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A note on the direct proof for ex-post stability implies well-ordered ex-post stability*

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Abstract

We provide a novel and direct proof of the following result: each ex-post stable probabilistic assignment can be decomposed into a collection of deterministic stable assignments, which can be ordered in such a way that each man's welfare is non-increasing and each woman's welfare is non-decreasing as we follow the assignments from the first to the last.

Key words: Marriage problem, pr obabilistic assignments, ex-post stability.

1. Introduction

A "marriage problem" is an assignment problem consisting of two groups of agents: men and women, workers and firms, students and schools, interns and hospitals, etc. Throughout this paper we will refer to these groups as men and women. Each man has o rdinal preferences over women and vice versa. A deterministic assignment matches each man to a woman, and each woman to a man in such a way that no one has two different mates. As opposed to assigning agents to each other in a deterministic way, one can think of probabilistic assignments, namely lotteries over deterministic assignments. Here, we are concerned with such assignments and in particular with their stability.

A central robustness criterion for deterministic assignments is "stability", which requires that there is no unmatched man-woman pair who prefer each other to their current matches. The theory of stable assignments without transfers (NTU-model) was first developed by Gale and Shapley (1962) [5]. Given the preferences

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of agents, the algorithm proposed by Gale and Shapley selects a stable assignment which is optimal from the view point of the proposing side. We analyze the stability of probabilistic assignments. The counterpart of stability for probabilistic assignments is "ex-post" stability (Roth et al. (1993) [7]), which requires that there is at least one decomposition of the probabilistic assignment into stable deterministic assignments. A probabilistic assignment is ex-post stable if it can decomposed into stable deterministic assignments.

Dogan and Yildiz (2016) [4] observe an interesting property of ex-post stable assignments related to the lattice structure of stable deterministic assignments (see Knuth (1976) [6], pp. 92-93, who attributes the discovery of this lattice structure to J. H. Conway).[1]

In their Proposition 1, Dogan and Yildiz show that each ex-post stable probabilistic assignment can be decomposed into a collection of deterministic stable assignments, which can be ordered in such a way that each man's welfare is non-increasing and each woman's welfare is non-decreasing as we follow the assignments from the first to the last. This result is shown via the *rounding approach* due to Sethuraman and Teo (1998) [9]. This proof also relies on previous results by Rothblum (1992) [8] and Roth et al. (1993) [7].[2] Therefore, the essence of the result is rather opaque to grasp directly. In our Theorem 1, we provide a substantially different, novel and self-contained proof of this result.[3]

2. Preliminaries

Let M be a set of n men, and W be a set of n women. Each $i \in M$ has preferences over W, and each $j \in W$ has preferences over M. Let $N = M \cup W$. For each $i \in N$, the preferences of i, which we denote by R_i, is a **weak order**, i.e.it is transitive and complete, with associated strict preference relation P_i, and indifference relation I_i. Let $\mathcal{R}_i$ denote the set of all possible preference relations for i, and let $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$ denote the set of all possible preference profiles.

A **deterministic assignment** is a one-to-one function from M to W. A deterministic assignment can be represented by an $n \times n$ matrix with rows indexed by men and columns indexed by women, and having entries in $\{0,1\}$ such that each row and each column has exactly one 1. Such a matrix is called a **permutation matrix**. For each $(m, w) \in M \times W$, having 1 in the entry corresponding to row m and column w indicates that m is assigned to w. A **probabilistic assignment** is a probability distribution over deterministic assignments. A probabilistic assignment

1 See also Alkan (2001) [1] and Alkan and Gale (2003) [2] who show that the lattice structure of stable matchings are preserved in two-sided markets with general choice functions.

2 See Lemma 2 in Dogan and Yildiz (2016) [4].

3 An incomplete version of this proof appears in the PhD thesis by Yildiz (2013) [11].

can be represented by an $n \times n$ matrix having entries in $[0,1]$ such that the sum of the entries in each row and each column is 1. Such a matrix is called a **doubly stochastic matrix**. For each probabilistic assignment π, and each pair $(m, w) \in M \times W$, the entry π_{mw} indicates the probability that m is assigned to w at π. Since each doubly stochastic matrix can be represented as a convex combination of permutation matrices (Birkhoff [3] and von Neumann [10]), the set of all doubly stochastic matrices can be thought of as the set of all probabilistic assignments. Let Π be the set of all doubly stochastic matrices.

3. Result

Let M denote the set of deterministic assignments. An assignment $\mu \in M$ **is stable at** $R \in \mathcal{R}$ if there is no $(m, w) \in M \times W$ such that $m P_w \mu(w)$, $w P_m \mu(m)$. An assignment $\pi \in \Pi$ **is ex-post stable** if it can be expressed as a convex combination of stable deterministic assignments.

For each $i \in N$, let $\mathcal{P}_i \subset \mathcal{R}_i$ be the set of all transtive, anti-symmetric, and complete preference relations for i. Let $\mathcal{P} = \times_{i \in N} \mathcal{P}_i$ be the set of all possible strict preference profiles. Let $P \in \mathcal{P}$. Let P_M and P_W denote the common preferences of men and women over deterministic assignments induced by P, defined as follows. For each $\mu, \mu' \in M$, $\mu P_M \mu'$ if and only if for each $m \in M$, $\mu P_m \mu'$. The relation P_W is defined similarly.

An assignment $\pi \in \Pi$ **is well-ordered ex-post stable at** $P \in \mathcal{P}$ if it has a decomposition into stable assignments $\mu_1, \ldots, \mu_T$ such that for each $t, t' \in \{1, \ldots, T\}$ with $t < t'$, we have $\mu_t P_M \mu_{t'}$ and $\mu_{t'} P_M \mu_t$. Next, we show that in fact well-ordered ex-post stability is equivalent to ex-post stability. The proof is based on the following two lemmas.

Lemma 1 *Let μ_1 and μ_2 be stable deterministic assignments at $P \in \mathcal{P}$ and let $\pi = \lambda_1 \mu_1 + \lambda_2 \mu_2$ such that $\lambda_1 \leq \lambda_2$. Let $\mu_1 \vee \mu_2$ ($\mu_1 \wedge \mu_2$) denote the matching where each man is matched with his best (worst) mate among the women he is matched in μ_1 or μ_2. Then, π can be decomposed into well ordered stable assignments as follows;*

$$\pi = \lambda_1(\mu_1 \wedge \mu_2) + (\lambda_2 - \lambda_1)\mu_2 + \lambda_1(\mu_1 \vee \mu_2) \tag{1}$$

Proof. Since μ_1 and μ_2 are stable, it is well known that $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are stable as well. One can easily verify that (1) holds.

Lemma 2 *Let* $\{\mu, \mu_1, \ldots, \mu_n\}$ *be a collection of stable deterministic assignments at* $P \in \mathcal{P}$ *such that* $\mu_n P_M \cdots P_M \mu_1$. *If* $\pi \in \Pi$ *can be decomposed into* $\{\mu, \mu_1, \ldots, \mu_n\}$, *then there exists a well-ordered stable decomposition of* π.

Proof. Let $\pi = \lambda\mu + \lambda_1\mu_1 + \cdots + \lambda_n\mu_n$. Let $\pi_0 = \lambda_1\mu_1 + \cdots + \lambda_n\mu_n$. Suppose that coefficients,$\{\lambda_i\}_{i=1}^{n}$ are ordered as $\lambda^1 \leq \cdots \leq \lambda^n$. Notice that $\lambda^1 \in min_{i\in\{1,\ldots,n\}}\lambda_i$, and is not necessarily equal to λ_1.

Step 1: We will show that if $\lambda \leq \lambda^1$, then $\lambda\mu + \pi_0$ has a well-ordered stable decomposition (wosd). To see this, first consider μ and μ_1. Since $\lambda \leq \lambda_1$, it follows from Lemma 1 that $\lambda\mu + \lambda_1\mu_1 = \lambda(\mu \wedge \mu_1) + (\lambda_1 - \lambda)\mu_1 + \lambda(\mu \vee \mu_1)$. Next, consider $\mu \vee \mu_1$ and μ_2. By the same reasoning we obtain $\lambda(\mu \vee \mu_1) + \lambda_2\mu_2 = \lambda[(\mu \vee \mu_1) \wedge \mu_2] + (\lambda_2 - \lambda)\mu_2 + \lambda[(\mu \vee \mu_1) \vee \mu_2]$. Notice that, since $\mu_2 P_M \mu_1$, we have $[(\mu \vee \mu_1) \wedge \mu_2] P_M \mu_1$ as well. Hence we obtain,

$$[(\mu \vee \mu_1) \vee \mu_2] P_M \mu_2 P_M [(\mu \vee \mu_1) \wedge \mu_2] P_M \mu_1 P_M (\mu \wedge \mu_1)$$

So far, we obtain a wosd for $\lambda\mu + \lambda_1\mu_1 + \lambda_2\mu_2$. Now, suppose we continue with $[(\mu \vee \mu_1) \vee \mu_2]$ and μ_3, and repeat what we have done so far, then we obtain a wosd for $\lambda\mu + \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3$. Thus, by proceeding similarly in at most n steps we obtain a wosd for $\lambda\mu + \pi_0$.

Step 2: By using Step 1 recursively, we will argue that if $\lambda^1 < \lambda \leq \lambda^n$, then $\lambda\mu + \pi_0$ has a wosd. Let $\epsilon_1 = \lambda^1$ and $\pi_1 = \epsilon_1\mu + \pi_0$. It follows from Step 1 that π_1 has a wosd. Let $\{\mu_i\}_{i\in I_1}$ be this wosd and let $\epsilon_2 = min_{i\in I_1}\lambda_i$. Notice that $\epsilon_2 \in \{\lambda^1, \lambda^2 - \lambda^1\}$. Let $\pi_2 = \epsilon_2\mu + \pi_1$. It follows from Step 1 that π_2 has a wosd, $\{\mu_i\}_{i\in I_2}$. Notice that as we continue to define ϵ_k and π_k similarly, at some step K we will necessarily obtain that $\epsilon_K = \lambda_2 - (K-1)\lambda_1$. Suppose we continue to construct ϵ_k and π_k similarly. First observe that, by construction, each π_k has a wosd. Second, for each $i \in \{1, \ldots, n\}$ there exists some k^i such that $\Sigma_{k=1}^{k^i}\epsilon_k = \lambda^i$ Now, consider $\pi_{k^n} = \epsilon_{k^n}\mu + \pi_{k^n}$. Notice that by the recursive construction, $\pi_{k^n} = \Sigma_{k=1}^{k^n}\epsilon_k\mu + \pi_0$ where $\Sigma_{k=1}^{k^i}\epsilon_k = \lambda^n$. Hence, we obtain that $\lambda^n\mu + \pi_0$ has a wosd. If $\lambda < \lambda^n$, then we obtain the desired decomposition by ending the construction once a probability mass amounting to λ is obtained for μ.

Before proceeding to Step 3, recall that for a given preference profile P, there are finitely many stable assignments. Let s be this finite number. Notice that $\lambda^n \geq \frac{1-\lambda}{s}$, which can be observed only at the uniform mixture of all the stable deterministic assignments. Combined with Step 2, it follows that for each $\lambda \leq \frac{1-\lambda}{s}$, $\lambda\mu + \pi_0$ has a wosd.

Step 3: By using Step 2 recursively, we will show that if $\frac{1-\lambda}{s} < \lambda \leq 1$, then $\lambda\mu + \pi_0$ has a wosd. To see this, let $\pi^1 = \lambda^n\mu + \pi_0$. It follows from Step 2 that π^1 has a wosd. Let λ^{max} be the greatest coefficient in this wosd. By using Step 2 we can conclude that $\lambda^{max}\mu + \pi^1$ has a wosd. By definition $\lambda^{max}\mu + \pi^1 = (\lambda^n + \lambda^{max})\mu + \pi_0$. Notice that $\lambda^{max} \geq \frac{1-\lambda-\lambda^n}{s} \geq \frac{1-\lambda}{s}$. Hence it follows that if $\lambda \leq 2\frac{1-\lambda}{s}$, then our claim holds. Since for some integer T, one must have $(T-1)\frac{1-\lambda}{s} \leq \lambda \leq T\frac{1-\lambda}{s}$, by proceeding similarly we reach a wosd of $\lambda\mu + \pi_0$.

Theorem 1 *If $\pi \in \Pi$ is ex-post stable at $P \in \mathcal{P}$, then π is well-ordered ex-post stable P.*

Proof. Let $\pi = \lambda_1\mu_1 + \cdots + \lambda_T\mu_T$ such that for each $i \in \{1 \ldots T\}$, μ_i is stable. We claim that π has a wosd. We show that our claim holds by induction on T, the number of the assignments in the given stable decomposition of π. First, suppose that $\pi = \lambda_1\mu_1 + \lambda_2\mu_2$ where $\lambda_1 \leq \lambda_2$. It follows from Lemma 1 that π has a wosd.

Next, suppose our claim holds for any $\pi' \in \Pi$ which has a decomposition into at most T distinct deterministic stable assignments. Now, consider any $\pi \in \Pi$ such that $\pi = \lambda_1\mu_1 + \cdots + \lambda_T\mu_{T+1}$. We argue that our claim holds for π as well. To see this, let $\pi_0 = (1-\lambda_{T+1})\lambda_1\mu_1 + \cdots + (1-\lambda_{T+1})\lambda_T\mu_T$. Notice that $\pi = \lambda_{T+1}\mu_{T+1} + (1/1-\lambda_{T+1})\pi_0$. Moreover, since π^0 has a decomposition into T deterministic stable assignments, it follows from the induction assumption that π_0 has a wosd, $\{\mu'_i\}_{i=1}^n$. Now, consider $\{\mu_{T+1}, \mu'_1, \ldots, \mu'_n\}$. Since $\mu'_n P_M \cdots P_M \mu'_1$, it directly follows from Lemma 2 that π has a wosd.

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Özet

Ardıl kararlılığın düzgün-sıralı ardıl kararlılığı gerektirdiğinin doğrudan ispatına ilişkin bir not

Takip eden sonucun yeni ve doğrudan bir ispatını sunmaktayız: Her olasılıksal ardıl kararlı olasılıksal eşleştirme bir kararlı belirlenimci eşleştirmeler kümesine ayrıştıralabilir, öyle ki bu kümedeki eşleştirmeler, baştan sona, hiçbir erkeğin refahı azalmayacak ve hiçbir kadının refahı artmacak biçimde dizilebilmektedir.

Anahtar kelimeler: Evlendirme problemi, olasılıksal eşleştirme, ardıl kararlılık.