Exact Solutions of Effective Mass Dirac Equation with non-PT-Symmetric and non-Hermitian Exponential-type Potentials

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Abstract

By using two-component approach to the one-dimensional effective mass Dirac equation bound states are investigated under the effect of two new non-PT-symmetric, and non-Hermitian, exponential type potentials. It is observed that the Dirac equation can be mapped into a Schrödinger-like equation by rescaling one of the two Dirac wave functions in the case of the position dependent mass. The energy levels, and the corresponding Dirac eigenfunctions are found analytically.

Keywords: Dirac equation, Position-Dependent Mass, non-PT-symmetric potential

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The investigation of the quantum systems having so-called non-Hermitian Hamiltonians has been a great interest because of its theoretical contributions to quantum mechanics [1-15]. The study of such quantum systems has been received many applications, especially in quantum field theories [16-18], and nuclear theory [19]. The solutions of the non-relativistic and relativistic equations with non-Hermitian Hamiltonians having real or complex energy spectra have also been studied by many authors by using different methods [3, 6, 11, 20]. Non-Hermitian Hamiltonians satisfy the condition given by $\hat{O}H\hat{O}^{-1} = \hat{O}H\hat{O} = H$, where the operator $\hat{O} \equiv PT$ is an operator combined with parity, and time-reversal transformations, respectively. So they are related with PT-symmetry [8]. If a potential $V(x)$ has the condition written as $V(-x) = V^{\star}(x)$ under the transformation of $x \to -x$, and $i \to -i$, then it is said that the potential is a PT-symmetric.

The relativistic wave equations, especially Dirac equation, give numerous important results, and explanations in the view of quantum mechanics. One of them is the spin-orbit coupling, and related topics, such as the Hall effect, spin torque, Zitterbewegung, etc [21-23]. Many authors have also been studied the solution of the Dirac equation with different potentials in different theoretical backgrounds [24, 25].

Recently, the generalization of the solutions of the relativistic and non-relativistic wave equations with PT-symmetric potentials in the case of the constant mass to the case of the position-dependent mass become very attractive research topic. Many authors have studied the effects of the position-dependent mass on the solutions of the above equations [26-36]. The new formalism based on the position-dependent mass is a useful ground to study the properties of some physical systems, such as quantum dots [37], semiconductor heterostructures [38], quantum liquids [39].

In this letter, we intend to study the effects of the position-dependent mass on the solutions of the one-dimensional Dirac equation by using the map between the mass, and potential functions without the spin-effects [43]. Our aim, by taking two examples of the mass distributions, is to show that the two-component approach [40, 41] to the one-dimensional Dirac equation can be used in the position-dependent mass formalism. We obtain two new non-PT-symmetric, and non-Hermitian potential functions produced by using the map between the mass, and potential functions.

The relativistic Dirac equation in the absence of an external potential can be written as

$$\left(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}-m\right)\Psi=0\,, \tag{1}$$

where m is the rest mass of the particle, and γ^{μ} ($\mu=0,1,2,3$) are gamma matrices ($\hbar=c=1$). The Dirac equation for a particle moving in an external potential $V(x)$ in one-dimension can be written as [40]

$$[\alpha\,.\,p+\beta m(x)-(E-V(x))]\,\psi(x)=0\,, \tag{2}$$

where E is the relativistic energy of the particle, p is the momentum operator, $m(x)$ is the mass function and α,β are 2×2 matrices which are set to Pauli matrices σ_3 and σ_1, respectively. The Dirac wave function, $\psi(x)$, is decomposed into an upper $\psi^{+}(x)$, and lower component $\psi^{-}(x)$, so that

$$\psi(x)=\begin{pmatrix}\psi^{+}(x)\\ \psi^{-}(x)\end{pmatrix}, \tag{3}$$

We have the following set of two coupled differential equations by inserting Eq. (3) into Eq. (2)

$$\frac{d\psi^{+}(x)}{dx}+[m(x)+(E-V(x))]\psi^{-}(x)=0\,, \tag{4}$$

$$\frac{d\psi^{-}(x)}{dx}+[m(x)-(E-V(x))]\psi^{+}(x)=0\,. \tag{5}$$

We define the wave functions $f(x)$ and $g(x)$ in two-component approach which makes possible to obtain the solutions easily without the spin effects [40, 41] as

$$\begin{pmatrix}f(x)\\ g(x)\end{pmatrix}=\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\begin{pmatrix}\psi^{+}(x)\\ \psi^{-}(x)\end{pmatrix}, \tag{6}$$

and insert into Eqs. (4) and (5), we have

$$\frac{df(x)}{dx}-i[E-V(x)]f(x)+im(x)g(x)=0\,, \tag{7}$$

$$\frac{dg(x)}{dx}+i[E-V(x)]g(x)-im(x)f(x)=0\,. \tag{8}$$

By using the transformation written in terms of mass function $m(x)$ in Eq. (7)

$$f(x)=\sqrt{m(x)}\;\phi(x)\,, \tag{9}$$

one gets

$$i\,\frac{d\phi(x)}{dx}+\left[\frac{im'(x)}{2m(x)}+(E-V(x))\right]\phi(x)-\sqrt{m(x)}\,g(x)=0\,. \tag{10}$$

where prime denotes the derivative to the spatial coordinate. To obtain a Schrödinger-like equation for $\phi(x)$, one sets the potential term $V(x)$ in Eq. (10) as

$$V(x)=i\frac{m'(x)}{2m(x)}\,, \tag{11}$$

which gives

$$\frac{d^2\phi(x)}{dx^2}-V_{eff}(x)\phi(x)=-E^2\phi(x)\,. \tag{12}$$

This is the Schrödinger equation having the energy eigenvalue E^2 under the effect of the potential $V_{eff}(x)=m^2(x)$. By choosing a suitable mass function, one can construct a potential $V(x)$ from Eq. (11). By solving the Schrödinger-like equation given by Eq. (12), one obtains the energy spectra and corresponding spinors of the Dirac equation. The Schrödinger and Schrödinger-like equations can be solved by using the calculation procedure, which suggests a hypergemetric type equation in the following form [42]

$$\frac{d^2\psi(z)}{dz^2}+\frac{\tilde{\tau}(z)}{\sigma(z)}\frac{d\psi(z)}{dz}+\frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z)=0\,, \tag{13}$$

where $\sigma(z)$, $\tilde{\sigma}(z)$ are polynomials at most second degree and $\tilde{\tau}(z)$ is a first degree polynomial.

By using the transformation $\psi(z)=\chi(z)y(z)$, one gets

$$\sigma(z)\,\frac{d^2y(z)}{dz^2}+\tau(z)\,\frac{dy(z)}{dz}+\lambda y(z)=0\,, \tag{14}$$

The first part of the total wave function $\chi(z)$ is defined as [42]

$$\frac{\chi'(z)}{\chi(z)}=\frac{\pi(z)}{\sigma(z)}\,, \tag{15}$$

and the other part of the solution $y(z)$ is given by the Rodrigues relation

$$y_n(z)=\frac{a_n}{\rho(z)}\frac{d^n}{dz^n}\left[\sigma^n(z)\;\rho(z)\right], \tag{16}$$

where a_n is a normalization constant, and $\rho(z)$ is the weight function obtained from the following relation [42]

$$\frac{d}{dz}\left[\sigma(z)\;\rho(z)\right]=\tau(z)\;\rho(z). \tag{17}$$

The polynomial $\pi(z)$ in Eq. (15), and the parameter λ required for the method are given by

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)}\,. \tag{18}$$

$$\lambda = k + \pi'(z)\,. \tag{19}$$

The discriminant of the expression under the square root in the polynomial $\pi(z)$ in Eq. (18) must be zero, which defines the constant k. Thus, a new eigenvalue equation becomes

$$\lambda = \lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\,\sigma''(z)\,. \tag{20}$$

where $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ has a negative derivative. The energy eigenvalues are obtained from the Eqs. (19) and (20).

We prefer the following mass function as a first case

$$m(x) = m_0(1 + e^{-\delta x})\,, \tag{21}$$

which gives an exponential type, non-PT-symmetric, non-Hermitian potential

$$V(x) = -\frac{i\delta}{2}\frac{e^{-\delta x}}{1 + e^{-\delta x}}\,. \tag{22}$$

By using the mass function, Eq. (12) gives

$$\frac{d^2\phi(x)}{dx^2} + \left\{E^2 - m_0^2 - 2m_0^2 e^{-\delta x} - m_0^2 e^{-2\delta x}\right\}\phi(x) = 0\,, \tag{23}$$

By inserting the new variable $z^{-1} = e^{\delta x}$, one obtains the following equation

$$\frac{d^2\phi(z)}{dz^2} + \frac{1}{z}\frac{d\phi(z)}{dz} + \frac{1}{z^2}\left\{\tilde{E}^2 - \alpha^2 - 2\alpha^2 z - \alpha^2 z^2\right\}\phi(z) = 0\,. \tag{24}$$

where $\tilde{E}^2 = E^2/\delta^2$, and $\alpha^2 = m_0^2/\delta^2$. Comparing Eqs. (24) and Eq. (13), and using the parameters $-a_1 = -\alpha^2$, $-a_2 = -2\alpha^2$, and $-\epsilon = \tilde{E}^2 - \alpha^2$, we obtain

$$\tilde{\tau}(z) = 1 \;\;;\;\; \sigma(z) = z \;\;;\;\; \tilde{\sigma}(z) = -a_1 z^2 - a_2 z - \epsilon\,. \tag{25}$$

To obtain the polynomial $\pi(z)$, one inserts these polynomials into Eq. (18), with $\sigma'(z) = 1$, then we get

$$\pi(z) = \pm\sqrt{a_1 z^2 + (k + a_2)z + \epsilon}\,, \tag{26}$$

We obtain two solutions for k as $k_{1,2} = -a_2 \pm 2\sqrt{a_1 \epsilon}$, from the condition that the discriminant of the expression under the square root has to be zero. Eq. (18) gives

$$\pi(z) = \begin{cases} \pm[\sqrt{a_1}\,z + \sqrt{\epsilon}\,], & k \to k_1\,, \\ \pm[\sqrt{a_1}\,z - \sqrt{\epsilon}\,], & k \to k_2\,. \end{cases}$$

We find the polynomial $\tau(z)$ by choosing k_2 as

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z) = 1 + 2\sqrt{\epsilon} - 2\sqrt{a_1}\,z\,, \tag{27}$$

which gives $\tau'(z) = -2\sqrt{a_1} < 0$. Thus, an energy eigenvalue equation is obtained from Eqs. (19) and (20) as

$$(2n+1)\sqrt{a_1} + 2\sqrt{a_1 \epsilon} + a_2 = 0\,, \tag{28}$$

$$E = \pm\sqrt{m_0^2 - \frac{\delta^2}{4}\left[2n + 1 + \frac{2m_0}{\delta}\right]^2}\,. \tag{29}$$

It can be seen that the energy levels of particle and antiparticle are symmetric about zero. We observe that it is obtained zero energy in the case of the limit $\delta \to 0$ which is the energy level of the following equation derived from Eq. (23)

$$\frac{d^2\phi(x)}{dx^2} + \omega_0^2\phi(x) = 0\,, \tag{30}$$

where $\omega_0^2 = E^2 - m_0^2$. The Dirac Hamiltonian with a general scalar potential, $V'(x)$, gives always zero-energy solutions in the ultrarelativistic limit [43], where the upper and lower components given in Eq. (2) are written as

$$\psi^+(x) \sim e^{-(mx+h(x))}\,, \tag{31}$$

$$\psi^-(x) \sim e^{+(mx+h(x))}\,. \tag{32}$$

with $h(x) = \int^x V'(y)dy$, m denotes the rest mass of particle. The normalization conditions for zero-energy eigenstates are discussed in Ref. [43]. The solutions of Eq. (30) have the same form with the ones given in Eqs. (31) and (32) for the limit $\delta \to 0$, i.e., $h(x) = 0$ for $m \to 2m_0$. One has to find first the weight function $\rho(z)$ to obtain the Dirac wave function $\phi(z)$. Using Eq. (17), we get

$$\rho(z) = e^{-2\sqrt{a_1}\,z}\, z^{2\sqrt{\epsilon}} \tag{33}$$

and from Eq. (16), we obtain

$$y_n(z) = \frac{a_n}{e^{-2\sqrt{a_1}\,z}\, z^{2\sqrt{\epsilon}}} \frac{d^n}{dz^n}\left[e^{-2\sqrt{a_1}\,z}\, z^{n+2\sqrt{\epsilon}}\right]. \tag{34}$$

From the last equation, we can write $y_n(z)$ in terms of the generalized Laguerre polynomials as $y_n(z) \simeq L_n^{(2\sqrt{\epsilon})}(z)$ [43]. The other part of the solution is obtained from Eq. (15) as

$$\chi(z) = e^{-\sqrt{a_1}\,z}\, z^{\sqrt{\epsilon}}\,. \tag{35}$$

Finally, we obtain the unnormalized wave function as

$$\phi_n(z) = e^{-\sqrt{a_1}\,z}\, z^{\sqrt{\epsilon}}\, L_n^{(2\sqrt{\epsilon})}(z)\,. \tag{36}$$

Thus, we write the upper component from Eq. (9) as

$$f_n(z) = \sqrt{m_0(1+z)}\; e^{-\sqrt{a_1}\,z}\, z^{\sqrt{\epsilon}}\, L_n^{(2\sqrt{\epsilon})}(z) \tag{37}$$

and the lower component from Eq. (8) as

$$g_n(z) = \frac{e^{-\sqrt{a_1}\,z}\, z^{\sqrt{\epsilon}}\, L_n^{(2\sqrt{\epsilon})}(z)}{\sqrt{m_0(1+z)}} \left\{ -i\delta z\left[\frac{\sqrt{\epsilon}}{z} - \sqrt{a_1} - \frac{L_{n-1}^{(2\sqrt{\epsilon})}(z)}{L_n^{(2\sqrt{\epsilon})}(z)} + E\right]\right\}. \tag{38}$$

It is seen that the mass of the Dirac particle contributes to the eigenfunctions, and the upper component depends on the energy of the particle.

We choose the second mass distribution

$$m(x) = \frac{m_0}{1+e^{-\delta x}}\,, \tag{39}$$

which gives following exponential type, non-PT-symmetric and non-Hermitian potential from Eq. (11) as

$$V(x) = \frac{i\delta}{2}\frac{1}{1+e^{\delta x}}\,. \tag{40}$$

It has the form of the Woods-Saxon potential for $i\delta/2 \to -V_0$ [45]. Substituting Eq. (39) into Eq. (12), and using the new variable $z = 1/(1+e^{-\delta x})$, we obtain

$$\frac{d^2\phi(z)}{dz^2} + \frac{1-2z}{z(1-z)}\frac{d\phi(z)}{dz} - \left\{\frac{\tilde{E}^2}{z^2(1-z)^2} + \frac{\alpha^2}{(1-z)^2}\right\}\phi(z) = 0\,, \tag{41}$$

where $-\tilde{E}^2 = E^2/\delta^2$. Comparing the Eqs. (41) and Eq. (13), we get

$$\tilde{\tau}(z) = 1-2z\ ;\quad \sigma(z) = z(1-z)\ ;\ \tilde{\sigma}(z) = -\tilde{E}^2 - \alpha^2 z^2\,. \tag{42}$$

Following the same procedure as in last section, we get the energy spectra

$$E = \pm \frac{1}{4}\sqrt{8m_0^2 - [\sqrt{\delta^2 + 4m_0^2} - \delta(2n+1)]^2 - \Big(\frac{4m_0^2}{\sqrt{\delta^2 + 4m_0^2} - \delta(2n+1)}\Big)^2}\,. \tag{43}$$

and the corresponding wave functions are written in terms of the Jacobi polynomials

$$\phi_n(z) = z^{\tilde{E}}\,(1-z)^{-M}\,P_n^{(2\tilde{E}\,,\,-2M)}(1-2z)\,. \tag{44}$$

The lower and upper components $g_n(z)$ and $f_n(z)$ are written by using Eqs. (44) and (8) respectively as

$$f_n(z) = \sqrt{m_0}\, z^{\tilde{E}+1/2}\,(1-z)^{-M}\,P_n^{(2\tilde{E}\,,\,-2M)}(1-2z)\,, \tag{45}$$

and

$$\begin{aligned} g_n(z) \;=\; & z^{\tilde{E}-1/2}\,(1-z)^{-M}\,P_n^{(2\tilde{E}\,,\,-2M)}(1-2z)\,\Big\{\frac{i\delta}{\sqrt{m_0}}\Big[\frac{1}{2}(1-z) + \tilde{E}(1-z) \\ +\; & Mz - z(1-z)(n+1+2\tilde{E}-2M)\,\frac{P_{n-1}^{(1+2\tilde{E}\,,\,1-2M)}(1-2z)}{P_n^{(2\tilde{E}\,,\,-2M)}(1-2z)}\Big] + \frac{E}{\sqrt{m_0}}\Big\}\,. \end{aligned} \tag{46}$$

where $M = \sqrt{\tilde{E}^2 + \alpha^2}\,$.

The energy levels of the particles and antiparticles are symmetric about zero in Eq. (43). The non-PT-symmetric and non-Hermitian potential given by Eq. (40) has a real energy spectrum under the condition that $8m_0^2 > [\sqrt{\delta^2 + 4m_0^2} - \delta(2n+1)]^2 + \Big(\frac{4m_0^2}{\sqrt{\delta^2+4m_0^2}-\delta(2n+1)}\Big)^2$. The energy expression in Eq. (43) gives zero energy in the limit of $\delta \to 0$ as in the first mass distribution case. Both of the eigenfunctions are dependent on the rest mass of the particle.

In summary, we have investigated the position-dependent mass energy spectra, and the corresponding wave functions of the Dirac equation in two-component approach. We have transformed it into a Schrödinger-like equation with the help of a transformation applied on one of the Dirac wave functions by using two different mass distributions. Thus two new non-PT-symmetric and non-Hermitian complex potentials are produced. We have found that the Dirac equation can be turned into a Schrödinger-like equation by using a suitable transformation depending on the mass function $m(x)$ in two-component approach. This formalism gives us two new non-PT-symmetric and non-Hermitian exponential and inverse-exponential potentials. We have obtained a complex energy spectrum for the exponentially potential given by Eq. (22), and the potential given by Eq. (43) has a real energy levels if $8m_0^2 > [\sqrt{\delta^2 + 4m_0^2} - \delta(2n+1)]^2 + \Big(\frac{4m_0^2}{\sqrt{\delta^2+4m_0^2}-\delta(2n+1)}\Big)^2$.

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