

An alternative simple solution of the sextic anharmonic oscillator and perturbed Coulomb problems

Sameer M. Ikhdaïr* and Ramazan Sever†

**Department of Physics, Near East University, Nicosia, North Cyprus, Mersin-10, Turkey*

†*Department of Physics, Middle East Technical University, 06531 Ankara, Turkey.*

(September 26, 2018)

Abstract

Utilizing an appropriate ansatz to the wave function, we reproduce the exact bound-state solutions of the radial Schrödinger equation to various exactly solvable sextic anharmonic oscillator and confining perturbed Coulomb models in D -dimensions. We show that the perturbed Coulomb problem with eigenvalue E can be transformed to a sextic anharmonic oscillator problem with eigenvalue \hat{E} . We also check the explicit relevance of these two related problems in higher-space dimensions. It is shown that exact solutions of these potentials exist when their coupling parameters with $k = D + 2\ell$ appearing in the wave equation satisfy certain constraints.

Keywords: Sextic anharmonic oscillator problem, perturbed Coulomb problem, bound states, wave functions, Schrödinger equation

PACS number: 03.65.-w; 03.65.Fd; 03.65.Ge.

*sikhdaïr@neu.edu.tr

†sever@metu.edu.tr

I. INTRODUCTION

The solution of the fundamental dynamical equations is an interesting phenomenon because of its importance in quantum-field theory, molecular physics, solid-state and statistical physics. To obtain the exact ℓ -state solutions of the Schrödinger equation (SE) to various quantum mechanical problems are possible only for few potentials and hence approximation methods are used to obtain their solutions. According to the Schrödinger formulation of quantum mechanics, a total wave function provides implicitly all relevant information about the behaviour of a physical system. Hence if it is exactly solvable for a given potential, the wave function can describe such a system completely. Until now, many efforts have been made to solve the stationary SE with a sextic anharmonic oscillator and perturbed Coulomb potentials in one to three dimensions through the Hill determinant matrix method [1-4]. The study of the SE with these potentials provides us with insight into the physical problem under consideration. Further, the study of SE with some of these potentials in the arbitrary dimensions D is presented [3].

The purpose of this paper is to carry out the analytical solutions of the D -dimensional radial SE with exactly-solvable Coulomb plus linear plus harmonic (CLH) $V_1(r) = -a/r + br + cr^2$ and sextic anharmonic oscillator (AHO) $V_2(r) = \mu r^2 + \lambda r^4 + \eta r^6$ potentials through an appropriate ansatz to the wave function. In cases of exactly solvable models, the wave function can be expressed as a finite power series polynomial multiplied by an appropriate reference function (usually, the asymptotic form) to reproduce the exact solutions. The analytical solution of the Schrödinger equation for the energy levels with a class of confining potentials of type V_1 have been studied by Datta and Mukherjee [5] by using Kato-Rellich perturbation theory for linear operations. It is well known that this confinement potential has been used for calculation of $q\bar{q}$ bound-state masses [6]. Killingbeck [7] has calculated the energy eigenvalues of the confinement potential by using hypervirial relations. Exact solutions [8] of potentials of type V_1 and V_2 are obtained by a number of authors in three-dimensional space when the coupling parameters satisfy certain relations. Chaudhuri and

Mondal [3] studied the D -dimensional sextic AHO and CLH problems within the framework of supersymmetric quantum mechanics (SUSYQM) and Hill determinant method [4,9-11]. Chaudhuri and Mondal have shown that SUSYQM yields exact solutions for a single state only for the quasi-exactly-solvable potentials of type V_1 and V_2 in D -dimensions with some constraints on the coupling constants [3]. They have also obtained numerical results throughout the Hill determinant method. The ideas of supersymmetric quantum mechanics have been used for the study of atomic systems [11], the evaluation of the eigenvalues of a bistable potential [12], the improvement of the large- N expansion [13], the analysis of all known shape invariant potentials [10,14], and the development of a more accurate WKB approximation [14]. Tymczak et al [4] devised a highly accurate quantization procedures for the inner product representation both in configuration and momentum space for various AHO potentials in one and two dimensions. Additionally, Dobrovolska and Tutik [15] studied the bound-state problem within the framework of the SE through the logarithmic perturbation theory. Recently, they also extended the formalism to the bound-state problem for spherical oscillator of type V_2 with its subsequent application to the doubly anharmonic oscillator [16]. Furthermore, a simple formalism [17] based on a suitable choice of the wave function ansatz has been proposed for reproducing exact bound-state energy eigenvalues and eigenfunctions for exactly solvable model within the framework of the SE. Very recently, this simple approach has also been applied [18] with remarkable success, to various molecular quantum mechanical problems in D -dimensions [19,20].

The object of this paper is to extend the above simple approach to reproduce bound-state exact solutions for potentials of type V_1 and V_2 in D -dimensions with some constraints on the coupling constants. For a certain choice of parameters the method provides exactly-solvable potentials for the D -dimensional sextic AHO and CLH problems. We then compare our results with the exact ones obtained from the transformation of LHO into sextic AHO problem.

This paper is organized as follows. In Section II, we solve analytically the D -dimensional radial Schrödinger equation for the sextic AHO and CLH problems by a suitable choice of a

wave function ansatz to each exactly-solvable problem. On the other hand, the exact energy eigenvalues of sextic AHO are obtained by transforming radial wave SE in D -dimensions with angular momentum ℓ to another problem in $(2D - 4)$ -dimensions with angular momentum $2\ell + 1$. The results and conclusion will be given in Section III.

II. THE D -DIMENSIONAL RADIAL SCHRÖDINGER EQUATION

In the D -dimensional Hilbert space, the reduced radial wave Schrödinger equation (with $\hbar = m = 1$ units) for a spherically symmetric potential $V(r)$ takes the form [21]

$$\left[\frac{d^2}{dr^2} + \frac{(D-1)}{r} \frac{d}{dr} - \frac{\ell(\ell+D-2)}{r^2} + 2(E - V(r)) \right] \psi(r) = 0, \quad (1)$$

where the interaction potential is chosen to be of type V_1 or V_2 and E stands for its eigenvalues. Further, equation (1) can be simply transformed to the form [21]

$$\left\{ \frac{d^2}{dr^2} - \frac{[(k-1)(k-3)]}{4r^2} + 2(E - V(r)) \right\} R(r) = 0, \quad (2)$$

where $R(r)$, the reduced radial wave function, is defined by

$$R(r) = r^{(D-1)/2} \psi(r), \quad (3)$$

and

$$k = D + 2\ell, \quad (4)$$

which is a parameter depends on a linear combination of the spatial dimensions D and the angular momentum quantum number ℓ [21].

We substitute $r = \gamma\rho^2/2$ and $\psi = \chi(\rho)/\rho$ to transform Eq.(1) into another Schrödinger-type equation in $(D' = 2D - 4)$ -dimensional space with angular momentum $L = 2\ell + 1$,

$$\left[\frac{d^2}{d\rho^2} + \frac{(D'-1)}{\rho} \frac{d}{d\rho} - \frac{L(L+D'-2)}{\rho^2} + 2(\hat{E} - \hat{V}(\rho)) \right] \chi(\rho) = 0, \quad (5)$$

where

$$\hat{E} = \gamma^2 \rho^2 E, \quad \hat{V}(\rho) = \gamma^2 \rho^2 V(\gamma \rho^2/2), \quad \gamma = 1/(-E)^{1/2}. \quad (6)$$

It is seen that through this transformation, the D -dimensional radial wave Schrödinger equation (1) with angular momentum ℓ can be transformed to a $(D' = 2D - 4)$ -dimensional problem with new angular momentum $L = 2\ell + 1$. In particular, under this transformation, CLH problem of type V_1 with eigenvalue E can be transformed to a sextic AHO problem of type V_2 with eigenvalue \hat{E} as

$$\hat{V}(\rho) = \mu\rho^2 + \lambda\rho^4 + \eta\rho^6, \quad (7)$$

with coupling constants given by

$$\mu = 1, \quad \lambda = \frac{b}{2(-E)^{3/2}}, \quad \eta = \frac{c}{4(-E)^2}, \quad (8)$$

and eigenvalue

$$\hat{E} = \frac{2a}{(-E)^{1/2}}. \quad (9)$$

A. Confining perturbed Coulomb problem

We attempt to solve the wave equation (2) of reduced radial wave $R(r)$ in the D -dimensions for a spherically symmetric potential of confining CLH form [3,22]:

$$V(r) = -\frac{a}{r} + br + cr^2, \quad c > 0. \quad (10)$$

In particular, for $c = 0$ and $b > 0$, such a potential reduces to the well known quarkonium Cornell potential (cf. Refs. [21] and references therein). Apart from its relevance in heavy quarkonium spectroscopy (cf [21] and references therein), this class of potentials with $c = 0$ has important applications in atomic physics. For exactly solvable problems such as CLH, the representation of the radial portion of wave function ansatz, containing an appropriate reference function (usually, asymptotic form), is

$$R(r) = \exp[p(\alpha, \beta, r)] \sum_{n=0} a_n r^{2n+(k-1)/2}, \quad (11)$$

provided that the power of the reference function has the following selection:

$$p(\alpha, \beta, r) = \frac{1}{2}\alpha r^2 + \beta r, \quad (12)$$

should fall faster than the asymptotic form of the wave function. Upon substituting Eq. (11) into Eq. (2) and equating the coefficients of $r^{2n+(k-1)/2}$ to zero, we readily arrive at the following relation

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0 \quad (13)$$

where

$$A_n = 2E + \beta^2 + \alpha(4n + k), \quad (14)$$

$$B_n = 2a + \beta(4n + k - 1), \quad (15)$$

$$C_n = 4n^2 + 2n(k - 2), \quad (16)$$

and the value of the parameters for $p(\alpha, \beta, r)$ can be evaluated as

$$\alpha = \pm\sqrt{2c}, \quad \beta = \pm\frac{b}{\sqrt{2c}}. \quad (17)$$

To obtain a well-behaved solution at the origin and infinity, it is more convenient to take $\alpha = -\sqrt{2c}$ and $\beta = -\frac{b}{\sqrt{2c}}$ which ensure that wave function ansatz representation in (11), be finite for all r and convergent at ∞ . Further, for a given p , if $a_p \neq 0$, but $a_{p+1} = a_{p+2} = a_{p+3} = \dots = 0$, we then obtain $A_p = 0$ from Eq. (14), i.e.,

$$E_p^D = -\frac{b^2}{4c} + \sqrt{\frac{c}{2}}(4p + 2\ell + D), \quad p = 0, 1, 2, \dots \quad (18)$$

Carrying through a parallel analysis to Refs [17,18], A_n , B_n and C_n must satisfy the determinant relation for a nontrivial solution

$$Det \begin{vmatrix} B_0 & C_1 & \cdots & \cdots & \cdots & 0 \\ A_0 & B_1 & C_2 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_{p-1} & B_p \end{vmatrix} = 0. \quad (19)$$

To utilize this method showing the simplicity of this approach, we pursue determinant analysis to present the exact solution for $p = 0, 1$ as follows.

Case (1): when $p = 0$, we get from Eq. (18), the exact bound-state solution of the CLH problem in D -dimensions. So that the eigenvalue (ground state) is given by [17,18]

$$E_0^D = -\frac{b^2}{4c} + \sqrt{\frac{c}{2}}(2\ell + D). \quad (20)$$

Further, it is shown from Eq. (19) that $B_0 = 0$, which leads to the following constraint on the coupling parameters as

$$b(k-1) = 2a\sqrt{2c}. \quad (21)$$

which consequently, from Eqs. (20) and (21), the perturbed Coulomb potential admits an exact ground eigen energy:

$$E_0^k = \frac{1}{2} \left[\frac{b(k-1)^2}{2a} + \frac{b(k-1)}{2a} - \frac{4a^2}{(k-1)^2} \right], \quad (22)$$

which is consistent with Refs [3,22]. Further, the corresponding wave function (unnormalized):

$$\psi_0^{(k)}(r) = a_0 r^\ell \exp \left[-\frac{2ar}{(k-1)} - \frac{b(k-1)r^2}{4a} \right]. \quad (23)$$

Case (2): When $p = 1$, the exact energy spectrum becomes

$$E_1^k = \frac{1}{2} \left[\frac{b(k+1)^2}{2a} + \frac{b(k+1)}{2a} - \frac{4a^2}{(k+1)^2} \right], \quad (24)$$

and the corresponding wave function (unnormalized) can be readily found as:

$$\psi_1^{(k)}(r) = (a_0 + a_1 r) r^\ell \exp \left[-\frac{2ar}{(k+1)} - \frac{b(k+1)r^2}{4a} \right], \quad (25)$$

where a_0 and a_1 are normalization constants. The relation between them can be determined through the relation $B_0 a_0 + C_1 a_1 = 0$ to be

$$a_1 = \left[\frac{b}{\sqrt{2c}} - \frac{2a}{(k-1)} \right] a_0. \quad (26)$$

Furthermore, we have the following recurrence relation from Eq. (19), that is, $B_0B_1 = A_0C_1$, which consequently provides the following constraint on the coupling constants of potential:

$$4a^2 - \frac{b^2}{2c}(k-1)(k+1) - \frac{4ab}{\sqrt{2c}}k = \frac{2b}{a}k(k-1). \quad (27)$$

Following this approach, we can further generate a class of exact solutions through setting $p = 0, 1, 2, \dots$, etc. Generally speaking, if $a_p \neq 0$, $a_{p+1} = a_{p+2} = \dots = 0$, from which we can obtain the energy spectra (cf. determinant (19)). For the generalization, one needs to use the shape invariance property and the relation between supersymmetric partners [22,23] to find the general solution of energy eigenvalues as

$$E_n^k = \frac{1}{2} \left[\frac{b(2n+k-1)^2}{2a} + \frac{b(2n+k-1)}{2a} - \frac{4a^2}{(2n+k-1)^2} \right], \quad (28)$$

with the corresponding wave functions

$$\psi^{(p)}(r) = (a_0 + a_1r + \dots + a_pr^p) r^\ell \exp \left[-\frac{2ar}{(2n+k-1)} - \frac{b(2n+k-1)r^2}{4a} \right], \quad (29)$$

where a_i ($i = 0, 1, 2, \dots, p$) are normalization constants.

Finally, considering the following exactly solvable potentials. In particular: (i) Harmonic oscillator: when $a = b = 0$ and $c = \frac{1}{2}\omega^2$ are inserted to Eq. (10), giving $\alpha = -\omega$ and $\beta = 0$. Thus, we can readily obtain the energy eigenvalues through using Eqs. (14)-(16) as [16,18,24]

$$E_{n\ell} = \frac{\omega}{2} (4n + D + 2\ell), \quad n, \ell = 0, 1, 2, \dots, \quad (30)$$

and the corresponding radial wave function becomes

$$\psi^{(n)}(r) = (a_0 + a_1r^2 + \dots + a_nr^{2n}) r^\ell \exp \left[-\frac{\omega}{2}r^2 \right]. \quad (31)$$

where a_i with $i = 0, 1, 2, \dots, n$ are normalization constants.

(ii) Coulomb problem: when $b = c = 0$, and $a = Z$, implies that $\alpha = 0$ and $\beta = -\frac{2Z}{(2n+D+2\ell-1)}$, from which we readily obtain the exact eigenvalues as [25]

$$E_{n\ell} = -\frac{2Z^2}{(2n + D + 2\ell - 1)^2}, \quad n, \ell = 0, 1, 2, \dots, \quad (32)$$

and radial wave function becomes

$$\psi^{(n)}(r) = (a_0 + a_1r^2 + \dots + a_nr^{2n}) r^\ell \exp \left[-\frac{2Z}{(2n + D + 2\ell - 1)}r \right], \quad (33)$$

where a_i with $i = 0, 1, 2, \dots, n$ are normalization constants

B. The sextic anharmonic oscillator problem

This version of sextic AHO potential

$$V(r) = \mu r^2 + \lambda r^4 + \eta r^6, \quad \eta > 0, \quad (34)$$

has been studied in the D dimensions through Hill determinant method [3]. We want to solve the radial SE, Eq. (2), with Eq. (34) by selecting the following representation of ansatz to the radial portion of wave function

$$R(r) = \exp [p(\alpha, \beta, r)] \sum_{n=0} a_n r^{2n+(k-1)/2}, \quad (35)$$

provided that the power of the reference function has the following selection:

$$p(\alpha, \beta, r) = \frac{1}{2}\alpha r^2 + \frac{1}{4}\beta r^4. \quad (36)$$

Implementing the present method on the representation (35), and taking the coefficients of $r^{2n+(k+1)/2}$ to zero, we obtain the relation

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0, \quad (37)$$

where

$$A_n = \alpha^2 + \beta(4n + k + 2) - 2\mu, \quad (38)$$

$$B_n = 2E + \alpha(4n + k), \quad (39)$$

$$C_n = 2n(2n + k - 2), \quad (40)$$

and the value of the parameters for $p(\alpha, \beta, r)$ can be evaluated as

$$\beta = \pm\sqrt{2\eta}, \quad \alpha = \pm\frac{\lambda}{\sqrt{2\eta}}. \quad (41)$$

To obtain a well-behaved solution at the origin and infinity, we must set $\beta = -\sqrt{2\eta}$ and $\alpha = -\frac{\lambda}{\sqrt{2\eta}}$ which ensures that wave function ansatz, Eq. (35), be finite for all r . Further,

for a given p , if $a_p \neq 0$, but $a_{p+1} = a_{p+2} = a_{p+3} = \dots = 0$, we then obtain $A_p = 0$ from Eq. (38), which leads to the following constraint on the coupling parameters of sextic AHO problem as

$$2\mu + \sqrt{2\eta}(4n + k + 2) - \frac{\lambda^2}{2\eta} = 0. \quad (42)$$

Case (1): when $p = 0$, it is shown from Eq. (19) that $B_0 = 0$, which leads to the following energy eigenvalue (ground state):

$$E_0^k = \frac{\lambda k}{2\sqrt{2\eta}}, \quad (43)$$

and the corresponding wave function (unnormalized):

$$\psi_0(r) = a_0 r^\ell \exp \left[-\frac{\lambda r^2 + \eta r^4}{\sqrt{2\eta}} \right], \quad (44)$$

Case (2): when $p = 1$, it is shown from Eq. (19) that $B_0 B_1 = A_0 C_1$, which leads to the constraints on the coupling parameters of the potential as

$$E_1^k = \frac{\lambda}{2\sqrt{2\eta}}(k + 2) + \sqrt{\frac{\lambda^2}{4\eta}(k + 2) - \frac{k}{2} \left[\sqrt{2\eta}(k + 2) + 2\mu \right]}. \quad (45)$$

In particular, an important version of the above sextic AHO is the harmonic oscillator problem

$$V(r) = \mu r^2, \quad \mu > 0. \quad (46)$$

Selecting the following representation of the wave function ansatz::

$$R(r) = \exp [p(\alpha, r)] \sum_{n=0} a_n r^{2n+(k-1)/2}, \quad (47)$$

with

$$p(\alpha, r) = \frac{1}{2}\alpha r^2, \quad (48)$$

and iterating the previous steps, one gets:

$$A_n = \alpha^2 - 2\mu, \quad (49)$$

$$B_n = 2E + \alpha(4n + k), \quad (50)$$

$$C_n = 2n(2n + k - 2), \quad (51)$$

which leads to choosing $\alpha = -\sqrt{2\mu}$.

Case (1): when $p = 0$, it is shown from Eq. (19) that $B_0 = 0$, which leads to the following energy eigenvalue (ground state):

$$E_0^k = \frac{\sqrt{2\mu}k}{2}, \quad (52)$$

and the corresponding wave function (unnormalized):

$$\psi_0^{(k)}(r) = a_0 r^\ell \exp\left[-\frac{\sqrt{2\mu}}{2}r^2\right], \quad (53)$$

Case (2): when $p = 1$, it is shown from Eq. (19) that $B_0 B_1 = A_0 C_1$, which leads to the restriction on k and the parameters of the potential as

$$E_1^k = \frac{\sqrt{2\mu}}{2}(k + 2) + \sqrt{2\mu}. \quad (54)$$

Generally speaking, the energy eigenvalues are

$$E_n^k = \frac{\sqrt{2\mu}}{2}(4n + k), \quad (55)$$

and the corresponding wave function can be read

$$\psi_n(r) = (a_0 + a_1 r + \cdots + a_p r^{2n}) r^\ell \exp\left[-\frac{\sqrt{2\mu}}{2}r^2\right], \quad (56)$$

where a_i ($i = 0, 1, 2, \dots, n$) are normalization constants.

III. CONCLUDING REMARKS

We applied the wavefunction ansatz method to the confining CLH and sextic AHO interactions. Table 1 shows the calculated energies of the CLH type potential together with those obtained by the SUSYQM and Hill determinant method for high values of parameters

in three- and four-dimensions. We know from (8) and (9) that the CLH problem with exactly solvable potentials $V_1^{(1),(2),(3)}(r)$ in four-dimensions can be transformed to the sextic AHO problem (7) in four-dimensions with the exact eigenvalues given through (9).

We compute the energy eigenvalues of the following conjugate AHO potentials: $\widehat{V}_1^{(4)}(r) = r^2 + [1/2(7.625)^{3/2}]r^4 + [1/4(32)(7.625)^2]r^6$, $\widehat{V}_1^{(5)}(r) = r^2 + [1/2(7.375)^{3/2}]r^4 + [1/4(32)(7.375)^2]r^6$ and $\widehat{V}_1^{(6)}(r) = r^2 + [1/2(7.125)^{3/2}]r^4 + [1/4(32)(7.125)^2]r^6$ in two-dimensions by the present simple method and compute our results in Table 2 with the exact values given by (9). A class of AHO may be constructed from (43) that admits exact solutions. We also compute the energy eigenvalues of the following conjugate AHO potentials: $\widehat{V}_1^{(4)}(r) = r^2 + [1/2(7.5)^{3/2}]r^4 + [1/4(32)(7.5)^2]r^6$, $\widehat{V}_1^{(5)}(r) = r^2 + [1/2(7.25)^{3/2}]r^4 + [1/4(32)(7.25)^2]r^6$ and $\widehat{V}_1^{(6)}(r) = r^2 + [1/2(7.0)^{3/2}]r^4 + [1/4(32)(7.0)^2]r^6$ in four-dimensions by the method and compute our results in Table 3 with the exact values given by (9). These eigenvalues are checked by other methods.

This method yields exact solutions for a single state only for a potential of type V_2 and many states of type V_2 with some constraints on the coupling parameters. Our method is applicable to many general CLH or AHO and produces excellent results for the low-lying states. It gives the exact solutions of the Coulomb and the harmonic oscillator in D -dimensions. The CLH and sextic AHO in D -dimensions are related through Eq.(9) and are verified in Table 2 by this method.

A class of conjugate AHO having exact eigenvalues may be constructed from the transformation of CLH potential. It is found that the eigenvalues of central potential $V(r)$ are identical for $D = 2, \ell = 2$, $D = 4, \ell = 1$ and $D = 6, \ell = 0$ states. This is because $k = D + 2\ell$ remains unaltered for these states.

ACKNOWLEDGMENTS

S.M. Ikhdair is grateful to his family for love and assistance.

REFERENCES

- [1] R. N. Chaudhuri and M. Mondal, *Phys. Rev. A* 40 (1989) 6080; 43 (1991) 3241.
- [2] R. K. Agrawal and V. S. Varma, *Phys. Rev. A* 48 (1993) 1921.
- [3] R. N. Chaudhuri and M. Mondal, *Phys. Rev. A* 52 (1995) 1850.
- [4] C. J. Tymczak, G. S. Japaridze, C. R. Handy and Xiao-Qian Wang, *Phys. Rev. Lett.* 80 (1998) 3673.
- [5] D. P. Datta and S. Mukherjee, *J. Phys. A* 15 (1982) 2369.
- [6] A. Datta, J. Dey, M. Dey and P. Ghose, *Phys. Lett.* 106 B (1981) 505; R. N. Chaudhuri, M. Tater and M. Znojil, *J. Phys. A* 20 (1987) 1401.
- [7] J. Killingbeck, *Phys. Lett.* 65 A (1978) 87; 67 A (1978) 13.
- [8] R. P. Saxena and V. S. Varma, *J. Phys. A* 15 (1982) L221; G. P. Flessas and K. P. Das, *Phys. Lett.* 78 A (1980) 19; A. Khare, *ibid.* 83 A (1981) 237.
- [9] E. Witten, *Nucl. Phys. B* 185 (1981) 513; F. Cooper and B. Freedman, *Ann. Phys.* (N.Y.) 146 (1983) 262; C. V. Sukumar, *J. Phys. A* 18 (1985) 2917.
- [10] R. Dutt, A. Khare and U. Sukhatme, *Am. J. Phys.* 56 (1988) 163.
- [11] V. A. Kostelecky and M. M. Nieto, *Phys. Rev. Lett.* 53 (1984) 2285; *Phys. Rev. A* 32 (1985) 1293.
- [12] M. Bernstein and L. S. Brown, *Phys. Rev. Lett.* 52 (1984) 1933.
- [13] T. Imbo and U. Sukhatme, *Phys. Rev. Lett.* 54 (1985) 2184.
- [14] A. Khare, *Phys. Lett* 161 B (1985) 131, R. Dutt, A. Khare and U. Sukhatme, *Phys. Lett. B* 181 (1986) 295.
- [15] I. V. Dobrovolska and R. S. Tutik, *Int. J. Mod. Phys. A* 16 (2001) 2493; *ibid.* *J. Phys. A* 32 (1999) 563.

- [16] I. V. Dobrovolska and R. S. Tutik, [quant-ph/0611064].
- [17] S.-H. Dong, *Physica Scripta* 65 (2002) 289.
- [18] S. M. Ikhdaïr and R. Sever, [quant-ph/0702052] submitted to *J. Molec. Spectrosc.*
- [19] S.M. Ikhdaïr and R. Sever, *J. Mol. Struct.: THEOCHEM* 806 (2007) 155.
- [20] S. M. Ikhdaïr and R. Sever, [arXiv:quant-ph/0611065]: to appear in *Int. J. Mod. Phys. E*.
- [21] S. M. Ikhdaïr and R. Sever, *Z. Phys. C* 56 (1992) 155; C 58 (1993) 153; D 28 (1993) 1; *Hadronic J.* 15 (1992) 389; *Int. J. Mod. Phys. A* 18 (2003) 4215; A 19 (2004) 1771; A 20 (2005) 4035; A 20 (2005) 6509; A 21 (2006) 2191; A 21 (2006) 3989; A 21 (2006) 6699; [arXiv:hep-ph/0504176] (in press) *Int. J. Mod. Phys. E*; S. Ikhdaïr, O. Mustafa and R. Sever, *Tr. J. Phys.* 16 (1992) 510; 17 (1993) 474.
- [22] O. Özer and B. Gönül, *Mod. Phys. Lett. A* 18 (2003) 2581.
- [23] F. Cooper, A. Khare and U. Sukhatme, *Phys. Rep.* 251 (1995) 267.
- [24] S. Flügge, *Practical Quantum Mechanics 1* (Berlin-Heidelberg-New York: springer-verlag, 1971).
- [25] S. M. Ikhdaïr and R. Sever, *Int. J. Mod. Phys. A* 21 (2006) 6465; [arXiv:quant-ph/0604073]: DOI 10.1007/s10910-006-9080-2, to appear in *J. Math. Chem.*; [arXiv:quant-ph/0604078]: DOI 10.1007/s10910-006-9115-8, to appear in *J. Math. Chem.*; [arXiv:quant-ph/0508009] to appear in *J. Math. Chem* ; [arXiv:quant-ph/0603205] *J. Mol. Struct. (THEOCHEM)* (2007) (DOI:10.1016/j.theochem.2007.01.019).

TABLES

TABLE I. The lowest energy eigenvalues of the CLH potential for $\ell = 0, 1, 2$ in three- and four-dimensions.

a^a	b	c	ℓ	D	Present method	SUSYQM [3]
4	1	$\frac{1}{32}$	0	3	-7.625	-7.625
8	1	$\frac{1}{32}$	1	3	-7.375	-7.375
12	1	$\frac{1}{32}$	2	3	-7.125	-7.125
6	1	$\frac{1}{32}$	0	4	-7.500	-7.500
10	1	$\frac{1}{32}$	1	4	-7.250	-7.250
14	1	$\frac{1}{32}$	2	4	-7.000	-7.000

^aThe parameter values here are taken from [3].

TABLE II. The eigenvalues of the conjugate sextic anharmonic oscillators in two-dimensions are compared with the exact values.

Conjugate sextic AHO ^a	ℓ	Present work	Exact Value, Eq.(9)
$\widehat{V}_1^{(4)}(r)$	1	2.8971438733606	2.8971438733606
$\widehat{V}_1^{(5)}(r)$	3	5.8916775545493	5.8916775545493
$\widehat{V}_1^{(6)}(r)$	5	8.9912237911843	8.9912237911846

^aThe parameter values in constructing the sextic AHO are taken from Table 1.

TABLE III. The eigenvalues of the conjugate sextic anharmonic oscillators in four-dimensions are compared with the other works and the exact values.

Conjugate sextic AHO ^a	ℓ	Present work	Hill Determinant [3]	Exact Value, Eq.(9)
$\widehat{V}_1^{(4)}(r)$	1	4.3817804600412	4.381780461	4.381780459
$\widehat{V}_1^{(5)}(r)$	3	7.427813527082	7.427813527	7.427813526
$\widehat{V}_1^{(6)}(r)$	5	10.583005244257	10.583005244	10.583005240

^aThe parameter values in constructing the sextic AHO are taken from Table 1.