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A model of intra-group tug-of-war*

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Abstract

In a standard tug-of-war model, two players engage in a component battle each period and a player wins the contest if her number of battle victories exceeds the other contestant's by a certain number. In this paper, we introduce a multi-player model of intra-group tug-of-war played by two groups of two players, where a player wins the contest if she wins a certain number of battles more than the other player from her group before either player from the other group achieves the same against one another. We characterize the unique subgame perfect Nash equilibrium of the model and further analyze an asymmetric case with different number of players in the competing groups. Our results indicate an extreme discouragement effect for the laggards and a strong momentum effect for the winner of the first battle.

Key words: Contest, multi-battle contest, tug-of-war, intra-group tug-of-war, Tullock contest, subgame perfect Nash equilibrium

JEL codes: C72, D74

1. Introduction

Following the seminal works by Tullock (1980) and Lazear and Rosen (1981), there has been an increased interest in the theoretical and experimental investigation of contest-like situations (see Corchón, 2007; Konrad, 2009; Dechenaux et al., 2015 among others). A *contest* game is a competition between a number of players such that each player exerts costly efforts in order to win a specific prize. The effort costs are irreversible meaning that a player would always

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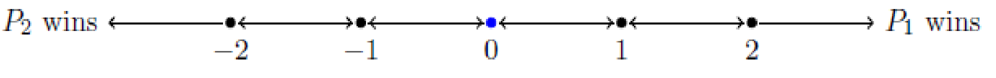
incur the cost independent of whether she wins or loses the contest. There are several real-life examples: sports, warfare, political campaigns, firm competition.

Although most of the contest models in the literature are one-shot games where players choose their efforts simultaneously, there also are sequential models in which the contending parties compete in a component battle each period such that either the winner in each period gets a prize after that period, or the winner in the whole contest gets a prize at the end of the contest, or both. Perhaps the most common example to such *multi-battle* contests is a *race* in which whoever reaches a certain number of battle victories wins. Klumpp and Polborn (2006) studied a two-player race where the outcome of each component battle is determined by a Tullock contest success function (CSF), so that a player's probability of winning a battle is given by the ratio of the player's effort to the total effort exerted by all players. Later, Konrad and Kovenock (2009) analyzed a two-player race employing an all-pay auction CSF, which stipulates that whoever exerts the greatest effort wins the battle for sure. And recently, utilizing a Tullock CSF, Doğan et al. (2018) generalized the race model to include any number of players.

Another common multi-battle contest is a *tug-of-war*, which has a single difference from a race: a player wins the contest if her number of battle victories exceeds the other contestant's number of battle victories by a certain number. More formally, in a two-player tug-of-war, the game starts from the node 0 as specified in Figure 1. In each period, (i) a win by Player 1 results in a move towards the right; and (ii) a win by Player 2 results in a move towards the left. There is a difference threshold $T \geq 2$ such that Player $i \in \{1,2\}$ wins the contest if she wins T more battles than Player $j \neq i$ does. For example, the difference threshold is assumed to be $T = 3$ in Figure 1, so that Player $i \in \{1,2\}$ wins the contest once she obtains three battle victories more than what Player $j \neq i$ has, which would move the game onto her preferred *terminal* node indicating P_i wins.

Figure 1

A Standard Tug-of-war with Two Players



As a modeling tool, tug-of-war has a wide range of application areas in several disciplines, such as economics, political science, and biology. For instance, Organski and Lust-Okar (1997) referred to the struggl e about the status of Jerusalem, while Yoo (2001) described the relation between the United States and

North Korea as examples of a tug-of-war. In biological sciences, Larsson et al. (2004) and Zhou et al. (2004) referred to the interaction between viruses and some parts of the immune system as a tug-of-war. Utilizing a tug-of-war model, Schaub (1995) represented the conflict over food that occurs between long-tailed macaque females. Finally, Bradley et al. (2005) argued that a tug-of-war takes place between male members of wild mountain gorilla groups for the control over reproduction.

Harris and Vickers (1987) were the first to analyze a model of tug-of-war. In their model, there are two players and each battle outcome is determined by a Tullock CSF. Recently, Karagözoğlu et al. (2018) studied a similar model and completely characterized its unique Markov perfect equilibrium. They reported perseverance showing that players exert positive efforts until the very last battle, which is in stark contrast to a previous result in the literature. According to this previous result by Konrad and Kovenock (2005), who studied a model of tug-of-war employing an all-pay auction CSF, there are at most two nodes on which players exert positive efforts while they exert zero effort on all of the other nodes. This indicates an extreme case of *the discouragement effect*.[1]

To the best of our knowledge, tug-of-war has never been generalized to include more than two players in the literature. Such a multi-player model would be very similar to the multi-player race studied by Doğan et al. (2018). These authors characterized the unique subgame perfect Nash equilibrium of their model and reported another form of extreme discouragement effect. Accordingly, once a player falls behind two other players, she would prefer quitting the game, by exerting zero effort from that point onwards. This converts the rest of the game into a two-player race. As we have shown in Appendix A below, these equilibrium results almost trivially carry over to a standard multi-player tug-of-war.

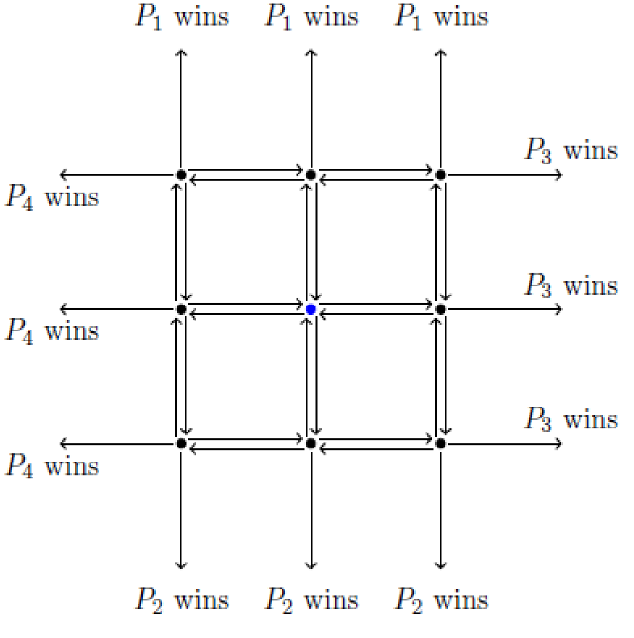
In this paper, we introduce an *intra-group* tug-of-war illustrated in Figure 2. In our baseline model, there are four players and the difference threshold is two. The contest starts from the center node. In each period, (i) a win by Player 1 results in a move upwards; (ii) a win by Player 2 results in a move downwards; (iii) a win by Player 3 results in a move towards the right; and (iv) a win by Player 4 results in a move towards the left. Each node with the label "P_i wins" is referred to as a terminal node; and if one of these terminal nodes is reached, the respective

[1] There are several other earlier works on tug-of-war. Agastya and McAfee (2006) studied a version where a component battle may result in a draw and showed that there is an equilibrium with players choosing not to fight. Häfner and Konrad (2016) analyzed a tug-of-war team contest model and showed that *eternal peace*, as reported by Agastya and McAfee (2006), is no longer observed in any equilibrium. Häfner (2017) also studied a team contest model and characterized its unique Markov perfect equilibrium. Deck and Sheremeta (2015) conducted the first lab experiment on tug-of-war and reported that subjects behave very differently from the predictions by Konrad and Kovenock (2005).

player wins the contest. Notice that although this is seemingly a four-player contest, there are two groups, $\{1,2\}$ and $\{3,4\}$, engaged in *hidden* sub-contests, such that if Player i from group $\{i, j\}$ is to win this contest, she must defeat Player j in the respective sub-contest before one of the players in the other group defeats her opponent in their own sub-contest. For example, Player 1 wins this contest, if she wins two more rounds than Player 2 does, before either Player 3 or 4 wins two more rounds than the other player does.

Figure 2

An Intra-group Tug-of-war with Four Players



Although the current paper is interested in this model of intra-group tug-of-war because of its theoretical appeal, we now present a motivation for our model to highlight which type of strategic interactions it can represent. Our motivation also includes a behavioral flavor, implementing a well-known phenomenon, *the decoy effect*. This phenomenon suggests that when an individual cannot decide between two alternatives, A and C, her preferences would change in favor of A, when a third alternative B, which is completely dominated by A but not by C, becomes available. The third alternative here is called a *decoy* (see Huber et al., 1982, 2014; Bateman et al., 2008; Frederick et al., 2014; Yang and Lynn, 2014 among others). That being said, we now assume a consumer considering to buy one of four products: A, B, C, and D. Assume that A and B have similar characteristics, C and D have similar characteristics, but there is no other such similarity between any other pair of products.[2] The suppliers of these products compete against each other in different aspects to influence the consumer. Now, for example, if A wins two more battles than B does, i.e., if A shows that it dominates B in two additional aspects, before C or D achieves a similar domination over the other, then B starts to be seen as a decoy for A, which would change the consumer's preferences in favor of A, so that the consumer would choose A over the other alternatives. Although this type of consumer behavior is not *explicitly* modeled here, we simply argue that our model of intra-group tug-of-war captures the essence of this story.

In this paper, we characterize the equilibrium strategies in the baseline model as well as those in an asymmetric model with different number of players in the competing groups. In both models, it is shown that any player would start exerting zero effort once she falls behind the other player in her group. This indicates an extreme discouragement effect for such players. We also find a strong momentum effect, suggesting that the winner of the first component battle would have a very high chance of winning the tug-of-war. We complete our analysis by comparing these observations with the results reported in earlier works on multi-battle contests. The implications on and about *rent dissipation* are further discussed.

The rest of the paper is organized as follows. Section 2 analyzes the baseline model with four players and a difference threshold of two, and Section 3 investigates an asymmetric version with different number of players in two groups. Section 4 concludes.

2. The model

Consider the set of players, $N = \{1,2,3,4\}$, who are competing in a model of intra-group tug-of-war as specified in Figure 2 above. There is a component battle each period, and each player aims to win these battles in order to move the game towards her preferred terminal node. In every period t, each player $i \in N$ chooses an effort $e_i^t \in [0,\infty)$ to exert in the respective component battle. The probability that Player i wins this component battle is given by the following Tullock contest success function (CSF):

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2 Assuming that these alternatives are houses, suppose that A and B are located in the city center, whereas C and D are located outside city center. Similarly, assuming that the consumer considers buying a new car, suppose that A and B are fuel-efficient cars, whereas C and D are sports cars. More solid examples can be provided.

$$p_i(e_1^t, e_2^t, e_3^t, e_4^t) = \frac{e_i^t}{e_1^t + e_2^t + e_3^t + e_4^t}.$$

No prize is given to the winner of a component battle. The winner of the whole game gets a prize of $V > 0$, whereas the losers do not receive any prize. Finally, the cost function is assumed to be linear:

$$c_i(e_i^t) = ke_i^t \qquad \textit{for some } k > 0.$$

Let the node $(1,0,1,0)$ represent the case in which Players 1 and 3 have the same amount of battle victories and Players 2 and 4 have one less battle victories. We can denote the other nodes in a similar manner. The following proposition reveals the unique subgame perfect Nash equilibrium (SPNE) of this baseline model.

Proposition 1 *In an intra-group tug-of-war with four players and a difference threshold of two, the unique SPNE is such that [i] when there are two players who are one battle victory away from winning the game (e.g., at a node (1,0,1,0)), those players exert* $V/4k$ *, whereas the other players exert* 0 *; [ii] when there is only one player who is one battle victory away from winning the game (e.g., at a node (1,0,0,0)), that player exerts* $15V/98k$ *, her competitor from her group exerts* 0 *, and the other players exert* $3V/98k$ *; and [iii] at the initial node, all players exert* $370V/3136k$.

Proof. We analyze equilibrium via backward induction. We start with the node $T_2 \equiv (1,0,1,0)$. At this node Player 1 maximizes

$$p_1\left(e_1^{T_2}, e_2^{T_2}, e_3^{T_2}, e_4^{T_2}\right)V + p_2\left(e_1^{T_2}, e_2^{T_2}, e_3^{T_2}, e_4^{T_2}\right)V_{\approx} + p_4\left(e_1^{T_2}, e_2^{T_2}, e_3^{T_2}, e_4^{T_2}\right)V_{+} - ke_1^{T_2}.$$

where $V_{\approx}$ is the equilibrium expected prize value for Player 1 seen from the node $(0,0,1,0)$, and V_{+} is the equilibrium expected prize value for Player 1 seen from the node $(1,0,0,0)$. The first order condition with respect to $e_1^{T_2}$:

$$\frac{e_2^{T_2} + e_3^{T_2} + e_4^{T_2}}{\left(e_1^{T_2} + e_2^{T_2} + e_3^{T_2} + e_4^{T_2}\right)^2}V - \frac{e_2^{T_2}}{\left(e_1^{T_2} + e_2^{T_2} + e_3^{T_2} + e_4^{T_2}\right)^2}V_{\approx} - \frac{e_4^{T_2}}{\left(e_1^{T_2} + e_2^{T_2} + e_3^{T_2} + e_4^{T_2}\right)^2}V_{+} - k = 0.$$

Player 3 has a symmetric first order condition. Moreover, at this particular node Player 2 maximizes

$$p_2\left(e_1^{T_2}, e_2^{T_2}, e_3^{T_2}, e_4^{T_2}\right)V_{\approx} + p_4\left(e_1^{T_2}, e_2^{T_2}, e_3^{T_2}, e_4^{T_2}\right)V_{-} - ke_2^{T_2}.$$

where V_{-} is the equilibrium expected prize value for Player 1 seen from the node $(0,1,0,0)$.[3] The first order condition with respect to $e_2^{T_2}$:

$$\frac{e_1^{T_2} + e_3^{T_2} + e_4^{T_2}}{\left(e_1^{T_2} + e_2^{T_2} + e_3^{T_2} + e_4^{T_2}\right)^2}V_{\approx} - \frac{e_4^{T_2}}{\left(e_1^{T_2} + e_2^{T_2} + e_3^{T_2} + e_4^{T_2}\right)^2}V_{-} - k = 0.$$

Player 4 has a symmetric first order condition. Now, we can see that $e_2^{T_2 *} = e_4^{T_2 *}$ at the equilibrium. Utilizing this information, we also find that $e_1^{T_2 *} = e_3^{T_2 *}$ at the equilibrium. Then equalizing the first order conditions with respect to $e_1^{T_2}$ and $e_2^{T_2}$:

$$\left(e_1^{T_2} + 2e_2^{T_2}\right)V - \left(2e_1^{T_2} + e_2^{T_2}\right)V_{\approx} = e_2^{T_2}(V_{+} + V_{:} - V_{-}).$$

This yields

$$e_2^{T_2} = e_1^{T_2}\,\frac{2V_{\approx} - V}{V_{-} + 2V - 2V_{\approx} - V_{+}}.$$

In the symmetric equilibrium, since Player 1's probability of winning the contest would be lower than 1 at the node $(1,0,0,0)$ and lower than $1/2$ at the node $(0,0,1,0)$, we can argue that $V_{+} < V$ and $2V_{\approx} < V$. Accordingly, $e_2^{T_2} < 0$. This means that $e_2^{T_2 *} = e_4^{T_2 *} = 0$, so that Players 2 and 4 quit the game at this node. This is labeled as the discouragement effect. We then conclude that $e_1^{T_2 *} = e_3^{T_2 *} = V/4k$.

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3 Due to symmetry, $V_{\approx}$ is the equilibrium expected prize value for Player 2 seen from the node $(0,0,1,0)$, and V_{-} is the equilibrium expected prize value for Player 2 seen from the node $(1,0,0,0)$.

This yields an expected value of $V/4$ to Players 1 and 3, whereas an expected value of 0 to Players 2 and 4.

All of the other nodes with two players who are one battle victory away from winning the game, which are $(1,0,0,1)$, $(0,1,1,0)$, and $(0,1,0,1)$, have symmetric results.

We proceed to the analysis of the node $T_1 \equiv (1,0,0,0)$. At this node Player 1 maximizes

$$p_1\left(e_1^{T_1},e_2^{T_1},e_3^{T_1},e_4^{T_1}\right)V + p_2\left(e_1^{T_1},e_2^{T_1},e_3^{T_1},e_4^{T_1}\right)V^0 + \left[p_3\left(e_1^{T_1},e_2^{T_1},e_3^{T_1},e_4^{T_1}\right) + p_4\left(e_1^{T_1},e_2^{T_1},e_3^{T_1},e_4^{T_1}\right)\right]\frac{V}{4} - ke_1^{T_1}.$$

where V^0 is the equilibrium expected prize value for Player 1 seen from the node $(0,0,0,0)$. The first order condition with respect to $e_1^{T_1}$:

$$\frac{e_2^{T_1}+e_3^{T_1}+e_4^{T_1}}{\left(e_1^{T_1}+e_2^{T_1}+e_3^{T_1}+e_4^{T_1}\right)^2}V - \frac{e_2^{T_1}}{\left(e_1^{T_1}+e_2^{T_1}+e_3^{T_1}+e_4^{T_1}\right)^2}V^0 - \frac{e_3^{T_1}+e_4^{T_1}}{\left(e_1^{T_1}+e_2^{T_1}+e_3^{T_1}+e_4^{T_1}\right)^2}\frac{V}{4} - k = 0.$$

At this node Player 3 maximizes

$$p_2\left(e_1^{T_1},e_2^{T_1},e_3^{T_1},e_4^{T_1}\right)V^0 + p_3\left(e_1^{T_1},e_2^{T_1},e_3^{T_1},e_4^{T_1}\right)\frac{V}{4} - ke_3^{T_1}.$$

The first order condition with respect to $e_3^{T_1}$:

$$\frac{e_1^{T_1}+e_2^{T_1}+e_4^{T_1}}{\left(e_1^{T_1}+e_2^{T_1}+e_3^{T_1}+e_4^{T_1}\right)^2}\frac{V}{4} - \frac{e_2^{T_1}}{\left(e_1^{T_1}+e_2^{T_1}+e_3^{T_1}+e_4^{T_1}\right)^2}V^0 - k = 0.$$

Player 4 has a symmetric first order condition. Then, we can see that $e_3^{T_1*} = e_4^{T_1*}$ at the equilibrium. Moreover, at this particular node Player 2 maximizes[4]

$$p_2\left(e_1^{T_1},e_2^{T_1},e_3^{T_1},e_4^{T_1}\right)V^0 - ke_2^{T_1}.$$

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4 If Player 1 wins, the contest ends for sure; and if either Player 3 or 4 wins, then Player 2 would be discouraged in the SPNE.

The first order condition with respect to $e_2^{T_1}$:

$$\frac{e_1^{T_1}+e_3^{T_1}+e_4^{T_1}}{\left(e_1^{T_1}+e_2^{T_1}+e_3^{T_1}+e_4^{T_1}\right)^2}V^0-k=0.$$

From here we can find that $e_2^{T_1*}=0$.[5] This is labeled as the discouragement effect as well. Now, utilizing $e_3^{T_1*}=e_4^{T_1*}$ and equalizing the first order conditions with respect to $e_1^{T_1}$ and $e_3^{T_1}$:

$$\left(e_1^{T_1}+e_3^{T_1}\right)\frac{V}{4}=e_3^{T_1}\frac{3V}{2}.$$

We then obtain $e_1^{T_1*}=5e_3^{T_1*}=5e_4^{T_1*}$. Furthermore, $e_1^{T_1*}=15v/98k$ and $e_3^{T_1*}=e_4^{T_1*}=3V/98k$. This yields an expected value of $124V/196$ to Player 1, an expected value of 0 to Player 2, and an expected value of $V/196$ to each of the other players.

All of the other nodes with one player who is one battle victory away from winning the game, which are $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$, have symmetric results.

Finally, at the node $0\equiv(0,0,0,0)$ Player 1 maximizes

$$p_1\left(e_1^0,e_2^0,e_3^0,e_4^0\right)\frac{124V}{196}+\left[p_3\left(e_1^0,e_2^0,e_3^0,e_4^0\right)+p_4\left(e_1^0,e_2^0,e_3^0,e_4^0\right)\right]\frac{V}{196}-ke_1^0.$$

The first order condition with respect to e_1^0:

$$\frac{e_2^0+e_3^0+e_4^0}{\left(e_1^0+e_2^0+e_3^0+e_4^0\right)^2}\frac{124V}{196}-\frac{e_3^0+e_4^0}{\left(e_1^0+e_2^0+e_3^0+e_4^0\right)^2}\frac{V}{196}-k=0.$$

Since we also have symmetric first order conditions for the other players, we should have $e_1^{0*}=e_2^{0*}=e_3^{0*}=e_4^{0*}$ at the equilibrium. Then we find that each player

[5] See Appendix B for this analysis.

exerts $370V/3136k$. And this yields an expected value of $134V/3136$ to all players.

Our first observation is the *extreme* discouragement effect. If Player $i \in N$ from group $\{i, j\}$ wins the component battle in the first period, then Player j would be completely discouraged in the next period. If Player i also wins the component battle next period, then the game ends; but if Player i fails to win, then there appears another discouraged player, which means that the game proceeds to a final round in which both of the laggards are discouraged. This implies that the game surely ends in three periods. All these equilibrium outcomes are similar to the results reported by Doğan et al. (2018) for a multi-player race, which are shown to almost trivially carry over to a standard multi-player tug-of-war (see Appendix A). On a related note, Konrad and Kovenock (2005) reported that all players are completely discouraged on most of the nodes in a two-player tug-of-war with an all-pay auction CSF; whereas Karagözoğlu et al. (2018) reported that both players exert positive efforts until the very last battle in a two-player tug-of-war with a Tullock CSF. Here we see that our results lie in between these two rather extreme cases.

The second observation is that the winner of the component battle in the first period gains a *momentum* in the sense that she would have a very high chance of winning the tug-of-war. In particular, there is a $5/7$ chance that the same player wins the next component battle, directly becoming the ultimate winner. With the remaining probability, the player would lose the second battle; but then there is a $1/2$ chance that she wins the third battle. This corresponds to a total winning probability of $6/7$. This is labeled as a *strong* momentum effect. To compare with earlier results, we can further note that both Doğan et al. (2018) and Karagözoğlu et al. (2018) reported similar results on the momentum effect.

From a contest design perspective, a social planner would be interested in the amount of *rent dissipation*, which is defined as the prize value minus the total expected value in the equilibrium. Given that the efforts are not productive, it is commonly discussed in the literature that exerting efforts to gain a chance of earning an exogenously given prize is inefficient, so that rent dissipation is not desirable. For a social planner who prefers a lower amount of rent dissipation, our model of intra-group tug-of-war appears to be the better alternative. We can see that in a four-player race or in a standard four-player tug-of-war, each player has an expected value of $0.0401V$, which corresponds to a total expected value of $0.1604V$, so that almost 83.96% of the rent is dissipated. On the other hand, with an expected value of $0.0427V$ for each player, only 82.92% of the rent is dissipated in our model.

3. An asymmetric case

The difference between a two-player race and a two-player tug-of-war is obvious. In both models, a component battle takes place each period; and a race ends when one player collects a certain number of battle victories, whereas a tug-of-war ends when the difference between the two players' battle victories reaches a certain number. As a consequence, a race always moves forward with no return, whereas a tug-of-war might return to a previously visited node, precisely when the laggard wins a component battle. Similar definitions and implications apply to multi-player versions of these models. Given the extreme discouragement result by Doğan et al. (2018) for a multi-player race, and given our equilibrium analysis in Appendix A suggesting that the results carry over to a standard multi-player tug-of-war, we can see that a tug-of-war would never return to a previously visited node along the equilibrium path. Therefore, it can be argued that a multi-player race and a multi-player tug-of-war reduce to the same model in the unique SPNE.

As the results by Doğan et al. (2018) apply to any n-player race, it is natural to expect for an intra-group tug-of-war that if there is an increase in the number of groups or a symmetric increase in the number of players within groups, then most of the results provided for the baseline model would carry over. However, if there appears an asymmetry in the model with one group being more crowded than the other, then one might expect to observe some differences in the equilibrium strategies. Here we consider such an asymmetric case where $N = \{1,2,3,4,5\}$ with groups $\{1,2\}$ and $\{3,4,5\}$.

Let $((1,0),(1,1,0))$ represent the case in which Players 1, 3, and 4 have the same amount of battle victories and Players 2 and 5 have one less battle victories. We can denote the other nodes in a similar manner. The following proposition describes a SPNE of this asymmetric model.

Proposition 2 *In an intra-group tug-of-war with five players and a difference threshold of two, there exists a SPNE such that*

- at the node $((1,0),(1,1,0))$: $\left(\frac{2V}{9k},0,\frac{2V}{9k},\frac{2V}{9k},0\right)$;

- at the node $((0,0),(1,1,0))$: $\left(0,0,\frac{V}{4k},\frac{V}{4k},0\right)$;

• at the node $((1,0),(1,0,0))$: $\left(\frac{V}{4k},0,\frac{V}{4k},0,0\right)$;

• at the node $((1,0),(0,0,0))$: $\left(\frac{63V}{400k},0,\frac{9V}{400k},\frac{9V}{400k},\frac{9V}{400k}\right)$;

• at the node $((0,0),(1,0,0))$: $\left(\frac{3V}{169k},\frac{3V}{169k},\frac{27V}{169k},\frac{3V}{169k},\frac{3V}{169k},\right)$; and

• at the initial node: Players 1 and 2 approximately exert $0.1388V/k$, whereas Players 3, 4, and 5 approximately exert $0.0427V/k$.

Proof. See Appendix B.

According to the equilibrium strategies specified in Proposition 2, an *extreme* discouragement effect appears also in this asymmetric model. We can see that if the game is still on after the first two component battles, the laggards would be totally discouraged. However, the discouragement effect is now slightly weaker, because in the current model, in case the component battle in the first period is won by a player from the latter group with three players, all players would stay in the game for one more period. Furthermore, similar to the baseline model, these equilibrium strategies imply that the game surely ends in three periods.

As for the momentum effect, we report that if Player 1 or 2 wins the component battle in the first period, there is a $7/10$ chance that she wins the second battle, and there is a $17/20$ chance that she wins the whole contest; whereas if Player 3, 4, or 5 wins the component battle in the first period, there is a $9/13$ chance that she wins the second battle, and there is a $11/13$ chance that she wins the whole contest. This indicates a *strong* momentum effect, which is relatively stronger for a player from the former group with two players. Comparing to the baseline model, we can further state that the momentum effect is now slightly weaker, due to the fact that there are now more players competing for the prize.

Interestingly, players exert *very* asymmetric efforts in the first component battle. Although exerting more effort comes with additional costs, it also increases the probability of winning the first battle for Players 1 and 2; and as a result, it turns out that

$$V_1^0 = V_2^0 \approx 0.0729V \quad \textit{and} \quad V_3^0 = V_4^0 = V_5^0 \approx 0.0073V.$$

This can be interpreted as “it is almost ten times more beneficial to be in the two-player group rather than in the three-player group". This observation is quite intuitive, because although all players compete against each other in the component battles, each player actually tries to defeat the other member(s) of her group. And obviously, a smaller group implies a lower within-group competition, hence a higher expected value.

As for rent dissipation, we can conclude that a social planner who prefers a lower amount of rent dissipation would design an intra-group tug-of-war between five players rather than a race or a standard tug-of-war. This is because the total expected value is approximately $2 \cdot 0.0729V + 3 \cdot 0.0073V = 0.1677V$ in our model, which indicates that 83.23% of the rent is dissipated, whereas 87.1% of the rent would be dissipated in a five-player race or in a standard five-player tug-of-war.

4. Conclusion

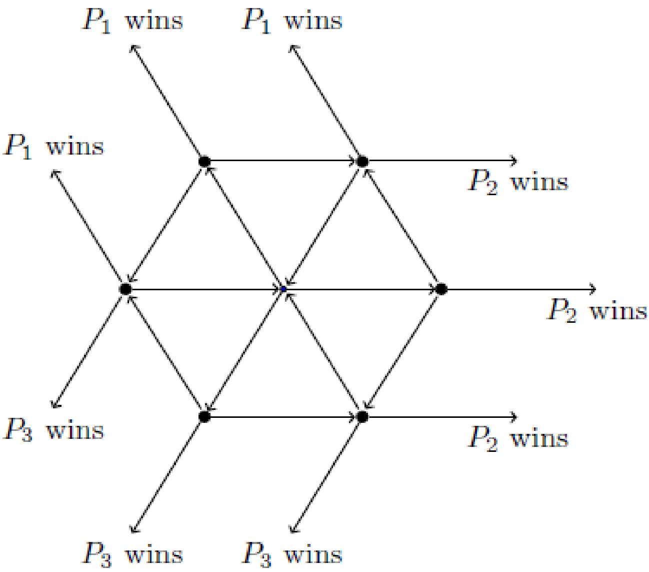
We have introduced a model of intra-group tug-of-war in this paper. Characterizing its unique SPNE, we have shown that there is an extreme discouragement effect for the laggards and there is a strong momentum effect for the winner of the first component battle. Then, we have also studied an asymmetric case where the competing groups have different number of players and investigated the respective changes observed in the equilibrium strategies. All these equilibrium results are compared to the previous results reported in the literature on multi-battle contests.

Finally, it can be argued that our results can be generalized to the cases with any number of players in any number of groups and with any value of difference threshold. Most importantly, the extreme discouragement result would be preserved, suggesting that once a player falls behind the other players in her group, she would start exerting zero effort from that point onwards. This reduces the rest of the game into a two-player race and simplifies the rest of the equilibrium analysis.

Appendix A

In this Appendix, we analyze a standard multi-player tug-of-war with three players and a difference threshold of two. The game is illustrated in Figure 3.

Figure 3
A Standard Tug-of-war with Three Players



In this contest, (i) a win by Player 1 results in a move towards the upper left; (ii) a win by Player 2 results in a move towards the right; while (iii) a win by Player 3 results in a move towards the lower left. Player $i \in \{1,2,3\}$ wins the contest once her preferred terminal node is reached, i.e., as soon as she wins two component battles more than the worst-performing player does.

As assumed in our baseline model of intra-group tug-of-war, the outcome of each component battle is determined by a Tullock CSF, the winner of the whole game gets a prize of $V > 0$, and each player has a linear cost function.

Proposition 3 *In a standard tug-of-war with three players and a difference threshold of two, the unique SPNE is such that [i] when there are two players who are one battle victory away from winning the game (e.g., at a node (1,1,0)), those players exert* $V/4k$ *, whereas the other players exert* 0 *; [ii] when there is only one player who is one battle victory away from winning the game (e.g., at a node*

(1,0,0)), that player exerts $15V/98k$ *, whereas the other players exert* $3V/98k$ *; and [iii] at the initial node, all players exert* $41V/294k$.

Proof. Let the node $T_2 \equiv (1,1,0)$ represent the case in which Players 1 and 2 have the same amount of victories and Player 3 has one victory less. At this node Player 1 maximizes

$$p_1\left(e_1^{T_2}, e_2^{T_2}, e_3^{T_2}\right)V + p_3\left(e_1^{T_2}, e_2^{T_2}, e_3^{T_2}\right)V^0 - ke_1^{T_2}.$$

where V^0 is the equilibrium expected prize value for Player 1 seen from the node $(0,0,0)$. The first order condition with respect to $e_1^{T_2}$:

$$\frac{e_2^{T_2} + e_3^{T_2}}{\left(e_1^{T_2} + e_2^{T_2} + e_3^{T_2}\right)^2}V - \frac{e_3^{T_2}}{\left(e_1^{T_2} + e_2^{T_2} + e_3^{T_2}\right)^2}V^0 - k = 0.$$

Player 2 has a symmetric first order condition. We can see that $e_1^{T_2 *} = e_2^{T_2 *}$ at the equilibrium. Moreover, at this particular node Player 3 maximizes

$$p_3\left(e_1^{T_2}, e_2^{T_2}, e_3^{T_2}\right)V^0 - ke_3^{T_2}.$$

The first order condition with respect to $e_3^{T_2}$:

$$\frac{e_1^{T_2} + e_2^{T_2}}{\left(e_1^{T_2} + e_2^{T_2} + e_3^{T_2}\right)^2}V^0 - k = 0.$$

Utilizing $e_1^{T_2 *} = e_2^{T_2 *}$ and equalizing the first order conditions with respect to $e_1^{T_2}$ and $e_3^{T_2}$:

$$2e_1^{T_2}V^0 = e_1^{T_2}V + e_3^{T_2}(V - V^0).$$

This yields

$$e_3^{T_2} = e_1^{T_2}\frac{2V^0 - V}{V - V^0}.$$

We can argue that $V^0 < V/3$ at the equilibrium. Accordingly, when $e_1^{T_2} > 0$, we have $e_3^{T_2} < 0$. This means that $e_3^{T_2 *} = 0$, so that Player 3 is discouraged. We then conclude that $e_1^{T_2 *} = e_2^{T_2 *} = V/4k$. This yields an expected value of $V/4$ to Players 1 and 2, and an expected value of 0 to Player 3.

Noting that the equilibrium strategies on the nodes $(1,0,1)$ and $(0,1,1)$ would be symmetric, we now analyze the node $T_1 \equiv (1,0,0)$. At this node Player 1 maximizes

$$p_1\left(e_1^{T_1}, e_2^{T_1}, e_3^{T_1}\right)V + \left(1 - p_1\left(e_1^{T_1}, e_2^{T_1}, e_3^{T_1}\right)\right)\frac{V}{4} - ke_1^{T_1}.$$

The first order condition with respect to $e_1^{T_1}$:

$$\frac{e_2^{T_1} + e_3^{T_1}}{\left(e_1^{T_1} + e_2^{T_1} + e_3^{T_1}\right)^2}V - \frac{e_2^{T_1} + e_3^{T_1}}{\left(e_1^{T_1} + e_2^{T_1} + e_3^{T_1}\right)^2}\frac{V}{4} - k = 0.$$

At this node Player 2 maximizes

$$p_2\left(e_1^{T_1}, e_2^{T_1}, e_3^{T_1}\right)\frac{V}{4} - ke_2^{T_1}.$$

The first order condition with respect to $e_2^{T_1}$:

$$\frac{e_1^{T_1} + e_3^{T_1}}{\left(e_1^{T_1} + e_2^{T_1} + e_3^{T_1}\right)^2}\frac{V}{4} - k = 0.$$

Player 3 has a symmetric first order condition. Now, we can see that $e_2^{T_1 *} = e_3^{T_1 *}$ at the equilibrium. Utilizing this and equalizing the first order conditions with respect to $e_1^{T_1}$ and $e_2^{T_1}$:

$$(e_1^{T_1} + e_2^{T_1})\frac{V}{4} = e_2^{T_1}\frac{3V}{2}.$$

We then have $e_1^{T_1 *} = 5e_2^{T_1 *} = 5e_3^{T_1 *}$. Furthermore, $e_1^{T_1 *} = 15V/98k$ and $e_2^{T_1 *} = e_3^{T_1 *} = 3V/98k$. This yields an expected value of $124V/196$ to Player 1, and an expected value of $V/196$ to Players 2 and 3.

Once again, we note that the equilibrium strategies on the nodes $(0,1,0)$ and $(0,0,1)$ are symmetric. Finally, at the node $0 \equiv (0,0,0)$ Player 1 maximizes

$$p_1\left(e_1^0, e_2^0, e_3^0\right)\frac{124V}{196} + \left(1 - p_1\left(e_1^0, e_2^0, e_3^0\right)\right)\frac{V}{196} - ke_1^0.$$

The first order condition with respect to e_1^0:

$$\frac{e_2^0 + e_3^0}{\left(e_1^0 + e_2^0 + e_3^0\right)^2}\frac{124V}{196} - \frac{e_2^0 + e_3^0}{\left(e_1^0 + e_2^0 + e_3^0\right)^2}\frac{V}{196} - k = 0.$$

Since we have symmetric first order conditions for the other players, we should have $e_1^{0*} = e_2^{0*} = e_3^{0*}$ at the equilibrium. Then we find that each player exerts $41V/294k$. And this yields an expected value of $22V/294$ to all players. □

These results are the same with those reported by Doğan et al. (2018) for a multi-player race. Although we have provided an analysis for the three-player version here, observing that the equilibrium analyses in both models follow almost identically, it can be argued that the equilibrium results for any generalized model of multi-player race would carry over to the corresponding model of tug-of-war.

Appendix B

Proof of Proposition 1: In the proof above, we have claimed that $e_2^{T_1 *} = 0$ at the node $(1,0,0,0)$. Here we prove this claim. Recall that the first order conditions can be written as

$$\left(e_1^{T_1} + 2e_3^{T_1}\right)V^0 = k\left(e_1^{T_1} + e_2^{T_1} + e_3^{T_1} + e_4^{T_1}\right)^2,$$
$$\left(e_1^{T_1} + e_2^{T_1} + e_3^{T_1}\right)\frac{V}{4} - e_2^{T_1}V^0 = k\left(e_1^{T_1} + e_2^{T_1} + e_3^{T_1} + e_4^{T_1}\right)^2,$$
$$e_2^{T_1}(V - V^0) + 3e_3^{T_1}\frac{V}{2} = k\left(e_1^{T_1} + e_2^{T_1} + e_3^{T_1} + e_4^{T_1}\right)^2.$$

Then we find $e_1^{T_1} = 3e_2^{T_1} + 5e_3^{T_1}$ and

$$e_3^{T_1}\frac{V - 5V^0}{4V^0 - V} = e_2^{T_1}. \tag{1}$$

Moreover, we can write $V_{\approx}$ and V_{-} in terms of $e_2^{T_1}$, $e_3^{T_1}$, and V^0:

$$V_{\approx} = \frac{e_2^{T_1}}{4e_2^{T_1} + 7e_3^{T_1}}V^0 + \frac{e_3^{T_1}}{4e_2^{T_1} + 7e_3^{T_1}}\frac{V}{4} - e_3^{T_1}k$$

and

$$V_{-} = \frac{e_2^{T_1}}{4e_2^{T_1} + 7e_3^{T_1}}V^0 - e_2^{T_1}k.$$

We can argue that $V^0 < V/4$ at the symmetric equilibrium. Then

$$V_{\approx} < \frac{e_2^{T_1} + e_3^{T_1}}{4e_2^{T_1} + 7e_3^{T_1}}\frac{V}{4} \quad \text{and} \quad V_{-} < \frac{e_2^{T_1}}{4e_2^{T_1} + 7e_3^{T_1}}\frac{V}{4}$$

Finally, we write the first order condition for Player 1 with respect to e_1^0 at the node $0 \equiv (0,0,0,0)$ in terms of V_{+}, $V_{\approx}$, and V_{-}:

$$\frac{e_2^0+e_3^0+e_4^0}{\left(e_1^0+e_2^0+e_3^0+e_4^0\right)^2}V_+ + \frac{e_3^0+e_4^0}{\left(e_1^0+e_2^0+e_3^0+e_4^0\right)^2}V_\approx - \frac{e_2^0}{\left(e_1^0+e_2^0+e_3^0+e_4^0\right)^2}V_- .$$

Considering the symmetric first order conditions for the other players, we have $e_1^{0*}=e_2^{0*}=e_3^{0*}=e_4^{0*}$ at the equilibrium. Then we find

$$e_1^{0*}=\frac{3V_+-2V_\approx-V_-}{16k}$$

and

$$V^0=\frac{1}{4}V_++\frac{1}{2}V_\approx+\frac{1}{4}V_--\frac{3V_+-2V_\approx-V_-}{16}=\frac{V_++10V_\approx+5V_-}{16}.$$

From this equation, we can see that

$$5V^0=\frac{5V_++50V_\approx+25V_-}{16}<\frac{5V+50\frac{e_2^{T_1}+e_3^{T_1}}{4e_2^{T_1}+7e_3^{T_1}}\frac{V}{4}+25\frac{e_2^{T_1}}{4e_2^{T_1}+7e_3^{T_1}}\frac{V}{4}}{16}=\frac{5V+75\frac{e_2^{T_1}}{4e_2^{T_1}+7e_3^{T_1}}\frac{V}{4}+50\frac{e_3^{T_1}}{4e_2^{T_1}+7e_3^{T_1}}\frac{V}{4}}{16}<V$$

Since $5V^0<V$, the equation (1) implies that $e_2^{T_1}<0$. Accordingly, we conclude that $e_2^{T_1*}=0$.

Proof of Proposition 2: We analyze equilibrium via backward induction. Throughout this analysis, without loss of generality, we consider the nodes where players with a lower index number is more advantaged than the others in their respective groups.

Utilizing the insights obtained from the baseline model, we start our analysis of the node $T_{1,2}\equiv((1,0),(1,1,0))$ under the assumption that $e_2^*=e_5^*=0$.[6] At this node Player 1 maximizes

6 This assumption implies that we analyze whether there exists a SPNE in which the disadvantageous players at the furthest node are discouraged. At the end of our equilibrium analysis, we should and will check whether this assumption is consistent with the corresponding equilibrium expected values. Although we cannot guarantee the uniqueness of such a SPNE, we conjecture that there is no SPNE of other sort.

$$p_1\left(e_1^{T_{1,2}},\ldots,e_5^{T_{1,2}}\right)V-ke_1^{T_{1,2}}.$$

The first order condition with respect to $e_1^{T_{1,2}}$:

$$\frac{e_3^{T_{1,2}}+e_4^{T_{1,2}}}{\left(e_1^{T_{1,2}}+e_3^{T_{1,2}}+e_4^{T_{1,2}}\right)^2}V-k=0.$$

Since we have symmetric objective function and symmetric first order conditions for the other players,[7] we should have $e_1^{T_{1,2}*}=e_3^{T_{1,2}*}=e_4^{T_{1,2}*}$ at the equilibrium. Then we find that $e_1^{T_{1,2}*}=e_3^{T_{1,2}*}=e_4^{T_{1,2}*}=2V/9k$. This yields an expected value of $V/9$ to Players 1, 3, and 4, whereas an expected value of 0 to Players 2 and 5.

We continue with the node $T_{0,2}\equiv((0,0),(1,1,0))$. At this node Player 1 maximizes

$$p_1\left(e_1^{T_{0,2}},\ldots,e_5^{T_{0,2}}\right)\frac{V}{9}+p_5\left(e_1^{T_{0,2}},\ldots,e_5^{T_{0,2}}\right)V_1^0-ke_1^{T_{0,2}}.$$

where V_i^0 is the equilibrium expected prize value for Player i seen from the node $((0,0),(0,0,0))$. The first order condition with respect to $e_1^{T_{0,2}}$:

$$\frac{e_2^{T_{0,2}}+e_3^{T_{0,2}}+e_4^{T_{0,2}}+e_5^{T_{0,2}}}{\left(e_1^{T_{0,2}}+\ldots+e_5^{T_{0,2}}\right)^2}\frac{V}{9}-\frac{e_5^{T_{0,2}}}{\left(e_1^{T_{0,2}}+\ldots+e_5^{T_{0,2}}\right)^2}V_1^0-k=0.$$

Player 2 has a symmetric first order condition. We can then see that $e_1^{T_{0,2}*}=e_2^{T_{0,2}*}$ at the equilibrium. At this particular node Player 3 maximizes

7 For example, although there is asymmetry in group sizes, Player 3 maximizes $p_3\left(e_1^{T_{1,2}},\ldots,e_5^{T_{1,2}}\right)V-ke_3^{T_{1,2}}$ at this node. Taking the first order condition with respect to $e_3^{T_{1,2}}$ yields a symmetric first order condition.

$$\left[p_1\left(e_1^{T_{0,2}},\ldots,e_5^{T_{0,2}}\right)+p_2\left(e_1^{T_{0,2}},\ldots,e_5^{T_{0,2}}\right)\right]\frac{V}{9}+p_3\left(e_1^{T_{0,2}},\ldots,e_5^{T_{0,2}}\right)V+p_5\left(e_1^{T_{0,2}},\ldots,e_5^{T_{0,2}}\right)V_3^0-ke_3^{T_{0,2}}.$$

The first order condition with respect to $e_3^{T_{0,2}}$:

$$\frac{e_1^{T_{0,2}}+e_2^{T_{0,2}}+e_4^{T_{0,2}}+e_5^{T_{0,2}}}{\left(e_1^{T_{0,2}}+\ldots+e_5^{T_{0,2}}\right)^2}V-\frac{e_1^{T_{0,2}}+e_2^{T_{0,2}}}{\left(e_1^{T_{0,2}}+\ldots+e_5^{T_{0,2}}\right)^2}\frac{V}{9}-\frac{e_5^{T_{0,2}}}{\left(e_1^{T_{0,2}}+\ldots+e_5^{T_{0,2}}\right)^2}V_3^0-k=0.$$

Player 4 has a symmetric first order condition. We can then see that $e_3^{T_{0,2}*}=e_4^{T_{0,2}*}$ at the equilibrium. Moreover, at this node Player 5 maximizes

$$p_5\left(e_1^{T_{0,2}},\ldots,e_5^{T_{0,2}}\right)V_5^0-ke_5^{T_{0,2}}.$$

The first order condition with respect to $e_5^{T_{0,2}}$:

$$\frac{e_1^{T_{0,2}}+e_2^{T_{0,2}}+e_3^{T_{0,2}}+e_4^{T_{0,2}}}{\left(e_1^{T_{0,2}}+\ldots+e_5^{T_{0,2}}\right)^2}V_5^0-k=0.$$

Our analysis yields $e_1^{T_{0,2}*}=e_2^{T_{0,2}*}=e_5^{T_{0,2}*}=0$ at the equilibrium, since each of the other cases leads to a contradiction. Accordingly, $e_3^{T_{0,2}*}=e_4^{T_{0,2}*}=V/4k$. This yields an expected value of $V/4$ to Players 3 and 4, and an expected value of 0 to each of the other players.

We continue with the node $T_{1,1}\equiv((1,0),(1,0,0))$. At this node Player 1 maximizes

$$p_1\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)V+p_2\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)\tilde{V}_1+\left[p_4\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)+p_5\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)\right]\frac{V}{9}-ke_1^{T_{1,1}}.$$

where $\tilde{V}_i$ represents the equilibrium expected prize value for Player i seen from the node $((0,0),(1,0,0))$. The first order condition with respect to $e_1^{T_{1,1}}$:

$$\frac{e_2^{T_{1,1}}+e_3^{T_{1,1}}+e_4^{T_{1,1}}+e_5^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}V-\frac{e_2^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}\tilde{V}_1-\frac{e_4^{T_{1,1}}+e_5^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}\frac{V}{9}-k=0.$$

At this particular node Player 2 maximizes

$$p_2\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)\tilde{V}_2-ke_2^{T_{1,1}}.$$

The first order condition with respect to $e_2^{T_{1,1}}$:

$$\frac{e_1^{T_{1,1}}+e_3^{T_{1,1}}+e_4^{T_{1,1}}+e_5^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}\tilde{V}_2-k=0.$$

At this particular node Player 3 maximizes

$$p_2\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)\tilde{V}_3+p_3\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)V+\left[p_4\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)+p_5\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)\right]\frac{V}{9}-ke_3^{T_{1,1}}.$$

The first order condition with respect to $e_3^{T_{1,1}}$:

$$\frac{e_1^{T_{1,1}}+e_2^{T_{1,1}}+e_4^{T_{1,1}}+e_5^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}V-\frac{e_2^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}\tilde{V}_3-\frac{e_4^{T_{1,1}}+e_5^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}\frac{V}{9}-k=0.$$

Moreover, at this node Player 4 maximizes

$$p_2\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)\tilde{V}_4+p_4\left(e_1^{T_{1,1}},\ldots,e_5^{T_{1,1}}\right)\frac{V}{9}-ke_4^{T_{1,1}}.$$

The first order condition with respect to $e_4^{T_{1,1}}$:

$$\frac{e_1^{T_{1,1}}+e_2^{T_{1,1}}+e_3^{T_{1,1}}+e_5^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}\frac{V}{9}-\frac{e_2^{T_{1,1}}}{\left(e_1^{T_{1,1}}+\ldots+e_5^{T_{1,1}}\right)^2}\tilde{V}_4-k=0.$$

Player 5 has a symmetric first order condition. We can see that $e_4^{T_{1,1}^*}=e_5^{T_{1,1}^*}$ at the equilibrium. Our analysis also yields $e_2^{T_{1,1}^*}=e_4^{T_{1,1}^*}=e_5^{T_{1,1}^*}=0$, since each of the other cases leads to a contradiction. Accordingly, $e_1^{T_{1,1}^*}=e_3^{T_{1,1}^*}=V/4k$. This yields an expected value of $V/4$ to Players 1 and 3, and an expected value of 0 to each of the other players.

We then proceed to the analysis of the node $T_{1,0}\equiv((1,0),(0,0,0))$ at which Player 1 maximizes

$$p_1\left(e_1^{T_{1,0}},\ldots,e_5^{T_{1,0}}\right)V+p_2\left(e_1^{T_{1,0}},\ldots,e_5^{T_{1,0}}\right)V_1^0+\left[1-p_1\left(e_1^{T_{1,0}},\ldots,e_5^{T_{1,0}}\right)-p_2\left(e_1^{T_{1,0}},\ldots,e_5^{T_{1,0}}\right)\right]\frac{V}{4}-ke_1^{T_{1,0}}.$$

The first order condition with respect to $e_1^{T_{1,0}}$:

$$\frac{e_2^{T_{1,0}}+e_3^{T_{1,0}}+e_4^{T_{1,0}}+e_5^{T_{1,0}}}{\left(e_1^{T_{1,0}}+\ldots+e_5^{T_{1,0}}\right)^2}V-\frac{e_2^{T_{1,0}}}{\left(e_1^{T_{1,0}}+\ldots+e_5^{T_{1,0}}\right)^2}V_1^0-\frac{e_3^{T_{1,0}}+e_4^{T_{1,0}}+e_5^{T_{1,0}}}{\left(e_1^{T_{1,0}}+\ldots+e_5^{T_{1,0}}\right)^2}\frac{V}{4}-k=0.$$

At this particular node Player 2 maximizes

$$p_2\left(e_1^{T_{1,0}},\ldots,e_5^{T_{1,0}}\right)V_2^0-ke_2^{T_{1,0}}.$$

The first order condition with respect to $e_2^{T_{1,0}}$:

$$\frac{e_1^{T_{1,0}}+e_3^{T_{1,0}}+e_4^{T_{1,0}}+e_5^{T_{1,0}}}{\left(e_1^{T_{1,0}}+\ldots+e_5^{T_{1,0}}\right)^2}V_2^0-k=0.$$

Moreover, at this node Player 3 maximizes

$$p_2\left(e_1^{T_{1,0}},\ldots,e_5^{T_{1,0}}\right)V_3^0+p_3\left(e_1^{T_{1,0}},\ldots,e_5^{T_{1,0}}\right)\frac{V}{4}-ke_3^{T_{1,0}}.$$

The first order condition with respect to $e_3^{T_{1,0}}$:

$$\frac{e_1^{T_{1,0}} + e_2^{T_{1,0}} + e_4^{T_{1,0}} + e_5^{T_{1,0}}}{\left(e_1^{T_{1,0}} + \ldots + e_5^{T_{1,0}}\right)^2} \frac{V}{4} - \frac{e_2^{T_{1,0}}}{\left(e_1^{T_{1,0}} + \ldots + e_5^{T_{1,0}}\right)^2} V_3^0 - k = 0.$$

Considering the symmetric first order conditions for Players 4 and 5, we have $e_3^{T_{1,0}^*} = e_4^{T_{1,0}^*} = e_5^{T_{1,0}^*}$ at the equilibrium. Our analysis also yields $e_2^{T_{1,0}^*} = 0$ at the equilibrium. Utilizing all these information and equalizing the first order conditions with respect to $e_1^{T_{1,0}}$ and $e_3^{T_{1,0}}$:

$$\frac{9V}{4} e_3^{T_{1,0}} = \left(e_1^{T_{1,0}} + 2e_3^{T_{1,0}}\right)\frac{V}{4}.$$

Accordingly, we have $e_1^{T_{1,0}^*} = 7e_3^{T_{1,0}^*} = 7_4^{T_{1,0}^*} = 7e_5^{T_{1,0}^*}$. Furthermore, $e_1^{T_{1,0}^*} = 63V/400k$ and $e_3^{T_{1,0}^*} = e_4^{T_{1,0}^*} = e_5^{T_{1,0}^*} = 9V/400k$. This yields an expected value of $247V/400$ to Player 1, an expected value of 0 to Player 2, and an expected value of $V/400$ to Players 3, 4, and 5.

Then, we proceed to the analysis of the node $T_{0,1} \equiv ((0,0),(1,0,0))$ at which Player 1 maximizes

$$p_1\left(e_1^{T_{0,1}}, \ldots, e_5^{T_{0,1}}\right)\frac{V}{4} - ke_1^{T_{0,1}}.$$

The first order condition with respect to $e_1^{T_{0,1}}$:

$$\frac{e_2^{T_{0,1}} + e_3^{T_{0,1}} + e_4^{T_{0,1}} + e_5^{T_{0,1}}}{\left(e_1^{T_{0,1}} + \ldots + e_5^{T_{0,1}}\right)^2} \frac{V}{4} - k = 0.$$

Considering the symmetric objective functions and symmetric first order conditions for Players 2, 4, and 5, we have $e_1^{T_{0,1}^*} = e_2^{T_{0,1}^*} = e_4^{T_{0,1}^*} = e_5^{T_{0,1}^*}$ at the equilibrium. Moreover, at this node Player 3 maximizes

$$p_3\left(e_1^{T_{0,1}},\ldots,e_5^{T_{0,1}}\right)V+\left[1-p_3\left(e_1^{T_{0,1}},\ldots,e_5^{T_{0,1}}\right)\right]\frac{V}{4}-ke_3^{T_{0,1}}.$$

The first order condition with respect to $e_3^{T_{0,1}}$:

$$\frac{e_1^{T_{0,1}}+e_2^{T_{0,1}}+e_4^{T_{0,1}}+e_5^{T_{0,1}}}{\left(e_1^{T_{0,1}}+\ldots+e_5^{T_{0,1}}\right)^2}V-\frac{e_1^{T_{0,1}}+e_2^{T_{0,1}}+e_4^{T_{0,1}}+e_5^{T_{0,1}}}{\left(e_1^{T_{0,1}}+\ldots+e_5^{T_{0,1}}\right)^2}\frac{V}{4}-k=0.$$

Utilizing $e_1^{T_{0,1}*}=e_2^{T_{0,1}*}=e_4^{T_{0,1}*}=e_5^{T_{0,1}*}$ and equalizing the first order conditions with respect to $e_1^{T_{0,1}}$ and $e_3^{T_{0,1}}$:

$$3Ve_1^{T_{0,1}}=\left(e_3^{T_{0,1}}+3e_1^{T_{0,1}}\right)\frac{V}{4}.$$

We then have $9e_1^{T_{0,1}*}=9e_2^{T_{0,1}*}=e_3^{T_{0,1}*}=9e_4^{T_{0,1}*}=9e_5^{T_{0,1}*}$. Furthermore, $e_1^{T_{0,1}*}=e_2^{T_{0,1}*}=3V/169k$, $e_3^{T_{0,1}*}=27V169k$, and $e_4^{T_{0,1}*}=e_5^{T_{0,1}*}=3V/169k$. This yields an expected value of $308V/676$ to Player 3, and an expected value of $V/676$ to each of the other players.

$$EV^{T_{0,1}*}=\left(\frac{V}{676},\frac{V}{676},\frac{308V}{676},\frac{V}{676},\frac{V}{676}\right).$$

Last, we analyze the node $T_{0,0}\equiv 0\equiv((0,0),(0,0,0))$ at which Player 1 maximizes

$$p_1\left(e_1^0,\ldots,e_5^0\right)\frac{247V}{400}+\left[1-p_1\left(e_1^0,\ldots,e_5^0\right)-p_2\left(e_1^0,\ldots,e_5^0\right)\right]\frac{V}{676}-ke_1^0.$$

The first order condition with respect to e_1^0:

$$\frac{e_2^0+e_3^0+e_4^0+e_5^0}{\left(e_1^0+\ldots+e_5^0\right)^2}\frac{247V}{400}-\frac{e_3^0+e_4^0+e_5^0}{\left(e_1^0+\ldots+e_5^0\right)^2}\frac{V}{676}-k=0.$$

At this particular node Player 3 maximizes

$$\left[p_1\left(e_1^0,\ldots,e_5^0\right)+p_2\left(e_1^0,\ldots,e_5^0\right)\right]\frac{V}{400}+p_3\left(e_1^0,\ldots,e_5^0\right)\frac{308V}{676}+\left[p_4\left(e_1^0,\ldots,e_5^0\right)+p_5\left(e_1^0,\ldots,e_5^0\right)\right]\frac{V}{676}-ke_1^0.$$

The first order condition with respect to e_3^0:

$$\frac{e_1^0+e_2^0+e_4^0+e_5^0}{\left(e_1^0+\ldots+e_5^0\right)^2}\frac{308V}{676}-\frac{e_1^0+e_2^0}{\left(e_1^0+\ldots+e_5^0\right)^2}\frac{V}{400}-\frac{e_4^0+e_5^0}{\left(e_1^0+\ldots+e_5^0\right)^2}\frac{V}{676}-k=0.$$

Considering the respective symmetric first order conditions for the other players, we have $e_1^{0*}=e_2^{0*}$ and $e_3^{0*}=e_4^{0*}=e_5^{0*}$ at the equilibrium. Then we find $19519\,e_1^{0*}=63529\,e_3^{0*}$ and

$$e_1^{0*}=e_2^{0*}=\frac{1616933817\ 99371}{1164508974\ 005000}\frac{V}{k}\approx\frac{0.1388V}{k}$$

$$e_3^{0*}=e_4^{0*}=e_5^{0*}=\frac{4967956554\ 2381}{1164508974\ 005000}\frac{V}{k}\approx\frac{0.0427V}{k}$$

Finally, seeing that the respective equilibrium expected values are consistent with our assumption that the laggards at the furthest node are discouraged, we verify that the strategies described above constitute a SPNE.

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Özet

Bir grup-içi halat çekme yarışı modeli

Standart bir halat çekme yarışı modelinin her periyodunda iki oyuncu birbirleri ile savaşır, ve periyot zaferi sayısı bakımından diğer oyuncuya karşı belirli bir sayı farkla üstünlük kuran oyuncu halat çekme yarışını kazanır. Bu makalemizde, ikişer oyunculu iki grup ile oynanan çok oyunculu bir grup-içi halat çekme yarışı modeli ortaya koyuyoruz; öyle ki, her oyuncunun amacı grubundaki diğer oyuncuya karşı periyot zaferi sayısı bakımından belirli bir sayı farkla üstünlük kurmak, fakat bunu bunun aynısını diğer gruptaki oyunculardan birisi o gruptaki diğer oyuncuya karşı yapmadan başarmış olmak. Önce modelimizdeki alt-oyun mükemmel Nash dengesini karakterize ediyoruz, sonra da gruplardaki oyuncu sayılarının farklı olduğu asimetrik bir durumu çalışıyoruz. Sonuçlarımız ilk periyottaki savaşı kazanmanın güçlü bir momentum etkisi yarattığını ve ilk savaşları kaybetmenin son derece motivasyon kırıcı bir etkisi olduğunu gösteriyor.

Anahtar kelimeler: Yarışma, çok savaşlı yarışma, halat çekme yarışı, grup-içi halat çekme yarışı, Tullock yarışması, alt-oyun mükemmel Nash dengesi.

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