Improved analytical approximation to arbitrary l-state solutions of the Schrödinger equation for the hyperbolical potentials

Sameer M. Ikhdair[1, *] and Ramazan Sever[2, †]

^{1}*Department of Physics, Near East University, Nicosia, North Cyprus, Turkey*

^{2}*Department of Physics, Middle East Technical University, 06800 Ankara,Turkey*

(Dated: September 13, 2018)

Abstract

A new approximation scheme to the centrifugal term is proposed to obtain the $l \neq 0$ bound-state solutions of the Schrödinger equation for an exponential-type potential in the framework of the hypergeometric method. The corresponding normalized wave functions are also found in terms of the Jacobi polynomials. To show the accuracy of the new proposed approximation scheme, we calculate the energy eigenvalues numerically for arbitrary quantum numbers n and l with two different values of the potential parameter σ_0. Our numerical results are of high accuracy like the other numerical results obtained by using program based on a numerical integration procedure for short-range and long-range potentials. The energy bound-state solutions for the s-wave ($l = 0$) and $\sigma_0 = 1$ cases are given.

Keywords: Energy eigenvalues and eigenfunctions, Exponential-type potentials, Hypergeometric method, Approximation schemes

PACS numbers: 03.65.Ge; 34.20.Cf

---

*E-mail: sikhdair@neu.edu.tr

†E-mail: sever@metu.edu.tr

I. INTRODUCTION

The exact analytic solutions of the wave equations (nonrelativistic and relativistic) are only possible for certain potentials of physical interest under consideration since they contain all the necessary information on the quantum system. It is well known that the exact solutions of these wave equations are only possible in a few simple cases such as the Coulomb, the harmonic oscillator, pseudoharmonic potentials and others [1-5]. Recently, the analytic exact solutions of the wave equation with some exponential-type potentials are impossible for $l \neq 0$ states. Approximation methods have to be used to deal with the centrifugal term like the Pekeris approximation [6-8] and the approximated scheme suggested by Greene and Aldrich [9]. Some of these exponential-type potentials include the Morse potential [10], the Hulthén potential [11], the Pöschl-Teller [12], the Woods-Saxon potential [13], the Kratzer-type and pseudoharmonic potentials [14], the Rosen-Morse-type potentials [15], the Manning-Rosen potential [15-22], other multiparameter exponential-type potentials [23,24] and hyperbolical potential [25-27].

In this work, we attempt to study another exponential-type potential called the hyperbolical potential [25-27]

$$V(r) = D\left[1 - \sigma_0 \coth(\alpha r)\right], \tag{1}$$

where D, α and σ_0 are three positive parameters. It is indicated in [25] that this exponential-type potential is closely related to the Morse, the Kratzer, the Coulomb, the harmonic oscillator and other potential functions. The properties and applications of this potential are given in [25,26]. It is known that for this potential the Schrödinger equation (SE) can be solved for the s-wave, angular momentum quantum number $l = 0$. However, for a general solution, it is need to include some approximations if one wants to obtain analytical or semianalytical solutions to the SE. For the $l \neq 0$ case, the potential (1) can not be solved exactly without an approximation to the centrifugal term [22]. Hence, in the previous papers, several approximations have been developed to find better analytical formulas for the hyperbolical potential [27].

Our aim in this work is to attempt to study the arbitrary l-state solutions of the Schrödinger equation for the hyperbolical potential. In order to improve the accuracy of our previous approximation [20,21], we propose and apply a new approximation scheme for

the centrifugal term in the form:

$$\frac{1}{r^2} \approx 4\alpha^2 \left[c_0 + \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} + \left(\frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} \right)^2 \right], \tag{2}$$

where c_0 is a proper shift to be found by the expansion procedures. Thus, with this new approximation scheme, we calculate the $l \neq 0$ energy levels and wavefunctions of the hyperbolical potential using the hypergeometric approach (Nikiforov and Uvarov) N-U method. This method has shown its power in calculating the exact energy levels for some solvable quantum systems [13,14,20-22]. The approximation given by (2) has proved its power and accuracy over the other currently used approximations in literature [16-21]. It has been applied recently on the Manning-Rosen potential [22] and has also proved its power and efficiency when compared with the other numerical simulations for the non-approximated problem used to calculate the energy bound states. It provides good results which are in agreement with the numerical integration method by Lucha and Schöberl [28].

The paper is organized as follows: In Section II we breifly present the N-U method. In Section III, we present the new proposed approximation scheme and apply it to calculate the l-wave eigensolutions of the SE for the hyperbolical potential by the N-U method. In Section IV, we present our numerical results for energy eigenvalues numerically for arbitrary quantum numbers n and l with two different values of the potential parameter σ_0. Section V, is devoted to study two special cases, the s-wave $(l = 0)$ case and the $\sigma_0 = 1$ exponential-type potential. Finally, we make a few concluding remarks in Section VI.

II.   THE NIKIFORV-UVAROV METHOD

The Nikiforov-Uvarov (N-U) method is based on solving the second-order linear differential equation by reducing it to a generalized equation of hypergeometric type [29]. In this method after employing an appropriate coordinate transformation $z = z(r)$, the Schrödinger equation can be written in the following form:

$$\psi_n''(z) + \frac{\widetilde{\tau}(z)}{\sigma(z)} \psi_n'(z) + \frac{\widetilde{\sigma}(z)}{\sigma^2(z)} \psi_n(z) = 0, \tag{3}$$

where $\sigma(z)$ and $\widetilde{\sigma}(z)$ are the polynomials with at most of second-degree, and $\widetilde{\tau}(s)$ is a first-degree polynomial. The special orthogonal polynomials [29] reduce Eq. (3) to a simple

form by employing $\psi_n(z) = \phi_n(z)y_n(z)$, and choosing an appropriate function $\phi_n(z)$. Consequently, Eq. (3) can be reduced into an equation of the following hypergeometric type:

$$\sigma(z)y_n''(z) + \tau(z)y_n'(z) + \lambda y_n(z) = 0, \tag{4}$$

where $\tau(z) = \widetilde{\tau}(z) + 2\pi(z)$ (its derivative must be negative) and λ is a constant given in the form

$$\lambda = \lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z), \quad n = 0, 1, 2, ... \tag{5}$$

It is worthwhile to note that λ or λ_n are obtained from a particular solution of the form $y(z) = y_n(z)$ which is a polynomial of degree n. Further, $y_n(z)$ is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(z) = \frac{B_n}{\rho(z)}\frac{d^n}{dz^n}[\sigma^n(z)\rho(z)], \tag{6}$$

where B_n is the normalization constant and the weight function $\rho(z)$ must satisfy the differential equation: [29]

$$w'(z) - \left(\frac{\tau(z)}{\sigma(z)}\right)w(z) = 0, \ w(z) = \sigma(z)\rho(z). \tag{7}$$

In order to determine the weight function given in Eq. (7), we must obtain the following polynomial:

$$\pi(z) = \frac{\sigma'(z) - \widetilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \widetilde{\tau}(z)}{2}\right)^2 - \widetilde{\sigma}(z) + k\sigma(z)}. \tag{8}$$

In principle, the expression under the square root sign in Eq. (8) can be arranged as the square of a polynomial. This is possible only if its discriminant is zero. In this case, an equation for k is obtained. After solving this equation, the obtained values of k are included in the N-U method and here there is a relationship between λ and k by $k = \lambda - \pi'(z)$. After this point an appropriate $\phi_n(z)$ can be extracted from the differential equation:

$$\phi'(z) - \left(\frac{\pi(z)}{\sigma(z)}\right)\phi(z) = 0. \tag{9}$$

III.   ANALYTICAL SOLUTIONS

A.   An Impoved Approximation Scheme

The approximation is based on the expansion of the centrifugal term in a series of exponentials depending on the intermolecular distance r and keeping terms up to second order.

4

Therefore, instead of using the approximation in [9,11,19], we use this choice of approximation:

$$\frac{1}{r^2} \approx (2\alpha)^2 \left[c_0 + v(r) + v^2(r) \right], \ v(r) = \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}},$$

$$\frac{1}{r^2} \approx (2\alpha)^2 \left[c_0 + \frac{1}{e^{2\alpha r} - 1} + \frac{1}{(e^{2\alpha r} - 1)^2} \right], \tag{10}$$

which has a similar form of the hyperbolical potential. Changing the coordinate to x by using $x = (r - r_0)/r_0$, one obtains

$$(1 + x)^{-2} = \gamma^2 \left[c_0 + \frac{1}{e^{\gamma(1+x)} - 1} + \frac{1}{(e^{\gamma(1+x)} - 1)^2} \right], \ \gamma = 2\alpha r_0 \tag{11}$$

and *expanding* Eq. (11) around $r = r_0$ ($x = 0$), we obtain the following Taylor's expansion:

$$1 - 2x + O(x^2) = \gamma^2 \left(c_0 + \frac{1}{e^\gamma - 1} + \frac{1}{(e^\gamma - 1)^2} \right)$$

$$- \gamma^3 \left(\frac{1}{e^\gamma - 1} + \frac{3}{(e^\gamma - 1)^2} + \frac{2}{(e^\gamma - 1)^3} \right) x + O(x^2), \tag{12}$$

from which we obtain

$$\gamma^2 \left[c_0 + \frac{1}{e^\gamma - 1} + \frac{1}{(e^\gamma - 1)^2} \right] = 1,$$

$$\gamma^3 \left(\frac{1}{e^\gamma - 1} + \frac{3}{(e^\gamma - 1)^2} + \frac{2}{(e^\gamma - 1)^3} \right) = 2. \tag{13}$$

Therefore the shifting paramete, c_0, can be found from the solution of the above two equations as:

$$c_0 = \frac{1}{\gamma^2} - \frac{1}{e^\gamma - 1} - \frac{1}{(e^\gamma - 1)^2} = 0.0823058167837972, \tag{14}$$

where e is the base of the natural logarithms, $e = 2.718281828459045$ and the parameter $\gamma = 0.4990429999$. Hence, we have the following substitution for the centrifugal term:

$$\frac{1}{r^2} = \lim_{\alpha \to 0} 4\alpha^2 \left[\frac{1}{\gamma^2} - \frac{1}{e^\gamma - 1} - \frac{1}{(e^\gamma - 1)^2} + \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} + \left(\frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} \right)^2 \right]. \tag{15}$$

Finally, it is worth to note that, in the case if $c_0 = 0$, the approximation given in Eq. (10) is identical to the commonly used approximation in the previous works [9,11,19-21].

B. Energy Eigenvalues and Eigenfunctions Solution

To study any quantum physical system characterized by the empirical potential given in Eq. (1), we solve the original SE which is given in the well known textbooks [1,2]

$$\left(\frac{p^2}{2m} + V(r)\right)\psi(\mathbf{r},\theta,\phi) = E\psi(\mathbf{r},\theta,\phi), \tag{16}$$

where the potential $V(r)$ is taken as the hyperbolical potential (1). Using the separation method with the wavefunction $\psi(\mathbf{r},\theta,\phi) = r^{-1}R(r)Y_{lm}(\theta,\phi)$, we obtain the following radial Schrödinger eqauation as

$$\frac{d^2R_{nl}(r)}{dr^2} + \left\{\frac{2\mu E_{nl}}{\hbar^2} - \frac{2\mu D}{\hbar^2}\left[1 - \sigma_0\left(\frac{e^{\alpha r} + e^{-\alpha r}}{e^{\alpha r} - e^{-\alpha r}}\right)\right]^2 - 4\alpha^2 l(l+1)\left(c_0 + \frac{e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}\right)\right\}$$

$$\times R_{nl}(r) = 0. \tag{17}$$

Since the SE with the above hyperbolical potential has no analytical solution for l-waves, we have used the approximation to the centrifugal term given by case 1. The other approximations will be left for future investigations. To solve it by the N-U method, we need to recast Eq. (17) into the form of Eq. (3) changing the variables $r \to z$ through the mapping function $r = f(z)$ and making the following definitions:

$$z = e^{-2\alpha r}, \ \nu = \frac{\mu D}{2\alpha^2\hbar^2}, \ \varepsilon' = \sqrt{-\frac{\mu E_{nl}}{2\alpha^2\hbar^2} + \Delta E_l}, \ E_{nl} < \frac{2\alpha^2\hbar^2}{\mu}\Delta E_l, \ \Delta E_l = l(l+1)c_0, \tag{18}$$

we obtain the following hypergeometric equation:

$$\frac{d^2R(z)}{dz^2} + \frac{(1-z)}{z(1-z)}\frac{dR(z)}{dz} + \frac{1}{[z(1-z)]^2}$$

$$\times \left\{-\varepsilon'^2 - \nu(1-\sigma_0)^2 + \left[2\nu(1-\sigma_0^2) + 2\varepsilon'^2 - l(l+1)\right]z - \left[\nu(1+\sigma_0)^2 + \varepsilon'^2\right]z^2\right\}R(z) = 0. \tag{19}$$

It is shown from Eq. (18) that for bound state (real) solutions, we require:

$$z = \begin{cases} 0, & \text{when } r \to \infty, \\ 1, & \text{when } r \to 0, \end{cases} \tag{20}$$

and as a result the radial wavefunctions $R_{nl}(z) \to 0$ for the values of z given in Eq. (18). To apply the N-U method, we compare Eq. (19) with Eq. (3) and obtain the following values for the parameters:

$$\widetilde{\tau}(z) = 1 - z, \ \sigma(z) = z - z^2,$$

6

$$\tilde{\sigma}(z) = -\left[\nu\left(1+\sigma_0\right)^2 + \varepsilon'^2\right]z^2 + \left[2\nu\left(1-\sigma_0^2\right) + 2\varepsilon'^2 - l(l+1)\right]z - \varepsilon'^2 - \nu\left(1-\sigma_0\right)^2. \quad (21)$$

If one inserts these values of parameters into Eq. (8), with $\sigma'(z) = 1 - 2z$, the following linear function is obtained

$$\pi(z) = -\frac{z}{2} \pm \frac{1}{2}\sqrt{a_2 z^2 + a_1 z + a_0}, \quad (22)$$

where $a_2 = 1 + 4\left[\varepsilon'^2 + \nu\left(1+\sigma_0\right)^2 - k\right]$, $a_1 = 4\left\{k + l(l+1) - 2\nu\left(1-\sigma_0^2\right) - 2\varepsilon'^2\right\}$ and $a_0 = 4\left[\varepsilon'^2 + \nu\left(1-\sigma_0\right)^2\right]$. According to this method the expression in the square root has to be set equal to zero, that is, $\Delta = a_2 z^2 + a_1 z + a_0 = 0$. Thus the constant k can be determined as

$$k = -\frac{1}{4}\left[(1+2l)^2 + 16\nu\sigma_0(\sigma_0 - 1) - 1\right] \pm \beta(1+2\delta), \quad (23)$$

where

$$\beta = \sqrt{\varepsilon'^2 + \nu\left(1-\sigma_0\right)^2}, \quad \delta = \frac{1}{2}\left[-1 + \sqrt{16\nu\sigma_0^2 + (1+2l)^2}\right]. \quad (24)$$

In this regard, we can find four possible functions for $\pi(z)$ as

$$\pi(z) = -\frac{z}{2} \pm \frac{1}{2}$$

$$\times \begin{cases} (2\beta - 2\delta - 1)\,z - 2\beta, & \text{for } k = -\left[(1+2l)^2 + 16\nu\sigma_0(\sigma_0 - 1) - 1\right]/4 + \beta(1+2\delta), \\ (2\beta + 2\delta + 1)\,z - 2\beta; & \text{for } k = -\left[(1+2l)^2 + 16\nu\sigma_0(\sigma_0 - 1) - 1\right]/4 - \beta(1+2\delta). \end{cases}$$
$$(25)$$

We must select

$$k = -\frac{z}{2} - \frac{1}{2}\left[(2\beta + 2\delta + 1)\,z - 2\beta\right], \quad (26)$$

in order to obtain the polynomial, $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ having negative derivative as

$$\tau(z) = 1 - 2z - \left[2\beta + 2\delta + 1\right]z - 2\beta, \quad \tau'(z) = -(2\beta + 2\delta + 3). \quad (27)$$

We can also write the values of $\lambda = k + \pi'(z)$ and $\lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z)$, $n = 0, 1, 2, ...$ as

$$\lambda = -\frac{1}{4}\left[(1+2l)^2 + 16\nu\sigma_0(\sigma_0 - 1) - 1\right] - \left[\beta + \delta + 1\right], \quad (28)$$

$$\lambda_n = n(n + 2\beta + 2\delta + 2), \quad n = 0, 1, 2, \cdots \quad (29)$$

respectively. Additionally, using the definition of $\lambda = \lambda_n$ and solving the resulting equation for ε', allows one to obtain

$$\beta = -\frac{(n+1)^2 + l(l+1) + (2n+1)\delta - 4\nu\sigma_0(1-\sigma_0)}{2(n+\delta+1)} = \sqrt{-\frac{\mu E_{nl}}{2\alpha^2\hbar^2} + \Delta E_l + \nu\left(1-\sigma_0\right)^2}. \quad (30)$$

Hence, we obtain analytically the following discrete bound-energy levels

$$E_{nl} = (1 - \sigma_0)^2 D + \frac{2\alpha^2 \hbar^2 l(l+1)}{\mu} \left[\frac{1}{\gamma^2} - \frac{1}{e^\gamma - 1} - \frac{1}{(e^\gamma - 1)^2} \right]$$

$$- \frac{2\alpha^2 \hbar^2}{\mu} \left[\frac{(n+1)^2 + l(l+1) + (2n+1)\delta - 4\nu\sigma_0(1-\sigma_0)}{2(n+\delta+1)} \right]^2, \quad 0 \le n, l < \infty \qquad (31)$$

where $n = 0, 1, 2, \cdots$ and l signify the usual radial and angular momentum quantum numbers, respectively.

Let us now find the corresponding radial part of the normalized wave functions. Using $\sigma(z)$ and $\pi(z)$ in Eqs. (21) and (26), we obtain

$$\phi(z) = z^\beta (1-z)^{\delta+1}, \qquad (32)$$

$$\rho(z) = z^{2\beta}(1-z)^{2\delta+1}, \qquad (33)$$

$$y_{nl}(z) = C_n z^{-2\beta}(1-z)^{-(2\delta+1)} \frac{d^n}{dz^n} \left[z^{n+2\beta}(1-z)^{n+2\delta+1} \right]. \qquad (34)$$

The functions $y_{nl}(z)$, up to a numerical factor, are in the form of Jacobi polynomials, i.e., $y_{nl}(z) \simeq P_n^{(2\beta, 2\delta+1)}(1-2z)$, valid physically in the interval $(0 \le r < \infty \to 0 \le z \le 1)$ [30]. Therefore, the radial part of the wave functions can be found by substituting Eqs. (32) and (34) into $R_{nl}(z) = \phi(z)y_{nl}(z)$ as

$$R_{nl}(r) = N_{nl} e^{-2\alpha\beta r}(1 - e^{-2\alpha r})^{1+\delta} P_n^{(2\beta, 2\delta+1)}(1 - 2e^{-2\alpha r}), \qquad (35)$$

where β and δ are given in Eq. (24) and N_{nl} is a normalization factor to be determined from the normalization condition:which gives [20-22]

$$N_{nl} = \frac{1}{\sqrt{s(n)}},$$

$$s(n) = \frac{(-1)^n}{2\alpha} \frac{\Gamma(n+2\delta+2)\Gamma(n+2\beta+1)^2}{\Gamma(n+2\beta+2\delta+2)}$$

$$\times \sum_{p,r=0}^{n} \frac{(-1)^{p+r}\Gamma(n+2\beta+r-p+1)(p+2\delta+2)}{p!r!(n-p)!(n-r)!\Gamma(n+2\beta-p+1)\Gamma(2\beta+r+1)(n+2\beta+r+2\delta+2)}. \qquad (36)$$

8

IV. NUMERICAL RESULTS

To show the accuracy of the new approximation scheme, we calculate the energy eigenvalues for various n and l quantum numbers with two different values of the parameter σ_0. The results calculated by Eq. (31) are compared with those obtained by a MATHEMATICA package programmed by Lucha and Schöberl [28] as shown in Table 1 for short-range potential (small α) and long-range potential (large α). It provides that the new proposed approximation scheme to the centrifugal term in Eq. (2), even when the potential parameter α becomes large, produces energy eigenvalues of high accuracy like the other numerical methods [28]. Consequently, this is also an illustration to assess the validity and usefulness of our present approximation. Further, it is quite simple, computationally efficient, reliable and accurate.

V. DISCUSSIONS

We have used the hypergeometric method (N-U) to solve the radial SE with the exponential-type potentials for arbitrary l-states. We have derived the binding energy spectra in Eq. (31) and their corresponding normalized wave functions in Eq. (35).

Firstly, let us attempt to study the s-wave case ($l = 0$). Hence, the energy eigenvalue and the radial eigen function solutions; (31) and (35), reduce to the following forms:

$$E_n = D(1 - \sigma_0)^2 - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{(n+1)^2 + (2n+1)\delta_1 - 4\nu\sigma_0(1-\sigma_0)}{2(n+\delta_1+1)}\right]^2, \quad 0 \leq n < \infty, \quad (37)$$

and

$$R_{nl}(r) = N_{nl}e^{-2\alpha\beta_1 r}(1 - e^{-2\alpha r})^{1+\delta_1}P_n^{(2\beta_1, 2\delta_1+1)}(1 - 2e^{-2\alpha r}), \quad (38)$$

respectively, with

$$\beta_1 = \sqrt{-\frac{\mu E_n}{2\alpha^2\hbar^2} + \frac{\mu D}{2\alpha^2\hbar^2}(1-\sigma_0)^2}, \quad \delta_1 = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{8\mu D\sigma_0^2}{\alpha^2\hbar^2}}\right], \quad (39)$$

and N_{nl} is given in Eq. (36). This result is in agreement with Ref. [27].

Secondly, we further discuss another special case $\sigma_0 = 1$. As a result, the potential (1) reduces to

$$V(r) = \frac{4De^{-4\alpha r}}{(1 - e^{-2\alpha r})^2}. \quad (40)$$

9

The corresponding energy levels and radial wave functions are given by

$$E_{nl} = \frac{2\alpha^2\hbar^2 l(l+1)}{\mu}\left[\frac{1}{\gamma^2} - \frac{1}{e^\gamma - 1} - \frac{1}{(e^\gamma - 1)^2}\right] - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{(n+1)^2 + l(l+1) + (2n+1)\delta_2}{2(n+\delta_2+1)}\right]^2,$$

(41)

and

$$R_{nl}(r) = N_{nl}e^{-2\alpha\beta_2 r}(1 - e^{-2\alpha r})^{1+\delta_2}P_n^{(2\beta_2,2\delta_2+1)}(1 - 2e^{-2\alpha r}),$$

(42)

respectively, with

$$\beta_2 = \sqrt{-\frac{\mu E_{nl}}{2\alpha^2\hbar^2} + \frac{l(l+1)}{\gamma^2} - \frac{l(l+1)}{e^\gamma - 1} - \frac{l(l+1)}{(e^\gamma - 1)^2}}, \quad \delta_2 = \frac{1}{2}\left[-1 + \sqrt{\frac{8\mu D}{\alpha^2\hbar^2} + (1+2l)^2}\right],$$

(43)

and N_{nl} is given in Eq. (36). This result essentially coincides with that of Manning-Rosen potential with the special case $A = 0$ and $D = \frac{\alpha^3(1-\alpha)\hbar^2}{2\mu}$ as shown in our recent work [22].

VI. COCLUDING REMARKS

The arbirary l-wave solutions of the SE with an exponential-type potential have been obtained approximately by proposing an improved shifted approximation to the centrifugal term. It is found that the normalized wave functions can be expressed by means of the Jacobi polynomials. With this approximation scheme, we can easily build an analytic formulations [Eqs. (31) and (35)] and use it to evaluate eigenvalues and eigenfunctions. This approximation scheme has also been used with great success in problems which do not have exact solutions for $l \neq 0$ case with exponential-type potentials like Manning-Rosen potential and hyperbolic potential. Essentially, two special cases for the s-wave case ($l = 0$) and $\sigma_0 = 1$ and found that these results have been reduced to those given in [25-27]. To show the accuracy of our results, we have calculated the eigenvalues numerically for arbitrary n and l with two different values of the parameter σ_0. We found that the results obtained by (31) are in good agreement with those obtained by using the MATHEMATICA program based on the numerical integration procedure for short-range potential (small α) and long-range potential (large α) [28]. As a demonstration of the accuracy of our results, Table 1 shows that the estimated energy eigenvalues can be computed up to $0.001 - 0.13$ %. However, the accuracy of the energy eigenvalues in [27] is computed up to $0.051 - 1.0$ %. Therefore, the accuracy of the present model reaches up to $10 - 50$ times better than the estimations provided by [27].

10

Acknowledgments

[1] L. I. Schiff, Quantum Mechanics 3rd edn. (McGraw-Hill Book Co., New York, 1968).

[2] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, Non-relativistic Theory, 3rd edn. (Pergamon, New York, 1977).

[3] M. M. Neito, Am. J. Phys. **47** (1979) 1067.

[4] S. Ikhdair and R. Sever, J. Mol. Struct.-Theochem 806 (2007) 155.

[5] S.M. Ikhdair and R. Sever, J. Mol. Struct.-Theochem 855 (2008) 13.

[6] C.L. Pekeris, Phys. Rev. 45 (1934) 98; C. Berkdemir, Nucl. Phys. A 770 (2006) 32.

[7] W.-C. Qiang and S.-H. Dong, Phys. Lett. A 363 (2007) 169.

[8] C. Berkdemir and J. Han, Chem. Phys. Lett. 409 (2005) 203; C. Berkdemir, A. Berkdemir and J. Han, Chem. Phys. Lett. 417 (2006) 326.

[9] R.L. Greene and C. Aldrich, Phys. Rev. A 14 (1976) 2363; B. Gönül and İ. Zorba, Phys. Lett. A 269 (2000) 83.

[10] P.M. Morse, Phys. Rev. 34 (1929) 57.

[11] L. Hulthén, Ark. Mat. Astron. Fys. A 28 (1942) 5; S.M. Ikhdair and R. Sever, J. Math. Chem. 42 (3) (2007) 461; S.M. Ikhdair, "Bound-States of the Klein-Gordon Equation with Vector and Scalar General Hulthen Potentials in D-Dimensions", to appear in Int. J. Mod. Phys. C.

[12] Ö. Yeşiltaş, Phys. Scr. 75 (2007) 41.

[13] S.M. Ikhdair and R. Sever, Ann. Phys. (Leipzig) 16 (2007) 218; *ibid.* Int. J. Theor. Phys. 46 (6) (2007) 1643; *ibid.* 46 (10) (2007) 2384; *ibid.* Int. J. Mod. Phys. E 17 (6) (2008) 1107.

[14] S.M. Ikhdair, Chin. J. Phys. 46 (2008) 291; S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. C 18 (10) (2007) 1571; *ibid.* C 19 (2) (2008) 221; Cent. Eur. J. Phys. 5 (4) (2007) 516; *ibid.* 6 (1) (2008) 141; *ibid.* 6 (3) (2008) 685; *ibid.* 6 (3) (2008) 697; *ibid.* arXiv:0801.4857, to appear in the Int. J. Mod. Phys. C; S.M. Ikhdair and R. Sever, DOI:10.1007/s10910-008-9438-8 to appear in J. Math. Chem.

[15] N. Rosen and P.M. Morse, Phys. Rev. 42 (1932) 210.

[16] M.F. Manning, Phys. Rev. 44 (1933) 951; M.F. Manning and N. Rosen, Phys. Rev. 44 (1933) 953.

[17] A. Diaf, A. Chouchaoui and R.L. Lombard, Ann. Phys. (Paris) 317 (2005) 354.

[18] S.-H. Dong and J. Garcia-Ravelo, Phys. Scr. 75 (2007) 307.

[19] W.-C. Qiang and S. H. Dong, Phys. Lett. A 368 (2007) 13.

[20] S.M. Ikhdair and R. Sever, arXiv:0801.4271, submitted to Int. J. Mod. Phys. E.

[21] S.M. Ikhdair and R. Sever, arXiv:0801.3518, DOI:10.1002/andp.200810322 to appear in Ann. Phys. (Berlin).

[22] S.M. Ikhdair and R. Sever, arXiv:0807.2085, submitted to J. Math. Chem.

[23] C.-S. Jia *et al.*, J. Phys. A: Math. Gen. 37 (2004) 11275; C.-S. Jia *et al.*, Phys. Lett. A 311 (2003) 115.

[24] H. Eğrifes, D. Demirhan and F. Büyükkılıç, Phys. Lett. A 275 (2000) 229.

[25] D. Schiöberg, Mol. Phys. 59 (1986) 1123.

[26] J. Lu, H. X. Qian, L. M. Li and F. L. Liu, Chin. J. Phys. 14 (2005) 2402.

[27] Shishan Dong, J. Garcia-Ravelo and Shi-Hai Dong, Phys. Scr. 76 (2007) 393.

[28] W. Lucha and F.F. Schöberl, Int. J. Mod. Phys. C 10 (1999) 607.

[29] A.F. Nikiforov and V.B. Uvarov, Special Functions of Mathematical Physics (Birkhauser, Bassel, 1988).

[30] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1964).

TABLE I: Energy eigenvalues as a function of the parameter α for $2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f,$ $5g, 6p, 6d, 6f$ and $6g$ states for $\sigma_0 = 0.1$, 0.2 and $D = 10$ in atomic units ($\hbar = \mu = 1$).

states	α	$\sigma_0 = 0.1$			$\sigma_0 = 0.2$		
		present	Dong et al [27]	Lucha et al [28]	present	Dong et al [27]	Lucha et al [28]
$2p$	0.10	2.61874	2.61556	2.61935	1.20876	1.20559	1.20903
	0.15	3.90544	3.89830	3.90645	1.86636	1.85922	1.86689
	0.20	5.00331	4.99062	5.00457	2.52000	2.50731	2.52080
	0.25	5.88594	5.86611	5.88725	3.14666	3.12683	3.14766
$3p$	0.10	4.73540	4.73223	4.73638	2.68308	2.67990	2.68358
	0.15	6.04543	6.03829	6.04649	3.67127	3.66413	3.67198
	0.20	6.91663	6.90394	6.91733	4.46516	4.45247	4.46579
	0.25	7.48400	7.46417	7.48358	5.09231	5.07247	5.09235
$3d$	0.10	3.62699	3.61747	3.62769	1.57873	1.56921	1.57920
	0.15	5.29404	5.27263	5.29510	2.54773	2.52631	2.54859
	0.20	6.47492	6.43684	6.47598	3.48119	3.44311	3.48228
$4p$	0.10	6.00287	5.99969	6.00390	3.75692	3.75375	3.75758
	0.15	7.11526	7.10812	7.11589	4.81215	4.80501	4.81274
	0.20	7.71903	7.70634	7.71826	5.53111	5.51842	5.53087
$4d$	0.10	5.33129	5.32177	5.33216	2.95257	2.94305	2.95317
	0.15	6.73583	6.71441	6.73642	4.10410	4.08268	4.10470
	0.20	7.54480	7.50672	7.54331	5.00179	4.96371	5.00137
$4f$	0.10	4.68965	4.67061	4.69058	2.07342	2.05438	2.07417
	0.15	6.42992	6.38708	6.43112	3.35622	3.31338	3.35742
	0.20	7.43397	7.35782	7.43334	4.47408	4.39793	4.47486
$5p$	0.10	6.80345	6.80027	6.80432	4.54946	4.54628	4.55015
$5d$	0.10	6.37762	6.36810	6.37842	3.95677	3.94725	3.95740
$5f$	0.10	5.98063	5.96159	5.98147	3.31497	3.29593	3.31567
$5g$	0.10	5.62805	5.59631	5.62926	2.64017	2.60844	2.64124
$6p$	0.10	7.32416	7.32099	7.32476	5.13763	5.13446	5.13824
$6d$	0.10	7.04824	7.03872	7.04873	4.69929	4.68977	4.69979
$6f$	0.10	6.79479	6.77575	6.79528	4.22654	4.20751	4.22706
$6g$	0.10	6.57377	6.54204	6.57452	3.73301	3.70128	3.73378