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ON GENERAL FORM OF THE TANH METHOD AND ITS APPLICATION TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The tanh method is used to compute travelling waves solutions of one-dimensional nonlinear wave and evolution equations. The technique is based on seeking travelling wave solutions in the form of a finite series in tanh. In this article, we introduce a new general form of tanh transformation and solve well-known nonlinear partial differential equations in which tanh method becomes weaker in the sense of obtaining general form of solutions.

1. **Introduction.** Nonlinear partial differential equations are encountered in various fields of mathematics, physics (fluid dynamics [14]), chemistry, and mathematical biology (population dynamics [11]), and numerous applications. Exact (closed–form) solutions of differential equations play a crucial role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Because of the increasing interest in finding exact solutions for those problems, many powerful methods have been developed, such as, tanh-function method (see, e.g, [5, 7, 8, 9, 10]), inverse scattering method (see, e.g, [1]) and direct algebraic method (see, e.g, [2, 3, 4]). Here, we briefly discuss the tanh method, which was pioneered by Malfliet [10]. In this work, we investigate the nonlinear wave and evolution equations (one dimensional) which are written as

$$A(u, u_t, u_x, u_{xx}, ...) = 0 \quad \text{or} \quad A(u, u_{tt}, u_x, u_{xx}, ...) = 0. \tag{1}$$

We assume that these equations admit exact travelling wave solutions. The first step is to construct a new variable ξ in terms of our independent variables, x and t, $\xi = k(x-Vt)$, which is the travelling frame of reference. Here, k and V represent the wave number and velocity of the travelling wave, respectively. We assume that the wave number is greater than zero. Accordingly, the quantity $u(x,t)$ is replaced by $U(\xi)$, so that we deal with ODEs rather than with PDEs. Therefore, the equations 1 are transformed into

$$-kV\frac{dU}{d\xi} = A(U, k\frac{dU}{d\xi}, ...) \quad \text{or} \quad k^2V^2\frac{d^2U}{d\xi^2} = A(U, k\frac{dU}{d\xi}, ...). \tag{2}$$

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In the next step, we introduce a new independent variable in terms of ξ,

$$Y = \frac{e^{\xi} + \eta e^{\eta\xi}}{e^{\xi} + e^{\eta\xi}} \qquad \text{where } \eta \in \mathbb{R}, \tag{3}$$

which is the general form[1] of tanh function for every fixed $\eta \in \mathbb{R}$.
In the following section, we see that η generally[2] changes itself depending on the problem whose travelling wave solution not only depends on travelling frame ξ but also on η, which makes it more general than the one obtained from tanh method.

Now if we put our new form into 2, coefficients of 2 then only depends on Y. Moreover, observe that

$$\frac{d}{d\xi} = (-Y^2 + (\eta+1)Y - \eta)\frac{d}{dY}.$$

Therefore, it makes sense to attempt to find solution(s), $U(\xi) = F(Y)$, as a finite power series in Y,

$$F(Y) = \sum_{n=0}^{N} a_n Y^n, \tag{4}$$

which incorporates solitary-wave and shock-wave profiles (see e.g., [5, 7, 10]). We determine the degree (N) of the power series by using the balancing procedure (see [5]).

The rest of the paper is organized as follows. Section 2 describes several examples of nonlinear partial differential equations and their exact form of solutions in terms of general tanh form. For each of the given problems, we firstly use an independent variable ξ, to convert the nonlinear PDEs to ordinary differential equations (ODEs). Then, we introduce a new independent variable 3, and the solution(s) we are looking for would be written as a finite power series in terms of 3. In section 3, we compare our method with the tanh method and give several insights about the future work related with the subject.

2. **Examples.**

In this section, we use our new transformation in well-known examples of PDEs in detail.

2.1. **The Burgers Equation.**

The Burgers equations,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial x^2} = 0, \tag{5}$$

is one of the most crucial nonlinear diffusion equations which is used for describing wave processes in gas dynamics, hydrodynamics, and acoustics (see, e.g, [5, 9]). The positive parameter a is a dissipative effect. Here, we firstly transform Burgers equation 5 into

$$-kV\frac{dU}{d\xi} + kU\frac{dU}{d\xi} - ak^2\frac{d^2U}{d\xi^2} = 0. \tag{6}$$

Next, we introduce

$$Y = \frac{e^{\xi} + \eta e^{\eta\xi}}{e^{\xi} + e^{\eta\xi}}, \qquad \eta \in \mathbb{R}, \tag{7}$$

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[1] Observe that for $\eta = -1$, we obtain well-known tanh transformation.

[2] For some problems, η can take any value in $\mathbb{R}$ regardless of the problem (see Example 2.1 and Example 2.5).

which yields

$$-kV(-Y^2+(\eta+1)Y-\eta)\frac{dF(Y)}{dY}+kF(Y)(-Y^2+(\eta+1)Y-\eta)\frac{dF(Y)}{dY}$$
$$-ak^2(-Y^2+(\eta+1)Y-\eta)\frac{d}{dY}\Big[(-Y^2+(\eta+1)Y-\eta)\frac{dF}{dY}\Big]=0. \tag{8}$$

After substitution of 7 into 8, and balancing the highest power of Y, we obtain that $2N+1=N+2$, i.e., $N=1$. Namely, $F(Y)=b_1Y+b_0$. Substitute it into the equation 8 and after necessary simplifications are done, we have the following

$$b_1=-2ak, \qquad b_0=ak(\eta+1)+V, \qquad \forall\eta\in\mathbb{R},$$

or

$$u(x,t)=(ak(\eta+1)+V)-2ak\frac{e^{k(x-Vt)}+\eta e^{\eta k(x-Vt)}}{e^{k(x-Vt)}+e^{\eta k(x-Vt)}}$$

for $\eta\in\mathbb{R}$.

2.2. **The Fisher Equation.** The Fisher equation (see e.g., [5, 11, 12]):

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+au(1-u). \tag{9}$$

where the positive parameter a is a dissipative effect. To obtain the solution, we first need to transform 9 into

$$-kV\frac{dU(\xi)}{d\xi}=k^2\frac{d^2U(\xi)}{d\xi^2}+aU(\xi)(1-U(\xi)). \tag{10}$$

Now, using our transformation, we get the system in terms of Y,

$$kV(-Y^2+(\eta+1)Y-\eta)\frac{dF(Y)}{dY}+aF(Y)(1-F(Y))$$
$$+k^2(-Y^2+(\eta+1)Y-\eta)\Big[\frac{d}{dY}\Big[(-Y^2+(\eta+1)Y-\eta)\frac{dF(Y)}{dY}\Big]\Big]=0.$$

After balancing the highest power of Y, we get $2N=N+2$, i.e., $N=2$. Here, in this problem, we search our solution as $F(Y)=b_0(1-Y)^2$ (see e.g, [5]). After necessary arrangements, we have the following

$$Y^4(6b_0k^2-ab_0^2)+Y^3(-2b_0kV-10b_0k^2\eta-14b_0k^2+4ab_0^2)+Y^2(2b_0kV\eta$$
$$+4b_0kV+4b_0k^2\eta^2+22b_0k^2\eta+10b_0k^2-6ab_0^2+ab_0)+Y(-8b_0k^2\eta^2-14b_0k^2\eta$$
$$-2b_0k^2+4ab_0^2-2ab_0-4b_0kV\eta-2b_0kV)+(2b_0kV\eta+4b_0k^2\eta^2+2b_0k^2\eta-ab_0^2$$
$$+ab_0)=0.$$

The equality holds if and only if each coefficient of powers of Y vanishes. After required computations, we have the following two cases:

$$\eta=1-\frac{\sqrt{6a}}{6k}, \qquad V=\frac{5\sqrt{a}}{\sqrt{6}}, \qquad b_0=\frac{6k^2}{a}, \qquad k\neq 0. \tag{11}$$

and

$$\eta=1+\frac{\sqrt{6a}}{6k}, \qquad V=-\frac{5\sqrt{a}}{\sqrt{6}}, \qquad b_0=\frac{6k^2}{a}, \qquad k\neq 0. \tag{12}$$

2.3. **The Korteweg-de Vries (KdV) Equation.** KdV equation is used in many applications of nonlinear mechanics and theoretical physics for describing one - dimensional nonlinear dispersive nondissipative waves. Here, we consider the following KdV equation (see, e.g., [5]):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} = 0. \tag{13}$$

The positive parameter b refers to a dispersive effect. Here, we transform 13 into

$$-kV\frac{dU}{d\xi} + kU\frac{dU}{d\xi} + bk^3\frac{d^3U}{d\xi^3} = 0. \tag{14}$$

Then, we introduce

$$Y = \frac{e^{\xi} + \eta e^{\eta\xi}}{e^{\xi} + e^{\eta\xi}}, \qquad \eta \in \mathbb{R}. \tag{15}$$

Now set $\psi = -Y^2 + (\eta+1)Y - \eta$ and convert 14 into

$$-kV\psi\frac{dF(Y)}{dY} + kF(Y)\psi\frac{dF(Y)}{dY} + bk^3\psi\frac{d}{dY}\Big\{\psi\frac{d}{dY}\Big[\psi\frac{dF(Y)}{dY}\Big]\Big\} = 0. \tag{16}$$

Substitute equation 15 into 16 and balance the highest powers of Y, we obtain $2N+1 = N+3$, or $N = 2$. Now, in this problem, we search our solution as $F(Y) = a_0 + a_2Y^2$, (i.e., we assume that the coefficient of Y is zero) and put this into the equation 16, and do necessary arrangements, at the end we have the following

$$\begin{aligned}
&Y^5(-24a_2bk^2 - 2a_2^2) + Y^4(54a_2bk^2(\eta+1) + 2a_2^2(\eta+1)) + \\
&+Y^3(-38a_2bk^2(\eta^2+1) - 116a_2bk^2\eta - 2a_2^2\eta - 2a_2a_0 + 2a_2V) + \\
&+Y^2(8a_2bk^2(\eta^3+1) + 76a_2bk^2\eta(\eta+1) + 2a_0a_2(\eta+1) - 2a_2(\eta+1)V) + \\
&+Y(-14a_2bk^2\eta(\eta^2+1) - 44a_2bk^2\eta^2 - 2a_2a_0\eta + 2a_2V\eta) + 6a_2bk^2\eta^2(\eta+1) \\
&= 0.
\end{aligned}$$

Above equality holds if each coefficient of the power of Y should be zero. As we realize from the last coefficient, $6a_1bk^2\eta^2(\eta+1) = 0$, i.e., $\eta = -1$ and from the first coefficient, we obtain $a_2 = -12bk^2$. Also, from the coefficient of Y^3, we get $a_0 = V + 8bk^2$. Hence, under these assumptions on η, a_2 and a_0 we have the equality. In this problem, we observe that $\eta = -1$ which makes our transformation to become tanh transformation. At the end we obtain our solution

$$u(x,t) = (V + 8bk^2) - 12bk^2\tanh^2(k(x - Vt)). \tag{17}$$

2.4. **The Fitzhugh-Nagumo Equation.** The Fitzhugh-Nagumo equation arises in population genetics and models the transmission of nerve impulses (see, e.g., [12])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(1-u)(a-u), \tag{18}$$

In this problem, we assume that $a^2 - a + 1 \neq 0$. We firstly convert the PDE 18 into ODE, by the same method

$$-kV\frac{dU(\xi)}{d\xi} = k^2\frac{d^2U(\xi)}{d\xi^2} - U(\xi)(1 - U(\xi))(a - U(\xi)). \tag{19}$$

Now, using our transformation, we get the following

$$kV(-Y^2+(\eta+1)Y-\eta)\frac{dF(Y)}{dY}-F(Y)(1-F(Y))(a-F(Y))+$$
$$+k^2(-Y^2+(\eta+1)Y-\eta)\Big[\frac{d}{dY}\Big[(-Y^2+(\eta+1)Y-\eta)\frac{dF(Y)}{dY}\Big]\Big]=0.$$

Balancing the highest powers of Y, we have $N+2=3N$, or $N=1$.

Hence we obtain $F(Y)=s_0+s_1Y$, i.e., we have the following

$$Y^3(-s_1^3+2k^2s_1)+Y^2(-3k^2s_1(\eta+1)+s_1^2(a+1)-3s_1^2s_0-kVs_1)+$$
$$+\,Y(k^2s_1(\eta^2+1)+4k^2\eta s_1+2s_1s_0(a+1)-as_1-3s_1s_0^2+kV(\eta+1)s_1)+$$
$$+\,(-k^2s_1\eta(\eta+1)+s_0^2(a+1)-s_0^3-as_0-kV\eta s_1)=0. \tag{20}$$

The equality 20 holds when each coefficient of power of Y vanishes. Hence, from the first coefficient, we obtain that $s_1^2=2k^2$ and for $\eta\neq-1$ (we consider the case in which $\eta=-1$ later), we get $V=-3k(\eta+1)$ and $s_0=(a+1)/3$ from second coefficient. By using these values, we obtain the following equality from the third coefficient

$$\frac{a^2-a+1}{6}=k^2(\eta^2+\eta+1). \tag{21}$$

From the last coefficient, we have

$$\frac{(a-2)(2a-1)(a+1)}{54k^2}=-s_1\eta(\eta+1). \tag{22}$$

We further assume that $a-2\neq0,\ 2a-1\neq0$ and $a+1\neq0$ to make the left hand side of 22 to be non-zero. From 21 and 22 together, the following cases are obtained:

Case 1. $\eta=\dfrac{1-2a}{a-2},\quad k=\pm\dfrac{(a-2)}{3\sqrt{2}},\quad s_1=\dfrac{2-a}{3}.$

Case 2. $\eta=\dfrac{a+1}{a-2},\quad k=\pm\dfrac{(a-2)}{3\sqrt{2}},\quad s_1=\dfrac{2-a}{3}.$

Case 3. $\eta=\dfrac{1-2a}{a+1},\quad k=\pm\dfrac{(a+1)}{3\sqrt{2}},\quad s_1=-\dfrac{a+1}{3}.$

Case 4. $\eta=\dfrac{a-2}{a+1},\quad k=\pm\dfrac{(a+1)}{3\sqrt{2}},\quad s_1=-\dfrac{a+1}{3}.$

Case 5. $\eta=\dfrac{a+1}{1-2a},\quad k=\pm\dfrac{(2a-1)}{3\sqrt{2}},\quad s_1=\dfrac{2a-1}{3}.$

Case 6. $\eta=\dfrac{a-2}{1-2a},\quad k=\pm\dfrac{(2a-1)}{3\sqrt{2}},\quad s_1=\dfrac{2a-1}{3}.$

Hence, from each case we obtain the solution of 18. Now let us consider the case $\eta=-1$, which is the case that the special transformation turns out to be tanh transformation. More precisely, if $\eta=-1$ in the equation 20, then after necessary computations, the following cases are obtained:

Case 1. $s_0 = \frac{1}{2}, \quad s_1 = \frac{1}{2}, \quad k = \pm\frac{\sqrt{2}}{4}, \quad V = \pm\frac{2a-1}{\sqrt{2}}.$

Case 2. $s_0 = \frac{a}{2}, \quad s_1 = \frac{a}{2}, \quad k = \pm\frac{\sqrt{2}a}{4}, \quad V = \pm\frac{a(2-a)}{\sqrt{2}}.$

Case 3. $s_0 = \frac{1}{2}(a+1), \quad s_1 = \frac{1}{2}(1-a), \quad k = \pm\frac{\sqrt{2}}{4}(1-a), \quad V = \pm\frac{a^2-1}{\sqrt{2}}.$

2.5. **The Landau-Ginburg-Higgs Equation.** Now, we consider the following nonlinear differential equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - m^2 u + n^2 u^3 = 0, \tag{23}$$

where m, n are real non-zero constants (e.g., see [6], [13]).

First of all, we convert the PDE 23 into ODE, i.e.,

$$k^2(V^2-1)\frac{d^2U(\xi)}{d\xi^2} - m^2U(\xi) + n^2U^3(\xi) = 0. \tag{24}$$

Now, using our transformation, we get the following

$$k^2(V^2-1)(-Y^2+(\eta+1)Y-\eta)\frac{d}{dY}\left[(-Y^2+(\eta+1)Y-\eta)\frac{dF}{dY}\right] - m^2F$$
$$+n^2F^3 = 0,$$

where $F := F(Y) = \sum_{n=0}^{N} a_n Y^n$. Balancing the highest powers of Y, we have $N+2 = 3N$, or $N = 1$. Namely, we obtain $F(Y) = a_0 + a_1 Y$. Putting $F(Y)$ into the above equation and do required computations, we have the following

$$\begin{aligned} &Y^3(a_1^3n^2 - 2a_1k^2(1-V^2)) + \\ &Y^2(3a_1k^2(\eta+1)(1-V^2) + 3a_0a_1^2n^2) + \\ &Y(3a_0^2a_1n^2 - a_1k^2(1-V^2)(4\eta+\eta^2+1) - a_1m^2) + \\ &(a_1k^2\eta(\eta+1)(1-V^2) - a_0m^2 + a_0^3n^2) = 0. \end{aligned} \tag{25}$$

Equality holds if and only if all coefficients of the equation 25 vanishes. Hence, after doing necessary calculations, we find that

$$a_1 = \pm\frac{2m}{n(\eta-1)}, \quad a_0 = \pm\frac{m(\eta+1)}{n(\eta-1)}, \quad k = \pm\frac{\sqrt{2}m}{(\eta-1)\sqrt{1-V^2}}, \quad \eta \neq 1,\ V \neq \pm 1.$$

Also, one can easily prove that, for the case $V = \pm 1$, we have constant solution for 23. However, when $\eta = 1$ and $V \neq \pm 1$ in the equation 25, we get $m = 0$ which is not the case. Hence, no solution exists when $\eta = 1$.

3. **Conclusion.** In this paper, we introduce a new technique, by constructing a new model which is the general form of tanh function, whereby we obtain exact solutions in case of the nonlinear PDEs. As it is known well, the tanh method gives us a solution which depends only on travelling frame ξ (see [5, 7, 8]). In our model, we see that the solution not only depends on travelling frame ξ, but also η which characterizes the travelling wave solution. In fact, for each η, the solution is changing its structure, which makes it more general than one that is obtained from tanh method.

In this work, we investigate well-known one dimensional nonlinear PDEs on which no boundary conditions are imposed. It would be interesting to analyze nonlinear

PDEs under well-defined boundary conditions (see [10]) and obtain exact solutions by using general tanh method. Moreover, the wave profiles obtained in this way may be useful for those who wish to tackle these problems exclusively in a numerical way.

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