Consistent S^3 reductions of six-dimensional supergravity

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We work out the consistent $\mathrm{AdS}_3 \times S^3$ truncations of the bosonic sectors of both the six-dimensional $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravity theories. They result in inequivalent three-dimensional half-maximal SO(4) gauged supergravities describing 32 propagating bosonic degrees of freedom apart from the nonpropagating supergravity multiplet. We present the full nonlinear Kaluza-Klein reduction formulas and illustrate them by explicitly uplifting a number of AdS_3 vacua.

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I. INTRODUCTION

Consistent sphere truncations have a long history in supergravity. Within maximal supergravity, this goes back to the seminal work of Ref. [1] on the consistent truncation of 11-dimensional supergravity on $\mathrm{AdS}_4 \times S^7$ to the lowest Kaluza-Klein multiplet, giving rise to four-dimensional SO(8) gauged supergravity. An analogous result for $\mathrm{AdS}_7 \times S^4$ was established in Ref. [2], while the proof of the consistent truncation of IIB supergravity on $\mathrm{AdS}_5 \times S^5$ was completed only recently [3]. Consistent truncations have led to a better comprehension of the structures of the theories of concern and the dualities they enjoy. Notably, these are not truncations in an effective field theory sense, with the massive Kaluza-Klein towers integrated out, yet every solution of the lower-dimensional theory lifts to a solution of the higher-dimensional theory. They are of particular importance in holographic applications, ensuring the validity of lower-dimensional supergravity computations, such as holographic correlators and renormalization group (RG) flows [4].

This work deals with consistent sphere compactifications in the context of $\mathrm{AdS}_3 \times S^3$, one of the central examples in the AdS/CFT correspondence [5] in which supergravity techniques have been successfully employed [6–11] in order to unravel the structure of the dual two-dimensional conformal field theories. Generic consistent S^3 truncations in (super)gravity have been discussed in Refs. [12–14], where the full nonlinear Kaluza-Klein *Ansätze* were constructed for a higher-dimensional theory that comprises the field content of the bosonic string. The resulting lower-dimensional theories are SO(4) gauged (super)gravities carrying gauge fields, a 2-form, and scalar fields, whose potential does not admit any stationary points. In the particular case of $\mathrm{AdS}_3 \times S^3$, the higher-dimensional theory is $D = 6$, $\mathcal{N} = (1,0)$ supergravity coupled to a single tensor multiplet that carries an anti-self-dual 2-form. In contrast to the higher-dimensional examples, the 2-forms in the resulting three-dimensional theory are auxiliary and can be integrated out, giving rise to an additional contribution to the scalar potential. This amended potential turns out to support a stable supersymmetric AdS_3 vacuum [15], corresponding to the supersymmetric $\mathrm{AdS}_3 \times S^3$ solution of the $D = 6$ theory. The nonlinear Kaluza-Klein *Ansätze* can be confirmed by direct computation.

More recently, new techniques have emerged for a more systematic understanding of consistent truncations within exceptional field theory (ExFT) and generalized geometry [16–21], see also [22,23] in the context of double field theory. Using the reformulation of $D = 6$, $\mathcal{N} = (1,0)$ supergravity as an ExFT based on the group SO(4, 4) [24], the nonlinear Kaluza-Klein *Ansätze* from Refs. [12,15] can straightforwardly be reproduced from the generalized Scherk-Schwarz twist matrices $\mathcal{U}$ in this framework. In this paper, we will extend the consistent S^3 truncations to the full $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravities in six dimensions. The relevant framework is an SO(8, 4) ExFT, which, depending on the solution of its section constraint, describes the aforementioned six-dimensional supergravities [24]. The resulting three-dimensional theories are SO(4) gauged supergravities coupled to four half-maximal scalar multiplets [25,26], i.e., with scalar target space given by $\mathrm{SO}(8,4)/(\mathrm{SO}(8) \times \mathrm{SO}(4))$. In the ExFT framework, the construction of the consistent truncations simply amounts to embedding the SO(4, 4) twist matrices $\mathcal{U}$ into the SO(8, 4) isometry group of the

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ungauged three-dimensional theory. Two inequivalent embeddings give rise to inequivalent three-dimensional gaugings, describing the truncation of the $\mathcal{N}=(1,1)$ and the $\mathcal{N}=(2,0)$ theory, respectively.

The paper is organized as follows: In Sec. II, we introduce the relevant three-dimensional supergravities, their gauge structure, and scalar potentials. We give an explicit parametrization of their scalar target space $\mathrm{SO}(8,4)/(\mathrm{SO}(8)\times\mathrm{SO}(4))$ and determine the full set of stationary points of their scalar potentials. In Sec. III, we review the framework of $\mathrm{SO}(8,4)$ ExFT. In particular, we discuss the two inequivalent solutions of its section constraint and establish the full dictionary of the ExFT fields into the six-dimensional fields of $\mathcal{N}=(1,1)$ and $\mathcal{N}=(2,0)$ supergravity, respectively. In Secs. IV and V, we use the explicit Scherk-Schwarz twist matrix $\mathcal{U}$ together with the ExFT-supergravity dictionary to work out the full nonlinear Kaluza-Klein *Ansätze* for all six-dimensional fields, defining the consistent truncation. As an illustration and a consistency check, we use these *Ansätze* in Sec. VI in order to give the explicit uplift of some of the three-dimensional AdS_3 vacua into full solutions of $D=6$ supergravity. We close with some comments in Sec. VII.

II. THE THREE-DIMENSIONAL SUPERGRAVITY

In this section, we collect the basic formulas of the relevant three-dimensional supergravities. In particular, we give an explicit parametrization of their scalar target space, which allows us to determine the full set of stationary points of the scalar potentials.

A. 3D gauged supergravity

Three-dimensional gauged supergravity with $\mathcal{N}=8$ (half-maximal) supersymmetry has been constructed in Refs. [25,26]. The theory is based on the coset space

$$\mathrm{G}/\mathrm{H}=\mathrm{SO}(8,4)/(\mathrm{SO}(8)\times\mathrm{SO}(4)), \tag{2.1}$$

with all couplings completely specified by the choice of a constant symmetric embedding tensor $\Theta_{\bar K\bar L,\bar M\bar N}$ of the form

$$\Theta_{\bar K\bar L,\bar M\bar N}=\theta_{\bar K\bar L\bar M\bar N}+\frac{1}{2}(\eta_{\bar M[\bar K}\theta_{\bar L]\bar N}-\eta_{\bar N[\bar K}\theta_{\bar L]\bar M})+\theta\eta_{\bar M[\bar K}\eta_{\bar L]\bar N}, \tag{2.2}$$

with antisymmetric $\theta_{\bar K\bar L\bar M\bar N}=\theta_{[\bar K\bar L\bar M\bar N]}$, symmetric $\theta_{\bar M\bar N}=\theta_{(\bar M\bar N)}$, and the $\mathrm{SO}(8,4)$ invariant tensor $\eta_{\bar M\bar N}$. Indices $\bar M,\bar N,\ldots$ label the vector representation of $\mathrm{G}=\mathrm{SO}(8,4)$, and are raised and lowered with $\eta_{\bar M\bar N}$. The embedding tensor encodes the minimal coupling of vector fields to scalars according to

$$D_\mu M_{\bar M\bar N}\equiv\partial_\mu M_{\bar M\bar N}+2A_\mu{}^{\bar P\bar Q}\Theta_{\bar P\bar Q,\bar K\bar L}(T^{\bar K\bar L})_{(\bar M}{}^{\bar R}M_{\bar N)\bar R}, \tag{2.3}$$

with the symmetric matrix $M_{\bar M\bar N}$ parametrizing the coset space [Eq. (2.1)]. By $T^{\bar M\bar N}$, we denote the generators of $\mathfrak{g}=\mathrm{Lie}\,\mathrm{G}$ acting by left multiplication with the algebra

$$[T^{\bar K\bar L},T^{\bar M\bar N}]=2(\eta^{\bar K[\bar M}T^{\bar N]\bar L}-\eta^{\bar L[\bar M}T^{\bar N]\bar K}). \tag{2.4}$$

The number of vector fields involved in the connection [Eq. (2.3)] is equal to the rank of $\Theta_{\bar K\bar L,\bar M\bar N}$ (taken as a $\dim\mathrm{G}\times\dim\mathrm{G}$ matrix).

The complete bosonic Lagrangian of the three-dimensional theory is given as a gravity-coupled Chern-Simons gauged G/H coset space σ-model:

$$e^{-1}\mathcal{L}=\frac{1}{4}R+\frac{1}{32}g^{\mu\nu}\partial_\mu M^{\bar M\bar N}\partial_\nu M_{\bar M\bar N}+e^{-1}\mathcal{L}_{\mathrm{CS}}-V, \tag{2.5}$$

with the three-dimensional metric $g_{\mu\nu}$ and $e\equiv\sqrt{|\det g_{\mu\nu}|}$. The Chern-Simons term is explicitly given by

$$\mathcal{L}_{\mathrm{CS}}=\frac{1}{4}\varepsilon^{\mu\nu\rho}A_\mu{}^{\bar K\bar L}\Theta_{\bar K\bar L,\bar M\bar N}\times\left(\partial_\nu A_\rho{}^{\bar M\bar N}+\frac{1}{3}f^{\bar M\bar N,\bar P\bar Q}{}_{\bar R\bar S}\Theta_{\bar P\bar Q,\bar U\bar V}A_\mu{}^{\bar R\bar S}A_\rho{}^{\bar U\bar V}\right), \tag{2.6}$$

in terms of the embedding tensor [Eq. (2.2)], with the $\mathrm{SO}(8,4)$ structure constants $f^{\bar M\bar N,\bar P\bar Q}{}_{\bar R\bar S}$ from Eq. (2.4). The form of the scalar potential V is determined by the embedding tensor and may be written in the form [27] (where we have corrected a typo in the second line)

$$\begin{aligned}V=&\frac{1}{48}\theta_{\bar K\bar L\bar M\bar N}\theta_{\bar P\bar Q\bar R\bar S}(M^{\bar K\bar P}M^{\bar L\bar Q}M^{\bar M\bar R}M^{\bar N\bar S}\\&-6M^{\bar K\bar P}M^{\bar L\bar Q}\eta^{\bar M\bar R}\eta^{\bar N\bar S}+8M^{\bar K\bar P}\eta^{\bar L\bar Q}\eta^{\bar M\bar R}\eta^{\bar N\bar S}\\&-3\eta^{\bar K\bar P}\eta^{\bar L\bar Q}\eta^{\bar M\bar R}\eta^{\bar N\bar S})\\&+\frac{1}{32}\theta_{\bar K\bar L}\theta_{\bar P\bar Q}(2M^{\bar K\bar P}M^{\bar L\bar Q}-2\eta^{\bar K\bar P}\eta^{\bar L\bar Q}\\&-M^{\bar K\bar L}M^{\bar P\bar Q})+\theta\theta_{\bar K\bar L}M^{\bar K\bar L}-8\theta^2.\end{aligned} \tag{2.7}$$

From the general expression of the scalar potential, we have omitted the term carrying a totally antisymmetric $M^{\bar K\bar L\bar M\bar N\bar P\bar Q\bar R\bar S}$, which drops out upon restriction to embedding tensors satisfying the additional constraint

$$\Theta_{[\bar K\bar L,\bar M\bar N}\Theta_{\bar P\bar Q,\bar R\bar S]}=0. \tag{2.8}$$

As pointed out in Ref. [24], consistent truncations obtained by generalized Scherk-Schwarz reduction necessarily lead to three-dimensional theories satisfying Eq. (2.8), and we will in the following restrict our analysis to such theories. For the fermionic completion of Eq. (2.5) and its full supersymmetry transformations, we refer to Refs. [25,26].

For the following, it will be convenient to choose a specific basis upon breaking:

$$\begin{aligned} \mathrm{SO}(8,4) &\to \mathrm{GL}(4)\times \mathrm{SO}(4), \\ X^{\bar M} &\to \{X^A, X_A, X^\alpha\}, \end{aligned} \tag{2.9}$$

with $A=1,\dots,4$ and $\alpha=1,\dots,4$ labeling the GL(4) and the SO(4) vector representations, respectively. In this basis, the GL(4) is embedded into an SO(4,4), such that the SO(8,4) invariant tensor is of the form

$$\eta_{\bar M\bar N}=\begin{pmatrix} 0 & \delta_A{}^B & 0\\ \delta_B{}^A & 0 & 0\\ 0 & 0 & -\delta_{\alpha\beta}\end{pmatrix}. \tag{2.10}$$

Specifically, we will be interested in the theories described by the two embedding tensors:

$$\begin{aligned} &(\mathrm{A}){:}\ \theta_{AB}=4\delta_{AB},\quad \theta_{ABCD}=-2\alpha\varepsilon_{ABCD},\\ &(\mathrm{B}){:}\ \theta_{ABC}{}^{D}=\varepsilon_{ABCE}\delta^{ED},\quad \theta_{ABCD}=-2\alpha\varepsilon_{ABCD}, \end{aligned} \tag{2.11}$$

with the totally antisymmetric ε_{ABCE}, a free constant α. These theories capture the S^3 reductions of $\mathcal{N}=(1,1)$ and $\mathcal{N}=(2,0)$ supergravity, respectively. In particular, the embedding tensors induce the gauge connections

$$-\frac{1}{4}A_\mu{}^{\bar K\bar L}\Theta_{\bar K\bar L,\bar M\bar N}T^{\bar M\bar N}=\begin{cases} A_\mu{}^{AB}T_B{}^A+(A_{\mu B}{}^A+\frac{\alpha}{2}\varepsilon_{ABCD}A_\mu{}^{CD})T^{AB} & (\mathrm{A})\\ \tilde A_\mu{}^{AB}T_B{}^A+(A_{\mu B}{}^A+\frac{\alpha}{2}\varepsilon_{ABCD}\tilde A_\mu{}^{CD})\tilde T^{AB} & (\mathrm{B})\end{cases}, \tag{2.12}$$

with

$$\begin{aligned} \tilde A_\mu{}^{AB}&=\frac{1}{2}\varepsilon^{ABCD}A_\mu{}^{CD},\\ \tilde T^{AB}&=\frac{1}{2}\varepsilon^{ABCD}T^{CD} \end{aligned} \tag{2.13}$$

in the second case. Both embedding tensors induce a gauge group of non-semi-simple type

$$\mathrm{G}_{\mathrm{gauge}}=\mathrm{SO}(4)\ltimes T^6, \tag{2.14}$$

with the Abelian generators $\{T^{AB}\}$ of T^6 transforming in the adjoint representation of SO(4). Chern-Simons gauge theories with a gauge group of type (2.14) and the T^6 generators realized as shift symmetries on scalar fields can be rewritten as SO(4) Yang-Mills theories upon integrating out the vectors $A_{\mu A}{}^B$ associated with the T^6 generators [26,28].

B. Parametrization of the SO(8,4)/(SO(8) × SO(4)) scalar coset

In order to study the structure of the scalar potential [Eq. (2.7)], it turns out to be useful to adopt particular parametrizations of the scalar matrix $M_{\bar M\bar N}$. To this end, we decompose the SO(8,4) generators according to Eq. (2.9), such that a coset element $\mathcal{V}\in \mathrm{SO}(8,4)/(\mathrm{SO}(8)\times\mathrm{SO}(4))$ can be parametrized in the triangular gauge as

$$\mathcal{V}=e^{\phi_{AB}T^{AB}}e^{\phi_{A\alpha}T^{A\alpha}}\mathcal{V}_{\mathrm{GL}(4)}, \tag{2.15}$$

with nilpotent generators $T^{AB}=T^{[AB]}$ and a GL(4) matrix $\mathcal{V}_{\mathrm{GL}(4)}$. Modula an SO(4) gauge freedom, this matrix carries the 32 physical scalar degrees of freedom. In the following, we will make use of the fact that the gauge groups we are studying include shift symmetries acting on the scalars ϕ_{AB}, cf. Eq. (2.12), which we may use to adopt a gauge in which $\phi_{AB}\to 0$. As a result, the gauge group (2.14) reduces to a standard SO(4).

Explicitly, we choose a representation such that

$$\mathcal{V}_{\bar M}{}^{\bar K}|_{\phi_{AB}\to 0}=\begin{pmatrix} \mathcal{V}_A{}^B & \frac{1}{2}\phi_{A\gamma}\phi_{C\gamma}(\mathcal{V}^{-1})_B{}^C & \phi_{A\gamma}\delta^{\gamma\beta}\\ 0 & (\mathcal{V}^{-1})_B{}^A & 0\\ 0 & \phi_{C\alpha}(\mathcal{V}^{-1})_B{}^C & \delta_\alpha{}^\beta\end{pmatrix} \tag{2.16}$$

in the basis (2.9), with $\mathcal{V}_A{}^B\equiv(\mathcal{V}_{\mathrm{GL}(4)})_A{}^B$. The symmetric positive definite matrix $M_{\bar M\bar N}=(\mathcal{V}\mathcal{V}^T)_{\bar M\bar N}$ then takes the form

$$M_{\bar M\bar N}=\begin{pmatrix} m_{AB}+\frac{1}{4}(\phi\phi)_{AC}m^{CD}(\phi\phi)_{DB}+(\phi\phi)_{AB} & \frac{1}{2}(\phi\phi)_{AC}m^{BC} & \frac{1}{2}(\phi\phi)_{AC}m^{CD}\phi_{D\beta}+\phi_{A\beta}\\ \frac{1}{2}(\phi\phi)_{BC}m^{AC} & m^{AB} & \phi_{C\beta}m^{AC}\\ \frac{1}{2}\phi_{C\alpha}m^{CD}(\phi\phi)_{BD}+\phi_{B\alpha} & \phi_{C\alpha}m^{BC} & \delta_{\alpha\beta}+\phi_{C\alpha}\phi_{D\beta}m^{CD}\end{pmatrix}, \tag{2.17}$$

with $m_{AB}\equiv\mathcal{V}_A{}^C\mathcal{V}_B{}^C$, m^{AB} denoting its inverse matrix $m_{AC}m^{CB}=\delta_A{}^B$, and where we have denoted $(\phi\phi)_{AB}\equiv\phi_{A\gamma}\phi_{B\gamma}$.

Evaluating the scalar kinetic term from Eq. (2.5) in this parametrization yields

$$\begin{aligned}\mathcal{L}_{\rm kin} &= -\frac{1}{32}{\rm Tr}[D_\mu M M^{-1} D^\mu M M^{-1}] \\ &= -\frac{1}{16}{\rm Tr}[D_\mu m m^{-1} D^\mu m m^{-1}] - \frac{1}{8} D_\mu \phi_{A\alpha} m^{AB} D^\mu \phi_{B\alpha} \\ &\quad + \frac{1}{64} Y_{\mu AB} m^{BC} Y^\mu{}_{CD} m^{DA}, \end{aligned} \tag{2.18}$$

with

$$Y_{\mu AB} = D_\mu \phi_{A\alpha} \phi_{B\alpha} - D_\mu \phi_{B\alpha} \phi_{A\alpha} \tag{2.19}$$

and SO(4) covariant derivatives D_μ. The first term in Eq. (2.18) represents a GL(4)/SO(4) σ model.

Let us finally evaluate the scalar potential for the two choices of embedding tensor [Eq. (2.11)]. For the embedding tensor (A), describing the $\mathcal{N} = (1,1)$ reduction on S^3, the potential (2.7) depends exclusively on the block M^{AB}. In the parametrization (2.17), the potential is thus independent of the scalars $\phi_{A\alpha}$. Explicitly, it takes the form

$$V_{\rm (A)} = 2\alpha^2 e^{4\varphi} + \frac{1}{2} e^{2\varphi} (2\tilde{m}^{AB}\tilde{m}^{AB} - \tilde{m}^{AA}\tilde{m}^{BB}), \tag{2.20}$$

where we have defined

$$m^{AB} = \tilde{m}^{AB} e^{\varphi}, \tag{2.21}$$

with $\det \tilde{m}^{AB} = 1$. This precisely agrees with the result of Ref. [15] as required by consistency, since the additional scalar fields $\phi_{A\alpha}$ do not show up in the potential. Note that rescaling $\varphi \to \varphi - \log|\alpha|$ turns the constant α into a global scaling factor in front of the potential, which is thus irrelevant for the existence of stationary points. Depending on the sign of α, there are, however, two different fermionic completions of the theory.

In contrast, for the embedding tensor (B) from Eq. (2.11), describing the $\mathcal{N} = (2,0)$ reduction on S^3, the potential after some computation takes the form

$$\begin{aligned} V_{\rm (B)} &= (\det m^{CD}) \Big(2\Big(\alpha - \frac{1}{4}(\phi\phi)_{AA}\Big)^2 \\ &\quad + m_{AB}\Big(m_{AB} + \frac{1}{2}(\phi\phi)_{AB}\Big) - \frac{1}{2} m_{AA} m_{BB} \Big) \\ &= 2\alpha^2 e^{4\varphi} + \frac{1}{2} e^{2\varphi}(2\tilde{m}_{AB}\tilde{m}_{AB} - \tilde{m}_{AA}\tilde{m}_{BB}) \\ &\quad - \alpha e^{4\varphi}(\phi\phi)_{CC} + \frac{1}{2} e^{3\varphi}\tilde{m}_{CD}(\phi\phi)_{CD} \\ &\quad + \frac{1}{8} e^{4\varphi} (\phi\phi)_{CC}(\phi\phi)_{DD}, \end{aligned} \tag{2.22}$$

where again we use the parametrization in Eq. (2.21). We note that for $\phi_{A\alpha} = 0$, this expression coincides with the potential (2.20) upon flipping

$$\tilde{m}^{AB} \leftrightarrow \tilde{m}_{AB}. \tag{2.23}$$

This is consistent with the fact that upon setting the $\phi_{A\alpha}$ fields to zero, both the $\mathcal{N} = (1,1)$ and the $\mathcal{N} = (2,0)$ theories reduce to the same $\mathcal{N} = (1,0)$ theory in six dimensions, which gives rise to the potential computed in Ref. [15]. Again, the constant α can be absorbed by shifting φ together with a rescaling of $\phi_{A\alpha}$. However, the presence of a term linear in α implies that there are two inequivalent theories depending on the sign of α which cannot be absorbed into a field redefinition. In the following, we will adopt the normalization $|\alpha| = 1$.

C. Extrema of the scalar potential

In this section, we derive the full set of extremal points of the scalar potentials [Eqs. (2.20) and (2.22)]. Since Eq. (2.20) sits within Eq. (2.22) as a truncation $\phi_{A\alpha} = 0$, it will be sufficient to analyze the extremal points of the latter. Below, we will then uplift some of these extremal points to solutions of the six-dimensional supergravities.

Variation of Eq. (2.22) with respect to the scalar field φ yields the condition

$$\begin{aligned} &\delta_\varphi V_{\rm (B)} \stackrel{!}{=} 0 \\ &\quad \Rightarrow 0 = 8\alpha^2 + e^{-2\varphi}(2\tilde{m}_{AB}\tilde{m}_{AB} - \tilde{m}_{AA}\tilde{m}_{BB}) \\ &\qquad - \alpha(\phi\phi)_{CC} - \frac{1}{4}(\phi\phi)_{CC}(\phi\phi)_{DD}. \end{aligned} \tag{2.24}$$

Next, let us consider the variation with respect to $\phi_{A\alpha}$, such that $\delta_\Sigma(\phi\phi)_{AB} = \Sigma_{A\alpha}\phi_{B\alpha} + \Sigma_{B\alpha}\phi_{A\alpha}$. Variation of the potential yields

$$\begin{aligned} \delta_\Sigma V_{\rm (B)} = \Sigma_{A\alpha}\Big(&-2\alpha e^{4\varphi}\phi_{A\alpha} + e^{3\varphi}\tilde{m}_{AB}\phi_{B\alpha} \\ &+ \frac{1}{2} e^{4\varphi}(\phi\phi)_{DD}\phi_{A\alpha}\Big), \end{aligned} \tag{2.25}$$

such that extremization leads to the eigenvector equation

$$\delta_\Sigma V_{\rm (B)} \stackrel{!}{=} 0 \Rightarrow \tilde{m}_{AB}\phi_{B\alpha} = e^{\varphi}\Big(2\alpha - \frac{1}{2}(\phi\phi)_{DD}\Big)\phi_{A\alpha}. \tag{2.26}$$

Finally, variation with respect to the SL(4) scalars according to $\delta_\Lambda \tilde{m}_{AB} = 2\Lambda_{(A}{}^C \tilde{m}_{B)C}$ with traceless $\Lambda_A{}^B$ gives rise to

$$\begin{aligned} \delta_\Lambda V_{\rm (B)} \stackrel{!}{=} 0 \Rightarrow 0 = (&4\tilde{m}_{CB}\tilde{m}_{AB} - 2\tilde{m}_{CA}\tilde{m}_{BB} \\ &+ e^{\varphi}\tilde{m}_{CD}(\phi\phi)_{AD})\Lambda_A{}^C. \end{aligned} \tag{2.27}$$

Upon reducing the last term by means of Eq. (2.26), this equation can be solved for $(\phi\phi)_{AB}$ as

$$(\phi\phi)_{AB} = e^{-2\varphi}\chi^{-1}\left(2\tilde{m}_{AB}\tilde{m}_{CC} - 4\tilde{m}_{AC}\tilde{m}_{BC} - \frac{1}{2}\delta_{AB}\tilde{m}_{CC}\tilde{m}_{DD} + \delta_{AB}\tilde{m}_{CD}\tilde{m}_{CD}\right) + \frac{1}{4}\delta_{AB}(\phi\phi)_{CC}, \qquad (2.28)$$

with $\chi = (2\alpha - \frac{1}{2}(\phi\phi)_{DD})$. Plugging this expression back into the eigenvector equation (2.26) eventually implies

$$\begin{aligned} 0 &= e^{-2\varphi}\chi^{-1}\left(2\tilde{m}_{AB}\tilde{m}_{BC}\tilde{m}_{DD} - 4\tilde{m}_{AB}\tilde{m}_{BD}\tilde{m}_{CD} - \frac{1}{2}\tilde{m}_{AC}\tilde{m}_{EE}\tilde{m}_{DD} + \tilde{m}_{AC}\tilde{m}_{DE}\tilde{m}_{DE}\right) \\ &\quad - e^{-\varphi}\left(2\tilde{m}_{AC}\tilde{m}_{DD} - 4\tilde{m}_{AD}\tilde{m}_{CD} - \frac{1}{2}\delta_{AC}\tilde{m}_{EE}\tilde{m}_{DD} + \delta_{AC}\tilde{m}_{DE}\tilde{m}_{DE}\right) + \frac{1}{4}\tilde{m}_{AC}(\phi\phi)_{DD} - \frac{1}{4}e^{\varphi}\chi\delta_{AC}(\phi\phi)_{DD}. \end{aligned} \qquad (2.29)$$

The conditions for stationary points thus boil down to solving Eqs. (2.24) and (2.29). The value of the potential at an extremal point is computed by evaluating Eq. (2.22) using Eqs. (2.24) and (2.26):

$$V_{(\mathrm{B}),0} = -\alpha e^{4\varphi}\left(2\alpha - \frac{1}{2}(\phi\phi)_{CC}\right), \qquad (2.30)$$

which corresponds to a three-dimensional AdS length

$$\ell^2 = \frac{2}{|V_0|}, \qquad \mathring{R}_{\mu\nu} = -\frac{2}{\ell^2}\mathring{g}_{\mu\nu}, \qquad (2.31)$$

with $\mathring{g}_{\mu\nu}$ denoting the AdS_3 metric.

Let us first consider the sector $\phi_{A\alpha} = 0$, which is a consistent truncation of the potential (2.22) and contains the stationary points common to Eqs. (2.20) and (2.22). In this case, Eq. (2.26) is trivially satisfied. Solutions of the remaining equations (2.24) and (2.27) are most conveniently found in a basis in which $\tilde{M}_{AB}$ is diagonal. Inspection reveals a one-parameter family of solutions given by

$$m^{AB} = \mathrm{diag}\{e^{\eta}, e^{\eta}, e^{-\eta}, e^{-\eta}\}, \qquad \phi_{A\alpha} = 0. \qquad (2.32)$$

The existence of this flat direction in the scalar potential has already been noted in Ref. [29]. The potential for these families remains fixed at $V_{(\mathrm{B}),0} = -2$, and the scalar spectrum is given by

$$m^2\ell^2:\ 0\,[5], \quad 8\,[1], \quad 4e^{2\eta} - 4\,[2], \quad 4e^{-2\eta} - 4\,[2], \qquad (2.33)$$

completed by

$$\left\{\begin{array}{ccc} 0\,[16] & : & \text{potential (2.20)} \\ e^{2\eta} - 2e^{\eta}(\mathrm{sgn}\alpha)\,[8], e^{-2\eta} - 2e^{-\eta}(\mathrm{sgn}\alpha)\,[8] & : & \text{potential (2.22)} \end{array}\right\} \qquad (2.34)$$

for the different potentials. These spectra are stable (in the Breitenlohner-Freedman sense, $m^2\ell^2 \geq -1$ [30]) for

$$\frac{1}{2}\sqrt{3} \leq e^{\eta} \leq \frac{2}{3}\sqrt{3}. \qquad (2.35)$$

The vector spectrum is given by

$$m\ell:\ \pm 2\,[1+1], \qquad 1 \pm \sqrt{2\cosh(2\eta) - 1}\,[2+2], \qquad -1 \pm \sqrt{2\cosh(2\eta) - 1}\,[2+2], \qquad (2.36)$$

reflecting the unbroken $\mathrm{SO}(2) \times \mathrm{SO}(2) \subset \mathrm{SO}(4)$. Finally, the gravitino spectrum is given by

$$m\ell:\ \pm\frac{1}{2}(2\cosh\eta + (\mathrm{sgn}\alpha))\ [4+4], \qquad (2.37)$$

showing that only for $\alpha = -1$, the vacuum at $\eta = 0$ is supersymmetric, preserving $\mathcal{N} = (4,4)$ supersymmetry. This corresponds to the six-dimensional supersymmetric background $\mathrm{AdS}_3 \times S^3$. The $\alpha = +1$ solution is not supersymmetric, but it may correspond to a supersymmetric solution in an $\mathcal{N} = (1,0)$ theory coupled to tensor multiplets.

The potential (2.22) allows for additional stationary points with $\phi_{A\alpha} \neq 0$. In this case, the remaining equations (2.24) and (2.29) again are most conveniently solved in a basis in which $\tilde{m}_{AB}$ is diagonal, where we find four discrete solutions. They all necessitate positive $\alpha = +1$, with the potential taking the values

$$V_{(\mathrm{B}),0} = \begin{cases} -\frac{27}{8} & \text{(i)} \\ -\frac{8788}{3125} & \text{(ii)} \\ -4 & \text{(iii)} \\ -\frac{25}{8} & \text{(iv)} \end{cases}. \qquad (2.38)$$

All these stationary points fully break supersymmetry and SO(4) gauge symmetry, and they all contain unstable scalars with masses below the Breitenlohner-Freedman bound $m^2\ell^2 = -1$. For later checks, let us only note the location of solution (i):

$$m^{AB} = \mathfrak{m}\delta^{AB} = \frac{3}{2}\delta^{AB}, \qquad \phi_{A\alpha} = \frac{\sqrt{2}}{\sqrt{3}}\delta_{A\alpha}, \tag{2.39}$$

with the scalar mass spectrum given by

$$m^2\ell^2\colon\ \frac{2}{3} \pm 2\,[9+9], \quad 6 \pm 2\,[1+1], \quad 0\,[6], \tag{2.40}$$

in units of the AdS length $\ell = \frac{4}{3\sqrt{3}}$.

III. SO(8,4) EXCEPTIONAL FIELD THEORY

In this section, we review the structure of SO(8, 4) ExFT, constructed in Ref. [24], to which we refer for details. This theory provides the manifestly duality covariant formulation of the 6D supergravity theories relevant for our consistent truncations. We discuss the inequivalent solution to its section constraints and establish the dictionary of the ExFT fields to the 6D fields of $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravity theories.

A. Lagrangian

Similar to the three-dimensional supergravities reviewed in Sec. II, SO(8, 4) ExFT is based on the coset space [Eq. (2.1)], which we parametrize by a symmetric positive definite matrix $\mathcal{M}_{MN}$. In contrast to the matrix of Eq. (2.17), this matrix depends not only on three external coordinates x^μ, but in addition on (dim SO(8, 4)) coordinates Y^{MN}, with the latter dependence strongly constrained by the section conditions

$$\partial_{[MN} \otimes \partial_{KL]} = 0 = \eta^{NK}\partial_{MN} \otimes \partial_{KL}, \tag{3.1}$$

which restrict the fields to live on sections of dimension three (at most). Depending on the choice of these sections, the theory describes the 6D $\mathcal{N} = (1,1)$ or $\mathcal{N} = (2,0)$ supergravity, respectively. The theory is invariant under generalized internal diffeomorphisms, acting as

$$\mathcal{L}_{\Lambda,\Sigma}\mathcal{M}^{MN} = \Lambda^{KL}\partial_{KL}\mathcal{M}^{MN} + 4(\partial^{K(M}\Lambda_{KL} - \partial_{KL}\Lambda^{K(M} + 2\Sigma^{(M}{}_L)\mathcal{M}^{N)L} \tag{3.2}$$

on the scalar matrix. Here, the gauge parameters Σ_{MN} are subject to algebraic constraints analogous to Eq. (3.1), i.e.,

$$\Sigma_{[MN} \otimes \Sigma_{KL]} = 0 = \eta^{NK}\Sigma_{MN} \otimes \Sigma_{KL}, \tag{3.3}$$

as well as compatibility with the partial derivatives as

$$\Sigma_{[MN} \otimes \partial_{KL]} = 0 = \eta^{NK}\Sigma_{MN} \otimes \partial_{KL}. \tag{3.4}$$

Invariance under local internal diffeomorphisms [Eq. (3.2)] is ensured by minimal couplings to gauge fields $(\mathcal{A}_\mu{}^{MN}, \mathcal{B}_{\mu MN})$ via covariant external derivatives:

$$D_\mu = \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu,\mathcal{B}_\mu}. \tag{3.5}$$

The full Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{\rm EH} + \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm CS} - \sqrt{-g}V_{\rm ExFT}, \tag{3.6}$$

each term being separately invariant under generalized internal diffeomorphisms [Eq. (3.2)]. The modified Einstein-Hilbert term and the scalar kinetic term have the forms

$$\begin{aligned}\mathcal{L}_{\rm EH} &= \sqrt{-g}e_a{}^\mu e_b{}^\nu(R_{\mu\nu}{}^{ab} + F_{\mu\nu}{}^{MN}e^{a\rho}\partial_{MN}e_\rho{}^b) \equiv \sqrt{-g}\hat{R},\\ \mathcal{L}_{\rm kin} &= \frac{1}{8}\sqrt{-g}g^{\mu\nu}D_\mu\mathcal{M}^{MN}D_\nu\mathcal{M}_{MN},\end{aligned} \tag{3.7}$$

with the covariant derivatives [Eq. (3.5)], the Yang-Mills field strength $F_{\mu\nu}{}^{MN}$, and the Riemann tensor $R_{\mu\nu}{}^{ab}$ computed from the external vielbein $e_\mu{}^a$ with derivatives covariantized under internal diffeomorphisms, under which $e_\mu{}^a$ transforms as a scalar density (of weight $\lambda = 1$). The gauge fields couple with a Chern-Simons term that takes the explicit form

$$\begin{aligned}\mathcal{L}_{\rm CS} = \sqrt{2}\varepsilon^{\mu\nu\rho}\Big(&F_{\mu\nu}{}^{MN}\mathcal{B}_{\rho MN} + \partial_\mu\mathcal{A}_{\nu N}{}^K\partial_{KM}\mathcal{A}_\rho{}^{MN} - \frac{2}{3}\partial_{MN}\partial_{KL}\mathcal{A}_\mu{}^{KP}\mathcal{A}_\nu{}^{MN}\mathcal{A}_{\rho P}{}^L\\ &+\frac{2}{3}\mathcal{A}_\mu{}^{LN}\partial_{MN}\mathcal{A}_\nu{}^M{}_P\partial_{KL}\mathcal{A}_\rho{}^{PK} - \frac{4}{3}\mathcal{A}_\mu{}^{LN}\partial_{MP}\mathcal{A}_\nu{}^M{}_N\partial_{KL}\mathcal{A}_\rho{}^{PK}\Big).\end{aligned} \tag{3.8}$$

Finally, the last term in Eq. (3.6) carries only internal derivatives ∂_{MN} and is given by

$$\begin{aligned}V_{\rm ExFT} \equiv &-\frac{1}{8}\mathcal{M}^{KP}\mathcal{M}^{LQ}\partial_{KL}\mathcal{M}_{MN}\partial_{PQ}\mathcal{M}^{MN} - \frac{1}{2}\partial_{MK}\mathcal{M}^{NP}\partial_{NL}\mathcal{M}^{MQ}\mathcal{M}^{KL}\mathcal{M}_{PQ} - \frac{1}{4}\partial_{MN}\mathcal{M}^{PK}\partial_{KL}\mathcal{M}^{QM}\mathcal{M}_P{}^L\mathcal{M}_Q{}^N\\ &+2\partial_{MK}\mathcal{M}^{NK}\partial_{NL}\mathcal{M}^{ML} - g^{-1}\partial_{MN}g\partial_{KL}\mathcal{M}^{MK}\mathcal{M}^{NL} - \frac{1}{4}\mathcal{M}^{MK}\mathcal{M}^{NL}g^{-2}\partial_{MN}g\partial_{KL}g - \frac{1}{4}\mathcal{M}^{MK}\mathcal{M}^{NL}\partial_{MN}g^{\mu\nu}\partial_{KL}g_{\mu\nu}.\end{aligned} \tag{3.9}$$

Depending on the solution of the section constraints [Eq. (3.1)], the action [Eq. (3.6)] describes 6D $\mathcal{N} = (1,1)$ or $\mathcal{N} = (2,0)$ supergravity. In the next two subsections, we review the two inequivalent solutions to the section constraints and the associated dictionaries of the ExFT fields into the 6D supergravity fields.

B. $\mathcal{N}=(1,1)$ solution of section constraint

Consider the decomposition of SO(8, 4) under its subgroup

$$\mathrm{GL}(3)\times\mathrm{SO}(1,1)\times\mathrm{SO}(4)\subset\mathrm{SO}(4,4)\times\mathrm{SO}(4)\subset\mathrm{SO}(8,4), \tag{3.10}$$

such that the fundamental vector of SO(8, 4) decomposes as

$$\{V^M\}\to\{(V^i)_{(-1)},(V_i)_{(+1)},(V^0)_{(-3)},(V_0)_{(+3)},(V^\alpha)_{(0)}\}, \tag{3.11}$$

where subscripts refer to the sum of the $\mathrm{GL}(1)\subset\mathrm{GL}(3)$ charge and the SO(1, 1) charge, defining the grading associated with the higher-dimensional origin of these fields. Here $i=1,2,3$ and $\alpha=1,\ldots,4$. The invariant tensor η_{MN} decomposes accordingly:

$$\eta_{MN}=\begin{pmatrix} 0 & \delta_i{}^j & 0 & 0 & 0\\ \delta_j{}^i & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -\delta_{\alpha\beta}\end{pmatrix}. \tag{3.12}$$

The $\mathcal{N}=(1,1)$ solution to the section constraints [Eq. (3.1)] is given by decomposing coordinates Y^{MN} according to Eq. (3.11) and restricting the internal coordinate dependence of all fields to the coordinates Y^{0i}, such that the only nonvanishing internal derivatives are

$$\partial_i\equiv\frac{1}{\sqrt{2}}\partial_{0i}, \tag{3.13}$$

providing a solution to Eq. (3.1). Breaking the ExFT fields according to Eq. (3.11) then matches the field content of the 6D $\mathcal{N}=(1,1)$ supergravity, which, in addition to the metric and the dilaton, contains four vector fields and a (non-chiral) 2-form gauge field.

Specifically, the ExFT vector fields transform in the adjoint representation of SO(8, 4). Under Eq. (3.11), they decompose into

$$\mathcal{A}_\mu{}^{MN}\to\begin{cases} -4\ :\ \mathcal{A}_\mu{}^{0i} & \subset\ \text{6D metric}\\ -3\ :\ \mathcal{A}_\mu{}^{0\alpha} & \subset\ \text{6D vectors}\\ -2\ :\ \mathcal{A}_{\mu i}{}^{0},\mathcal{A}_\mu{}^{ij} & \subset\ \text{6D 2-form and dual}\\ -1\ :\ \mathcal{A}_\mu{}^{i\alpha} & \subset\ \text{6D dual 3-form}\\ \ \ 0\ :\ \mathcal{A}_{\mu i}{}^{j},\mathcal{A}_{\mu 0}{}^{0},\mathcal{A}_\mu{}^{\alpha\beta} & \subset\ \text{6D dual graviton, etc.}\\ +1\ :\ \ldots & \end{cases}, \tag{3.14}$$

allowing us to identify the higher-dimensional origin of the various components. The fields of positive grading do not enter the action [Eq. (3.6)]. Similarly, one decomposes the scalar fields, parametrizing the coset SO(8,4)/(SO(8)× SO(4)) into

$$\begin{cases} +4\ :\ \phi_{0i} & \subset\ \text{6D dual graviton}\\ +3\ :\ \phi_{\alpha 0} & \subset\ \text{6D dual 3-form}\\ +2\ :\ \phi^i{}_0,\phi_{ij} & \subset\ \text{6D 2-form and its dual}\\ +1\ :\ \phi_{i\alpha} & \subset\ \text{6D vectors}\\ \ \ 0\ :\ g_{ij},\varphi & \subset\ \text{6D metric and dilaton}\end{cases}. \tag{3.15}$$

In order to identify their location within the scalar matrix $\mathcal{M}_{MN}$, it is useful to determine the action of a generalized diffeomorphism [Eq. (3.2)] on the various components of the matrix $\mathcal{M}^{MN}$. In the decomposition (3.11), this gives particularly neat expressions when acting on some specific combinations:

$$\begin{aligned}
\mathcal{L}_{\Lambda,\Sigma}\mathcal{M}^{00}&=L_\lambda\mathcal{M}^{00}-2(\partial_k\lambda^k)\mathcal{M}^{00},\\
\mathcal{L}_{\Lambda,\Sigma}\mathcal{M}^{0i}&=L_\lambda\mathcal{M}^{0i}-(\partial_k\lambda^k)\mathcal{M}^{0i}-\varepsilon^{ijk}\partial_j\xi_k\mathcal{M}^{00},\\
\mathcal{L}_{\Lambda,\Sigma}(\mathcal{M}^{00}\mathcal{M}^{ij}-\mathcal{M}^{0i}\mathcal{M}^{0j})&=L_\lambda(\mathcal{M}^{00}\mathcal{M}^{ij}-\mathcal{M}^{0i}\mathcal{M}^{0j})-2(\partial_m\lambda^m)(\mathcal{M}^{00}\mathcal{M}^{ij}-\mathcal{M}^{0i}\mathcal{M}^{0j}),\\
\mathcal{L}_{\Lambda,\Sigma}(\mathcal{M}^{00}\mathcal{M}^{\alpha i}-\mathcal{M}^{0\alpha}\mathcal{M}^{0i})&=L_\lambda(\mathcal{M}^{00}\mathcal{M}^{\alpha i}-\mathcal{M}^{0\alpha}\mathcal{M}^{0i})+\partial_j\Lambda^\alpha(\mathcal{M}^{00}\mathcal{M}^{ij}-\mathcal{M}^{0i}\mathcal{M}^{0j})-2(\partial_m\lambda^m)(\mathcal{M}^{00}\mathcal{M}^{\alpha i}-\mathcal{M}^{0\alpha}\mathcal{M}^{0i}),\\
\mathcal{L}_{\Lambda,\Sigma}(\mathcal{M}^{00}\mathcal{M}^{i}{}_j-\mathcal{M}^{0i}\mathcal{M}^{0}{}_j)&=L_\lambda(\mathcal{M}^{00}\mathcal{M}^{i}{}_j-\mathcal{M}^{0i}\mathcal{M}^{0}{}_j)+\partial_j\Lambda^\alpha(\mathcal{M}^{00}\mathcal{M}^{\alpha i}-\mathcal{M}^{0\alpha}\mathcal{M}^{0i})\\
&\quad+(\partial_k\tilde{\xi}_j-\partial_j\tilde{\xi}_k)(\mathcal{M}^{00}\mathcal{M}^{ik}-\mathcal{M}^{0i}\mathcal{M}^{0k})-2(\partial_m\lambda^m)(\mathcal{M}^{00}\mathcal{M}^{i}{}_j-\mathcal{M}^{0i}\mathcal{M}^{0}{}_j),
\end{aligned} \tag{3.16}$$

where we have redefined the gauge parameters as

$$\Lambda^{0i}=\frac{1}{\sqrt{2}}\lambda^i,\qquad \Lambda^{ij}=\frac{1}{\sqrt{2}}\varepsilon^{ijk}\xi_k,\qquad \Lambda_i{}^0=\frac{1}{\sqrt{2}}\tilde{\xi}_i,\qquad \Lambda^{\alpha 0}=\frac{1}{\sqrt{2}}\Lambda^\alpha,\qquad \Lambda^{\alpha i}=\frac{1}{2\sqrt{2}}\varepsilon^{ijk}\Lambda_{jk}{}^\alpha, \tag{3.17}$$

with the totally antisymmetric ε^{ijk}. Here, L_λ denotes the standard Lie derivative along the vector field λ^k. Identifying the higher-dimensional origin of the gauge parameters among internal 6D diffeomorphisms and gauge transformations according to the identification of the vector fields [Eq. (3.14)] then allows us to read off the dictionary between the components of $\mathcal{M}^{MN}$ and the internal components of the 6D fields:

$$\begin{aligned}
\mathcal{M}^{00} &= g^{-1}e^{\phi},\\
\mathcal{M}^{0i} &= -\frac{1}{2}\mathcal{M}^{00}\varepsilon^{ijk}B_{jk},\\
\mathcal{M}^{00}\mathcal{M}^{ij}-\mathcal{M}^{0i}\mathcal{M}^{0j} &= g^{-1}g^{ij},\\
\mathcal{M}^{00}\mathcal{M}^{\alpha i}-\mathcal{M}^{0\alpha}\mathcal{M}^{0i} &= g^{-1}g^{ij}A_j{}^{\alpha},\\
\mathcal{M}^{00}\mathcal{M}^{i}{}_{j}-\mathcal{M}^{0i}\mathcal{M}^{0}{}_{j} &= g^{-1}g^{ik}\tilde{B}_{kj}+\frac{1}{2}g^{-1}g^{ik}A_k{}^{\alpha}A_j{}^{\alpha}.
\end{aligned} \tag{3.18}$$

The dictionary is such that the generalized diffeomorphisms [Eq. (3.2)] reproduce the gauge transformations,

$$\begin{aligned}
&\delta A_i{}^{\alpha}=\partial_i\Lambda^{\alpha},\\
&\delta B_{ij}=2\partial_{[i}\xi_{j]},\\
&\delta\tilde{B}_{ij}=2\partial_{[i}\tilde{\xi}_{j]}+\partial_{[i}\Lambda^{\alpha}A_{j]}{}^{\alpha}=2\partial_{[i}\left(\tilde{\xi}_{j]}+\frac{1}{2}\Lambda^{\alpha}A_{j]}{}^{\alpha}\right)-\frac{1}{2}\Lambda^{\alpha}F_{ij}{}^{\alpha},
\end{aligned} \tag{3.19}$$

of the 6D vector fields $A_i{}^{\alpha}$, 2-form B_{ij}, and its dual $\tilde{B}_{ij}$. Finally, using the dictionary [Eq. (3.18)], we may also consider

$$\begin{aligned}
\mathcal{L}_{\Lambda,\Sigma}((\mathcal{M}^{00})^{-1}\mathcal{M}^{0\alpha}) = L_{\lambda}&((\mathcal{M}^{00})^{-1}\mathcal{M}^{0\alpha})\\
&+(\partial_k\lambda^k)((\mathcal{M}^{00})^{-1}\mathcal{M}^{0\alpha})\\
&-\frac{1}{2}\varepsilon^{ijk}\partial_i\Lambda_{jk}{}^{\alpha}-\frac{1}{2}\partial_i\Lambda^{\alpha}\varepsilon^{ijk}B_{jk},
\end{aligned} \tag{3.20}$$

from which we infer the identification

$$(\mathcal{M}^{00})^{-1}\mathcal{M}^{0\alpha}=-\frac{1}{6}\varepsilon^{ijk}a_{ijk}{}^{\alpha}, \tag{3.21}$$

with the scalars $a_{ijk}{}^{\alpha}$ from the dual 3-forms in six dimensions. These transform as

$$\delta a_{ijk}{}^{\alpha}=3\partial_{[i}\Lambda_{jk]}{}^{\alpha}+3B_{[ij}\partial_{k]}\Lambda^{\alpha}, \tag{3.22}$$

under 6D gauge transformations.

C. $\mathcal{N}=(2,0)$ solution of section constraint

Consider the decomposition of $SO(8,4)$ under its subgroup $GL(3)\times SO(1,5)$, such that the fundamental $SO(8,4)$ vector decomposes as

$$\{V^M\}\to\{V_{i(-2)},V^{i}{}_{(+2)},V^0_{(0)},V^a_{(0)}\}, \tag{3.23}$$

with subscripts referring to $GL(1)$ charges. Here $i=1,2,3$ and $a=1,\ldots,5$. The invariant metric η_{MN} decomposes as

$$\eta_{MN}=\begin{pmatrix} 0 & \delta^i{}_j & 0 & 0\\ \delta^j{}_i & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -\delta_{ab}\end{pmatrix}. \tag{3.24}$$

Later on, we will also decompose $a\to\{\bar{0},\alpha\}$. The $\mathcal{N}=(2,0)$ solution to the section constraints in Eq. (3.1) is given by decomposing coordinates Y^{MN} according to Eq. (3.23) and restricting the internal coordinate dependence of all fields to the coordinates Y_{ij}, such that the only nonvanishing internal derivatives are

$$\partial_i\equiv\frac{1}{2}\varepsilon_{ijk}\partial^{jk}, \tag{3.25}$$

providing a solution to Eq. (3.1). Breaking the ExFT fields according to Eq. (3.23) then matches the field content of the 6D $\mathcal{N}=(2,0)$ supergravity coupled to a tensor multiplet, which contains five self-dual and one anti-self-dual 2-form gauge fields, together with five scalar fields parametrizing the coset space $SO(1,5)/SO(5)$.

Specifically, the ExFT vector fields transform in the adjoint representation of $SO(8,4)$. Under Eq. (3.23), they decompose into

$$\mathcal{A}_{\mu}{}^{MN}\to\begin{cases} -4\ :\ \mathcal{A}_{\mu ij} & \subset \text{6D metric}\\ -2\ :\ \mathcal{A}_{\mu i}{}^{a},\mathcal{A}_{\mu i}{}^{0} & \subset \text{6D 2-forms}\\ \ \ 0\ :\ \mathcal{A}_{\mu i}{}^{j},\mathcal{A}_{\mu}{}^{ab},\mathcal{A}_{\mu}{}^{a0} & \subset \text{6D dual graviton, etc.}\\ \ \ 2\ :\ \ldots \end{cases}, \tag{3.26}$$

allowing us to identify the higher-dimensional origin of the various components. The fields of positive grading do not enter the action [Eq. (3.6)]. Similarly, one decomposes the scalar fields, parametrizing the coset $SO(8,4)/(SO(8)\times SO(4))$ into

$$\begin{cases} +4\ :\ \phi^{ij} & \subset \text{6D dual graviton}\\ +2\ :\ \phi_a{}^{i},\phi_0{}^{i} & \subset \text{6D 2-forms}\\ \ \ 0\ :\ m_{ij},\phi,m_{\bar{a}\bar{b}} & \subset \text{6D metric and scalars}\end{cases}, \tag{3.27}$$

where $\bar{a}=0,\ldots,5$, and m_{ij} and $m_{\bar{a}\bar{b}}$ parametrize the coset spaces $SL(3)/SO(3)$ and $SO(1,5)/SO(5)$, respectively. For the latter, we have the invariant tensor

$$\eta_{\bar{a}\bar{b}}=\begin{pmatrix}1 & 0\\ 0 & -\delta_{ab}\end{pmatrix}. \tag{3.28}$$

In order to identify the precise location of scalar fields within the scalar matrix $\mathcal{M}_{MN}$, it is useful to determine the action of a generalized diffeomorphism (3.2) on the various components of the matrix $\mathcal{M}^{MN}$. In the decomposition (3.23), this takes the form

$$\begin{aligned}\mathcal{L}_{\Lambda,\Sigma}\mathcal{M}_{ij} &= L_\lambda \mathcal{M}_{ij} - 2(\partial_k\lambda^k)\mathcal{M}_{ij},\\ \mathcal{L}_{\Lambda,\Sigma}\mathcal{M}_i{}^{\bar a} &= L_\lambda\mathcal{M}^{i\bar a} - (\partial_k\lambda^k)\mathcal{M}_i{}^{\bar a} + 2\varepsilon^{kmn}\mathcal{M}_{ik}\partial_m\Lambda^{\bar a}{}_n,\\ \mathcal{L}_{\Lambda,\Sigma}\mathcal{M}^{\bar a\bar b} &= L_\lambda\mathcal{M}^{\bar a\bar b} + 4\varepsilon^{ijk}\mathcal{M}_k{}^{(\bar a}\partial_i\Lambda^{\bar b)}{}_j,\end{aligned} \tag{3.29}$$

with the gauge parameter relabeled as

$$\Lambda_{ij} \equiv \frac{1}{2}\varepsilon_{ijk}\lambda^k. \tag{3.30}$$

These let us infer the dictionary

$$\begin{aligned}\mathcal{M}_{ij} &= g^{-1}g_{ij},\\ \mathcal{M}_i{}^{\bar a} &= g^{-1}g_{ij}\varepsilon^{jkl}B_{kl}{}^{\bar a},\\ \mathcal{M}^{\bar a\bar b} &= M^{\bar a\bar b} + 2B_{ij}{}^{\bar a}B_{kl}{}^{\bar b}g^{ik}g^{jl} = M^{\bar a\bar b} + gg^{ij}\mathcal{M}_i{}^{\bar a}\mathcal{M}_j{}^{\bar b},\end{aligned} \tag{3.31}$$

with $g \equiv \det g_{ij}$, and the components $B_{ij}{}^{\bar a}$ transform under tensor gauge transformations as $\delta B_{ij}{}^{\bar a} = 2\partial_{[i}\Lambda^{\bar a}{}_{j]}$.

D. Generalized Scherk-Schwarz reduction

Consistent truncations of $\mathrm{SO}(8,4)$ ExFT can be defined by a generalized Scherk-Schwarz compactification *Ansatz* [17,18], in which the dependence of the ExFT fields on the internal coordinates is carried by an $\mathrm{SO}(8,4)$ twist matrix $\mathcal{U}_M{}^{\bar N}$ and a scalar factor ρ. Specifically, the ExFT fields take the factorized form [24]

$$\begin{aligned}g_{\mu\nu}(x,Y) &= \rho(Y)^{-2}g_{\mu\nu}(x),\\ \mathcal{M}_{MN}(x,Y) &= \mathcal{U}_M{}^{\bar M}(Y)M_{\bar M\bar N}(x)\mathcal{U}_N{}^{\bar N}(Y),\\ \mathcal{A}_\mu{}^{MN}(x,Y) &= \rho(Y)^{-1}\mathcal{U}^M{}_{\bar M}(Y)\mathcal{U}^N{}_{\bar N}(Y)A_\mu{}^{\bar M\bar N}(x),\\ \mathcal{B}_{\mu KL}(x,Y) &= -\frac{1}{4}\rho(Y)^{-1}\mathcal{U}_{M\bar N}(Y)\partial_{KL}\mathcal{U}^M{}_{\bar M}(Y)A_\mu{}^{\bar M\bar N}(x),\end{aligned} \tag{3.32}$$

in terms of the x-dependent fields of 3D gauged supergravity reviewed in Sec. II above. The embedding tensor [Eq. (2.2)] of the 3D theory is given in terms of the twist matrix as

$$\begin{aligned}\theta_{\bar K\bar L\bar P\bar Q} &= 6\rho^{-1}\partial_{LP}\mathcal{U}_{N[\bar K}\mathcal{U}^N{}_{\bar L}\mathcal{U}^L{}_{\bar P}\mathcal{U}^P{}_{\bar Q]},\\ \theta_{\bar P\bar Q} &= 4\rho^{-1}\mathcal{U}^K{}_{\bar P}\partial_{KL}\mathcal{U}^L{}_{\bar Q} - \frac{\rho^{-1}}{3}\eta_{\bar P\bar Q}\mathcal{U}^{K\bar L}\partial_{KL}\mathcal{U}^L{}_{\bar L}\\ &\quad - 4\rho^{-2}\partial_{\bar P\bar Q}\rho,\\ \theta &= \frac{\rho^{-1}}{3}\mathcal{U}^{K\bar L}\partial_{KL}\mathcal{U}^L{}_{\bar L},\end{aligned} \tag{3.33}$$

and the truncation is consistent if all three objects in Eq. (3.33) are actually Y-independent. Using the twist matrices from Ref. [24] we will, in the following, use the generalized Scherk-Schwarz *Ansatz* in order to derive the explicit reduction formulas for the 6D consistent truncations.

IV. $\mathcal{N} = (1,1)$ UPLIFT FORMULAS

In this and the following section, we will review from Ref. [24] the twist matrices inducing the embedding tensors [Eq. (2.11)]. Combining them with the *Ansatz* [Eq. (3.32)] and the supergravity dictionaries worked out in Secs. III B and III C above, we deduce the six-dimensional $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ reduction formulas.

A. Twist matrix

The $\mathrm{SO}(8,4)$ twist matrix $\mathcal{U}_M{}^{\bar M}$ describing the consistent S^3 truncation of 6D $\mathcal{N} = (1,1)$ supergravity has been constructed in Ref. [24]. Let us recall that the coordinates y^i relevant for 6D $\mathcal{N} = (1,1)$ supergravity have been identified among the Y^{MN} via Eq. (3.13). The associated twist matrix is given in terms of the elementary S^3 sphere harmonics $\mathcal{Y}^A$ (satisfying $\mathcal{Y}^A\mathcal{Y}^A = 1$), the round S^3 metric $\mathring{g}_{ij} = \partial_i\mathcal{Y}^A\partial_j\mathcal{Y}^A$ (with determinant $\mathring{g}$), and the vector field $\mathring{\zeta}^i$ defined by $\mathring{\nabla}_i\mathring{\zeta}^i = 1$. By $\mathring{\omega}_{ijk} \equiv \mathring{g}^{1/2}\varepsilon_{ijk}$, we denote the associated volume form. We refer to the Appendix for further identities among these objects. After some rewriting, the twist matrix of Ref. [24] takes the explicit form

$$\mathcal{U}_M{}^{\bar M} = \begin{pmatrix} \mathcal{U}_0{}^A & \mathcal{U}_{0A} & 0\\ \mathcal{U}_i{}^A & \mathcal{U}_{iA} & 0\\ \mathcal{U}^{0A} & \mathcal{U}^0{}_A & 0\\ \mathcal{U}^{iA} & \mathcal{U}^i{}_A & 0\\ 0 & 0 & \delta_\alpha{}^\beta\end{pmatrix} = \begin{pmatrix} \mathring{g}^{1/2}(\mathcal{Y}^A - 2\mathring{\zeta}^i\partial_i\mathcal{Y}^A) & 0 & 0\\ \partial_i\mathcal{Y}^A & 2\alpha\mathring{g}^{jl}\partial_l\mathcal{Y}^A\mathring{\omega}_{ijk}\mathring{\zeta}^k & 0\\ 0 & \mathring{g}^{-1/2}\mathcal{Y}^A & 0\\ 0 & 2\mathring{\zeta}^i\mathcal{Y}^A + \mathring{g}^{ij}\partial_j\mathcal{Y}^A & 0\\ 0 & 0 & \delta_\alpha{}^\beta\end{pmatrix}, \tag{4.1}$$

in a basis where the "curved index" M is decomposed according to Eq. (3.11), and the "flat index" $\bar{M}$ is decomposed in the basis (2.9), suitable for the fields of 3D supergravity. The free parameter α can (up to sign) be absorbed into a shift of the 6D dilaton.

B. Uplift formulas

According to the dictionary [Eq. (3.18)], the 6D dilaton is identified within the component $\mathcal{M}^{00}$ of the ExFT scalar matrix, such that its reduction formula is obtained via Eq. (3.32) as

$$e^{\phi} = g\mathcal{M}^{00} = g\mathcal{U}^0{}_A\mathcal{U}^0{}_B m^{AB} = \Delta^2\mathcal{Y}^A\mathcal{Y}^B m^{AB}, \tag{4.2}$$

where we have defined the warp factor

$$\Delta \equiv \frac{g^{1/2}}{\mathring{g}^{1/2}}. \tag{4.3}$$

Similarly, we identify the internal components of the 6D 2-form as

$$\begin{aligned} -\frac{1}{2}\varepsilon^{ijk}B_{jk} &= (\mathcal{M}^{00})^{-1}\mathcal{M}^{0i} \\ &= (\mathcal{M}^{00})^{-1}\mathring{g}^{-1/2}(2\mathring{\zeta}^i\mathcal{Y}^B + \mathring{g}^{ij}\partial_j\mathcal{Y}^B)\mathcal{Y}^A m^{AB}, \end{aligned} \tag{4.4}$$

giving rise to

$$B_{ij} = -\mathring{\omega}_{ijk}\left(2\mathring{\zeta}^k + \frac{1}{2}\mathring{g}^{kl}\partial_l \log\left(\mathcal{Y}^A\mathcal{Y}^B m^{AB}\right)\right). \tag{4.5}$$

Further computation yields the 6D internal (inverse) metric

$$\begin{aligned} g^{ij} &= g(\mathcal{M}^{00}\mathcal{M}^{ij} - \mathcal{M}^{0i}\mathcal{M}^{0j}) \\ &= \Delta^2(\mathring{g}^{ik}\partial_k\mathcal{Y}^A)(\mathring{g}^{jl}\partial_l\mathcal{Y}^B)\mathcal{Y}^C\mathcal{Y}^D(m^{AB}m^{CD} - m^{AC}m^{BD}). \end{aligned} \tag{4.6}$$

Identifying the SO(4) Killing vectors $\mathcal{K}_{AB}{}^i = \mathring{g}^{ij}\partial_j\mathcal{Y}_{[A}\mathcal{Y}_{B]}$ on the right-hand side, this result reproduces the standard Kaluza-Klein *Ansatz* for the internal metric [31]. Using sphere harmonics identities collected in the Appendix, we may deduce the internal metric

$$g_{ij} = \frac{\Delta^{-2}}{(\mathcal{Y}^A\mathcal{Y}^B m^{AB})}\partial_i\mathcal{Y}^C\partial_j\mathcal{Y}^D m_{CD}, \tag{4.7}$$

together with a compact expression for the warp factor [Eq. (4.3)]

$$\Delta = (e^{-\varphi/2})(\mathcal{Y}^A\mathcal{Y}^B m^{AB})^{-1/4}, \tag{4.8}$$

where we recall the definition [Eq. (2.21)] of the 3D scalar φ. The latter may be used to simplify the reduction formulas in Eqs. (4.2)–(4.7) as

$$\begin{aligned} e^{\phi} &= \Delta^{-2}e^{-2\varphi}, \\ B_{ij} &= -2\mathring{\omega}_{ijk}(\mathring{\zeta}^k - \mathring{g}^{kl}\partial_l\log\Delta), \\ g_{ij} &= \Delta^2 e^{\varphi}\partial_i\mathcal{Y}^C\partial_j\mathcal{Y}^D\tilde{m}_{CD}. \end{aligned} \tag{4.9}$$

We thus obtain a compact form of the full 6D metric:

$$\begin{aligned} ds_6^2 = e^{\varphi}(&\Delta^{-2}e^{-\varphi}g_{\mu\nu}(x)dx^\mu dx^\nu \\ &+ \Delta^2 e^{\varphi}\partial_i\mathcal{Y}^C\partial_j\mathcal{Y}^D m_{CD}dy^i dy^j), \end{aligned} \tag{4.10}$$

and we may also compute the internal component of the 3-form field strength:

$$3\partial_{[i}B_{jk]} = \mathring{\omega}_{ijk}(2e^{4\varphi}\Delta^8\mathcal{Y}^A m^{AC}m^{CB}\mathcal{Y}^B - e^{2\varphi}\Delta^4 m^{AA}). \tag{4.11}$$

We may compare these results to the reduction formulas found in Refs. [12,15] for the $\mathcal{N} = (1,0)$ subsector and find precise agreement upon applying the dictionary:

$$\begin{aligned} \mathcal{Y}^A &\longleftrightarrow \mu^i, \\ m^{AB} &\longleftrightarrow T_{ij}, \\ e^{\varphi} &\longleftrightarrow (\det T)^{1/4}, \\ \Delta^{-2}e^{-\varphi} &\longleftrightarrow \Delta^{1/2}, \\ e^{\phi} &\longleftrightarrow e^{-\sqrt{2}\varphi/2}. \end{aligned} \tag{4.12}$$

The present construction extends these formulas to the full $\mathcal{N} = (1,1)$ theory. The additional matter is made from $\mathcal{N} = (1,0)$ vector multiplets, whose reduction formulas are extracted from Eq. (3.18) as

$$g^{-1}g^{ij}A_j{}^\alpha = \mathring{g}^{-1}\mathring{g}^{ij}(\partial_j\mathcal{Y}^C)\mathcal{Y}^A\mathcal{Y}^B(m^{AB}M^{C\alpha} - m^{AC}M^{B\alpha}), \tag{4.13}$$

which upon combination with Eq. (4.9) and after some computation reduces to the simple formula

$$A_i{}^\alpha = (\partial_i\mathcal{Y}^A)\phi_A{}^\alpha, \tag{4.14}$$

showing that, in particular, the internal field strengths vanish:

$$F_{ij}{}^\alpha = 2\partial_{[i}A_{j]}{}^\alpha = 0. \tag{4.15}$$

Similarly, we extract the reduction formula for the dual 2-form $\tilde{B}_{ij}$ upon combining Eq. (3.18) with all previously obtained reduction formulas, and we find

$$\tilde{B}_{ij} = -2\alpha\mathring{\omega}_{ijk}\mathring{\zeta}^k \Rightarrow 3\partial_{[i}\tilde{B}_{jk]} = -2\alpha\mathring{\omega}_{ijk}. \tag{4.16}$$

Finally, we may work out the reduction formula for the internal components of the 6D 3-form (dual to the 6D vector fields) as

$$a_{ijk}{}^\alpha = -\mathring{\omega}_{ijk}e^{2\varphi}\Delta^4\mathcal{Y}^A m^{AB}\phi_B{}^\alpha. \tag{4.17}$$

Formulas (4.14) and (4.17) show that in the case of 3D constant scalar solutions, all 6D vector field strengths vanish, such that the embedding of the $\mathcal{N}=(1,0)$ theory into the $\mathcal{N}=(1,1)$ theory remains rather trivial. This reflects the fact that the potential [Eq. (2.20)] does not carry the additional 3D scalar fields $\phi_{A\alpha}$ and thus coincides with the potential of the truncation to the quarter-maximal theory of Refs. [12,15]. In contrast, for solutions with running scalars, such as 3D RG flows in the potential [Eq. (2.20)], these formulas describe nontrivial 6D gauge fields.

V. $\mathcal{N}=(2,0)$ UPLIFT FORMULAS

In this section, we repeat the analysis for the reduction of the $\mathcal{N}=(2,0)$ theory. As already reflected by the richer structure of the 3D potential [Eq. (2.22)], in this case the uplift formulas to six dimensions constitute a rather nontrivial extension of the formulas [12,15] for the quarter-maximal truncation.

A. Twist matrix

The twist matrix describing the consistent truncation of 6D $\mathcal{N}=(2,0)$ supergravity has been given in Ref. [24] in terms of the same geometrical data introduced in Sec. IV A above. Let us recall that the coordinates y^i relevant for 6D $\mathcal{N}=(2,0)$ supergravity have been identified in Eq. (3.25) above. In a basis where the "curved index" M is decomposed according to Eq. (3.23), and the "flat index" $\bar M$ is decomposed in the basis (2.9), the associated twist matrix is given by

$$\mathcal{U}_M{}^{\bar M}=\begin{pmatrix} \mathcal{U}^{iA} & \mathcal{U}^i{}_A & 0\\ \mathcal{U}_i{}^A & \mathcal{U}_{iA} & 0\\ \mathcal{U}_0{}^A & \mathcal{U}_{0A} & 0\\ \mathcal{U}_{\bar 0}{}^A & \mathcal{U}_{\bar 0 A} & 0\\ 0 & 0 & \delta_{\alpha\beta}\end{pmatrix}=\begin{pmatrix} \mathring{g}^{1/2}(\mathring{g}^{ij}\partial_j\mathcal{Y}^A+2\mathring{\zeta}^i\mathcal{Y}^A) & 2\alpha\mathring{g}^{1/2}(\mathring{\zeta}^i\mathcal{Y}^A-2\mathring{\zeta}^i\mathring{\zeta}^j\partial_j\mathcal{Y}^A) & 0\\ 0 & \mathring{g}^{-1/2}\partial_i\mathcal{Y}^A & 0\\ \frac{1}{\sqrt2}\mathcal{Y}^A & \frac{1}{\sqrt2}(\mathcal{Y}^A-2(1+\alpha)\mathring{\zeta}^i\partial_i\mathcal{Y}^A) & 0\\ \frac{1}{\sqrt2}\mathcal{Y}^A & -\frac{1}{\sqrt2}(\mathcal{Y}^A-2(1-\alpha)\mathring{\zeta}^i\partial_i\mathcal{Y}^A) & 0\\ 0 & 0 & \delta_{\alpha\beta}\end{pmatrix}. \tag{5.1}$$

Again, the free parameter α can (up to sign) be absorbed into a shift of the 6D dilaton.

B. Uplift formulas for the 3D scalar sector

1. Metric

Combining the embedding [Eq. (3.31)] of the internal metric g_{ij} into the scalar matrix with the twist *Ansatz* [Eq. (3.32)] and the twist matrix [Eq. (5.1)], we read off

$$g_{ij}=\Delta^2\partial_i\mathcal{Y}^A\partial_j\mathcal{Y}^B m^{AB}, \tag{5.2}$$

where we have defined

$$\Delta=\Delta(x,y)\equiv\frac{g^{1/2}}{\mathring{g}^{1/2}}=e^{-\varphi}(\mathcal{Y}^A\mathcal{Y}^B m_{AB})^{-1/4}. \tag{5.3}$$

The matrix m^{AB} denotes the GL(4) matrix constituting a 4×4 block of the matrix $M_{\bar M\bar N}$ [Eq. (2.17)] parametrizing the 3D coset space [Eq. (2.1)]; the matrix m_{AB} is its inverse. Some algebraic manipulation [cf. Eq. (4.6) above] yields the explicit form of the inverse metric

$$\begin{aligned} g^{ij}=&\,e^{4\varphi}\Delta^2(\mathring{g}^{ik}\partial_k\mathcal{Y}^A)(\mathring{g}^{jl}\partial_l\mathcal{Y}^B)\mathcal{Y}^C\mathcal{Y}^D\\ &\times(m_{AB}m_{CD}-m_{AC}m_{BD}).\end{aligned} \tag{5.4}$$

Comparison to Eqs. (4.6), (4.8), and (4.9) above shows precise agreement with the reduction formulas obtained for the $\mathcal{N}=(1,1)$ theory upon redefinition [Eq. (2.23)] of the 3D fields.

2. 2-forms

In the same way, we extract the reduction formulas for the 6D 2-forms via the dictionary [Eq. (3.31)]. With the explicit form of the twist matrix [Eq. (5.1)], after some computation and use of the explicit formulas [Eqs. (5.2) and (5.4)], this gives rise to the expressions

$$B_{ij\alpha}=\frac12\Delta^{-2}\mathring{\omega}_{ijk}\mathring{g}^{kl}\partial_l(\Delta^2\mathcal{Y}^A\phi_{A\alpha}) \tag{5.5}$$

for the SO(4) vector of 2-forms, and

$$\begin{aligned} \sqrt2 B_{ij}{}^0=&-(1+\alpha)\mathring{\omega}_{ijk}\mathring{\zeta}^k+\mathring{\omega}_{ijk}\mathring{g}^{kl}\partial_l\log\Delta\\ &+\frac{1}{8\Delta^4}\mathring{\omega}_{ijk}\mathring{g}^{kl}\partial_l(\Delta^4(\phi\phi)_{AB}\mathcal{Y}^A\mathcal{Y}^B),\\ \sqrt2 B_{ij}{}^{\bar 0}=&-(1-\alpha)\mathring{\omega}_{ijk}\mathring{\zeta}^k+\mathring{\omega}_{ijk}\mathring{g}^{kl}\partial_l\log\Delta\\ &-\frac{1}{8\Delta^4}\mathring{\omega}_{ijk}\mathring{g}^{kl}\partial_l(\Delta^4(\phi\phi)_{AB}\mathcal{Y}^A\mathcal{Y}^B)\end{aligned} \tag{5.6}$$

for the remaining two 2-forms. For later use, it will be interesting to explicitly compute the associated field strengths $H_{ijk}{}^{\bar a}=3\partial_{[i}B_{jk]}{}^{\bar a}$:

$$
\begin{aligned}
H_{ijk\alpha} &= -\frac{1}{2}\mathring{\omega}_{ijk}\tilde{\Delta}^4(m_{BB}\mathcal{Y}^A + m_{AB}\mathcal{Y}^B - 2\tilde{\Delta}^4\mathcal{Y}^D m_{DC}m_{CB}\mathcal{Y}^B\mathcal{Y}^A)\phi_{A\alpha},\\
\sqrt{2}H_{ijk}{}^0 &= -\mathring{\omega}_{ijk}\left(\alpha + \frac{1}{2}\tilde{\Delta}^4 m_{AA} - \tilde{\Delta}^8\mathcal{Y}^A m_{AC}m_{CB}\mathcal{Y}^B\right) + \frac{1}{4}\mathring{\omega}_{ijk}(\delta^{AB} - \tilde{\Delta}^4(m_{CC}\mathcal{Y}^A + 2\mathcal{Y}^C m_{AC})\mathcal{Y}^B)(\phi\phi)_{AB}\\
&\quad + \frac{1}{2}\mathring{\omega}_{ijk}\tilde{\Delta}^8\mathcal{Y}^A\mathcal{Y}^B\mathcal{Y}^C\mathcal{Y}^E m_{CD}m_{DE}(\phi\phi)_{AB},\\
\sqrt{2}H_{ijk}{}^{\bar{0}} &= \mathring{\omega}_{ijk}\left(\alpha - \frac{1}{2}\tilde{\Delta}^4 m_{AA} + \tilde{\Delta}^8\mathcal{Y}^A m_{AC}m_{CB}\mathcal{Y}^B\right) - \frac{1}{4}\mathring{\omega}_{ijk}(\delta^{AB} - \tilde{\Delta}^4(m_{CC}\mathcal{Y}^A + 2\mathcal{Y}^C m_{AC})\mathcal{Y}^B)(\phi\phi)_{AB}\\
&\quad - \frac{1}{2}\mathring{\omega}_{ijk}\tilde{\Delta}^8\mathcal{Y}^A\mathcal{Y}^B\mathcal{Y}^C\mathcal{Y}^E m_{CD}m_{DE}(\phi\phi)_{AB},
\end{aligned}
\tag{5.7}
$$

where we have defined the rescaled $\tilde{\Delta} \equiv e^{\varphi}\Delta$.

3. Scalars

Eventually, we can compute the 6D scalar fields from the last line of Eq. (3.31) upon subtracting the B^2 term using explicit expressions from Eqs. (5.5) and (5.6) above. The five 6D scalars sit in a coset space $\mathrm{SO}(1,5)/\mathrm{SO}(5)$, which we parametrize by a symmetric positive definite matrix $M^{\bar{a}\bar{b}}$. Evaluation of Eq. (3.31) yields the various components of this matrix as

$$
\begin{aligned}
M^{00} &= \frac{1}{8}(4\tilde{\Delta}^{-4} + 4\mathcal{Y}^A(\phi\phi)_{AB}\mathcal{Y}^B\\
&\quad + \tilde{\Delta}^4(2 + \mathcal{Y}^A(\phi\phi)_{AB}\mathcal{Y}^B)^2),\\
M^{0\bar{0}} &= \frac{1}{8}(4\tilde{\Delta}^4 - (2\tilde{\Delta}^{-2} + \tilde{\Delta}^2\mathcal{Y}^A(\phi\phi)_{AB}\mathcal{Y}^B)^2),\\
M^{\bar{0}\bar{0}} &= \frac{1}{8}(4\tilde{\Delta}^{-4} + 4\mathcal{Y}^A(\phi\phi)_{AB}\mathcal{Y}^B\\
&\quad + \tilde{\Delta}^4(2 - \mathcal{Y}^A(\phi\phi)_{AB}\mathcal{Y}^B)^2)
\end{aligned}
\tag{5.8}
$$

for the 2×2 block in $(0,\bar{0})$ directions, and

$$
\begin{aligned}
M^0{}_\alpha &= \frac{1}{2\sqrt{2}}(2 + \tilde{\Delta}^4(2 + \mathcal{Y}^C(\phi\phi)_{CD}\mathcal{Y}^D))\mathcal{Y}^A\phi_{A\alpha},\\
M^{\bar{0}}{}_\alpha &= \frac{1}{2\sqrt{2}}(-2 + \tilde{\Delta}^4(2 - \mathcal{Y}^C(\phi\phi)_{CD}\mathcal{Y}^D))\mathcal{Y}^A\phi_{A\alpha},\\
M_{\alpha\beta} &= \delta_{\alpha\beta} + \tilde{\Delta}^4\mathcal{Y}^A\phi_{A\alpha}\phi_{B\beta}\mathcal{Y}^B
\end{aligned}
\tag{5.9}
$$

for the remaining components.

C. Uplift formulas for the 3D vector sector

Building on the dictionary [Eq. (3.26)], we may also give the uplift of the 3D vector fields. We recall from Sec. II A [in particular, Eq. (2.12)] that the 3D Lagrangian carries 12 vector fields: six $A_\mu{}^{AB}$ and the six antisymmetric combinations $A_{\mu A}{}^B - A_{\mu B}{}^A$. Moreover, in the 3D gauge we are using (in which scalars ϕ_{AB} are set to zero), the vector fields $A_\mu{}^A{}_B$ can be eliminated by means of their algebraic field equations in terms of scalar currents and the field strengths $\star F^{AB}$.

For the off-diagonal block of the 6D metric, we thus find

$$
g^{ij}g_{j\mu} = \frac{1}{2}\varepsilon^{ijk}\mathcal{A}_{\mu jk} = \frac{1}{2}\mathring{\omega}^{ijk}\partial_j\mathcal{Y}^A\partial_k\mathcal{Y}^B A_\mu{}^{AB} = \mathcal{K}_{AB}{}^i\tilde{A}_\mu{}^{AB}
\tag{5.10}
$$

in terms of the 3D vector fields from Eq. (2.13) and the SO(4) Killing vectors $\mathcal{K}_{AB}{}^i = \mathring{g}^{ij}\partial_j\mathcal{Y}_{[A}\mathcal{Y}_{B]}$, and where we have used the relation (A5). This consistently reproduces the standard Kaluza-Klein *Ansatz* for the vector fields [31], such that upon combination with the result of Sec. V B 1, the full 6D metric takes the form

$$
ds_6^2 = \Delta^{-2}g_{\mu\nu}(x)dx^\mu dx^\nu + g_{ij}Dy^iDy^j,
\tag{5.11}
$$

with

$$
Dy^i = dy^i + \tilde{A}_\mu{}^{AB}\mathcal{K}^i{}_{AB}dx^\mu.
\tag{5.12}
$$

Similarly, we can work out the reduction formulas for the off-diagonal blocks of the 6D 2-forms, leading to

$$
\begin{aligned}
B_{\mu i0} = \mathcal{A}_{\mu i0} &= \frac{1}{\sqrt{2}}(\partial_i\mathcal{Y}^A\mathcal{Y}^B(A_\mu{}^{AB} + A_\mu{}^A{}_B)\\
&\quad - 2(1+\alpha)\mathring{\zeta}^k\partial_i\mathcal{Y}^A\partial_k\mathcal{Y}^B A_\mu{}^{AB}),\\
B_{\mu i\bar{0}} = \mathcal{A}_{\mu i\bar{0}} &= \frac{1}{\sqrt{2}}(\partial_i\mathcal{Y}^A\mathcal{Y}^B(A_\mu{}^{AB} - A_\mu{}^A{}_B)\\
&\quad - 2(1-\alpha)\mathring{\zeta}^k\partial_i\mathcal{Y}^A\partial_k\mathcal{Y}^B A_\mu{}^{AB}),\\
B_{\mu i}{}^\alpha = \mathcal{A}_{\mu i}{}^\alpha &= \partial_i\mathcal{Y}^A A_\mu{}^{A\alpha}.
\end{aligned}
\tag{5.13}
$$

Note that the vector fields $A_\mu{}^{A\alpha}$ do not appear in the Lagrangian [Eq. (2.5)], and they can be defined on shell and subsequently be set to zero by a suitable (tensor) gauge transformation. The complete 6D 2-forms are then given by

$$
B_{\bar{a}} = \frac{1}{2}B_{ij\bar{a}}Dy^i\wedge Dy^j + B_{\mu i\bar{a}}dx^\mu\wedge Dy^i + \frac{1}{2}B_{\mu\nu\bar{a}}dx^\mu\wedge dx^\nu,
\tag{5.14}
$$

where the first two terms have been given in Eqs. (5.5), (5.6) and Eq. (5.13), respectively, while the missing components $B_{\mu\nu\bar{a}}$ are most conveniently obtained directly from the 6D tensor self-duality equations, which allow us to express their field strengths $H_{\mu\nu\rho\bar{a}}$ in terms of the associated $H_{ijk\bar{a}}$, computed in Eq. (5.7).

VI. SOME EXPLICIT UPLIFTS

In order to illustrate and check the nonlinear uplift formulas obtained, we will now use them to uplift some of the AdS_3 solutions corresponding to the stationary points of the 3D scalar potential to full solutions of 6D supergravity.

A. 6D $\mathcal{N} = (2,0)$ bosonic field equations

The field equations of 6D $\mathcal{N} = (2,0)$ supergravity, coupled to a tensor multiplet, have been given, e.g., in Refs. [32,33]. Apart from a metric, five self-dual and an anti-self-dual 2-form, they feature five scalars sitting in a coset space $\mathrm{SO}(1,5)/\mathrm{SO}(5)$ which parametrize a symmetric positive definite matrix $M^{\bar{a}\bar{b}}$. In our notation, the field equations read

$$\nabla_{\hat\mu}(M^{\bar{a}\bar{c}}\partial^{\hat\mu}M_{\bar{c}\bar{d}})\eta^{\bar{d}\bar{b}} = -\frac{1}{9}\sqrt{G}^{-1}\varepsilon^{\hat\mu\hat\nu\hat\rho\hat\lambda\hat\sigma\hat\tau}H_{\hat\mu\hat\nu\hat\rho}{}^{\bar{a}}H_{\hat\lambda\hat\sigma\hat\tau}{}^{\bar{b}},$$
$$R_{\hat\mu\hat\nu} - \frac{1}{2}R^{(6)}g_{\hat\mu\hat\nu} = -\frac{1}{8}\partial_{\hat\mu}M_{\bar{a}\bar{b}}\partial_{\hat\nu}M^{\bar{a}\bar{b}} + \frac{1}{16}g_{\hat\mu\hat\nu}\partial_{\hat\rho}M_{\bar{a}\bar{b}}\partial^{\hat\rho}M^{\bar{a}\bar{b}} + \frac{1}{2}H_{\hat\mu\hat\rho\hat\sigma}{}^{\bar{a}}H_{\hat\nu}{}^{\hat\rho\hat\sigma\bar{b}}M_{\bar{a}\bar{b}}, \qquad (6.1)$$

together with the 6D self-duality equations

$$\star H_{\bar{a}} = M_{\bar{a}\bar{b}}H^{\bar{b}}. \qquad (6.2)$$

We use indices $\hat\mu, \hat\nu, \ldots = 0, \ldots, 5$, to denote the curved 6D space-time indices.

B. One-parameter deformation of $\mathrm{AdS}_3 \times S^3$

As a first example, we give the 6D uplift of the non-supersymmetric but stable one-parameter family of AdS_3 solutions [Eq. (2.32)] located at

$$m^{AB} = \mathrm{diag}\{e^{\eta}, e^{\eta}, e^{-\eta}, e^{-\eta}\}, \qquad \phi_{A\alpha} = 0, \qquad (6.3)$$

into 6D $\mathcal{N} = (2,0)$ supergravity. With an explicit parametrization of the S^3 sphere harmonics as

$$\mathcal{Y}^A = \{u^\alpha \cos\theta, v^\alpha \sin\theta\}, \qquad u^\alpha u^\alpha = 1 = v^\alpha v^\alpha,$$
$$u^\alpha = (\cos\xi_1, \sin\xi_1), \qquad v^\alpha = (\cos\xi_2, \sin\xi_2), \qquad (6.4)$$

the warp factor Δ is given by [Eq. (5.3)]

$$\Delta = (\cosh\eta - \cos(2\theta)\sinh\eta)^{-1/4}. \qquad (6.5)$$

The six-dimensional metric is then obtained from Eq. (5.11) as a warped product of AdS_3 and a deformed sphere S^3:

$$ds_6^2 = \Delta^{-2}(ds^2_{\mathrm{AdS}_3} + d\theta^2) + e^{\eta}\Delta^2\cos^2\theta d\xi_1^2 + e^{-\eta}\Delta^2\sin^2\theta d\xi_2^2, \qquad (6.6)$$

with the two surviving U(1) isometries corresponding to rotations along ξ_1 and ξ_2. The full 6D curvature scalar follows as

$$R^{(6)} = \Delta^{10}\sin^2(2\theta)\sinh^2\eta. \qquad (6.7)$$

The SO(1,5) scalars are computed from Eq. (5.8) as

$$M^{00} = \frac{1}{2}(\Delta^4 + \Delta^{-4}) = M^{\bar{0}\bar{0}}, \qquad M^{0\bar{0}} = \frac{1}{2}(\Delta^4 - \Delta^{-4}),$$
$$M_{\alpha\beta} = \delta_{\alpha\beta}, \qquad M^0{}_\alpha = 0 = M^{\bar{0}}{}_\alpha, \qquad (6.8)$$

and the components of the 3-form field strengths along the S^3 directions follow from Eq. (5.7) to be

$$\sqrt{2}H_{ijk}{}^0 = -\mathring{\omega}_{ijk}(\Delta^8 + \alpha),$$
$$\sqrt{2}H_{ijk}{}^{\bar{0}} = -\mathring{\omega}_{ijk}(\Delta^8 - \alpha),$$
$$H_{ijk\alpha} = 0. \qquad (6.9)$$

The remaining components of the 6D field strengths can then be determined by imposing the 6D self-duality equations (6.2), giving rise to

$$H^0 = -\frac{1}{\sqrt{2}}((\Delta^8 + \alpha)\mathring{\omega}_{S^3} + (\alpha + 1)\mathring{\omega}_{\mathrm{AdS}}),$$
$$H^{\bar{0}} = -\frac{1}{\sqrt{2}}((\Delta^8 - \alpha)\mathring{\omega}_{S^3} + (\alpha - 1)\mathring{\omega}_{\mathrm{AdS}}), \qquad (6.10)$$

and vanishing H_α. The field strengths are given in terms of the volume forms $\mathring{\omega}_{S^3}$, $\mathring{\omega}_{\mathrm{AdS}}$ of unit-length S^3 and AdS_3, respectively. The Bianchi identities constitute a nontrivial consistency check of this result. Furthermore, it is straightforward to check that all 6D second-order field equations (6.1) are indeed satisfied for $\alpha^2 = 1$.

C. Uplift of an AdS_3 vacuum

As a second example, let us work out the 6D uplift of the stationary point (i) [Eq. (2.38)] of the potential [Eq. (2.22)]. Although this solution is unstable as an AdS_3 vacuum, and thus not of immediate interest, the fact that its uplift solves all 6D field equations constitutes a nontrivial consistency check to our uplift formulas. Recall that the location of this solution is specified by Eq. (2.39) with $\mathfrak{m} \equiv 3/2$. Using the explicit parametrization introduced earlier in Eq. (6.4) for the sphere harmonics, one now finds a constant warp factor

$\Delta = \mathfrak{m}^{-3/4}$. Then the six-dimensional metric is readily obtained as

$$ds_6^2 = \mathfrak{m}^{3/2} d\mathring{s}^2_{\mathrm{AdS}} + \mathfrak{m}^{-1/2} d\mathring{s}^2_{S^3}. \tag{6.11}$$

The Ricci tensor of this metric can be conveniently given as

$$\begin{aligned} R_{\hat\mu\hat\nu} dx^{\hat\mu} dx^{\hat\nu} &= -\frac{2}{\ell^2} d\mathring{s}^2_{\mathrm{AdS}} + 2 d\mathring{s}^2_{S^3}, \\ R^{(6)} &= 6\left(\mathfrak{m}^{1/2} - \frac{1}{\ell^2}\mathfrak{m}^{-3/2}\right), \end{aligned} \tag{6.12}$$

which leads to

$$\begin{aligned} &\left(R_{\hat\mu\hat\nu} - \frac{1}{2} R^{(6)} g_{\hat\mu\hat\nu}\right) dx^{\hat\mu} dx^{\hat\nu} \\ &= 3\mathfrak{m}^2 \left(\frac{1}{3\mathfrak{m}^2\ell^2} - 1\right) d\mathring{s}^2_{\mathrm{AdS}} + \left(\frac{3}{\mathfrak{m}^2\ell^2} - 1\right) d\mathring{s}^2_{S^3}, \end{aligned} \tag{6.13}$$

for the Einstein tensor. The scalars are obtained from Eqs. (5.8) and (5.9) as

$$\begin{aligned} M^{00} &= \frac{1}{2}\left(\frac{9}{4\mathfrak{m}} + 1 + \mathfrak{m}\right) = 2, \\ M^{0\bar 0} &= \frac{1}{2}\left(\mathfrak{m} - \frac{9}{4\mathfrak{m}}\right) = 0, \\ M^{\bar 0\bar 0} &= \frac{1}{2}\left(\frac{9}{4\mathfrak{m}} - 1 + \mathfrak{m}\right) = 1, \\ M^0{}_\alpha &= \frac{1}{\sqrt{2\mathfrak{m}}}\left(\mathfrak{m} + \frac{3}{2}\right)\mathcal{Y}^\alpha = \sqrt{3}\mathcal{Y}^\alpha, \\ M^{\bar 0}{}_\alpha &= \frac{1}{\sqrt{2\mathfrak{m}}}\left(\mathfrak{m} - \frac{3}{2}\right)\mathcal{Y}^\alpha = 0, \\ M_{\alpha\beta} &= \delta_{\alpha\beta} + \mathcal{Y}^\alpha\mathcal{Y}^\beta, \end{aligned}$$

whereas the 3-form field strengths along the S^3 directions computed from Eq. (5.7) are

$$\begin{aligned} H_{ijk}{}^0 &= -\sqrt{2}\mathring{\omega}_{ijk}, \qquad H_{ijk}{}^{\bar 0} = 0, \\ H_{ijk\alpha} &= -\frac{3}{2\sqrt{\mathfrak{m}}}\mathcal{Y}^\alpha\mathring{\omega}_{ijk} = -\frac{\sqrt{3}}{\sqrt{2}}\mathcal{Y}^\alpha\mathring{\omega}_{ijk}. \end{aligned} \tag{6.14}$$

The 6D self-duality equations (6.2) can be used to determine the full 6D field strengths as

$$H^0 = -\sqrt{2}\left(\mathring{\omega}_{S^3} + \frac{\mathfrak{m}^3}{2}\mathring{\omega}_{\mathrm{AdS}}\right), \quad H^{\bar 0} = 0, \quad H_\alpha = -\frac{\sqrt{3}}{\sqrt{2}}\mathcal{Y}^\alpha\mathring{\omega}_{S^3}. \tag{6.15}$$

The determination of the 3-form field strengths by self-duality requirement renders the Bianchi identities $dH^{\bar a} = 0$ nontrivial, and one verifies straightforwardly that they are satisfied. For this, it is crucial that $H_{\hat\mu\hat\nu\hat\rho\alpha}$ have no components along the AdS_3 directions, which is indeed the case.

Moreover, the different contributions to the energy-momentum tensor are given by

$$\begin{aligned} &\left(\partial_{\hat\mu} M_{\bar a\bar b}\partial_{\hat\nu} M^{\bar a\bar b} - \frac{1}{2} g_{\hat\mu\hat\nu}\partial_{\hat\rho} M_{\bar a\bar b}\partial^{\hat\rho} M^{\bar a\bar b}\right) dx^{\hat\mu} dx^{\hat\nu} \\ &\quad = 6\mathfrak{m}^2 d\mathring{s}^2_{\mathrm{AdS}} + 2 d\mathring{s}^2_{S^3}, \\ &\quad (H_{\hat\mu\hat\rho\hat\sigma}{}^{\bar a} H_{\hat\nu}{}^{\hat\rho\hat\sigma\bar b} M_{\bar a\bar b}) dx^{\hat\mu} dx^{\hat\nu} = -3\mathfrak{m}^2 d\mathring{s}^2_{\mathrm{AdS}} + 3 d\mathring{s}^2_{S^3}. \end{aligned} \tag{6.16}$$

From this, it follows immediately that the Einstein equations (6.1) are verified with $\mathfrak{m}^2\ell^2 = 4/3$.

VII. CONCLUSIONS

In this paper, we have used the framework of ExFT to work out the consistent truncations of 6D $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravity theories on $\mathrm{AdS}_3 \times S^3$. The resulting three-dimensional theories are SO(4) gauged supergravities coupled to four half-maximal scalar multiplets, describing the 32 bosonic degrees of freedom. Employing the Scherk-Schwarz twist matrices from Ref. [24] and establishing the explicit dictionary between ExFT fields and the 6D supergravity fields, it is straightforward to derive the nonlinear Kaluza-Klein reduction *Ansätze* for the various 6D fields. In the truncation to the common $\mathcal{N} = (1,0)$ sector, the formulas consistently reduce to the reduction formulas from Refs. [12,15]. The results nicely illustrate the power of the ExFT framework as a tool in the study of consistent truncations.

The three-dimensional scalar potentials allow for a number of stationary points, most of which, however, turn out to be unstable by the existence of scalar directions with negative mass squares below the Breitenlohner-Freedman bound. Interestingly, they admit a one-parameter family of nonsupersymmetric but stable AdS_3 solutions. We have given the explicit uplift of this family to six dimensions. Further direct applications of our uplift formulas may include three-dimensional solutions with nonconstant scalars such as holographic RG flows in the scalar potentials. On a more general note, the proof of the consistent truncation to particular three-dimensional gauged supergravities allows us to consistently restrict holographic supergravity calculations such as those in Refs. [8–11] to a closed subsector of fields.

An immediate generalization of the results reported here is their extension to six-dimensional supergravities with additional tensor multiplet couplings, which generically arise from reductions from ten dimensions. In the ExFT context, this corresponds to an embedding $\mathrm{SO}(8,4) \hookrightarrow \mathrm{SO}(8,4+n)$ of the exceptional field theories and the associated twist matrices. Upon working out the extended dictionary between ExFT and supergravity fields, the corresponding uplift formulas can be extracted in analogy to the results of this paper.

It would also be highly interesting to examine if similar techniques could be employed to construct consistent truncations involving higher massive Kaluza-Klein multiplets and leading to three-dimensional theories of the type constructed in Ref. [34]. This might require an extension of the present framework to more general embeddings in the spirit of Refs. [19,21].

Finally, it would be interesting to explore to which extent similar structures can be unveiled in the context of $\mathrm{AdS}_3 \times S^2$ truncations of the five-dimensional supergravities obtained from compactification of M-theory on Calabi-Yau 3-manifolds.

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APPENDIX: S^3 HARMONICS AND IDENTITIES

Here we list some of the useful identities that we used throughout the text. Consider a parametrization of the unit radius S^3 by some coordinates $\mathcal{Y}^A$ (with $A = 1, \ldots, 4$) as

$$\mathcal{Y}^A \mathcal{Y}^A = 1. \tag{A1}$$

The isometries of S^3 can be described in terms of the SO(4) Killing vectors

$$\mathcal{K}_{ABi} = \partial_i \mathcal{Y}_{[A} \mathcal{Y}_{B]}. \tag{A2}$$

Then the metric of the round S^3 can be written in the SO(4)-covariant form as

$$\mathring{g}_{ij} = 2\mathcal{K}_{ABi}\mathcal{K}_{ABj}. \tag{A3}$$

Using these and the inverse metric $\mathring{g}^{ij}$ of the round S^3, we find that

$$\mathring{g}^{ij}\partial_i \mathcal{Y}^A \partial_j \mathcal{Y}^B = \delta^{AB} - \mathcal{Y}^A \mathcal{Y}^B, \tag{A4}$$

which has proven to be of great value in the simplification of the uplift formulas throughout. The following was also of use for the derivation of (5.10):

$$\mathring{\omega}^{kij}\partial_i \mathcal{Y}^A \partial_j \mathcal{Y}^B = \varepsilon_{ABCD}\mathring{g}^{kl}\partial_l \mathcal{Y}^C \mathcal{Y}^D. \tag{A5}$$

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