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Bliss and optimal growth

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Abstract

This paper studies optimal growth when households have bliss points in their consumption sets. Optimal policies are characterized for a one-sector, discrete-time growth model with quadratic utility and linear technology. The optimal consumption function turns out to be non-linear in an essential way. Depending on the level of patience and initial capital, poverty traps, Solow-type growth and endogenous growth are possible outcomes.

1. Introduction

The linear-quadratic framework is extensively used in both theoretical and empirical macroeconomics. In an intertemporal optimization problem with quadratic utility, the optimal policies are postulated to be linear, which is convenient. However, the quadratic utility function exhibits a bliss point in the consumption set of a representative consumer. Unless the unrealistic institutional requirement of zero present value of wealth at infinity is imposed, it is not reasonable for the optimal consumption to be a linear and increasing function of wealth. Therefore one should make sure that the economy is operating far below the bliss level, if the linear policy functions are to be reasonable approximations. This observation has been made by Laroque and Lamaire (1997) on theoretical and by Lewbel (1987) on empirical grounds.

In the context of a life-cycle permanent income hypothesis model, Laroque and Lamaire (1997) show that under free disposal, the optimal consumption function is non-linear and tends to be saturated for higher levels of wealth. They assume, in a partial equilibrium setup, that the subjective and market discount rates on savings are equal.

Here we study the same phenomena in the context of the Brock and Mirman (1971) general equilibrium growth model. Our version of the model exhibits a linear production function in the single cumulative factor of production, capital, and a quadratic utility function for the representative consumer. This setup is essentially the same as that of Laroque and Lamaire (1997) under borrowing constraints. Our contribution is to allow the level of subjective discount rate to be below or above the reciprocal of the marginal productivity of capital.

In the deterministic setup, we first find a critical level of capital stock, above which growth is of a Rebelo-type for all economies. For sufficiently patient societies, which start below this level, Solow-type growth takes place towards the critical level. For relatively less patient societies, however, a low level of initial capital may end up in a poverty trap characterized by a decreasing output and consumption sequence.

In the stochastic case, a sharp level of critical capital stock does not exist. In this case, however, one can talk about the probability of take-off as a function of initial capital. Since the state space is unbounded and an ergodic set does not exist for this problem; we develop a novel numerical algorithm for calculating the optimal value function based on a result by Stokey and Lucas (1989). We then use the calculated optimal consumption function in Monte-Carlo simulations to estimate the probability of take-off in a given finite amount of time.

The model is described in section 2. The deterministic and stochastic cases are studied in sections 3 and 4. Section 5 concludes.

2. The model

We consider a model of endogenous growth to reflect the problem of a social planner trying repeatedly over an infinite horizon to allocate society's output between consumption and investment. We specify a linear production function following the "*Ak*-model" of endogenous growth where $A>1$ is a coefficient representing factors that determine the level of technology and k denotes the single factor of production representing capital stock in a broader sense, including both physical and human capital.[1]

Stochasticity is introduced through a random variable z_t, representing any favorable or unfavorable shock affecting the production process. z_t is defined from a constant probability space into real line, is assumed i.i.d. and enters the production function in a multiplicative way, i.e., $y_t=Ak_tz_t$.

The social planner ranks stochastic consumption sequences according to the expected utility of the representative consumer. The underlying utility function takes the additively separable form:

$$E[u(c_0, c_1, ...)] = E\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right]$$

Here, $\beta \in (0,1)$ is the discount factor, $E(.)$ denotes the expected value with respect to the probability distribution of the random variables $\{c_t\}_{t=0}^{\infty}$. In order to capture the idea of bliss, we adopt a convenient quadratic utility function of the form $U(c_t) = -ac_t^2 + bc_t$ with $a, b>0$ to assure strict concavity and $U(0)=0$. Such a utility function clearly exhibits a bliss point $\hat{c} = b/2a > 0$.

The problem can then be written as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} = Ak_tz_t$$

$$c_t, k_t \geq 0 \qquad t=0,1,2,\ldots$$

$$k_0 > 0, \text{ given}$$

3. Deterministic case

Under the deterministic scenario, we simply set $z_t=1$ for all possible states of nature. For such a problem, it is clear that for a sufficiently high level of initial capital stock, consuming always at the bliss point is feasible, and hence optimal. Proposition 3.1 states this fact.

For a relatively low level of initial capital stock, however, the relationship between the patience and productivity parameters bears importance. For relatively patient societies, a Solow-type growth towards

[1] *Ak*-type of production function has been used by Rebelo (1991) in order to show that positive rates of endogenous growth can be obtained despite the absence of increasing returns.

the bliss point takes place. In contrast, relatively impatient societies end up in a poverty trap, whenever their initial level of capital stock is below a threshold level. In between these two cases, there also exists a knife-edge society where both consumption and output are stagnant for low levels of initial capital. Propositions 3.2-3.4 characterize optimal policies for these three cases.

Without any further parameter restrictions, it is possible to find a critical value of capital stock over which the economy follows a sustained growth path.

Proposition 3.1.For ($k_0 > \hat{c}/(A-1)$) optimal will $\{k_t\}_{t=1}^{\infty}$ *be a strictly growing sequence, with* $\lim_{t\to\infty} k_t = \infty$ *at an asymptotic growth rate of A.*

Proof: We will show that the policy of always consuming at bliss level $c_t = \hat{c}$ for all t, is feasible and leads to sustained growth. Let $k_0 > \hat{c}/(A-1)$. Then $\hat{c} < k_0(A-1)$. Since $k_1 = Ak_0 - \hat{c}$, we have $k_1 > Ak_0 - k_0(A-1) = k_0$. Now, $k_2 = Ak_1 - \hat{c}$. Then

$k_2 > Ak_0 - \hat{c} = k_1$

$k_3 = Ak_2 - \hat{c} > Ak_1 - \hat{c} = k_2$

$k_4 = Ak_3 - \hat{c} > Ak_2 - \hat{c} = k_3$

…

$k_{t+1} > k_t \quad \forall t$

Since $k_t = Ak_{t-1} - \hat{c}$, we can write:

$$k_t = A^t k_0 - \hat{c}\left[\sum_{i=0}^{t-1} A^i\right]$$

$$k_t = A^t k_0 - A^{t-1}\hat{c}\left[\sum_{i=0}^{t-1} (1/A)^i\right]$$

It is possible to express the summation term in parenthesis as the difference of the two geometric series; then we have:

$$k_t = A^t\left(k_0 - \frac{\hat{c}}{A-1}\right) + \frac{\hat{c}}{A-1}$$

We have assumed ($k_0 > \hat{c}/(A-1)$) which implies $\lim_{t\to\infty} k_t = \infty$. □

The existence of such a critical capital stock level distinguishes a region where growth is ‘Rebelo-type’ in the sense that we witness endogenous sustained growth, from a region where growth is ‘Solow-type’ in the sense that we observe a zero rate of growth at the steady state.

To fully characterize the solution, we use a sequential approach below. Sufficiency of the Euler equations and the transversality condition for sequences of the state variable where the constraint correspondence is non-binding is a well-known result (Stokey and Lucas, 1989). We present in the Appendix an extension to this classical result in order to allow for non-monotone utility functions and the possibility of corner solutions along optimal paths.

If the discount rate β is equal to the inverse of the marginal product of capital, the Euler equation reduces to

$$c_t = c_{t+1} \qquad \forall t = 0,1\ldots \tag{1}$$

which leads to the following result.

Proposition 3.2. Let $\beta A = 1$. The social planner’s optimal consumption function is,

$$c_t = \begin{cases} (A-1)k_t & \text{if} \quad k_t \le \hat{c}/(A-1) \\ \hat{c} & \text{if} \quad k_t > \hat{c}/(A-1) \end{cases} \tag{2}$$

Proof: For the problem with $\beta A = 1$, the Euler equation

$$U'(c_t) = \beta A U'(c_{t+1})$$

and the transversality condition

$$\lim_{T\to\infty} -\beta^T U'(c_T)k_{T+1} = 0$$

are satisfied for the consumption plan

$$c_t = \begin{cases} (A-1)k_t & \text{if} \quad k_t \le \hat{c}/(A-1) \\ \hat{c} & \text{if} \quad k_t > \hat{c}/(A-1) \end{cases}$$

Then, by *Proposition A.3.1*, the plan, which in this case gives an interior solution for each *t*, is optimal. □

The above statement characterizes the optimal behaviour for an economy as follows: An economy starting from an initial level of capital stock below the critical level, $k_c = \hat{c}/(A-1)$, stays in its initial state forever (Note that $c_t = (A-1)k_t$ implies $k_{t+1} = k_t$). On the other hand, an economy starting from an initial capital stock which is above the critical value, will always consume at bliss level $\hat{c}$. Figure 1 shows the optimal consumption function for the knife-edge case of $\beta A = 1$. Figure 2 shows the investment function, which at the same time can be seen as the phase diagram for capital stock.

Figure 1

Consumption Function (βA = 1)

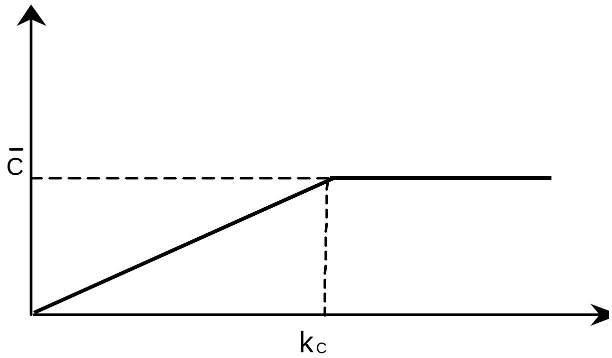
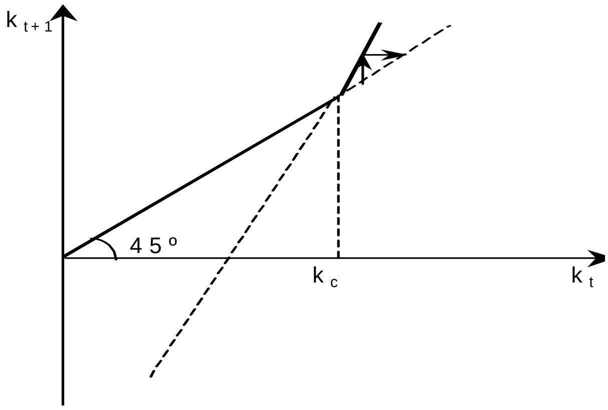


Figure 2

Investment Function (βA = 1)



For more patient societies, a drift towards the bliss consumption level is observed as follows:

Proposition 3.3. *Let $\beta A > 1$. The social planner's optimal consumption function is,*

$$c_t = \begin{cases} 0 & if \quad 0 \le k_t \le k_c\left((\beta A - 1)/(\beta A^2 - 1)\right) \\ (A - \frac{1}{\beta A})k_t + \hat{c}\left(\frac{1-\beta A}{\beta A(A-1)}\right) & if \quad k_c\left((\beta A - 1)/(\beta A^2 - 1)\right) < k_t < k_c \\ 0 & if \quad k_t \ge k_c \end{cases} \quad (3)$$

Proof: Under the given condition, $\beta A > 1$, for $k_t > k_c\left((\beta A-1)/(\beta A^2-1)\right)$, the Euler equation

$$U'(c_t) = \beta A U'(c_{t+1})$$

and for $k_t \leq k_c\left((\beta A-1)/(\beta A^2-1)\right)$, the modified Euler equation

$$U'(c_t) < \beta A U'(c_{t+1})$$

with $k_{t+1} = Ak_t$ with the transversality condition

$$\lim_{T\to\infty} -\beta^T U'(c_T)k_{T+1} = 0$$

hold for the consumption plan stated. Therefore *Proposition A.3.1* implies that the plan is optimal. □

The consumption function and the associated phase diagram are shown in Figures 3 and 4 respectively. For initial capital stock levels, Figure 4 clearly shows the presence of a unique steady state capital stock (k_{ss}) which actually equals the critical level that we have calculated before, i.e., $k_{ss} = k_c$. The optimal behaviour for an economy with $0 < k_0 \leq k_c$ is to move to the steady state; more sluggishly as the representative household gets more and more impatient, and then stay there forever consuming $\hat{c}$. On the other hand, a starting point $k_0 > k_c$ will provide the chance for always consuming at the bliss level.

Figure 3

Consumption Function ($\beta A > 1$)

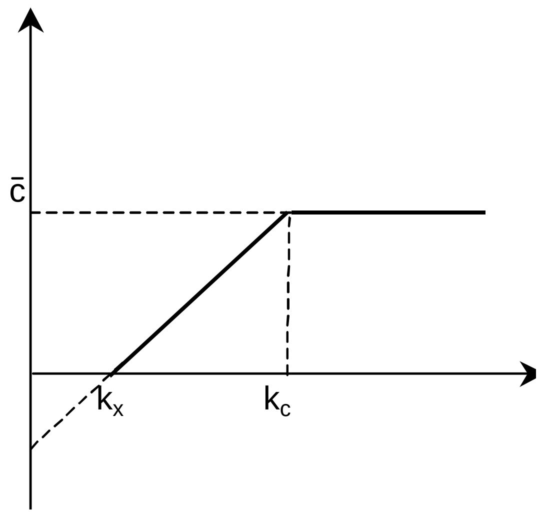
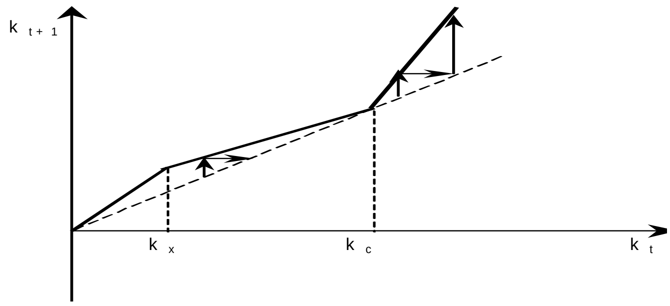


Figure 4

Investment Function ($\beta A > 1$)



When $\beta < 1/A$, the representative household, being impatient, is biased towards consuming today rather than tomorrow. This action results in a poverty trap.

Proposition 3.4. Let $\beta A < 1$. Given the initial capital stock k_0, the social planner's optimal consumption function satisfies,

$$c_t = \begin{cases} Ak_t & \text{if} \quad 0 \le k_t \le k_c(1-\beta A) \\ (A - \frac{1}{\beta A})\ k_t + \hat{c}\left(\frac{1-\beta A}{\beta A(A-1)}\right) & \text{if} \quad k_c(1-\beta A) < k_t < k_c \\ \hat{c} & \text{if} \quad k_t \ge k_c \end{cases} \tag{4}$$

Proof: For $\beta A < 1$, the consumption plan stated above satisfies the modified Euler equation

$U'(c_t) > \beta A U'(c_{t+1})$

with $k_{t+1} = 0$ for all $k_t \le k_c\ (1-\beta A)$, and the Euler equation

$U'(c_t) = \beta A U'(c_{t+1})$

for $k_t > k_c\ (1-\beta A)$, together with the transversality condition

$$\lim_{T\to\infty} -\beta^T U'(c_T) k_{T+1} = 0$$

Therefore, applying *Proposition A.3.1* once more, we arrive at the conclusion that the given plan is optimal. □

For the case of $\beta A < 1$ then, an economy starting with $k_0 \le k_c(1-\beta A)$ converges to zero by consuming all the production in the first period. An economy with $k_0 \in [k_c(1-\beta A),\ k_c]$ gradually moves towards the poverty trap to stay there for a lifetime. These observations can be made using Figures 5 and 6.

Figure 5
Consumption Function ($\beta A < 1$)

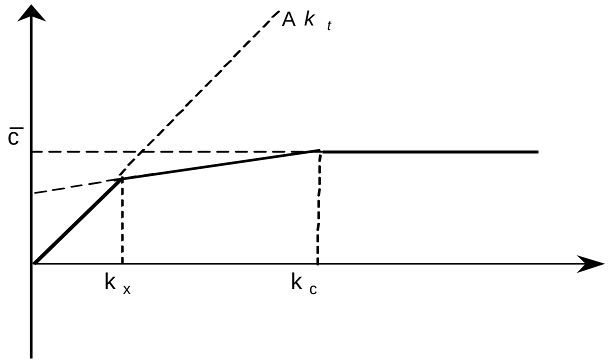
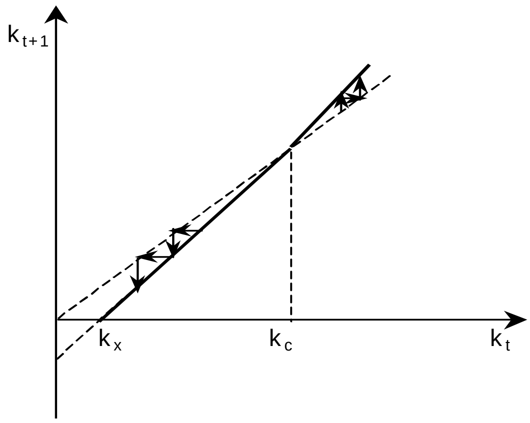


Figure 6
Investment Function ($\beta A < 1$)



4. Stochastic case

In the Brock-Mirman model of optimal growth, introducing uncertainty into the problem changes the long-run equilibrium concept. Rather than simply characterizing the optimal path as a fixed stationary capital stock level, the solution to the stochastic problem describes it as a (unique) probability distribution on an *ergodic set* of capital stocks. Likewise, in this section we analyze how the results of the deterministic case change in a stochastic environment. However, in our case, an ergodic set for the capital stock does not exist, since sustained growth is possible. This requires a computational novelty in finding a numerical solution to the corresponding dynamic programming problem.

The functional equation for the stochastic problem is:

$$v(k,z) = \max_{0 \le k' \le zf(k)} \left\{ U\ [zf(k) - k'] + \beta E[v\ (k', z') | k, z] \right\} \qquad (FE_s)$$

To characterize the behaviour under the stochastic scenario, we follow the monotone operator idea of Stokey and Lucas (1989, Thm.4.14) and use the corresponding numerical dynamic programming approach by recursive iterations on some function $\hat{v}$, an upper bound to the solution of the sequential problem.

We initialize the recursion with:

$$\hat{v}_0(k,z) = \frac{U(\hat{c})}{1-\beta} \quad \forall (k,z) \tag{5}$$

followed by application of the operator T to $\hat{v}_n$ ($n = 0,1,2,\ldots$), iterating down to a fixed point

$$T\ \hat{v}_n(.) = \hat{v}_{n+1} = \max \left\{ U(.) + \beta E\ [\hat{v}_n(.)] \right\} \tag{6}$$

Now, it turns out that the fixed point of T, which is $v = \lim_{n\to\infty}(T^n \hat{v})$ is the solution to the problem of the social planner of infinite horizon as well.[2]

As a simulation exercise, we solved the problem (FE_S) explicitly through the algorithm defined. While doing this, we have adopted a quadratic utility function $U(c) = -c^2+4c$, which has a bliss point of $\hat{c} = 2$. The production function has a technology parameter $A = 1.4$. For simplicity, we introduce two possible shock values, $z_l = 0.8$ with probability 0.5 and $z_h = 1.2$ with probability 0.5. We let $\beta = 0.7$ so that $E(Az\beta) = 0.98 < 1$. The maximum value of capital stock is taken as 40 in the implementation of the T map.

Under the stochastic setup, the critical value of the initial capital stock allowing for sustained growth, k_c of the deterministic case, leaves its place to a probability distribution over the initial values of the capital stock, k_0 denoting the probability of having a divergent sequence from each starting point. Figure 7 is obtained by calculating an optimal consumption function in Monte Carlo simulations where the economy is allowed to operate for 100 periods.

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[2] A slightly modified version of Theorem 4.11 of Stokey and Lucas (1989) to cover our simple stochastic setup is applicable in the proof of this statement.

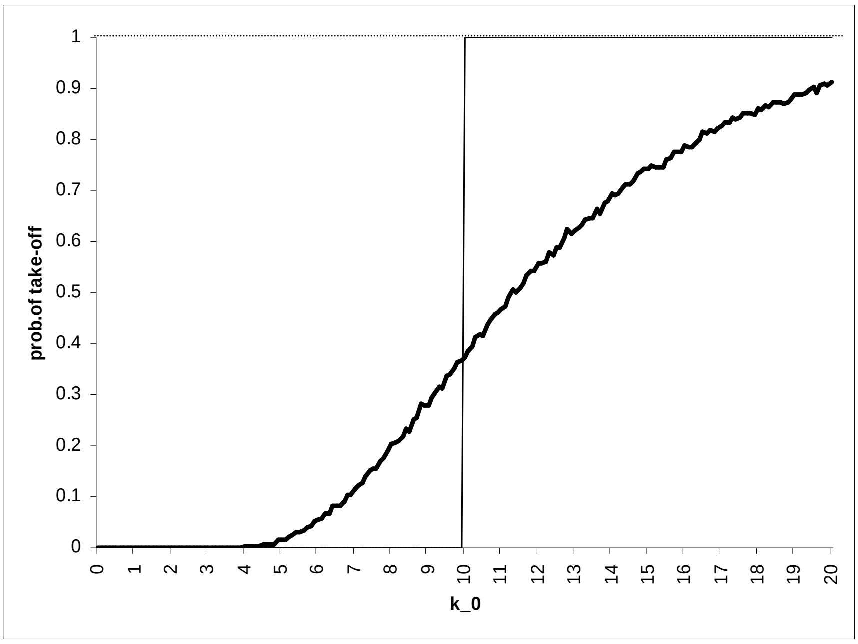
5. Concluding remarks

There has been a number of papers studying poverty traps and growth. Murphy, Shleifer and Vishny (1989), analyze the idea of *big push* in an imperfectly competitive economy with aggregate demand spillovers and show that it is possible to move from a bad to a good equilibrium. Azariadis and Drazen (1990) study the relation between poverty traps and growth with human capital. Abe (1995), in a model with public goods gives policy implications to achieve sustained growth. Zilibotti (1995)'s model of endogenous growth exhibiting aggregate non-convexities and thresholds separates a region where growth is Solow-type from a region where growth is Romer-type. Similarly Matsuyama (1999), combining the issues of capital accumulation and innovation, characterizes the notion of growing through cycles.

In contrast, our work has no increasing returns or externalities. Firstly, we show that only allowing for satiation on the consumption set may lead to results similar to those found in the studies mentioned above. Secondly, our study emphasizes the fact that care should be given when using a linear-quadratic setup in macro-models since especially in the stochastic setups, the optimal policy may turn out to be non-linear under the quite plausible assumption of free disposal.

Figure 7

Probability of Take-off Within 100 Periods
as a Function of Initial Capital Stock



Appendix

In this section we present a theorem for the sufficiency of a modified set of Euler inequalities and a transversality condition for a class of discrete time dynamic optimization problems where *corner solutions* along optimal paths are allowed. In the case of a quadratic felicity function, the $F_x > 0$ condition is violated. Hence, we relax this assumption as well. The theorem extends Stokey and Lucas (1989, Thm. 4.15).

1. The Problem

Let $F : \Re_+ x \Re_+ \rightarrow \Re$ be a concave, differentiable function. Also let $0 < \beta < 1$.

Maximize $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$

subject to

$x_{t+1} \in [l, b(x_t)]$

$x_0 \geq l$ given,

over all admissible sequences. Here l is a common lower bound on possible states in all periods and $b: \Re_+ \rightarrow \Re_+$ is a function determining an upper bound on next period's state. We assume $b(x) \geq l \;\; \forall x \in \boldsymbol{X}$.

2. Modified Euler and Transversality Conditions

Modified Euler Conditions (MEC)

$F_2(x_t,x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0$ and $x_{t+1} \in int[l,b(x_t)]$

$F_2(x_t,x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) > 0$ and $x_{t+1} = b(x_t)$

$F_2(x_t,x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) < 0$ and $x_{t+1} = l$

Transversality Condition (TVC)

$$\lim_{T \rightarrow \infty} \beta^T F_2(x_T, x_{T+1}) x_{T+1} = 0$$

3. Sufficiency Results

PROPOSITION A.3.1 *Let x be an admissible sequence satisfying the transversality condition and for each t the modified Euler equations.*

Moreover, suppose $F_2(x_t, x_{t+1}) \leq 0$ for all t. Then x solves the maximization problem.

Proof: Let x be an admissible sequence satisfying the transversality condition, and for each t the modified Euler equations, with $F_2(x_t, x_{t+1}) \leq 0$. Also let y be any other admissible sequence.

Define D as the difference between the lifetime utilities of x and y. It is now possible to follow steps analogous to those in Stokey and Lucas's (1989) proof of Theorem 4.15.

$$\begin{aligned} D &= \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F(x_t, x_{t+1}) - F(y_t, y_{t+1})] \\ &\geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F_1(x_t, x_{t+1})(x_t - y_t) + F_2(x_t, x_{t+1})(x_{t+1}, y_{t+1})] \end{aligned}$$

Since $x_0 = y_0$, rearranging the terms, we get:

$$D \geq \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t [F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2})](x_{t+1} - y_{t+1}) + \beta^T F_2(x_T, x_{T+1})(x_{T+1} - y_{T+1})$$

Since x satisfies *MEC*, the terms in the summation are all greater than or equal to zero.

Then,

$$\begin{aligned} D &\geq \lim_{T\to\infty} \beta^T F_2(x_T, x_{T+1})(x_{T+1} - y_{T+1}) \\ &\geq \lim_{T\to\infty} \beta^T F_2(x_T, x_{T+1}) x_{T+1} \end{aligned}$$

where the last line follows from $F_2(x_t, x_{t+1}) \leq 0$ and $y_t \geq 0$ for each t. It then follows from the *TVC* that $D \geq 0$ for each admissible y; x must be maximal in the admissible set.

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Özet

Doyum ve optimal büyüme

Bu makale, bireylerin tüketim kümelerinde bir doyum noktasına sahip olmaları hipotezi altında, optimal büyümeyi araştırmaktadır. Bu amaçla tek sektörlü, birinci dereceden homojen bir üretim ve parabolik fayda fonksiyonları altında, rassal bir endojen büyüme modeli içerisinde optimal politikalar karakterize edilmiştir. Modelin çözümü “tüketici sabırlılığı” ve “ekonominin başlangıç noktası”na bağlı ve kalitatif olarak çeşitlilik gösteren dengelerin varlığına işaret etmektedir. Bu sonuç, “yoksulluğun kısır döngüsü” ve “sürekli büyüme” konularının ortak bir analitik çerçevede işlenebilirliğini göstermektedir.