tau^{\mathcal{Y}} v(y)\, dy \\
&\text{for each } v \in W_{per}^{1,0}(\mathcal{Y})
\end{aligned}
}
\qquad (\textbf{VRCP2})
$$

in order to represent the solution $u_1 \in W_{per}^{1,0}(\mathcal{Y})$ of (14) in the form

$$
u_1(z, y) = \sum_{s=1}^{d} \chi_s(y) \cdot \frac{\partial}{\partial z_s} u_0(z) - \eta(y) \cdot u_0(z) + \tilde{u}_1(z),
$$

where $\tilde{u}_1$ (as an additive constant) is independent from $y$, i.e., $\tilde{u}_1 \in \dot{0}$ in $\mathcal{W}_{per}^1(\mathcal{Y})$.

By inserting $u_1$ in the second equation of the two-scale homogenized problem it follows for each $z \in \Omega$ that

$$-\sum_{s=1}^{d} \frac{\partial}{\partial z_s} \left[ \int_{\mathcal{Y}} \delta a^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_s \cdot \left( \nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) + \nabla_\tau^{\mathcal{Y}} u_1(z,y) \right) \right.$$

$$\left. - b^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_s \cdot u_0(z) \, dy \right]$$

$$+ \int_{\mathcal{Y}} c^0(z,y) \cdot \left( \nabla_z u_0(z) * \tau^{\mathcal{Y}}(y) + \nabla_\tau^{\mathcal{Y}} u_1(z,y) \right) + d^0(z,y) \cdot u_0(z) \, dy$$

$$= -\sum_{s=1}^{d} \frac{\partial}{\partial z_s} \left[ \int_{\mathcal{Y}} \delta a^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_s \cdot \left( \sum_{k=1}^{d} \frac{\partial}{\partial z_s} u_0(z) * \left( \tau^{\mathcal{Y}}(y) \right)_s \right. \right.$$

$$\left. + \nabla_\tau^{\mathcal{Y}} \left[ \sum_{k=1}^{d} \chi_k(y) \cdot \frac{\partial}{\partial z_s} u_0(z) - \eta(y) \cdot u_0(z) \right] \right)$$

$$\left. - b^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_s \cdot u_0(z) \, dy \right]$$

$$+ \int_{\mathcal{Y}} c^0(z,y) \cdot \left( \sum_{k=1}^{d} \frac{\partial}{\partial z_s} u_0(z) * \left( \tau^{\mathcal{Y}}(y) \right)_s \right.$$

$$\left. + \nabla_\tau^{Y} \left[ \sum_{k=1}^{d} \chi_k(y) \cdot \frac{\partial}{\partial z_s} u_0(z) - \eta(y) \cdot u_0(z) \right] \right)$$

$$+ d^0(z,y) \cdot u_0(z) \, dy$$

$$= -\sum_{s=1}^{d} \sum_{k=1}^{d} \frac{\partial}{\partial z_s} \left[ \int_{\mathcal{Y}} \delta a^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_s \cdot \left( \tau^{\mathcal{Y}}(y) \right)_k \right.$$

$$\left. + \delta a^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_s \cdot \nabla_\tau^{\mathcal{Y}} \chi_k(y) \, dy \right] \cdot \frac{\partial}{\partial z_k} u_0(z)$$

$$+ \sum_{k=1}^{d} \frac{\partial}{\partial z_s} \left[ \int_{\mathcal{Y}} b^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_k + \delta a^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_k \cdot \nabla_\tau^{\mathcal{Y}} \eta(y) \, dy \right] \cdot u_0(z)$$

$$+ \sum_{k=1}^{d} \left[ \int_{\mathcal{Y}} c^0(z,y) \cdot \left( \tau^{\mathcal{Y}}(y) \right)_k + c^0(z,y) \cdot \nabla_\tau^{\mathcal{Y}} \chi_k(y) \, dy \right] \cdot \frac{\partial}{\partial z_k} u_0(z)$$

$$+ \left[ \int_{\mathcal{Y}} d^0(z,y) - c^0(z,y) \cdot \nabla_\tau^{\mathcal{Y}} \eta(y) \, dy \right] \cdot u_0(z)$$

$$= \int\limits_{\mathcal{Y}} f^0(z,y)\, dy.$$

With the **homogenized coefficients**

$$
\begin{aligned}
a^{hom}_{sk}(z) &:= \int\limits_{\mathcal{Y}} \delta a^0(z,y) \cdot \left(\tau^{\mathcal{Y}}(y)\right)_s \cdot \left(\tau^{\mathcal{Y}}(y)\right)_k \\
&\qquad + \delta a^0(z,y) \cdot \left(\tau^{\mathcal{Y}}(y)\right)_s \cdot \nabla^{\mathcal{Y}}_\tau \chi_k(y)\, dy, \\[2ex]
b^{hom}_k(z) &:= \int\limits_{\mathcal{Y}} b^0(z,y) \cdot \left(\tau^{\mathcal{Y}}(y)\right)_k \\
&\qquad + \delta a^0(z,y) \cdot \left(\tau^{\mathcal{Y}}(y)\right)_k \cdot \nabla^{\mathcal{Y}}_\tau \eta(y)\, dy, \\[2ex]
c^{hom}_k(z) &:= \int\limits_{\mathcal{Y}} c^0(z,y) \cdot \left(\tau^{\mathcal{Y}}(y)\right)_k + c^0(z,y) \cdot \nabla^{\mathcal{Y}}_\tau \chi_k(y)\, dy, \\[2ex]
d^{hom}(z) &:= \int\limits_{\mathcal{Y}} d^0(z,y) - c^0(z,y) \cdot \nabla^{\mathcal{Y}}_\tau \eta(y)\, dy \\[2ex]
f^{hom}(z) &:= \int\limits_{\mathcal{Y}} f^0(z,y)\, dy
\end{aligned}
$$

we obtain the **homogenized equation**

$$
\begin{aligned}
&- \sum_{s,k=1}^d \frac{\partial}{\partial z_s}\left( a^{hom}_{sk}(z) \cdot \frac{\partial}{\partial z_k} u_0(z) \right) + \sum_{k=1}^d \frac{\partial}{\partial z_k}\left( b^{hom}_k(z) \cdot u_0(z) \right) \\
&+ \sum_{k=1}^d c^{hom}_k(z) \cdot \frac{\partial}{\partial z_k} u_0(z) + d^{hom}(z) \cdot u_0(z) = f^{hom}(z),\ z \in \Omega
\end{aligned}
$$

and, thus, the **homogenized problem**

$$
\begin{aligned}
&\text{Find } u_0 \in H^2(\Omega) \text{ such that} \\
&\qquad A_0 u_0(z),\quad z \in \Omega, \\
&\qquad u_0(z) = 0,\ z \in \partial\Omega.
\end{aligned}
\qquad (\mathbf{HP}_\delta)
$$

Now, we summarize the previous results.

**Theorem 7.7.** *The two-scale homogenized problem*

$$
-\nabla_\tau^{\mathcal{Y}}\Big(\delta a^0(z,y)\cdot\nabla_\tau^{\mathcal{Y}} u_1(z,y)\Big)
$$

$$
=\nabla_\tau^{\mathcal{Y}}\Big(\delta a^0(z,y)\cdot\nabla_z u_0(z)*\tau^{\mathcal{Y}}(y)-b^0(z,y)\cdot u_0(z)\Big),\ (z,y)\in\Omega\times\mathcal{Y}
$$

$$
-\sum_{s=1}^{d}\frac{\partial}{\partial z_s}\Big(\int_{\mathcal{Y}}\delta a^0(z,y)\cdot\big(\tau^{\mathcal{Y}}(y)\big)_s\cdot\big(\nabla_z u_0(z)*\tau^{\mathcal{Y}}(y)+\nabla_\tau^{\mathcal{Y}} u_1(z,y)\big)
$$

$$
-b^0(z,y)\cdot\big(\tau^{\mathcal{Y}}(y)\big)_s\cdot u_0(z)\,dy\Big)
$$

$$
+\int_{\mathcal{Y}} c^0(z,y)\cdot\Big(\nabla_z u_0(z)*\tau^{\mathcal{Y}}(y)+\nabla_\tau^{\mathcal{Y}} u_1(z,y)\Big)
$$

$$
+d^0(z,y)\cdot u_0(z)\,dy
$$

$$
=\int_{\mathcal{Y}} f^0(z,y)\,dy,\ (z,y)\in\Omega
$$

$$
u_0(z)=0,\ z\in\partial\Omega
$$

$$
u_1(z,\cdot)\ is\ \mathcal{Y}\text{-}periodic \tag{$\mathbf{TSHP}_{\boldsymbol{\delta}}$}
$$

*in combination with*

$$
u_1(z,y)=\sum_{s=1}^{d}\chi_s(y)\cdot\frac{\partial}{\partial z_s}u_0(z)-\eta(y)\cdot u_0(z)
$$

*is equivalent to the homogenized problem*

$$
-\sum_{s,k=1}^{d}\frac{\partial}{\partial z_s}\Big(a_{sk}^{hom}(z)\cdot\frac{\partial}{\partial z_k}u_0(z)\Big)+\sum_{k=1}^{d}\frac{\partial}{\partial z_k}\Big(b_k^{hom}(z)\cdot u_0(z)\Big)
$$
$$
+\sum_{k=1}^{d}c_k^{hom}(z)\cdot\frac{\partial}{\partial z_k}u_0(z)+d^{hom}(z)\cdot u_0(z)=f^{hom}(z),\ z\in\Omega. \tag{$\mathbf{HP}_{\boldsymbol{\delta}}$}
$$

*in combination with the reference cell problems*

$$
\begin{aligned}
&Find\ \chi_s\in W_{per}^{1,0}(\mathcal{Y}),\ such\ that\\
&\qquad A_{\mathcal{Y}}\chi_s(y)=\nabla_\tau^{\mathcal{Y}}\Big(\delta a^0(z,y)\cdot\big(\tau^{\mathcal{Y}}(y)\big)_s\Big),\ y\in\mathcal{Y}
\end{aligned} \tag{$\mathbf{RCP1}_{\boldsymbol{s}}$}
$$

*and*

$$\boxed{\begin{array}{l} \textit{Find } \eta \in W^{1,0}_{per}(\mathcal{Y}), \textit{ such that} \\[2mm] A_{\mathcal{Y}}\eta(y) = \nabla^{\mathcal{Y}}_{\tau}\Big(b^0(z,y)\Big), \ y \in \mathcal{Y}. \end{array}}$$ **(RCP2)**

8. **Conclusion and Outlook.** The homogenization of physical systems on very large periodic networks is a difficult task. Averaging techniques in combination with two-scale asymptotic expansions can be applied to derive a macroscopic model on the superior domain of the physical device. The coefficients of the macroscopic model provide a characterization of the effective behavior of the system on a global scale. In addition, the homogenized coefficients reflect the topology of the inherent microstructure. Despite the fact that the quality of the homogenized solution can be guaranteed by appropriate error results, the averaging approach is considered as a purely formal procedure. The method of two-scale asymptotic analysis does not rely on a particular notion of convergence. Extension operators, that transform a function defined on a subdomain into a function on the superior domain are not known for network functions. Instead, a notion of two-scale convergence based on a two-scale transform originally introduced by Lenczner et al. can be applied. A function on a varying network is transformed into a function on the fixed domain defined by the set product of the superior domain and the reference graph. In this way, both the global scale and the microscale are combined. In our previous studies, we have considered microscopic models in variational form. With the help of the notion of two-scale convergence for network functions a homogenized model in variational form has been identified. In particular, the sequence of solutions of the variational microscopic model as well as the corresponding sequence of tangential gradients are converging to a two-scale limit function. These limit functions directly depend on the solution of the variational macroscopic model. In this paper, we have further extended this approach. It is proved, that the homogenized macroscopic model can be expressed equivalently by a homogenized model defined on the superior domain and a certain number of reference cell problems. The reference cell problems integrate the geometry of the microscopic structure.

These new findings are bridging the gap between the homogenization result for homogenization problems in variational form on periodic networks and the results obtained with purely formal averaging technique. In this way, a mathematical rigorous justification of the homogenized models obtained with two-scale asymptotic expansions is achieved. This research opens promising avenues for future studies. For example, the influence of the microgeometry and the topology of the underlying network structures can be investigated. In particular, our findings can be an essential part of topology optimization studies [19]. Here, specific network topologies have to be identified such that certain global characteristics and a particular effective behaviour become optimized. Furthermore, future work should also address numerical solution schemes of topology optimization problems on networks. In particular, recent advances in model order reduction in homogenization theory can be applied. In [1], a finite element heterogeneous multi-scale method and a reduced basis technique for nonlinear problems are presented. This approach relies on the solutions of appropriate cell problems on sampling domains. These are selected greedy algorithm in an offline stage and assembled in a reduced basis. In an online

stage, the reduced basis is applied to solve two-scale problems. In addition, the homogenization of optimal control problems on periodic networks can be addressed. A very new field of research addresses homogenization strategies for stochastic systems on periodic networks. We intend to include the notion of almost periodicity in the sense of Bohr and Bochner into our studies [4, 5, 6]. Stochastic approaches of this type are important for applications in signal theory and for systems with microscopically periodic signals.

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