

A TWO-ECHELON INVENTORY MODEL WITH STOCK-DEPENDENT DEMAND AND VARIABLE HOLDING COST FOR DETERIORATING ITEMS

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ABSTRACT. In this study, we develop an inventory model for deteriorating items with stock dependent demand rate. Shortages are allowed to this model and when stock on hand is zero, then the retailer offers a price discount to customers who are willing to back-order their demands. Here, the supplier as well as the retailer adopt the trade credit policy for their customers in order to promote the market competition. The retailer can earn revenue and interest after the customer pays for the amount of purchasing cost to the retailer until the end of the trade credit period offered by the supplier. Besides this, we consider variable holding cost due to increase the stock of deteriorating items. Thereafter, we present an easy analytical closed-form solution to find the optimal order quantity so that the total cost per unit time is minimized. The results are discussed with the help of numerical examples to validate the proposed model. A sensitivity analysis of the optimal solutions for the parameters is also provided in order to stabilize our model. The paper ends with a conclusion and an outlook to possible future studies.

1. Introduction. In the classical *Economic Order Quantity (EOQ)* model, it is assumed that the retailer must pay for items upon receiving them. But in practice, suppliers allow a certain fixed period to settle the account for stimulating retailer's demand. During this credit period, the retailer can start to accumulate revenues on the sales and earn interest on that revenue through investing in share market or banking business, but beyond this period the supplier charges a higher interest if the payment is not settled at the end of the offered credit period. The same policy is applicable for the retailer to the customer. This is termed as two echelon trade credit financing. Hence, paying later indirectly reduces the cost of holding stock.

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On the other hand, trade credit is offered by the supplier which encourages the retailer to buy more and it is also a powerful promotional tool that attracts new customers, who consider it as an alternative incentive policy to quantity discounts. Hence, the trade credit can play a major role in inventory control for both the supplier as well as retailer.

In traditional inventory models, we assume that items preserve their physical characteristics while they are stored in inventory. But deterioration has a crucial impact in inventory control system. The term *deterioration* is defining a situation in which all items in the inventory system become obsolete at the end or during the prescribed planning horizon:

- Deterioration is defined as falling from a higher to a lower level in quality, also simply implies as change, decay, obsolescence, collapse, pinch, spoilage, vaporization, loss of utility or loss of marginal value of goods that results in decreasing of the usefulness of the original item.
- The rate of deterioration is nearly negligible for items like hardware, glassware, toys and steel; or, it is very much effective for products such as fruits, vegetables, medicines, volatile liquids, blood banks, high-tech products, etc.
- The deteriorated items have many physical features that most of the physical goods undergo through decay or deterioration over time, products such as fruits, vegetables, food items, metal parts, etc.
- For highly deteriorated items like gasoline, alcohol, turpentine, etc., the rates of consumption are very high and they deteriorate continuously through mortification.
- The deteriorated items like radioactive substances, electronic goods, grain, photographic film, etc., deplete through physical depletion over time through the process of evaporation and also deteriorate through a gradual loss of potential or utility with time to time.

Therefore, deterioration of items plays a vital role in the determination of an inventory model and must be taken into consideration.

In classical inventory models, we assume that the demand rate of an item be either time-dependent or constant. But in actual practice, it has been observed that the demand rate is usually influenced by the stock level. As an example, a large stock level in shelves can attract more customers with the idea of freshness, variety and popularity and, conversely, a low stock level can arise a question of freshness. So, a large pile of stocks displayed in a supermarket can pay more attention to the customer to buy more.

In recent days, changeable *holding cost* is attracting many researchers because maintaining inventory is very vital. In general, it is assumed that holding cost is known and is considered as constant. But when inventory is stored for future usage, then it is rather essential to maintain the physical status of the inventory at present situation. So, to be up-to date with the present market situation, variable holding cost is necessarily very important.

In recent years, many researchers have considered their observations and awareness regarding *price discount on backorders*, because they thought that price discount on backorders during a stock-out period makes customers more willing to wait for their desired items. When shortage occurs, some customers may prefer to accept backorders as, for example, for fashionable goods such as shoes, cosmetics and clothes. Under these situations, customers may prefer their demand to be back ordered. Generally, a supplier could offer a backorder price discount on the stockout

item to secure more backorders. In other words, the bigger the price discount on backorder is, the bigger the advantage to the customers.

The term *shortage* means a situation in which something needed which cannot be carried in sufficient amounts. In real-life situations, happening of shortage in inventory system is a vital situation. Shortages are of great importance for many models, especially, for any model where delay in payment is considered and due shortages. So, shortages can enjoy a profit obtained from delay in payment. Shortages of items may occur due to the withdrawal of imperfect items from the stock. In inventory models with shortages, the general condition is that the unfulfilled demand is either completely lost or completely backlogged. However, there is a little chance that while few customers leave, others are willing to wait until the fulfillment of their demand. As an example, suppose that a customer prefers a particular company for shopping and, in the mean time, the customer observes that there is a shortage for a particular product. But the customer is not willing to buy from another company. In this situation, the customer has to wait until the supply of the particular product will be given.

So, each term described above plays an important role in the design of our paper; the main motivations of our work are as follows:

- The selling items are perishable over time; such as gasoline, fruits, fresh fishes, photographic films, vegetables, etc., and they follow a distribution that allows an edge application scope.
- The parameters of holding cost are assumed as a linearly increasing function of time, because with changes in time, value of money and holding cost cannot remain constant.
- The rate of replenishment is finite.
- Stock-dependent demand approach is considered here, because a large number of items in shelves attract more customers, and this interesting fact reflects more real-life situations.
- Shortages are occurring in this paper to fit the model in more realistic senses.
- Price discount on backorders are offered because this is a new strategy to attract more customers.
- Here, the supplier as well as the retailer would offer a fixed-credit period which will encourage the supplier's selling and the retailer can take an advantage to reduce the cost and to increase the profit.

The residue is organized as follows: In Section 2, a motivation and a review on the research subject are presented. Section 3 consists of two subsections:

- (i) In Subsection 3.1, notations are presented.
- (ii) In Subsection 3.2, the assumptions are given.

Section 4 discusses the mathematical model of the proposed inventory problem. The solution procedure is derived in Section 5, which also includes two separate subsections:

- (i) Subsection 5.1 shows the optimal solution for different functions.
- (ii) Subsection 5.2 demonstrates how to find a decision criterion for the optimal replenishment cycle time.

Section 6 contains three numerical examples to illustrate the proposed problem and our methodology. Section 7 presents a sensitivity analysis related with our suggested approach. Section 8 presents concluding remarks related to our article and it proposes new pathways of future investigation.

2. Motivation and Review on Research. In practical situations, deterioration of items is a common phenomenon. So, deteriorating inventory models are improved constantly to obtain more real characteristics of an inventory system. During the replenishment period, the inventory decreases continuously due to the combined effect of demand and deterioration. Thus, while determining the EOQ model, loss may occur because of the deterioration which cannot be ignored. Ghare and Schrader [8] considered a no shortage inventory model with constant deterioration rate, while Harris [14] was the first to develop an EOQ model which was generalized by Wilson [34] and gave a formula to obtain an economic order quantity. Liao [17] developed an inventory control model with instantaneous receipt and exponential deteriorating item under two levels of trade-credit policy. Covert and Philip [6] extended the Levin model [15] and presented an EOQ model for variable deterioration rate by considering a two-parametric Weibull distribution deterioration. Whitin [33] considered deterioration for fashion goods in his model, but only after the end of a predetermined shortage period. Covert and Philip [6] extended Ghare-Schrader [8] model by assuming a two-parametric Weibull distribution and obtained an EOQ model for a variable rate of deterioration. Tripathy and Pandey [30] introduced an inventory model for deteriorating items with Weibull distribution deterioration and time-dependent demand under trade credit policy. Furthermore, Aggarwal and Jaggi [1] presented an inventory model under the condition of a permissible delay in payments, and the deterioration rate was constant. Liao et al. [16] proposed a deteriorated inventory model with permissible delay in payments. Pervin et al. [26] analyzed an inventory model with time-dependent demand and variable holding cost including stochastic deterioration.

In a classical inventory model, we assume that the demand rate is either constant or time-dependent. But in real-life situations, we see that the demand rate may go up or down with respect to the stock level. Several researchers are engaged to propose to the community works on stock-dependent demand rate. Levin et al. [15] observed that “large piles of consumer goods displayed in a supermarket will lead the customer to buy more”. Balkhi and Benkherouf [3] developed an inventory model for deteriorating items with stock-dependent and time-varying demand rates over a finite planning horizon. Datta and Paul [7] analyzed a multi-period EOQ model with stock-dependent and price-sensitive demand rate. Min et al. [20] developed an inventory model for deteriorating items under stock-dependent demand and two-level trade credit policy. Pal and Chandra [23] derived an inventory model with stock-dependent demand and permissible delay in payments.

In traditional inventory models, it is assumed that holding cost is pre-determined and constant with time. But holding cost may not always be constant. Goh [12] first considered a stock-dependent demand model with variable holding cost and assumed the unit holding cost as a nonlinear continuous function of time. In few inventory models like Giri et al. [11], Mishra et al. [21], the holding cost as well as the demand function were considered as time-dependent. Roy [27] developed a deteriorating inventory model where also the time-varying holding cost was introduced but the demand was price dependent. Swami et al. [29] formulated an inventory model with stock-dependent demand and variable holding cost.

In the real market, customer’s willingness is affected by many factors and this situation leads them to wait for backorders during the stock-out period. It happens severely for some fashionable goods such as branded bags, shoes, hi-fi equipment and clothes or for some products from a well-named company, and the customers

may prefer to wait for the next delivery. Besides, there is another cause to wait for backorder which is a price discount from the retailer to the customers; many authors like Pan and Hsiao [24], Chuang et al. [5], and Ouyang et al. [22] formulated their models on the basis of this fact.

In practice, suppliers usually offer their retailers a delay period in payment. During the period, there is no interest charged. Hence, retailers can earn the interest from sales revenue, meanwhile suppliers lose the interest earned during the same time. However, if the payment is not paid in full at the end of the permissible delay period, then suppliers charge retailers an interest on the outstanding amount. The permissible delay in payment produces two benefits to the supplier:

- It attracts new customers who consider it to be a type of price reduction.
- It may be applied as an alternative to price discount because it does not provoke competitors to reduce their prices and, thus, introduce lasting price reductions.

On the other hand, the policy of granting credit terms adds not only an additional cost, but also an additional dimension of default risk to the supplier.

In this regard, a number of research papers appeared which dealt with the *EOQ* problem under the condition of permissible delay in payments. Goyal [13] introduced first the *EOQ* inventory model under the condition of trade credit. Chand and Ward [4] extended Goyal's [13] model under assumptions of the classical economic-order quantity model, calculating different results. Shinn et al. [28] discussed Goyal's [13] model and considered quantity discount for freight cost. Liao et al. [17] analyzed an *EOQ* model for stock-dependent demand rate when a delay in payment is permissible. Mahata [18] designed an EPQ model for deteriorating items under trade credit policy. Pervin et al. [25] derived an inventory model for deteriorating items in a demand-declining market under trade credit policy. Recently, Ghoreishi et al. [9] considered an optimal pricing and ordering policy under inflation and customer returns for non-instantaneous deteriorating items. Ghoreishi et al. [10] also derived a work on delay in payments, inflation and selling price-dependent demand.

The occurrence of shortages in an inventory system is a natural phenomenon. We can easily notice that many products of famous brands or fashionable goods such as certain brand gum shoes, clothes, hi-fi equipment and jewelry may create a certain situation in which customers think that it will be better to wait until the receive of backorders at the time period when shortages occur. So, shortage cost or lost-sale cost should not be prohibited to smooth the feasibility condition. If holding cost for inventory is significantly higher than the shortage cost, then allowing shortage will be an excellent idea for business practice to reduce the total cost. To present the situation practically, Annadurai and Uthayakumar [2] considered a two-level trade credit policy in their decaying inventory model, where the demand rate is stock-dependent and shortages are allowed. Manna and Chaudhuri [19] developed an order-level inventory model with unit production cost, shortages and time-dependent demand for deteriorating items. Tripathi [31] developed an inventory model with stock-dependent demand and shortages under trade credit policy.

The research works of various authors related to this area are shown in Table 1.

Table 1: Research works of various authors related to this area.

Author(s)	Shortages	Trade credit policy	Stock dependent demand	Price discount on backorders	Deteriorations	Time varying costs
Ghare and Scharder (1963)					✓	
Giri et al. (1996)			✓	✓		
Manna and Chaudhuri (2001)	✓				✓	✓
Roy (2008)					✓	✓
Min et al. (2010)	✓		✓			
Mishra et al. (2013)	✓		✓			
Tripathi and Pandey (2013)			✓	✓		
Tripathi (2015)			✓	✓		
Annadurai and Uthayakumar (2015)	✓		✓			
Pervin et al. (2015)		✓		✓	✓	
Swami et al. (2015)			✓		✓	✓
Our paper	✓	✓	✓	✓	✓	✓

3. Notations and Assumptions. Based on the following notations and assumptions, we design our model.

3.1. Notations.

- T : length of cycle time;
- c : unit purchasing cost per item;
- A : ordering cost per order;
- δ : the fraction of the demand during the stock-out period that will be back ordered, where $0 \leq \delta \leq 1$;
- s : lost sale cost per unit;
- $I(t)$: the inventory level at time t ;
- I_0 : the maximum inventory level during $[0, T]$;
- $I_1(t)$: the inventory level that changes with time t during production period;
- $I_2(t)$: the inventory level that changes with time t during non-production period;
- $I_3(t)$: the inventory level that changes with time t during shortage period;
- I_e : interest, which can be earned per unit of time (i.e., per \$ per year) by the retailer;
- I_c : interest charges in stocks per unit of time (i.e., per \$ in stocks per year) by the supplier;
- θ : constant deterioration rate, where $0 \leq \theta < 1$;
- M : credit period in years offered by the supplier;
- N : trade credit period in years offered by the retailer;
- $\Pi(T)$: total relevant cost function per unit of time;
- $D(t)$: the demand rate is defined as follows:

$$D(t) = \begin{cases} \alpha + \beta I(t), & \text{if } I(t) > 0, \\ \alpha, & \text{if } I(t) \leq 0, \end{cases}$$

where α and β are positive constants, $\alpha > \beta$, $0 \leq t \leq T$;

- c_2 : shortage cost per unit per unit time, i.e., shortages are allowed to occur;
- $h(t)$: the holding cost per item per time-unit is time dependent, and it is assumed as $h(t) = a + bt$, where $a > 0$, $0 < b < 1$;
- Π_0 : marginal profit per unit;
- Π_1 : price discount on unit backordered offered;
- b_0 : upper bound on backorder ratio, $0 \leq b_0 \leq 1$.

3.2. Assumptions.

1. Annual demand, $D(t)$, is stock dependent.
2. The rate of replenishment is finite with rate k .
3. $I_c \geq I_e$.
4. The lead time is negligible.
5. Shortages are allowed and demand is partially backlogged during stock out period.
6. If $T \geq M$, then the retailer settles the account at time M and pays for the interest charges on items in stock with rate I_c over the interval $[M, T]$. If $T < M$, then the retailer settles the account at time M , and there is no interest charge in stock during the whole cycle. On the other hand, if $M > N$, then the retailer can accumulate revenue and earn interest during the period from N to M with rate I_e under the trade credit conditions.
7. There is no repair or replacement of deteriorated units during the planning horizon. The item will be withdrawn from the stock immediately as soon as it becomes deteriorated.
8. During the stock-out period, the fraction of the demand δ is directly proportional to the price discount Π_1 offered by the retailer. Thus, $\delta = \frac{b_0}{\Pi_0} \Pi_1$, where $0 \leq \Pi_1 \leq \Pi_0$.

4. Mathematical Formulation. Based on our prerequisite assumptions, the inventory system may be considered as detailed below: Initially, (i.e., at time $t = 0$), the cycle starts with a zero stock level at supply rate k . The replenishment or supply continues up to time t_1 . During the time period $[0, t_1]$, inventory piles up by adjusting the demand in the market. This accumulated inventory level at time t_1 gradually diminishes due to demand and deterioration during the period $[t_1, t_2]$ and ultimately falls to zero at time $t = t_2$. During the period $[t_2, T]$, shortages occur because of those reasons of market demand of items and ultimately falls to its lowest level at time $t = T$. After the scheduling period T , the cycle repeats itself.

Now, the differential equations involving the instantaneous state of the inventory level in the interval $[0, T]$, together with their initial values, are given by

$$\frac{dI_1(t)}{dt} + \theta(t)I_1(t) = k - D(t) = k - [\alpha + \beta I_1(t)] \quad (t \in [0, t_1]). \quad (1)$$

with $I_1(0) = 0$.

$$\frac{dI_2(t)}{dt} + \theta(t)I_2(t) = -D(t) = -[\alpha + \beta I_2(t)] \quad (t \in [t_1, t_2]). \quad (2)$$

with $I_2(t_2) = 0$.

$$\frac{dI_3(t)}{dt} = -\alpha\delta \quad (t \in [t_2, T]). \quad (3)$$

with $I_3(t_2) = 0$.

Now the solution of equation 1 using the initial condition becomes

$$I_1(t) = \frac{k - \alpha}{\beta + \theta} \left[e^{(\beta + \theta)(t_1 - t)} - 1 \right] \quad (t \in [0, t_1]).$$

Utilizing the new initial condition, the solution of equation 2 becomes

$$I_2(t) = \frac{\alpha}{\beta + \theta} \left[e^{(\beta + \theta)(t_2 - t)} - 1 \right] \quad (t \in [t_1, t_2]).$$

Solving equation 3 with the help of its just obtained initial condition, we get

$$I_3(t) = -\alpha\delta(T - t_2) \quad (t \in [t_2, T]).$$

So, the maximum inventory level is

$$I_0 = \frac{k - \alpha}{\theta + \beta} \left[e^{(\beta+\theta)t_1} - 1 \right]. \quad (4)$$

Here, Figure 1 represents the total inventory system of our proposed problem. The

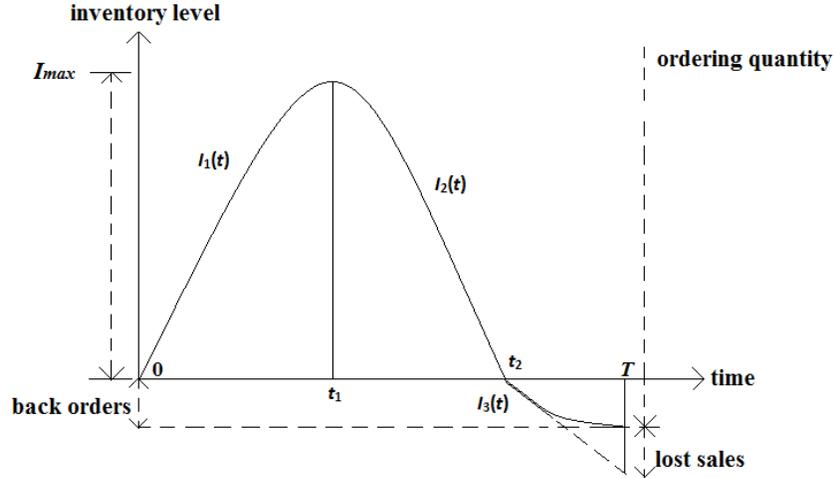


FIGURE 1. Graphical representation of our proposed Inventory control model

elements comprising the retailer's profit function per cycle are listed below:

1. The annual *ordering cost* OC is $= A$;
2. The total *holding cost*, HC , is computed for time interval $[0, t_1]$ and $[t_1, t_2]$, because only during this time period inventory is available in the system. So, the *annual stock holding cost* is represented as follows:

$$\begin{aligned} HC &= \int_0^{t_1} h(t)I_1(t)dt + \int_{t_1}^{t_2} h(t)I_2(t)dt \\ &= \int_0^{t_1} (a + bt) \left[\frac{k - \alpha}{\beta + \theta} \{ e^{(\beta+\theta)(t_1-t)} - 1 \} \right] dt \\ &+ \int_{t_1}^{t_2} (a + bt) \left[\frac{\alpha}{\beta + \theta} \{ e^{(\beta+\theta)(t_2-t)} - 1 \} \right] dt \\ &= \frac{a(k - \alpha)}{(\beta + \theta)^2} \left\{ e^{(\beta+\theta)t_1} - (\beta + \theta)t_1 - 1 \right\} + \frac{b(k - \alpha)}{(\beta + \theta)^3} \left\{ e^{(\beta+\theta)t_1} - 1 \right\} - \frac{b\alpha t_1}{(\beta + \theta)^2} \\ &+ \frac{a\alpha}{(\beta + \theta)^2} \left\{ e^{(\beta+\theta)(t_2-t_1)} - (\beta + \theta)(t_2 - t_1) - 1 \right\} \\ &+ \frac{b\alpha}{(\beta + \theta)^3} \left\{ e^{(\beta+\theta)(t_2-t_1)} - 1 \right\} - \frac{b\alpha(t_2 - t_1)}{(\beta + \theta)^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{a}{(\beta + \theta)^2} [(k - \alpha)\{e^{(\beta + \theta)t_1} - (\beta + \theta)t_1\} + \alpha\{e^{(\beta + \theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1)\} \\
&- k] + \frac{b}{(\beta + \theta)^3} [(k - \alpha)e^{(\beta + \theta)t_1} + \alpha e^{(\beta + \theta)(t_2 - t_1)} - k] - \frac{b\alpha t_2}{(\beta + \theta)^2}. \quad (5)
\end{aligned}$$

3. Shortage cost is accumulated during the time interval $[t_2, T]$; so, the *shortage cost*, SC , is expressed as:

$$\begin{aligned}
SC &= c_2 \int_{t_2}^T I_3(t) dt \\
&= c_2 \int_{t_2}^T [-\alpha\delta(T - t_2)] dt \\
&= \frac{c_2\alpha\delta}{2}(T - t_2)^2.
\end{aligned}$$

4. The *deteriorating cost*, DC , is represented by:

$$\begin{aligned}
DC &= c\theta \left[\int_0^{t_1} I_1(t) dt + \int_{t_1}^{t_2} I_2(t) dt \right] \\
&= c\theta \left[\int_0^{t_1} \left\{ \frac{k - \alpha}{\beta + \theta} [e^{(\beta + \theta)(t_1 - t)} - 1] \right\} dt + \int_{t_1}^{t_2} \left\{ \frac{\alpha}{\beta + \theta} [e^{(\beta + \theta)(t_2 - t)} - 1] \right\} dt \right] \\
&= \frac{c(k - \alpha)\theta}{(\theta + \beta)^2} [e^{(\beta + \theta)t_1} - (\beta + \theta)t_1 - 1] \\
&+ \frac{c\alpha\theta}{(\theta + \beta)^2} [e^{(\beta + \theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1) - 1] \\
&= \frac{c\theta}{(\theta + \beta)^2} [(k - \alpha)e^{(\beta + \theta)t_1} + \alpha e^{(\beta + \theta)(t_2 - t_1)} \\
&- (k - \alpha)(\beta + \theta)t_1 - \alpha(\beta + \theta)(t_2 - t_1) - k].
\end{aligned}$$

5. Due to shortage during time interval $[t_2, T]$, all the customers are not interested to wait for the coming lot size to arrive, which may occur loss in profit. Hence, the *lost sale cost*, LSC , is:

$$\begin{aligned}
LSC &= s \int_{t_2}^T (1 - \delta)D(t) dt \\
&= s \int_{t_2}^T (1 - \delta)\alpha dt \\
&= s\alpha(1 - \delta)(T - t_2).
\end{aligned}$$

6. *Purchase cost*, PC , during interval $[t_2, T]$ becomes:

$$\begin{aligned}
PC &= c \left(I_0 + \int_{t_2}^T \delta D(t) dt \right) \\
&= c \left[\frac{k - \alpha}{\theta + \beta} \left\{ e^{(\beta + \theta)t_1} - 1 \right\} + \alpha\delta(T - t_2) \right].
\end{aligned}$$

7. The *Backorder cost*, BC , is expressed as follows:

$$\begin{aligned} BC &= \delta \int_{t_2}^T I_3(t) dt \\ &= \delta \int_{t_2}^T [-\alpha\delta(T - t_2)] dt \\ &= \frac{\alpha\delta^2}{2}(T - t_2)^2. \end{aligned}$$

8. The *interest earned*, IE : From our considered assumptions and based on the values of N and M , there are three alternative cases arising in relation to interest earned. They are as follows:

Case 8.a: $T + N \geq M$, *Case 8.b:* $T + N < M$ and *Case 8.c:* $N \geq M$.

Case 8.a: $T + N \geq M$

In this case, the retailer accumulates a sales revenue on an account that earns sI_e units per year, starting from N through M . Hence, the *interest earned*, IE , per cycle is stated as below:

$$\begin{aligned} IE &= sI_e \int_N^M D(t)(M - t) dt \\ &= sI_e \int_N^M \{\alpha + \beta I_1(t)\}(M - t) dt \\ &= sI_e \left[\alpha + \frac{(k - \alpha)\theta\beta t_1}{\theta + \beta} + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right] (M - N)^2. \end{aligned}$$

Case 8.b: $T + N < M$

In this case, since the customer's last payment time $T + N$ is shorter than the supplier credit period M , hence, the *interest earned*, IE , per cycle is indicated as:

$$\begin{aligned} IE &= sI_e \left[\int_N^{T+N} \{D(t)(T + N - t) + D(t)T\{M - (T + N)\}\} dt \right] \\ &= sI_e \left[\int_N^{T+N} \{\alpha + \beta I_1(t)\}(T + N - t) dt \right. \\ &\quad \left. + \int_N^{T+N} \{\alpha + \beta I_1(t)\}(M - T - N) dt \right] \\ &= sI_e \left[\alpha + \frac{\theta(k - \alpha)\beta t_1}{\theta + \beta} + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right] (M - N - \frac{T}{2}). \end{aligned}$$

Case 8.c: $N \geq M$

In this case, the retailer trade credit period N is equal to or larger than the supplier credit period M . Consequently, there is no interest earned by the retailer. Therefore, the annual interest earned from time N to time M per cycle is equal to zero ($= 0$).

9. *Interest Charged*, IC : From our regarded assumptions and based on the values of N and M , there are three alternative cases arising in relation to interest charged. They are as follows:

Case 9.a: $T + N \geq M$, *Case 9.b:* $T + N < M$ and *Case 9.c:* $N \geq M$.

Case 9.a: $T + N \geq M$

In this case, the supplier's credit period M is shorter than the customer last payment time $T + N$. Hence, the retailer cannot pay off the purchase amount at time M , and he must finance all items sold after time $M - N$ at an interest charged I_c per dollar per year. Therefore, the *interest charged*, IC per cycle is stated as below:

$$\begin{aligned} IC &= cI_c \left[\int_M^{T+N} I_1(t) dt \right] \\ &= cI_c \int_M^{T+N} \left[\frac{k - \alpha}{\beta + \theta} \left\{ e^{(\beta + \theta)(t_1 - t)} - 1 \right\} \right] \\ &= cI_c \left[\frac{(k - \alpha)\theta t_1}{\theta + \beta} + \frac{(k - \alpha)\beta}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right] (T + N - M). \end{aligned}$$

Case 9.b: $T + N < M$

In this case, since the customer's last payment time $T + N$ is shorter than the supplier credit period M , the retailer faces no interest charged. So, the total interest accrued for this case is equal to zero ($= 0$).

Case 9.c: $N \geq M$

In this case, the retailer's trade credit period N is equal to or larger than the supplier credit period M . Therefore, the annual interest charged from time N to time M per cycle is stated as:

$$\begin{aligned} IC &= cI_c \left[(N - M)kt_1 + \int_N^{T+N} I_1(t) dt \right] \\ &= cI_c \left[(N - M)kt_1 + \int_N^{T+N} \left\{ \frac{k - \alpha}{\beta + \theta} \left\{ e^{(\beta + \theta)(t_1 - t)} - 1 \right\} \right\} \right] \\ &= cI_c \left[(N - M)kt_1 + \frac{\theta(k - \alpha)Tt_1}{\theta + \beta} + \frac{(k - \alpha)\beta T}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right]. \end{aligned}$$

From the above results, the annual total relevant cost per unit of time for the retailer can be expressed as:

$$\Pi(T) = [OC + HC + PC + DC + SC + BC + IC - IE - LSC]. \quad (6)$$

Therefore, by inserting the values of the above parameters, the following case-wise representations are achieved:

$$\begin{aligned} \Pi_1(T) &= \frac{A}{T} \quad (7) \\ &+ \frac{1}{T} \left[\frac{a}{(\beta + \theta)^2} [(k - \alpha)\{e^{(\beta + \theta)t_1} - (\beta + \theta)t_1\} + \alpha\{e^{(\beta + \theta)(t_2 - t_1)} \right. \\ &- (\beta + \theta)(t_2 - t_1)\} - k] + \frac{b}{(\beta + \theta)^3} [(k - \alpha)e^{(\beta + \theta)t_1} + \alpha e^{(\beta + \theta)(t_2 - t_1)} - k] \\ &- \frac{b\alpha t_2}{(\beta + \theta)^2} \left. \right] + \frac{c\theta}{T(\theta + \beta)^2} [(k - \alpha)e^{(\beta + \theta)t_1} + \alpha e^{(\beta + \theta)(t_2 - t_1)} - (k - \alpha)(\beta + \theta)t_1 \\ &- \alpha(\beta + \theta)(t_2 - t_1) - k] + \frac{\alpha\delta}{2T}(T - t_2)^2(c_2 + \delta) + \frac{c}{T} \left[\frac{k - \alpha}{\theta + \beta} \left\{ e^{(\beta + \theta)t_1} - 1 \right\} \right] \end{aligned}$$

$$\begin{aligned}
& +\alpha\delta(T-t_2)] + \frac{cI_c}{T} \left[\frac{(k-\alpha)\theta t_1}{\theta+\beta} + \frac{(k-\alpha)\beta}{(\theta+\beta)^2} (e^{(\theta+\beta)t_1} - 1) \right] (T+N-M) \\
& - \frac{sI_e}{T} \left[\alpha + \frac{\theta(k-\alpha)\beta t_1}{\theta+\beta} + \frac{(k-\alpha)\beta^2}{(\theta+\beta)^2} (e^{(\theta+\beta)t_1} - 1) \right] (M-N-\frac{T}{2}) \\
& - \frac{s\alpha}{T} (1-\delta)(T-t_2),
\end{aligned}$$

$\Pi_2(T)$

$$\begin{aligned}
& = \frac{A}{T} + \frac{1}{T} \left[\frac{a}{(\beta+\theta)^2} [(k-\alpha)\{e^{(\beta+\theta)t_1} - (\beta+\theta)t_1\} + \alpha\{e^{(\beta+\theta)(t_2-t_1)} \right. \\
& \left. - (\beta+\theta)(t_2-t_1)\} - k] + \frac{b}{(\beta+\theta)^3} [(k-\alpha)e^{(\beta+\theta)t_1} + \alpha e^{(\beta+\theta)(t_2-t_1)} - k] \right. \\
& \left. - \frac{bat_2}{(\beta+\theta)^2} \right] + \frac{c\theta}{T(\theta+\beta)^2} [(k-\alpha)e^{(\beta+\theta)t_1} + \alpha e^{(\beta+\theta)(t_2-t_1)} - (k-\alpha)(\beta+\theta)t_1 \quad (8) \\
& - \alpha(\beta+\theta)(t_2-t_1) - k] + \frac{\alpha\delta}{2T} (T-t_2)^2 (c_2 + \delta) + \frac{c}{T} \left[\frac{k-\alpha}{\theta+\beta} \{e^{(\beta+\theta)t_1} - 1\} \right. \\
& \left. + \alpha\delta(T-t_2) \right] - \frac{sI_e}{T} \left[\alpha + \frac{\theta(k-\alpha)\beta t_1}{\theta+\beta} + \frac{(k-\alpha)\beta^2}{(\theta+\beta)^2} (e^{(\theta+\beta)t_1} - 1) \right] \\
& (M-N-\frac{T}{2}) - \frac{s\alpha}{T} (1-\delta)(T-t_2),
\end{aligned}$$

$\Pi_3(T)$

$$\begin{aligned}
& = \frac{A}{T} + \frac{1}{T} \left[\frac{a}{(\beta+\theta)^2} [(k-\alpha)\{e^{(\beta+\theta)t_1} - (\beta+\theta)t_1\} + \alpha\{e^{(\beta+\theta)(t_2-t_1)} \right. \\
& \left. - (\beta+\theta)(t_2-t_1)\} - k] + \frac{b}{(\beta+\theta)^3} [(k-\alpha)e^{(\beta+\theta)t_1} + \alpha e^{(\beta+\theta)(t_2-t_1)} - k] \right. \\
& \left. - \frac{bat_2}{(\beta+\theta)^2} \right] + \frac{c\theta}{T(\theta+\beta)^2} [(k-\alpha)e^{(\beta+\theta)t_1} + \alpha e^{(\beta+\theta)(t_2-t_1)} \quad (9) \\
& - (k-\alpha)(\beta+\theta)t_1 - \alpha(\beta+\theta)(t_2-t_1) - k] + \frac{\alpha\delta}{2T} (T-t_2)^2 (c_2 + \delta) \\
& + \frac{c}{T} \left[\frac{k-\alpha}{\theta+\beta} \{e^{(\beta+\theta)t_1} - 1\} + \alpha\delta(T-t_2) \right] + \frac{cI_c}{T} [(N-M)kt_1 + \frac{\theta(k-\alpha)Tt_1}{\theta+\beta} \\
& + \frac{(k-\alpha)\beta T}{(\theta+\beta)^2} (e^{(\theta+\beta)t_1} - 1)] - \frac{s\alpha}{T} (1-\delta)(T-t_2).
\end{aligned}$$

5. Solution Procedure. For calculating the optimal solution of the proposed *EOQ* model, we consider the subsequent way.

Definition 5.1. A function h defined on an open interval (a, b) is said to be convex if for $x, y \in (a, b)$ and each $\lambda, 0 \leq \lambda \leq 1$, we have $h[\lambda x + (1-\lambda)y] \leq \lambda h(x) + (1-\lambda)h(y)$.

Theorem 5.2. Let l be a continuous function on the closed interval $[a, b]$ and let $l(a)l(b) < 0$. Then there exists a number $c \in (a, b)$ such that $l(c) = 0$.

Lemma 5.3. If $h(t)$ is a continuous function on (a, b) and if dh/dt is nondecreasing, then h is convex.

Proof. Given x, y with $a < x < y < b$, defining a function l on $[0, 1]$ by $l(t) = th(y) + (1-t)h(x) - h(ty + (1-t)x)$. We want to show that l is nonnegative on $[0, 1]$. Now, l is continuous and $l(0) = l(1) = 0$. Moreover,

$$dl(t)/dt = h(y) - h(x) - (y-x)dh/dt,$$

and $dl(t+m)/dt - dl(t)/dt = -(y-x)[dh(t+m)/dt - dh(t)/dt]$, for $t+m > t$.

Since $dh(t)/dt$ is nondecreasing, $dh(t+m)/dt - dh(t)/dt > 0$. It implies that $dl(t)/dt$ is nondecreasing on $[0, 1]$. Let c be a point where l assumes its minimum on $[0, 1]$. If $c = 1$, then $l(t) \geq l(1) = 0$ on $[0, 1]$. In this case l is nonnegative. Suppose that $c \in [0, 1]$. Since l has a local minimum at c , we have $dl(t)/dt \geq 0$ at $t = c$. But $dl(t)/dt$ was nondecreasing and so $dl(t)/dt \leq 0$ on $[0, c]$. Consequently g is nondecreasing on $[0, c]$ and hence $g(c) \leq l(0) = 0$ and, then, the minimum of l on $[0, 1]$ is nonnegative and so $l \geq 0$ on $[0, 1]$ \square

5.1. Optimal Solution for Different Functions. The first-order derivative of $\Pi_1(T)$ with respect to T is calculated as follows:

$$\frac{d\Pi_1(T)}{dT} = \frac{f_1(T)}{T^2}, \quad (10)$$

where

$$\begin{aligned} f_1(T) := & -A \\ & - \left[\frac{a}{(\beta + \theta)^2} [(k - \alpha) \{ T(\beta + \theta) \frac{dt_1}{dT} (e^{(\beta + \theta)t_1} - 1) - (e^{(\beta + \theta)t_1} - (\beta + \theta)t_1) \} \right. \\ & + \alpha \{ T(\beta + \theta) \left(\frac{dt_2}{dT} - \frac{dt_1}{dT} \right) (e^{(\beta + \theta)(t_2 - t_1)} - 1) - e^{(\beta + \theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1) \} \\ & + \frac{b}{(\beta + \theta)^3} [(k - \alpha) e^{(\beta + \theta)t_1} (T(\beta + \theta) \frac{dt_1}{dT} - 1) + \alpha e^{(\beta + \theta)(t_2 - t_1)} (T \left(\frac{dt_2}{dT} - \frac{dt_1}{dT} \right) \\ & - 1)] - \frac{b\alpha}{(\beta + \theta)^2} (T \frac{dt_2}{dT} - t_2)] - \frac{c\theta}{(\theta + \beta)^2} [(k - \alpha) e^{(\beta + \theta)t_1} \{ 1 - (\beta + \theta) T \frac{dt_1}{dT} \} \\ & + \alpha e^{(\beta + \theta)(t_2 - t_1)} \{ 1 - (\beta + \theta) T \left(\frac{dt_2}{dT} - \frac{dt_1}{dT} \right) \}] + (k - \alpha)(\beta + \theta) (T \frac{dt_1}{dT} - t_1) \\ & + \alpha(\beta + \theta) \{ T \left(\frac{dt_2}{dT} - \frac{dt_1}{dT} \right) - (t_2 - t_1) \} - \frac{\alpha\delta}{2} [(c_2 + \delta) \{ (T - t_2)^2 \\ & - T(T - t_2) \left(1 - \frac{dt_2}{dT} \right) \}] - c \left[\frac{k - \alpha}{\theta + \beta} \{ e^{(\beta + \theta)t_1} - 1 \} + \alpha \delta (T \frac{dt_2}{dT} - t_2) \right. \\ & - (k - \alpha) e^{(\beta + \theta)t_1} T \frac{dt_1}{dT}] - c I_c \left[\left\{ \frac{(k - \alpha)\theta}{\theta + \beta} (t_1 - T \frac{dt_1}{dT}) + \frac{(k - \alpha)\beta}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right. \right. \\ & - \left. \left. \frac{(k - \alpha)T\beta}{(\theta + \beta)} e^{(\theta + \beta)t_1} \frac{dt_1}{dT} \right\} (T + N - M) - \frac{(k - \alpha)\theta T t_1}{\theta + \beta} \right. \\ & - \left. \frac{(k - \alpha)\beta}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right] + s I_e \left[\left\{ \alpha \left(1 + \frac{T}{2} \right) + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right. \right. \\ & + \left. \left. \frac{\theta(k - \alpha)}{(\theta + \beta)} \left(t_1 - T \frac{dt_1}{dT} + \frac{T t_1}{2} \right) - \frac{(k - \alpha)T\beta^2}{(\theta + \beta)} e^{(\theta + \beta)t_1} \frac{dt_1}{dT} \right\} (M - N - \frac{T}{2}) \right. \\ & + \left. \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right] + s\alpha(1 - \delta) (T \frac{dt_2}{dT} - t_2). \end{aligned}$$

Therefore, $f_1(T)$ and $\Pi_1(T)$ both have the same sign and domain. The optimal values of T , say T_1^* , can be obtained by solving the following equation:

$$f_1(T) = 0. \quad (11)$$

We also have

$$\begin{aligned} \frac{df_1(T)}{dT} &= \frac{a}{(\beta + \theta)^2} [(k - \alpha) \{ T(\beta + \theta) \frac{d^2 t_1}{dT^2} (1 - e^{(\theta + \beta)t_1}) - T(\beta + \theta)^2 (\frac{dt_1}{dT})^2 e^{(\theta + \beta)t_1} \} \\ &\quad + \alpha \{ T(\beta + \theta)^2 e^{(\beta + \theta)(t_2 - t_1)} (\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 + T(\beta + \theta) (\frac{d^2 t_2}{dT^2} - \frac{d^2 t_1}{dT^2}) \\ &\quad (e^{(\beta + \theta)(t_2 - t_1)} - 1) \}] + \frac{b}{(\beta + \theta)^3} [(k - \alpha)(\beta + \theta)^2 e^{(\theta + \beta)t_1} (\frac{dt_1}{dT})^2 \\ &\quad + (k - \alpha)T(\beta + \theta)^2 e^{(\theta + \beta)t_1} \frac{d^2 t_1}{dT^2} + \alpha(\beta + \theta)e^{(\beta + \theta)(t_2 - t_1)} (\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 \\ &\quad + \alpha T e^{(\beta + \theta)(t_2 - t_1)} (\frac{d^2 t_2}{dT^2} - \frac{d^2 t_1}{dT^2})] \\ &\quad - \frac{bT\alpha}{(\beta + \theta)^2} \frac{d^2 t_2}{dT^2} - \frac{c\theta}{(\beta + \theta)^2} [(k - \alpha)(\beta + \theta)^2 \\ &\quad e^{(\theta + \beta)t_1} T \{ (\frac{dt_1}{dT})^2 - \frac{d^2 t_1}{dT^2} \} + \alpha T(\beta + \theta)^2 e^{(\beta + \theta)(t_2 - t_1)} (\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 \\ &\quad - \alpha(\beta + \theta) T e^{(\beta + \theta)(t_2 - t_1)} (\frac{d^2 t_2}{dT^2} - \frac{d^2 t_1}{dT^2}) + (k - \alpha) T(\beta + \theta) \frac{d^2 t_1}{dT^2} \\ &\quad - \alpha(\beta + \theta) T (\frac{d^2 t_2}{dT^2} - \frac{d^2 t_1}{dT^2})] + \alpha \delta [(c_2 + \delta) \{ T(1 - \frac{dt_2}{dT})^2 - T(T - t_2) \frac{d^2 t_2}{dT^2} \}] \\ &\quad + c [(k - \alpha) \{ (\beta + \theta) T (\frac{dt_1}{dT})^2 + \frac{d^2 t_1}{dT^2} \} - \alpha \delta (1 + \frac{d^2 t_2}{dT^2})] \\ &\quad + c I_c [\frac{\theta T (k - \alpha)}{(\theta + \beta)} \frac{d^2 t_1}{dT^2} + \frac{(k - \alpha) \beta T}{(\theta + \beta)} e^{(\beta + \theta)t_1} \{ (\frac{dt_1}{dT})^2 + \frac{d^2 t_1}{dT^2} \} \\ &\quad + \frac{(k - \alpha) \beta}{(\theta + \beta)^2} e^{(\beta + \theta)t_1} \{ (\frac{dt_1}{dT})^2 + \frac{d^2 t_1}{dT^2} \}] + s I_e [\frac{(k - \alpha) \beta^2}{(\theta + \beta)^2} e^{(\beta + \theta)t_1} \\ &\quad \{ (\frac{dt_1}{dT})^2 + \frac{d^2 t_1}{dT^2} \} - \frac{\alpha}{2} + \frac{\theta(k - \alpha) T}{(\theta + \beta)} (\frac{d^2 t_1}{dT^2} - \frac{1}{2} \frac{dt_1}{dT})] \\ &\quad + s \alpha (1 - \delta) \{ \frac{dt_2}{dT} + T \frac{d^2 t_2}{dT^2} \} > 0, \text{ if } T > 0. \end{aligned}$$

Hence, $f_1(T)$ is strictly monotonically increasing on $(0, \infty)$. From Lemma 5.3, $\Pi_1(T)$ is a convex function on $(0, \infty)$. Also we note

$$\lim_{T \rightarrow \infty} f_1(T) = \infty, \quad (12)$$

and

$$f_1(0) = - \left[A - \frac{2ka}{(\beta + \theta)^2} - \frac{bk}{(\beta + \theta)^3} - s I_e \alpha \right]. \quad (13)$$

Hence, we see that

$$\frac{d\Pi_1(T)}{dT} = \begin{cases} < 0, & \text{if } T \in (0, T_1^*), \\ = 0, & \text{if } T = T_1^*, \\ > 0, & \text{if } T \in (T_1^*, \infty), \end{cases}$$

provided that $f_1(0) < 0$. Based upon the above arguments, theorem 5.2 yields that the optimal solution, T_1^* , not only exists but also it is unique. A similar procedure described for $\Pi_1(T)$ can be applied to the next function, too.

The first-order derivative of $\Pi_2(T)$ with respect to T is defined as:

$$\frac{d\Pi_2(T)}{dT} = \frac{f_2(T)}{T^2}, \quad (14)$$

where

$$\begin{aligned} f_2(T) := & -A \\ & - \left[\frac{a}{(\beta + \theta)^2} [(k - \alpha)\{T(\beta + \theta)\frac{dt_1}{dT}(e^{(\beta+\theta)t_1} - 1) - (e^{(\beta+\theta)t_1} - (\beta + \theta)t_1)\} \right. \\ & + \alpha\{T(\beta + \theta)(\frac{dt_2}{dT} - \frac{dt_1}{dT})(e^{(\beta+\theta)(t_2-t_1)} - 1) - e^{(\beta+\theta)(t_2-t_1)} - (\beta + \theta)(t_2 - t_1)\} \\ & + \frac{b}{(\beta + \theta)^3} [(k - \alpha)e^{(\beta+\theta)t_1}(T(\beta + \theta)\frac{dt_1}{dT} - 1) + \alpha e^{(\beta+\theta)(t_2-t_1)}(T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - 1)] \\ & - \frac{b\alpha}{(\beta + \theta)^2} (T\frac{dt_2}{dT} - t_2)] - \frac{c\theta}{(\theta + \beta)^2} [(k - \alpha)e^{(\beta+\theta)t_1}\{1 - (\beta + \theta)T\frac{dt_1}{dT}\} \\ & + \alpha e^{(\beta+\theta)(t_2-t_1)}\{1 - (\beta + \theta)T(\frac{dt_2}{dT} - \frac{dt_1}{dT})\}] + (k - \alpha)(\beta + \theta)(T\frac{dt_1}{dT} - t_1) \\ & + \alpha(\beta + \theta)\{T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - (t_2 - t_1)\} - \frac{\alpha\delta}{2} [(c_2 + \delta)\{(T - t_2)^2 \\ & - T(T - t_2)(1 - \frac{dt_2}{dT})\}] - c[\frac{k - \alpha}{\theta + \beta}\{e^{(\beta+\theta)t_1} - 1\} + \alpha\delta(T\frac{dt_2}{dT} - t_2) \\ & - (k - \alpha)e^{(\beta+\theta)t_1}T\frac{dt_1}{dT}] + sI_e[\{\alpha(1 + \frac{T}{2}) + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1) \\ & + \frac{\theta(k - \alpha)}{(\theta + \beta)}(t_1 - T\frac{dt_1}{dT}) - \frac{(k - \alpha)\beta^2 T}{(\theta + \beta)}e^{(\theta+\beta)t_1}\frac{dt_1}{dT}\}(M - N - \frac{T}{2}) \\ & + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1)] + s\alpha(1 - \delta)(T\frac{dt_2}{dT} - t_2). \end{aligned}$$

Therefore, $f_2(T)$ and $\Pi_2(T)$ both have the same sign and domain. The optimal values of T , say T_2^* , can be obtained by solving the equation

$$f_2(T) = 0. \quad (15)$$

We also have

$$\begin{aligned} \frac{df_2(T)}{dT} = & \frac{a}{(\beta + \theta)^2} [(k - \alpha)\{T(\beta + \theta)\frac{d^2t_1}{dT^2}(1 - e^{(\theta+\beta)t_1}) - T(\beta + \theta)^2(\frac{dt_1}{dT})^2 e^{(\theta+\beta)t_1}\} \\ & + \alpha\{T(\beta + \theta)^2 e^{(\beta+\theta)(t_2-t_1)}(\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 + T(\beta + \theta)(\frac{d^2t_2}{dT^2} - \frac{d^2t_1}{dT^2}) \\ & (e^{(\beta+\theta)(t_2-t_1)} - 1)\}] + \frac{b}{(\beta + \theta)^3} [(k - \alpha)(\beta + \theta)^2 e^{(\theta+\beta)t_1}(\frac{dt_1}{dT})^2 \\ & + (k - \alpha)T(\beta + \theta)^2 e^{(\theta+\beta)t_1}\frac{d^2t_1}{dT^2} + \alpha(\beta + \theta)e^{(\beta+\theta)(t_2-t_1)}(\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 \\ & + \alpha T e^{(\beta+\theta)(t_2-t_1)}(\frac{d^2t_2}{dT^2} - \frac{d^2t_1}{dT^2})] - \frac{bT\alpha}{(\beta + \theta)^2} \frac{d^2t_2}{dT^2} - \frac{c\theta}{(\beta + \theta)^2} \end{aligned}$$

$$\begin{aligned}
& [(k - \alpha)(\beta + \theta)^2 e^{(\theta + \beta)t_1} T \{ (\frac{dt_1}{dT})^2 - \frac{d^2 t_1}{dT^2} \} + \alpha T (\beta + \theta)^2 e^{(\beta + \theta)(t_2 - t_1)} (\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 \\
& - \alpha(\beta + \theta) T e^{(\beta + \theta)(t_2 - t_1)} (\frac{d^2 t_2}{dT^2} - \frac{d^2 t_1}{dT^2}) + (k - \alpha) T (\beta + \theta) \frac{d^2 t_1}{dT^2} \\
& - \alpha(\beta + \theta) T (\frac{d^2 t_2}{dT^2} - \frac{d^2 t_1}{dT^2})] + \alpha \delta [(c_2 + \delta) \{ T (1 - \frac{dt_2}{dT})^2 - T(T - t_2) \frac{d^2 t_2}{dT^2} \}] \\
& + c [(k - \alpha) \{ (\beta + \theta) T (\frac{dt_1}{dT})^2 + \frac{d^2 t_1}{dT^2} \} - \alpha \delta (1 + \frac{d^2 t_2}{dT^2})] \\
& + s I_e [\frac{\alpha}{2} - \frac{\theta(k - \alpha)\beta T}{(\theta + \beta)} \frac{d^2 t_1}{dT^2} - (k - \alpha)\beta^2 T e^{(\theta + \beta)t_1} \{ (\frac{dt_1}{dT})^2 + \frac{d^2 t_1}{dT^2} \} \\
& + \frac{(k - \alpha)\beta^2}{(\theta + \beta)} e^{(\theta + \beta)t_1} \frac{dt_1}{dT}] + s \alpha (1 - \delta) \{ \frac{dt_2}{dT} + T \frac{d^2 t_2}{dT^2} \} > 0, \text{ if } T > 0.
\end{aligned}$$

Here, we observe that $f_2(T)$ is a increasing function on $(0, \infty)$. Hence, we can conclude from Lemma 5.3 that, $\Pi_2(T)$ is a convex function on $(0, \infty)$. Furthermore, we see that

$$\lim_{T \rightarrow \infty} f_2(T) = \infty, \quad (16)$$

and

$$f_2(0) = - \left[A - \frac{2ka}{(\beta + \theta)^2} - \frac{bk}{(\beta + \theta)^3} - s I_e \frac{\alpha}{2} \right]. \quad (17)$$

Hence, we note that

$$\frac{d\Pi_2(T)}{dT} = \begin{cases} < 0, & \text{if } T \in (0, T_2^*), \\ = 0, & \text{if } T = T_2^*, \\ > 0, & \text{if } T \in (T_2^*, \infty), \end{cases}$$

provided that $f_2(0) < 0$. Hence, with the help of theorem 5.2 and combining the above arguments, we draw the conclusion that the optimal solution, T_2^* , not only exists that it is also unique. Similarly, the first-order derivative of $\Pi_3(T)$ with respect to T is obtained as stated below:

$$\frac{d\Pi_3(T)}{dT} = \frac{f_3(T)}{T^2}, \quad (18)$$

where

$$\begin{aligned}
& f_3(T) := -A \\
& - \left[\frac{a}{(\beta + \theta)^2} [(k - \alpha) \{ T(\beta + \theta) \frac{dt_1}{dT} (e^{(\beta + \theta)t_1} - 1) - (e^{(\beta + \theta)t_1} - (\beta + \theta)t_1) \} \right. \\
& + \alpha \{ T(\beta + \theta) (\frac{dt_2}{dT} - \frac{dt_1}{dT}) (e^{(\beta + \theta)(t_2 - t_1)} - 1) - e^{(\beta + \theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1) \}] \\
& + \frac{b}{(\beta + \theta)^3} [(k - \alpha) e^{(\beta + \theta)t_1} (T(\beta + \theta) \frac{dt_1}{dT} - 1) + \alpha e^{(\beta + \theta)(t_2 - t_1)} (T (\frac{dt_2}{dT} - \frac{dt_1}{dT}) \\
& - 1)] - \frac{b\alpha}{(\beta + \theta)^2} (T \frac{dt_2}{dT} - t_2)] - \frac{c\theta}{(\theta + \beta)^2} [(k - \alpha) e^{(\beta + \theta)t_1} \{ 1 - (\beta + \theta) T \frac{dt_1}{dT} \} \\
& + \alpha e^{(\beta + \theta)(t_2 - t_1)} \{ 1 - (\beta + \theta) T (\frac{dt_2}{dT} - \frac{dt_1}{dT}) \}] + (k - \alpha)(\beta + \theta) (T \frac{dt_1}{dT} - t_1) \\
& + \alpha(\beta + \theta) \{ T (\frac{dt_2}{dT} - \frac{dt_1}{dT}) - (t_2 - t_1) \} - \frac{\alpha\delta}{2} [(c_2 + \delta) \{ (T - t_2)^2 - T(T - t_2) \\
& (1 - \frac{dt_2}{dT}) \}] - c \left[\frac{k - \alpha}{\theta + \beta} \{ e^{(\beta + \theta)t_1} - 1 \} + \alpha \delta (T \frac{dt_2}{dT} - t_2) - (k - \alpha) e^{(\beta + \theta)t_1} T \frac{dt_1}{dT} \right]
\end{aligned}$$

$$-cI_c[(N-M)k(t_1 - T\frac{dt_1}{dT}) - \frac{\theta(k-\alpha)kT}{\theta+\beta}\frac{dt_1}{dT} - \frac{(k-\alpha)T}{\theta+\beta}(\theta k + \beta e^{(\beta+\theta)t_1})\frac{dt_1}{dT}] + s\alpha(1-\delta)(T\frac{dt_2}{dT} - t_2).$$

Therefore, $f_3(T)$ and $\Pi_3(T)$ both have the same sign and domain. The optimal values of T , say T_3^* , can be evaluated from the equation

$$f_3(T) = 0. \quad (19)$$

We also have

$$\begin{aligned} \frac{df_3(T)}{dT} &= \frac{a}{(\beta+\theta)^2}[(k-\alpha)\{T(\beta+\theta)\frac{d^2t_1}{dT^2}(1-e^{(\theta+\beta)t_1}) - T(\beta+\theta)^2(\frac{dt_1}{dT})^2e^{(\theta+\beta)t_1}\} \\ &\quad + \alpha\{T(\beta+\theta)^2e^{(\beta+\theta)(t_2-t_1)}(\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 + T(\beta+\theta)(\frac{d^2t_2}{dT^2} - \frac{d^2t_1}{dT^2}) \\ &\quad (e^{(\beta+\theta)(t_2-t_1)} - 1)\}] + \frac{b}{(\beta+\theta)^3}[(k-\alpha)(\beta+\theta)^2e^{(\theta+\beta)t_1}(\frac{dt_1}{dT})^2 \\ &\quad + (k-\alpha)T(\beta+\theta)^2e^{(\theta+\beta)t_1}\frac{d^2t_1}{dT^2} + \alpha(\beta+\theta)e^{(\beta+\theta)(t_2-t_1)}(\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 \\ &\quad + \alpha T e^{(\beta+\theta)(t_2-t_1)}(\frac{d^2t_2}{dT^2} - \frac{d^2t_1}{dT^2})] - \frac{bT\alpha}{(\beta+\theta)^2}\frac{d^2t_2}{dT^2} \\ &\quad - \frac{c\theta}{(\beta+\theta)^2}[(k-\alpha)(\beta+\theta)^2e^{(\theta+\beta)t_1}T\{(\frac{dt_1}{dT})^2 - \frac{d^2t_1}{dT^2}\} \\ &\quad + \alpha T(\beta+\theta)^2e^{(\beta+\theta)(t_2-t_1)}(\frac{dt_2}{dT} - \frac{dt_1}{dT})^2 - \alpha(\beta+\theta)T e^{(\beta+\theta)(t_2-t_1)}(\frac{d^2t_2}{dT^2} - \frac{d^2t_1}{dT^2}) \\ &\quad + (k-\alpha)T(\beta+\theta)\frac{d^2t_1}{dT^2} - \alpha(\beta+\theta)T(\frac{d^2t_2}{dT^2} - \frac{d^2t_1}{dT^2})] \\ &\quad + \alpha\delta[(c_2+\delta)\{T(1 - \frac{dt_2}{dT})^2 - T(T-t_2)\frac{d^2t_2}{dT^2}\}] \\ &\quad + c[(k-\alpha)\{(\beta+\theta)T(\frac{dt_1}{dT})^2 + \frac{d^2t_1}{dT^2}\} - \alpha\delta(1 + \frac{d^2t_2}{dT^2})] \\ &\quad + cI_c[(N-M)kT\frac{d^2t_1}{dT^2} + \frac{\theta(k-\alpha)}{\theta+\beta}kT\frac{d^2t_1}{dT^2} + (k-\alpha)T e^{(\theta+\beta)t_1}(\frac{dt_1}{dT} + \frac{d^2t_1}{dT^2})] \\ &\quad + s\alpha(1-\delta)\{\frac{dt_2}{dT} + T\frac{d^2t_2}{dT^2}\} > 0, \text{ if } T > 0. \end{aligned}$$

Hence, $f_3(T)$ is increasing on $(0, \infty)$. With the help of Lemma 5.3, $\Pi_3(T)$ is a convex function on $(0, \infty)$. Also we conclude:

$$\lim_{T \rightarrow \infty} f_3(T) = \infty, \text{ and } f_3(0) = - \left[A - \frac{2ka}{(\beta+\theta)^2} - \frac{bk}{(\beta+\theta)^3} \right]. \quad (20)$$

Thus, we see that

$$\frac{d\Pi_3(T)}{dT} = \begin{cases} < 0, & \text{if } T \in (0, T_3^*), \\ = 0, & \text{if } T = T_3^*, \\ > 0, & \text{if } T \in (T_3^*, \infty), \end{cases}$$

provided that $f_3(0) < 0$. So, via theorem 5.2 and adding the above arguments, we reach on an agreement that the optimal solution, T_3^* , also exists uniquely. Now, we

let

$$\begin{aligned}\phi &:= A - \frac{2ka}{(\beta + \theta)^2} - \frac{bk}{(\beta + \theta)^3} - sI_e\alpha, \\ \psi &:= A - \frac{2ka}{(\beta + \theta)^2} - \frac{bk}{(\beta + \theta)^3} - sI_e\frac{\alpha}{2};\end{aligned}$$

we can easily find that $\phi < \psi$ and, combining the above cases, the following theorem is obtained.

Theorem 5.4. (a) If $\phi > 0$, then T_1^* is the unique optimal solution to the cost function $\Pi_1(T)$.

(b) If $\psi > 0$, then T_2^* is the unique optimal solution to the cost function $\Pi_2(T)$.

(c) $\Pi_3(T)$ has the unique optimal solution T_3^* on the non-negative interval $(0, \infty)$.

Proof. Proof of theorem 5.4 is obvious. \square

Here, the flowchart in figure 2 describes the solution procedure clearly.

5.2. Decision Criterion of the Optimal Replenishment Cycle Time T^* .

In this subsection, decision criterion are developed to find the retailer's optimal cycle time. From the definition of $\Pi(T)$, we have:

$$\Pi(T) = \begin{cases} \Pi_1(T), & \text{if } T \geq T_M, \\ \Pi_2(T), & \text{if } M \leq T \leq T_M, \\ \Pi_3(T), & \text{if } N \leq T \leq T_M, \end{cases} \quad (21)$$

where

$$\begin{aligned}T_M &= \frac{1}{\beta + \theta} \ln \left[\frac{\beta + \theta}{k - \alpha} \left(\frac{b(k - \alpha)}{\beta + \theta} - \frac{c\delta T}{\beta + \theta} e^{(\beta + \theta)T} \right) \right. \\ &\quad \left. - \left(\frac{a(k - \alpha)}{\beta + \theta} + \frac{b\theta\delta}{(\beta + \theta)^2} \right) (1 - e^{(\beta + \theta)T}) \right].\end{aligned}$$

We find that, at $T = T_M$, $\Pi_1(T_M) = \Pi_2(T_M)$, and at $T = M$, $\Pi_2(M) = \Pi_3(M)$; then $\Pi(T)$ is a continuous function and it is well-defined on $T > 0$. Since $\Pi(T)$ is a continuously differentiable function of T with a derivative that changes sign only once at T_i^* ($i = 1, 2, 3$) from negative to positive values, it follows that $\Pi(T)$ assumes its global minimum at the point T^* . However, a closed-form solution is not readily available from equation 21 and theorem 5.2. But, a fairly straightforward procedure is established subsequently to determine the optimal replenishment time to simplify the solution procedure. Let $T = T_M$; now it is obtained that

$$\begin{aligned}\Pi'_1(T_M) &= \frac{1}{T_M^2} [-A - \frac{a}{(\beta + \theta)^2} [(k - \alpha)\{T(\beta + \theta)\frac{dt_1}{dT}(e^{(\beta + \theta)t_1} - 1) \\ &\quad - (e^{(\beta + \theta)t_1} - (\beta + \theta)t_1)\} + \alpha\{T(\beta + \theta)(\frac{dt_2}{dT} - \frac{dt_1}{dT})(e^{(\beta + \theta)(t_2 - t_1)} - 1) \\ &\quad - e^{(\beta + \theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1)\}]_{T=T_M} + \frac{b}{(\beta + \theta)^3} [(k - \alpha)e^{(\beta + \theta)t_1} \\ &\quad (T(\beta + \theta)\frac{dt_1}{dT} - 1) + \alpha e^{(\beta + \theta)(t_2 - t_1)} (T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - 1)]_{T=T_M} \\ &\quad - \frac{b\alpha}{(\beta + \theta)^2} (T\frac{dt_2}{dT} - t_2)]_{T=T_M} - \frac{c\theta}{(\theta + \beta)^2} [(k - \alpha)e^{(\beta + \theta)t_1}\end{aligned}$$

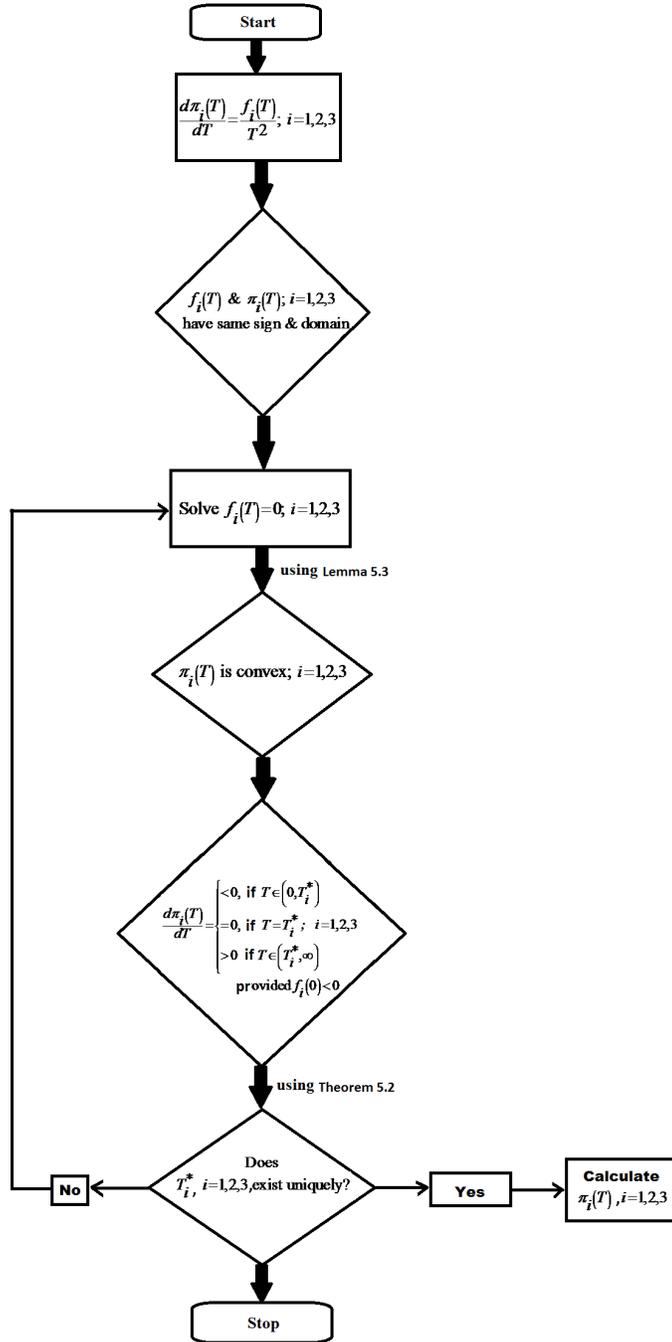


FIGURE 2. Flowchart of the solution procedure

$$\begin{aligned}
& \{1 - (\beta + \theta)T \frac{dt_1}{dT}\} + \alpha e^{(\beta+\theta)(t_2-t_1)} \{1 - (\beta + \theta)T(\frac{dt_2}{dT} - \frac{dt_1}{dT})\} \\
& + (k - \alpha)(\beta + \theta)(T \frac{dt_1}{dT} - t_1) + \alpha(\beta + \theta)\{T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - (t_2 - t_1)\}|_{T=T_M} \\
& - \frac{\alpha\delta}{2}[(c_2 + \delta)\{(T - t_2)^2 - T(T - t_2)(1 - \frac{dt_2}{dT})\}]|_{T=T_M} - c[\frac{k - \alpha}{\theta + \beta}\{e^{(\beta+\theta)t_1} - 1\} \\
& + \alpha\delta(T \frac{dt_2}{dT} - t_2) - (k - \alpha)e^{(\beta+\theta)t_1}T \frac{dt_1}{dT}]|_{T=T_M} - cI_c[\{\frac{(k - \alpha)\theta}{\theta + \beta}(t_1 - T \frac{dt_1}{dT}) \\
& + \frac{(k - \alpha)\beta}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1) - \frac{(k - \alpha)T\beta}{(\theta + \beta)}e^{(\theta+\beta)t_1} \frac{dt_1}{dT}\}(T + N - M) - \frac{(k - \alpha)\theta T t_1}{\theta + \beta} \\
& - \frac{(k - \alpha)\beta}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1)]|_{T=T_M} + sI_e[\{\alpha(1 + \frac{T}{2}) + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1) \\
& + \frac{\theta(k - \alpha)}{(\theta + \beta)}(t_1 - T \frac{dt_1}{dT} + \frac{T t_1}{2}) - \frac{(k - \alpha)T\beta^2}{(\theta + \beta)}e^{(\theta+\beta)t_1} \frac{dt_1}{dT}\}(M - N - \frac{T}{2}) \\
& + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1)]|_{T=T_M} + s\alpha(1 - \delta)(T \frac{dt_2}{dT} - t_2)|_{T=T_M}],
\end{aligned}$$

$\Pi'_2(T_M)$

$$\begin{aligned}
& = \frac{1}{T_M^2} [-A - [\frac{a}{(\beta + \theta)^2}[(k - \alpha)\{T(\beta + \theta) \frac{dt_1}{dT}(e^{(\beta+\theta)t_1} - 1) - (e^{(\beta+\theta)t_1} \\
& - (\beta + \theta)t_1)\} + \alpha\{T(\beta + \theta)(\frac{dt_2}{dT} - \frac{dt_1}{dT})(e^{(\beta+\theta)(t_2-t_1)} - 1) - e^{(\beta+\theta)(t_2-t_1)} \\
& - (\beta + \theta)(t_2 - t_1)\}]|_{T=T_M} + \frac{b}{(\beta + \theta)^3}[(k - \alpha)e^{(\beta+\theta)t_1}(T(\beta + \theta) \frac{dt_1}{dT} - 1) \\
& + \alpha e^{(\beta+\theta)(t_2-t_1)}(T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - 1)]|_{T=T_M} - \frac{b\alpha}{(\beta + \theta)^2}(T \frac{dt_2}{dT} - t_2)|_{T=T_M} \\
& - \frac{c\theta}{(\theta + \beta)^2}[(k - \alpha)e^{(\beta+\theta)t_1}\{1 - (\beta + \theta)T \frac{dt_1}{dT}\} + \alpha e^{(\beta+\theta)(t_2-t_1)}\{1 \\
& - (\beta + \theta)T(\frac{dt_2}{dT} - \frac{dt_1}{dT})\} + (k - \alpha)(\beta + \theta)(T \frac{dt_1}{dT} - t_1) \\
& + \alpha(\beta + \theta)\{T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - (t_2 - t_1)\}]|_{T=T_M} \\
& - \frac{\alpha\delta}{2}[(c_2 + \delta)\{(T - t_2)^2 - T(T - t_2)(1 - \frac{dt_2}{dT})\}]|_{T=T_M} \\
& - c[\frac{k - \alpha}{\theta + \beta}\{e^{(\beta+\theta)t_1} - 1\} + \alpha\delta(T \frac{dt_2}{dT} - t_2) - (k - \alpha)e^{(\beta+\theta)t_1}T \frac{dt_1}{dT}]|_{T=T_M} \\
& + sI_e[\{\alpha(1 + \frac{T}{2}) + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1) + \frac{\theta(k - \alpha)}{(\theta + \beta)}(t_1 - T \frac{dt_1}{dT}) \\
& - \frac{(k - \alpha)\beta^2 T}{(\theta + \beta)}e^{(\theta+\beta)t_1} \frac{dt_1}{dT}\}(M - N - \frac{T}{2}) + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1)]|_{T=T_M} \\
& + s\alpha(1 - \delta)(T \frac{dt_2}{dT} - t_2)|_{T=T_M}].
\end{aligned}$$

For convenience, let

$$\Delta_1 := -A - [\frac{a}{(\beta + \theta)^2}[(k - \alpha)\{T(\beta + \theta) \frac{dt_1}{dT}(e^{(\beta+\theta)t_1} - 1)$$

$$\begin{aligned}
& -(e^{(\beta+\theta)t_1} - (\beta + \theta)t_1) + \alpha\{T(\beta + \theta)(\frac{dt_2}{dT} - \frac{dt_1}{dT})(e^{(\beta+\theta)(t_2-t_1)} - 1) \\
& - e^{(\beta+\theta)(t_2-t_1)} - (\beta + \theta)(t_2 - t_1)\}|_{T=T_M} \\
& + \frac{b}{(\beta + \theta)^3}[(k - \alpha)e^{(\beta+\theta)t_1}(T(\beta + \theta)\frac{dt_1}{dT} - 1) \\
& + \alpha e^{(\beta+\theta)(t_2-t_1)}(T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - 1)]|_{T=T_M} - \frac{b\alpha}{(\beta + \theta)^2}(T\frac{dt_2}{dT} - t_2)|_{T=T_M} \\
& - \frac{c\theta}{(\theta + \beta)^2}[(k - \alpha)e^{(\beta+\theta)t_1}\{1 - (\beta + \theta)T\frac{dt_1}{dT}\} \\
& + \alpha e^{(\beta+\theta)(t_2-t_1)}\{1 - (\beta + \theta)T(\frac{dt_2}{dT} - \frac{dt_1}{dT})\} + (k - \alpha)(\beta + \theta)(T\frac{dt_1}{dT} - t_1) \\
& + \alpha(\beta + \theta)\{T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - (t_2 - t_1)\}]|_{T=T_M} \\
& - \frac{\alpha\delta}{2}[(c_2 + \delta)\{(T - t_2)^2 - T(T - t_2)(1 - \frac{dt_2}{dT})\}]|_{T=T_M} \\
& - c[\frac{k - \alpha}{\theta + \beta}\{e^{(\beta+\theta)t_1} - 1\} + \alpha\delta(T\frac{dt_2}{dT} - t_2) - (k - \alpha)e^{(\beta+\theta)t_1}T\frac{dt_1}{dT}]|_{T=T_M} \\
& - cI_c[\{\frac{(k - \alpha)\theta}{\theta + \beta}(t_1 - T\frac{dt_1}{dT}) + \frac{(k - \alpha)\beta}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1) \\
& - \frac{(k - \alpha)T\beta}{(\theta + \beta)}e^{(\theta+\beta)t_1}\frac{dt_1}{dT}\}(T + N - M) - \frac{(k - \alpha)\theta T t_1}{\theta + \beta} \\
& - \frac{(k - \alpha)\beta}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1)]|_{T=T_M} + sI_e[\{\alpha(1 + \frac{T}{2}) + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1) \\
& + \frac{\theta(k - \alpha)}{(\theta + \beta)}(t_1 - T\frac{dt_1}{dT} + \frac{T t_1}{2}) - \frac{(k - \alpha)T\beta^2}{(\theta + \beta)}e^{(\theta+\beta)t_1}\frac{dt_1}{dT}\}(M - N - \frac{T}{2}) \\
& + \frac{(k - \alpha)\beta^2}{(\theta + \beta)^2}(e^{(\theta+\beta)t_1} - 1)]|_{T=T_M} + s\alpha(1 - \delta)(T\frac{dt_2}{dT} - t_2)|_{T=T_M},
\end{aligned}$$

$$\begin{aligned}
\Delta_2 := & -A - [\frac{a}{(\beta + \theta)^2}[(k - \alpha)\{T(\beta + \theta)\frac{dt_1}{dT}(e^{(\beta+\theta)t_1} - 1) - (e^{(\beta+\theta)t_1} \\
& - (\beta + \theta)t_1)\} + \alpha\{T(\beta + \theta)(\frac{dt_2}{dT} - \frac{dt_1}{dT})(e^{(\beta+\theta)(t_2-t_1)} - 1) - e^{(\beta+\theta)(t_2-t_1)} \\
& - (\beta + \theta)(t_2 - t_1)\}]|_{T=T_M} + \frac{b}{(\beta + \theta)^3}[(k - \alpha)e^{(\beta+\theta)t_1}(T(\beta + \theta)\frac{dt_1}{dT} - 1) \\
& + \alpha e^{(\beta+\theta)(t_2-t_1)}(T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - 1)]|_{T=T_M} - \frac{b\alpha}{(\beta + \theta)^2}(T\frac{dt_2}{dT} - t_2)|_{T=T_M} \\
& - \frac{c\theta}{(\theta + \beta)^2}[(k - \alpha)e^{(\beta+\theta)t_1}\{1 - (\beta + \theta)T\frac{dt_1}{dT}\} + \alpha e^{(\beta+\theta)(t_2-t_1)} \\
& \{1 - (\beta + \theta)T(\frac{dt_2}{dT} - \frac{dt_1}{dT})\} + (k - \alpha)(\beta + \theta)(T\frac{dt_1}{dT} - t_1) \\
& + \alpha(\beta + \theta)\{T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - (t_2 - t_1)\}]|_{T=T_M} \\
& - \frac{\alpha\delta}{2}[(c_2 + \delta)\{(T - t_2)^2 - T(T - t_2)(1 - \frac{dt_2}{dT})\}]|_{T=T_M}
\end{aligned}$$

$$\begin{aligned}
& -c \left[\frac{k-\alpha}{\theta+\beta} \{e^{(\beta+\theta)t_1} - 1\} + \alpha \delta \left(T \frac{dt_2}{dT} - t_2 \right) - (k-\alpha) e^{(\beta+\theta)t_1} T \frac{dt_1}{dT} \right]_{|T=T_M} \\
& + s I_e \left[\left\{ \alpha \left(1 + \frac{T}{2} \right) + \frac{(k-\alpha)\beta^2}{(\theta+\beta)^2} (e^{(\theta+\beta)t_1} - 1) + \frac{\theta(k-\alpha)}{(\theta+\beta)} \left(t_1 - T \frac{dt_1}{dT} \right) \right. \right. \\
& \left. \left. - \frac{(k-\alpha)\beta^2 T}{(\theta+\beta)} e^{(\theta+\beta)t_1} \frac{dt_1}{dT} \right\} \left(M - N - \frac{T}{2} \right) + \frac{(k-\alpha)\beta^2}{(\theta+\beta)^2} (e^{(\theta+\beta)t_1} - 1) \right]_{|T=T_M} \\
& + s \alpha (1 - \delta) \left(T \frac{dt_2}{dT} - t_2 \right)_{|T=T_M}.
\end{aligned}$$

Then, we receive the following result.

Theorem 5.5. (a) If $\Delta_2 \geq 0$, then $\Pi(T^*) = \Pi(T_3^*)$ and $T^* = T_3^*$.
(b) If $\Delta_1 > 0$ and $\Delta_2 < 0$, then $\Pi(T^*) = \Pi(T_2^*)$ and $T^* = T_2^*$.
(c) If $\Delta_1 \leq 0$, then $\Pi(T^*) = \Pi(T_1^*)$ and $T^* = T_1^*$.

Proof. If $\Delta_2 \geq 0$, then $\Delta_1 > 0$; therefore, $\Pi'_1(T_M) = \Pi'_2(T_M) > 0$ and $\Pi'_2(M) = \Pi'_3(M) \geq 0$. Now, we get that

- $\Pi_1(T)$ is increasing on $[T_M, \infty)$.
- $\Pi_2(T)$ is increasing on $[M, T_M]$.
- $\Pi_3(T)$ is decreasing on $[N, T_3^*]$ and increasing on $[T_3^*, M]$.

Combining the above three criteria and equation 21, we understand that $\Pi(T)$ is decreasing on $(0, T_3^*]$ and increasing on $[T_3^*, \infty)$. Consequently, $T^* = T_3^*$. This proves the first part of the above theorem.

If $\Delta_1 > 0$ and $\Delta_2 < 0$, then $\Delta_3 < 0$, therefore, $T_2^* > M$, $T_1^* < T_M$ and $T_2^* < T_M$, we have $\Pi'_1(T_M) = \Pi'_2(T_M) > 0$ and $\Pi'_2(M) = \Pi'_1(M) < 0$. Then, from

$$\frac{d\Pi_i(T)}{dT} = \begin{cases} < 0, & \text{if } T \in (0, T_i^*), \\ = 0, & \text{if } T = T_i^*, \\ > 0, & \text{if } T \in (T_i^*, \infty), \end{cases}$$

we get that

- $\Pi_1(T)$ is increasing on $[T_M, \infty)$,
- $\Pi_2(T)$ is decreasing on $[M, T_2^*]$ and increasing on $[T_2^*, T_M]$.

Relating the above two criteria and equation 21, we see that $\Pi(T)$ is decreasing on $(0, T_2^*]$ and increasing on $[T_2^*, \infty)$. Consequently, $T^* = T_2^*$. This proves the second part of the theorem. For the last part, if $\Delta \leq 0$, then $T_2^* > M$, $T_1^* \geq T_M$ and $T_2^* \geq T_M$. Now, we have $\Pi'_1(T_M) = \Pi'_2(T_M) \leq 0$ and $\Pi'_2(M) = \Pi'_1(M) < 0$. So, from the above conclusion this implies that

- $\Pi_1(T)$ is decreasing on $[T_M, T_1^*]$ and increasing on $[T_1^*, \infty)$.
- $\Pi_2(T)$ is decreasing on $[M, T_M]$ and increasing on $[T_2^*, T_M]$.

Combining the above two criteria and equation 21, we have that $\Pi(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, \infty)$. Consequently, $T^* = T_1^*$. This proves the third part of the theorem and, hence, completes the proof. \square

Let us define Δ_T as

$$\begin{aligned}
\Delta_T := & A + \frac{(a+c\theta)(k-\alpha)}{(\beta+\theta)^2} \left[e^{(\theta+\beta)t} - (\theta+\beta)t - 1 \right] - \frac{b\alpha t}{(\beta+\theta)^2} + \alpha \delta (c_2 + \delta) \\
& - s I_e \left[\alpha + \frac{(k-\alpha)\beta^2}{(\beta+\theta)^2} (e^{(\theta+\beta)t} - 1) \right].
\end{aligned}$$

Now, we obtain the following theorem.

Theorem 5.6. (a) If $\Delta_T \geq 0$, then the retailer's optimal cycle time is T_2^* .
 (b) If $\Delta_T = 0$, then the retailer's optimal cycle time is $M - N$.
 (c) If $\Delta_T \leq 0$, then the retailer's optimal cycle time is T_1^* .

Proof. The arguments imply that the theorem holds. \square

6. Numerical Example. The following numerical examples are provided to illustrate our proposed method.

Example 1 Let us assume $A = \$300/\text{order}$, $s = \$20/\text{unit}$, $c = \$60/\text{unit}$, $a = \$0.7/\text{unit}/\text{year}$, $b = \$0.8/\text{unit}/\text{year}$, $M = 0.3$ years, $N = 0.5$ years, $I_c = \$0.11/\$/\text{year}$, $I_e = \$0.12/\$/\text{year}$, $k = 1000/\text{unit}/\text{year}$, $c_2 = \$40/\text{unit}$, $\alpha = 1.5$, $\beta = 0.8$, $\theta = 0.06$, $\delta = 0.05$. Then, $\Delta_1 = -2.5123 < 0$, $\Delta_2 = -1.8514 < 0$. Then, solving equation $f_1(T) = 0$ by Newton-Raphson method, we derive the values of T , t_1 and t_2 as $T^* = T_1^* = 3.1102$, $t_1 = 0.1974$, $t_2 = 0.1952$, which will satisfy all the above conditions and theorem 5.4. Now, utilizing the values of T , t_1 and t_2 in equation 7, we calculate the total cost of the inventory system with the help of Mathematica as $\Pi_1(T_1^*) = 225.4167$.

Example 2 Considering that, $A = \$250/\text{order}$, $s = \$30/\text{unit}$, $c = \$70/\text{unit}$, $a = \$0.8/\text{unit}/\text{year}$, $b = \$0.9/\text{unit}/\text{year}$, $M = 0.5$ years, $N = 0.6$ years, $I_c = \$0.21/\$/\text{year}$, $I_e = \$0.24/\$/\text{year}$, $k = 1200/\text{unit}/\text{year}$, $c_2 = \$50/\text{unit}$, $\alpha = 2.0$, $\beta = 1.5$, $\theta = 0.08$, $\delta = 0.06$. Now, $\Delta_1 = -2.2431 < 0$, $\Delta_2 = -2.0119 < 0$. Hence, solving equation $f_2(T) = 0$ by Newton-Raphson method, we obtain the values of T , t_1 and t_2 as $T^* = T_2^* = 3.2148$, $t_1 = 0.1783$, $t_2 = 0.1832$, and these values satisfy all the above conditions and theorem 5.4. Now, employing the values of T , t_1 and t_2 in equation 8, we derive the total cost of the inventory system by using Mathematica as $\Pi_2(T_2^*) = 230.1968$.

Example 3 Assuming, $A = \$280/\text{order}$, $s = \$40/\text{unit}$, $c = \$80/\text{unit}$, $a = \$1.1/\text{unit}/\text{year}$, $b = \$0.8/\text{unit}/\text{year}$, $M = 0.8$ years, $N = 0.9$ years, $I_c = \$0.18/\$/\text{year}$, $I_e = \$0.22/\$/\text{year}$, $k = 1100/\text{unit}/\text{year}$, $c_2 = \$60/\text{unit}$, $\alpha = 3.0$, $\beta = 2.0$, $\theta = 0.09$, $\delta = 0.07$. Then $\Delta_1 = -2.4518 < 0$, $\Delta_2 = -2.7012 < 0$. By solving equation $f_3(T) = 0$ with the help of Newton-Raphson method, we get the values of T , t_1 and t_2 as $T^* = T_3^* = 3.2207$, $t_1 = 0.2092$, $t_2 = 0.1654$, which will verify all the conditions and theorem 5.4. Now, using the values of T , t_1 and t_2 in equation 9, we obtain the total cost of the inventory system with the help of Mathematica as $\Pi_3(T_3^*) = 231.0221$.

7. Sensitivity Analysis. We now analyze the effects of changes in the system parameters A , s , c , a , b , M , N , I_e , α , β , γ , δ , and c_2 on the optimal values of T , t_1 , t_2 and the optimal cost $\Pi(T)$. The sensitivity analysis is performed by changing each of the parameters by +50%, +25%, +10%, -10%, -25% and -50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results based on **Example 1** are shown in Tables 2 and 3 and, on the basis of these results, the observations are taken into account. If we consider Examples 2 and 3, then, by applying the same procedure, we can likewise do a sensitivity analysis of the

parameters and can represent them by tables; but for the length of the paper, we only consider **Example 1** here.

Table 2: Sensitivity Analysis for different Parameters involved in **Example 1**.

Parameter	% change	value	T	t_1	t_2	TC
A	+50	450	3.9859	0.1978	0.1865	225.4871
	+25	375	3.9701	0.1975	0.1841	225.4991
	+10	330	3.9621	0.1968	0.1820	225.5123
	-10	270	3.9528	0.1042	0.1792	225.5472
	-25	225	3.9519	0.1040	0.1763	225.5612
	-50	150	3.9482	0.1037	0.1730	225.5860
s	+50	30	3.9883	0.1054	0.1579	225.6102
	+25	25	3.9865	0.1049	0.1556	225.6372
	+10	22	3.9851	0.1046	0.1534	225.6819
	-10	18	3.9840	0.1042	0.1518	225.6960
	-25	15	3.9832	0.1039	0.1475	225.7542
	-50	10	3.9818	0.1034	0.1455	225.7620
c	+50	90	3.9864	0.1043	0.1618	226.0171
	+25	75	3.9847	0.1039	0.1632	226.1261
	+10	66	3.9840	0.1044	0.1659	226.4189
	-10	54	3.9834	0.1103	0.1671	226.4703
	-25	45	3.9821	0.1111	0.1690	226.5100
	-50	30	3.9811	0.1235	0.1724	226.5275
a	+50	1.05	4.0854	0.1352	0.1858	225.8906
	+25	0.875	3.9981	0.1432	0.1822	225.8940
	+10	0.77	3.9702	0.1657	0.1805	225.8976
	-10	0.63	3.9453	0.1723	0.1778	225.9121
	-25	0.525	3.9321	0.1805	0.1751	225.9407
	-50	0.35	3.9161	0.1823	0.1736	225.9522
b	+50	1.2	3.9093	0.1834	0.1780	228.3131
	+25	1.0	3.924	0.1874	0.1799	228.4309
	+10	0.88	3.9398	0.1916	0.1827	228.4971
	-10	0.72	3.983	0.1396	0.1848	228.5102
	-25	0.60	4.0806	0.1668	0.1864	228.6524
	-50	0.40	3.1503	0.1625	0.1882	228.7601

Table 3: Sensitivity Analysis for different Parameters which are involved in **Example 1**.

Parameter	% change	value	T	t_1	t_2	TC
M	+50	0.45	3.9361	0.1790	0.1570	227.1092
	+25	0.375	3.9473	0.1796	0.1589	227.4121
	+10	0.33	3.9528	0.1853	0.1603	227.6708
	-10	0.27	3.9599	0.1854	0.1639	227.8211
	-25	0.225	3.9647	0.1861	0.1672	227.8355
	-50	0.15	3.9720	0.1864	0.1700	227.8708
N	+50	0.75	3.9471	0.1597	0.1968	226.9858
	+25	0.625	3.9510	0.1594	0.1940	226.9987
	+10	0.55	3.9540	0.1659	0.1903	227.6891
	-10	0.45	3.9593	0.1668	0.1881	227.6988
	-25	0.375	3.9549	0.1678	0.1854	227.7408
	-50	0.25	3.9601	0.1682	0.1826	227.1923
θ	+50	0.09	3.9840	0.1604	0.1725	227.83
	+25	0.075	3.9844	0.1629	0.1756	227.71
	+10	0.066	3.9847	0.1636	0.1791	227.16
	-10	0.054	3.9830	0.1646	0.1824	226.88
	-25	0.045	3.9871	0.1653	0.1847	226.41
	-50	0.03	3.9889	0.1664	0.1879	226.30
δ	+50	0.075	3.9840	0.1687	0.1763	225.74
	+25	0.0625	3.9844	0.1695	0.1735	225.63
	+10	0.055	3.9849	0.1736	0.1712	225.16
	-10	0.045	3.9850	0.1741	0.1675	224.98
	-25	0.0375	3.9867	0.1750	0.1633	224.47
	-50	0.025	3.9893	0.1769	0.1600	224.39

The following observations are made on the basis of Tables 2 and 3.

- (i) As ordering cost A , increases, the replenishment cycle time, T^* , increases as well as the total optimal cost, $\Pi(T^*)$, increases;
- (ii) TC , T , t_1 and t_2 are more sensitive with regard to change of the values of α , β and γ ;

- (iii) The larger the values of the unit selling price, s , and of the unit cost price, c , the smaller the values of the optimal cycle time, T^* , and of the optimal annual total cost, $\Pi(T^*)$;
- (iv) We can also see that under a higher value of the rate of interest earned, I_e , the annual total relevant cost, $\Pi(T^*)$, will be very much lower;
- (v) As holding cost, h , increases, the cycle time, T^* , decreases, whereas the optimal annual total cost, $\Pi(T^*)$, increases.
- (vi) The values of T , t_1 and t_2 increase with increasing the value of the backlogging rate, δ , and the value of total cost, TC , decreases with increasing this value.
- (vii) TC , T , t_1 and t_2 are less sensitive when changing the value of s .
- (viii) TC , T , t_1 and t_2 are moderately sensitive with respect to changes of the values of c and c_2 .

Here, Figure 3 and Figure 4 show the required inventory model based on our sensitivity analysis. The functions displayed in Figure 3 and Figure 4 are strictly convex functions, which is compatible with our assumptions. So, the figures indicate the existence of our proposed model.

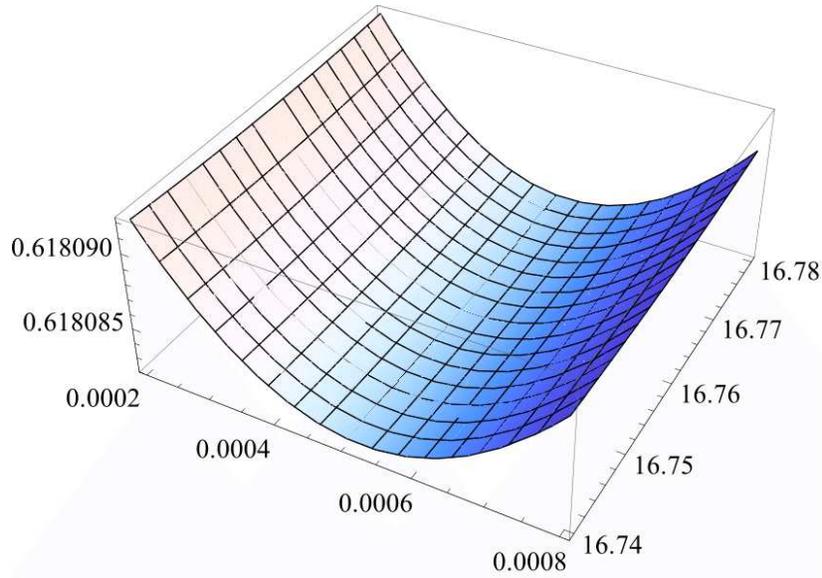


FIGURE 3. Graphical representation to show the convexity of total cost. The figure represents T , t_1 and the total cost $\Pi(T)$, along the x -axis, the y -axis and the z -axis, respectively

Here, the changes of the total cost with respect to the corresponding parameters are shown by the following figures (cf. Figures 5-8).

8. Concluding Remarks and Future Study. In this paper, we have formulated a production inventory model for deteriorating items with stock-dependent demand under two levels of trade credit policy. Shortages are allowed for this model and, during stock out period, price discount on backorders are allowed for those customers who are willing to wait until the fulfillment of their demand. Here, we have considered stock-dependent demand, because a large pile of stocks in shelves attracts the customers to buy more. As an example, there are many shopping malls (e.g., Big Bazar, City Life, City Mart, etc.), where the customers get many items at a time and, so, to save time, they shop everything from

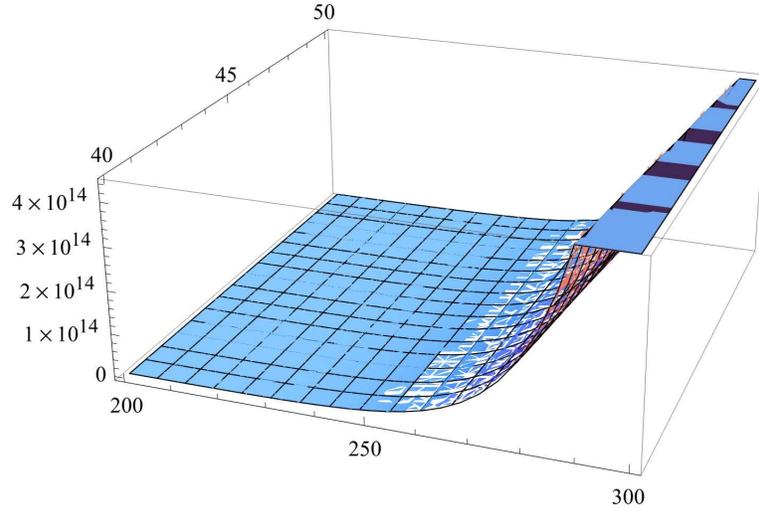


FIGURE 4. Graphical representation to show the convexity of total cost. The figure represents T , t_1 and the total cost $\Pi(T)$, along the x -axis, the y -axis and the z -axis, respectively

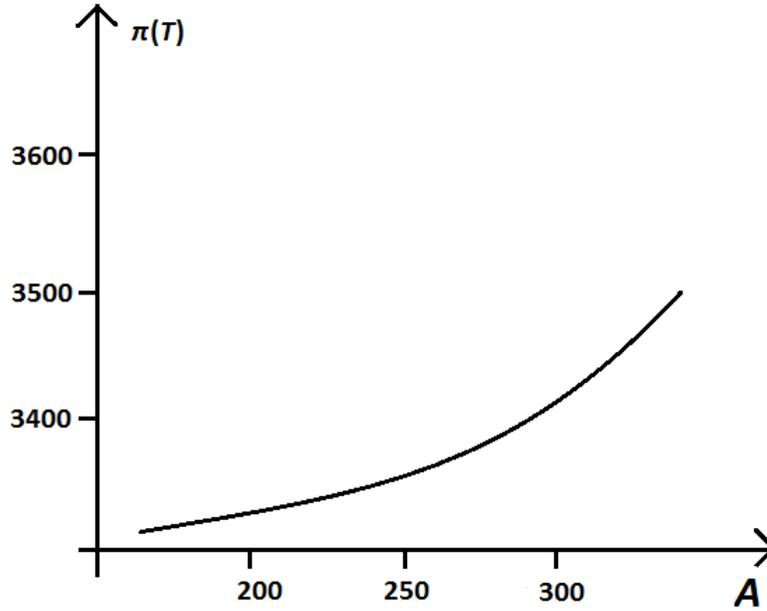


FIGURE 5. Change of total cost with respect to ordering cost, A , of our proposed model

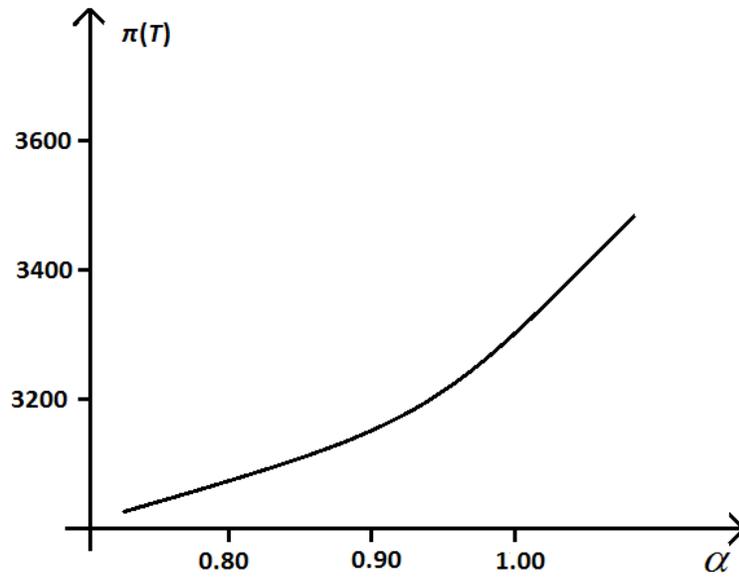


FIGURE 6. Change of total cost with respect to parameter α of our proposed model

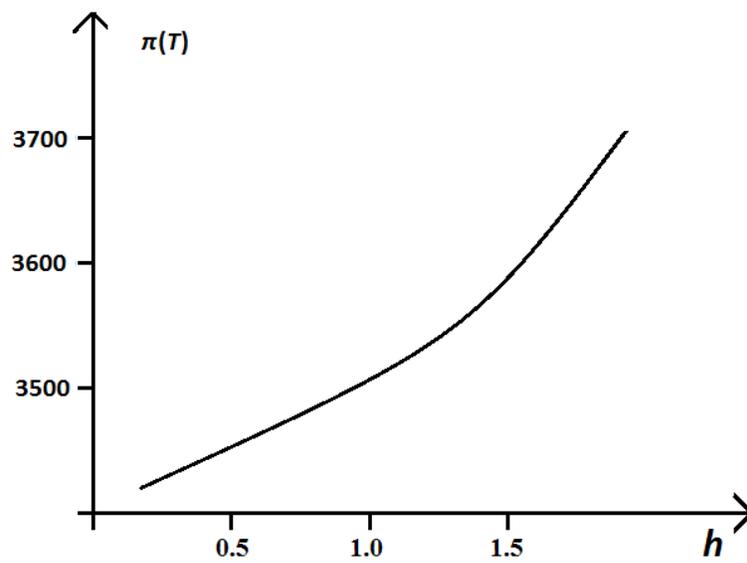


FIGURE 7. Change of total cost with respect to holding cost, h , of our proposed model

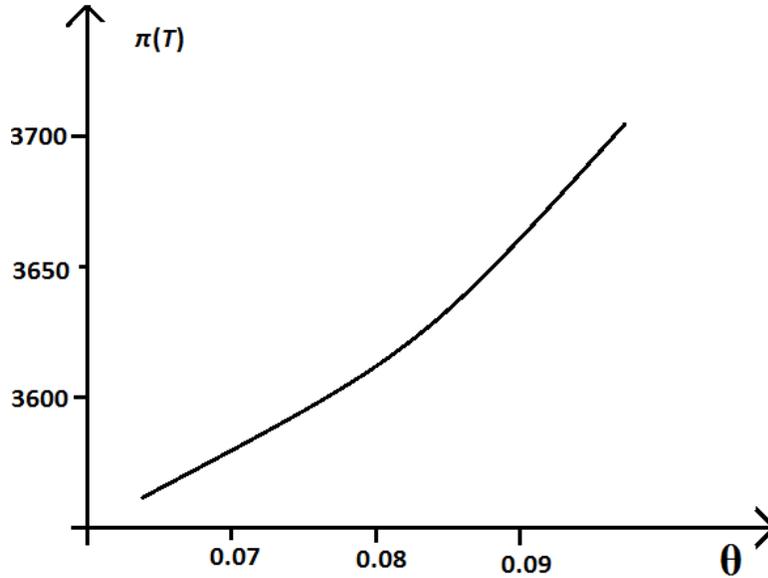


FIGURE 8. Change of total cost with respect to deteriorating cost, θ , of our proposed model

there, which increases the profit for the decision maker, which is the supplying company. Here, we have considered time-dependent holding cost because, as time increases, the rate of deterioration as well as the holding cost also increase.

From our sensitivity analysis, we can conclude that with the increase of purchasing cost, holding cost and shortage cost, the total average cost of the system increases; so to avoid increasing of total cost, we have to diminish the corresponding cost. When the ordering cost is increasing, then to reduce the number of orders, the retailer has to order more if the ordering cost has become high. Furthermore, we can conclude that for a higher rate of selling price, if the retailer wants a benefit from trade-credit policy, he(or she) has to order less. The retailer should get a higher benefit from any permissible delay if he(or she) earned a large rate of interest from the trade-credit policy. We can see that when the holding cost increases, the retailer shortens the cycle time and reduces the order quantity to maintain the profit. It is also found that for low backorder cost, it will be beneficial to the inventory manager to offer the customers a high discount on backorders. In our proposed approach, we have explored the fact that there exists a unique optimal replenishment time to minimize the total variable cost per unit time. We have also presented an optimal solution procedure to find the optimal replenishment policy. The effects of the model parameters on the optimal replenishment time and on the optimal total variable cost per unit time are investigated through numerical examples followed by a sensitivity analysis.

Besides this, we can say that our system is more important for industry because it provides a powerful tool for decision maker in planning and controlling the industry. This will also explain a useful model for many organizations that use the decision rule to improve their total operation costs. In this paper, the term deterioration is more emphasized and it will attract industrialists to update themselves and earn more profits from the system. The sensitivity analysis yields different conditions and solutions, which will provide an alternative and allow for a broader application scope of our system. Since the system is formulated from a retailer's perspective and the retailer has to decide to alter the

regular ordering pattern, it takes measure in preparation for the retailer to earn more profit. Finally, this paper considers a trade-credit which is an important and major part of inventory control and a powerful tool to improve sales and profit in an industry.

In a future study, the proposed model can be designed further in several ways. As an illustration, one can generalize variable deterioration rate as a stochastic deterioration rate. The demand function can be changed by a *probabilistic* demand function, *time-dependent* demand function instead of constant demand function. For a more practical situation, one can construct the model by introducing warehouses, quantity discounts, stochastic inflation, deteriorating cost, time-dependent deterioration rate and permissible delay in payments. In addition to the possible advances presented above, our model can provide an interesting research area with an additional impact by selling defective items at a lower price on demand.

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