Exact Solution of the Klein-Gordon Equation for the $\mathcal{PT}$-Symmetric Generalized Woods-Saxon Potential by the Nikiforov-Uvarov Method

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Abstract

The one-dimensional Klein-Gordon (KG) equation has been solved for the $\mathcal{PT}$-symmetric generalized Woods-Saxon (WS) potential. The Nikiforov-Uvarov (NU) method which is based on solving the second-order linear differential equations by reduction to a generalized equation of hypergeometric type is used to obtain exact energy eigenvalues and corresponding eigenfunctions. We have also investigated the positive and negative exact bound states of the s-states for different types of complex generalized WS potentials.

Keywords: Klein-Gordon equation, Energy Eigenvalues and Eigenfunctions; Woods-Saxon potential; $\mathcal{PT}$-symmetry, NU Method.

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I. INTRODUCTION

In the past few years there has been considerable work on non-Hermitian Hamiltonians. Among this kind of Hamiltonians, much attention has been focused on the investigation of properties of so-called $\mathcal{PT}$-symmetric Hamiltonians. Following the early studies of Bender *et al.* [1], the $\mathcal{PT}$-symmetry formulation has been successfuly utilized by many authors [2-8]. The $\mathcal{PT}$-symmetric but non-Hermitian Hamiltonians have real spectra whether the Hamiltonians are Hermitian or not. Non-Hermitian Hamiltonians with real or complex spectra have also been analyzed by using different methods [3-6,9]. Non-Hermitian but $\mathcal{PT}$-symmetric models have applications in different fields, such as optics [10], nuclear physics [11], condensed matter [12], quantum field theory [13] and population biology [14].

Exact solution of Schrödinger equation for central potentials has generated much interest in recent years. So far, some of these potentials are the parabolic type potential [15], the Eckart potential [16,17], the Fermi-step potential [16,17], the Rosen-Morse potential [18], the Ginocchio barrier [19], the Scarf barriers [20], the Morse potential [21] and a potential which interpolates between Morse and Eckart barriers [22]. Many authors have studied on exponential type potentials [23-26] and quasi exactly solvable quadratic potentials [27-29]. In addition, Schrödinger, Dirac, Klein-Gordon, and Duffin-Kemmer-Petiau equations for a Coulomb type potential are solved by using different method [30-34]. The exact solutions for these models have been obtained analytically.

Further, using the quantization of the boundary condition of the states at the origin, Znojil [35] studied another generalized Hulthén and other exponential potentials in non-relativistic and relativistic regions. Domingues-Adame [36] and Chetouani *et al.* [37] also studied relativistic bound states of the standard Hulthén potential. On the other hand, Rao and Kagali [38] investigated the relativistic bound states of the exponential-type screened Coulomb potential by means of the one-dimensional ($1D$) Klein-Gordon equation. However, it is well-known that for the exponential-type screened Coulomb potential there is no explicit form of the energy expression of bound-states for Schrödinger [39], KG [38] and

also Dirac [16] equations. Şimşek and Eğrifes [31] have presented the bound-state solutions of $1D$ Klein-Gordon (KG) equation for $\mathcal{PT}$-symmetric potentials with complex generalized Hulthén potential. In a latter study, Eğrifes and Sever [32] investigated the bound-state solutions of the $1D$ Dirac equation with $\mathcal{PT}$-symmetric and non-$\mathcal{PT}$-symmetric real and complex forms of the generalized Hulthén potential. Yi *et al.* [40] obtained the energy equations in the KG theory with equally mixed vector and scalar Rosen-Morse-type potentials. Berkdemir *et al.* [41] obtained the bound-state eigenvalues and eigenfunctions for the Schrödinger equation using a generalized form of Woods-Saxon (WS) potential by means of Nikoforov-Uvarov method [42]. In recent works, we have investigated the $\ell \neq 0$ bound-state solutions of the $1D$ Schrödinger equation with the real and complex forms of the modified Hulthén and WS potentials [43,44] for their energy spectra and correponding wave functions using the Nikiforov-Uvarov (NU) method. In addition, we have extended our study to relativistic models in solving the spinless Salpeter equation analytically for its exact bound-state spectra and wavefunctions for the real and complex forms of the $\mathcal{PT}$-symmetric generalized Hulthén potential [45].

In the present work, we investigate the bound-state solutions of the $1D$ Klein-Gordon (KG) equation with real and complex forms of the generalized Woods-Saxon (WS) potential. Hence, the objective of our work is to determine the exact energy levels of relativistic KG particles trapped in a spherically symmetric generalized WS potential which possesses a $\mathcal{PT}$-symmetry as well. In this context, it is possible to convert KG to a Schrödinger-like equation which takes the hypergeometric form to which one may apply the NU method.

The organization of this work is as follows. After a brief introductory discussion of the NU method in Section II, we discuss the KG problem and then obtain the exact bound-state energy spectra for real and complex cases of generalized WS potentials and their corresponding eigenfunctions in Section III. In Section IV, we discuss our results and mention briefly the scope of this relativistic study.

II. THE NIKIFOROV-UVAROV METHOD

In this section we outline the basic formulations of the method. The Schrödinger equation and other Schrödinger-type equations can be solved by using the Nikiforov-Uvarov (NU) method which is based on the solutions of general second-order linear differential equation with special orthogonal functions [42]. It is well known that for any given $1D$ radial potential, the Schrödinger equation can be reduced to a generalized equation of hypergeometric type with an appropriate transformation and it can be written in the following form

$$\psi_n''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\widetilde{\tau}(s)$ is of a first-degree polynomial. To find particular solution of Eq.(1) we apply the method of separation of variables using the transformation

$$\psi_n(s) = \phi_n(s)y_n(s), \tag{2}$$

which reduces Eq.(1) into a hypergeometric-type equation

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \tag{3}$$

whose polynomial solutions $y_n(s)$ of the hypergeometric type function are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \qquad (n = 0, 1, 2, ...) \tag{4}$$

where B_n is a normalizing constant and $\rho(s)$ is the weight function satisfying the condition [42]

$$\frac{d}{ds}w(s) = \frac{\tau(s)}{\sigma(s)}w(s), \tag{5}$$

where $w(s) = \sigma(s)\rho(s)$. On the other hand, the function $\phi(s)$ satisfies the condition

$$\frac{d}{ds}\phi(s) = \frac{\pi(s)}{\sigma(s)}\phi(s), \tag{6}$$

where the linear polynomial $\pi(s)$ is given by

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \qquad (7)$$

from which the root k is the essential point in the calculation of $\pi(s)$ is determined. Further, the parameter λ required for this method is defined as

$$\lambda = k + \pi'(s). \qquad (8)$$

Further, in order to find the value of k, the discriminant under the square root is being set equal to zero and the resulting second-order polynomial has to be solved for its roots $k_{+,-}$. Thus, a new eigenvalue equation for the SE becomes

$$\lambda_n + n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s) = 0, \qquad (n = 0, 1, 2, ...) \qquad (9)$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \qquad (10)$$

and it must have a negative derivative.

III. EXACT BOUND STATE SOLUTIONS OF THE GENERALIZED WOODS-SAXON POTENTIAL

In this section, we formulate the Nikiforov-Uvarov solution for the relativistic motion of a spin-zero particle bound in spherically symmetric well-known spherical symmetric generalized Woods-Saxon potential of the form [46]

$$V_q(r, R_0) = -V_0 \frac{e^{-\left(\frac{r-R_0}{a}\right)}}{1 + qe^{-\left(\frac{r-R_0}{a}\right)}}, \quad q \geq 0, \ R_0 \gg a, \qquad (11)$$

where r refers to the center-of-mass distance between the projectile and the target nuclei (the r range is from 0 to ∞). The relevant parameters of the nuclear potential are given as follows: $R_0 = r_0 A^{1/3}$ is to define the confinement barrier position value of the corresponding

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spherical nucleus or the width of the potential, A is the target mass number, r_0 is the radius parameter, V_0 controls the barrier height of the Coulombic part, a is the surface diffuseness parameter has to control its slope, which is usually adjusted to the experimental values of ionization energies. Further, q is a shape (deformation) parameter, the strength of the exponential part other than unity, set to determine the shape of potential and is arbitrarily taken to be a real constant within the potential. Further, we remark that the spatial coordinates in the potential are not deformed and thus the potential still remains spherical. Obviously, for some specific q values this potential reduces to the well-known types, such as for $q = 0$ to the exponential potential and for $q = -1$ and $a = \delta^{-1}$ to the modified Hulthén potential (cf. [43,44,45] and the references therein).

For a scalar particle of rest mass m and total energy E_{nl}, the radial part of the KG equationin in three-dimensional spherical coordinates [47]

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l\,(l+1)}{2mr^2} + \frac{1}{2mc^2}\left[m^2 c^4 - [E_{nl} - V(r)]^2\right]\right]\psi_{nl}(r) = 0, \tag{12}$$

where $\psi_{nl}(r)$ is the reduced radial wave function.

To calculate the energy eigenvalues and the corresponding eigenfunctions, we substitute the Hermitian real-valued 1D generalized WS potential (the r range is from 0 to ∞):

$$V_q(x) = -V_0 \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}}, \quad \alpha = 1/a, \; x = r - R_0 \tag{13}$$

into the 1D $\mathcal{PT}-$symmetrical Hermitian spinless KG equation (12) for $l = 0$ case (i.e., s-wave states) and then obtain

$$\frac{d^2\psi_{nq}(x)}{dx^2} + \frac{1}{\hbar^2 c^2}\left[(E_n^2 - m^2 c^4) + V_0^2 \frac{e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2} + 2V_0 E_n \frac{e^{-\alpha x}}{(1 + qe^{-\alpha x})}\right]\psi_{nq}(x) = 0. \tag{14}$$

We employ the following novel dimensionless transformation parameter, $s(x) = (1 + qe^{-\alpha x})^{-1}$, which is in real phase and found to maintain the transformed wavefunctions finite on the boundary conditions (i.e., $-\infty \le x \le \infty \to 0 \le s \le 1$) [44]. Hence, applying such transformation to the upper equation after setting $\hbar = c = 1$, it gives

$$\frac{d^2\psi_{nq}(s)}{ds^2} + \frac{1 - 2s}{s - s^2}\frac{d\psi_{nq}(s)}{ds} + \frac{1}{\alpha^2(s - s^2)^2}\left[(E_n^2 - m^2) + \tilde{V}_0^2(1-s)^2 + 2E_n\tilde{V}_0(1-s)\right]\psi_{nq}(s) = 0,$$

$$\tag{15}$$

where $\widetilde{V}_0 = V_0/q$. With the dimensionless definitions given by

$$-\epsilon^2 = \frac{(E_n^2 - m^2)}{\alpha^2} \geq 0, \ \beta^2 = \frac{2E_n\widetilde{V}_0}{\alpha^2}, \ \gamma^2 = \frac{\widetilde{V}_0^2}{\alpha^2}, \ (E_n^2 \leq m^2, \ \beta^2 > 0, \ \gamma^2 > 0 \ ), \qquad (16)$$

for bound-states, one can arrive at the simple hypergeometric equation given by

$$\psi''_{nq}(s) + \frac{1 - 2s}{(s - s^2)}\psi'_{nq}(s) + \frac{[s^2\gamma^2 - s(\beta^2 + 2\gamma^2) + \beta^2 + \gamma^2 - \epsilon^2]}{(s - s^2)^2}\psi_{nq}(s) = 0. \qquad (17)$$

Hence, comparing the last equation with the generalized hypergeometric type, Eq.(1), we obtain the associated polynomials as

$$\widetilde{\tau}(s) = 1 - 2s, \quad \sigma(s) = (s - s^2), \quad \widetilde{\sigma}(s) = s^2\gamma^2 - s\left(\beta^2 + 2\gamma^2\right) + \beta^2 + \gamma^2 - \epsilon^2. \qquad (18)$$

When these polynomials are substituted into Eq.(7), with $\sigma'(s) = 1 - 2s$, we obtain

$$\pi(s) = \pm\sqrt{-s^2(\gamma^2 + k) + s\left(\beta^2 + 2\gamma^2 + k\right) + \epsilon^2 - \beta^2 - \gamma^2}. \qquad (19)$$

Further, the discriminant of the upper expression under the square root has to be set equal to zero. Therefore, it becomes

$$\Delta = \left[\beta^2 + 2\gamma^2 + k\right]^2 - 4(\gamma^2 + k)(\beta^2 + \gamma^2 - \epsilon^2) = 0. \qquad (20)$$

Solving Eq.(20) for the constant k, we obtain the double roots as $k_{1,2} = \beta^2 - 2\epsilon^2 \pm 2\epsilon b$, where $b = \sqrt{\epsilon^2 - \beta^2 - \gamma^2} = \frac{1}{2}\left[\sqrt{1 - 4\gamma^2} - (2n + 1)\right] - \epsilon$ with $n = 0, 1, \cdots$. Thus, substituting these values for each k into Eq.(19), we obtain

$$\pi(s) = \pm \begin{cases} (b - \epsilon)\, s - b; & \text{for} \quad k_1 = \beta^2 - 2\epsilon^2 + 2\epsilon b, \\ (b + \epsilon)\, s - b; & \text{for} \quad k_2 = \beta^2 - 2\epsilon^2 - 2\epsilon b. \end{cases} \qquad (21)$$

Hence, making the following choice for the polynomial $\pi(s)$ as

$$\pi(s) = -(b + \epsilon)\, s + b, \qquad (22)$$

for $k_2 = \beta^2 - 2\epsilon^2 - 2\epsilon b$, giving the function:

$$\tau(s) = -2(1 + b + \epsilon)s + 1 + 2b, \qquad (23)$$

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which has a negative derivative of the form $\tau'(s) = -2(1 + b + \epsilon) = 2n - 1 - \sqrt{1 - 4\gamma^2}$. Thus, from Eqs.(8)-(9) and Eqs.(22)-(23), we find

$$\lambda = -\gamma^2 - (b + \epsilon)(b + \epsilon + 1), \tag{24}$$

and

$$\lambda_n = n^2 + n + 2n\,(\epsilon + b). \tag{25}$$

Therefore, after setting $\lambda_n = \lambda$ and solving for E_{nq}, we find the KG exact binding energy spectra as

$$E_{nq} = -\frac{V_0}{2q} \pm \xi \sqrt{\frac{m^2}{4V_0^2 + \xi^2} - \frac{1}{16q^2}}, \tag{26}$$

where

$$\xi = \sqrt{q^2\alpha^2 - 4V_0^2} - q\alpha(2n + 1), \tag{27}$$

providing that the condition $q^2\alpha^2 \geq 4V_0^2$ must be fulfilled for any possible bound-states. In this context, it is worthwhile to point out that Şimşek and Eğrifes [31] have recently obtained a similar expression for the Hulthén potential. We find the corresponding wavefunctions by applying the NU method to find the hypergeometric function $y_n(s)$ which is the polynomial solution of hypergeometric-type equation (3) described with the weight function [42]. By substituting $\pi(s)$ and $\sigma(s)$ in Eq.(6) and then solving the first-order differential equation, we find

$$\phi_n(s) = s^b(1 - s)^\epsilon. \tag{28}$$

To find the function $y_{nq}(s)$, which is the polynomial solution of hypergeometric-type equation, we multiply Eq.(3) by $\rho(s)$ so that it can be written in self-adjoint form [42]

$$(\sigma(s)\rho(s)y'_{nq}(s))' + \lambda\rho(s)y_{nq}(s) = 0, \tag{29}$$

where $\rho(s)$ satisfies the differential equation $(\sigma(s)\rho(s))' = \tau(s)\rho(s)$ which gives

$$\rho(s) = s^{2b}(1 - s)^{2\epsilon}. \tag{30}$$

We then obtain the eigenfunction of hypergeometric-type equation from the Rodrigues relation given by Eq.(4) as

$$y_{nq}(s) = D_{nq}s^{-2b}(1 - s)^{-2\epsilon}\frac{d^n}{ds^n}\left[s^{n+2b}(1 - s)^{n+2\epsilon}\right], \tag{31}$$

where D_{nq} is a normalizing constant. In the limit $q \to 1$, the polynomial solutions of $y_n(s)$ are expressed in terms of Jacobi Polynomials, which is one of the classical orthogonal polynomials, with weight function (30) in the closed interval $[0, 1]$, giving $y_{n,1}(s) \simeq P_n^{(2b,2\epsilon)}(1 - 2s)$ [42]. The radial wave function $\psi_{nq}(s)$ is obtained from the Jacobi polynomials in Eq.(31) and $\phi(s)$ in Eq.(28) for the s-wave functions could be determined as

$$\psi_{nq}(s) = N_{nq}s^{-b}(1 - s)^{-\epsilon}\frac{d^n}{ds^n}\left[s^{n+2b}(1 - s)^{n+2\epsilon}\right] = N_{nq}s^{b}(1 - s)^{\epsilon}P_n^{(2b,2\epsilon)}(1 - 2s), \tag{32}$$

where $b = \frac{\xi}{2q\alpha} - \frac{\sqrt{m^2 - E_n^2}}{\alpha}$, $s = (1 + qe^{-\alpha x})^{-1}$ and N_{nq} is a new normalization constant.

We now make use of the fact that the Jacobi polynomials can be explicitly written in two different ways [48]:

$$P_n^{(\rho,\nu)}(z) = 2^{-n}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+\rho}{p}\binom{n+\nu}{n-p}(1 - z)^{n-p}(1 + z)^{p}, \tag{33}$$

$$P_n^{(\rho,\nu)}(z) = \frac{\Gamma(n+\rho+1)}{n!\Gamma(n+\rho+\nu+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+\rho+\nu+r+1)}{\Gamma(r+\rho+1)}\left(\frac{z-1}{2}\right)^{r}, \tag{34}$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$. Using Eqs.(33)-(34), we obtain the explicit expressions for $P_n^{(2b,2\epsilon)}(1 - 2s)$

$$P_n^{(2b,2\epsilon)}(1 - 2s) = (-1)^n\Gamma(n+2b+1)\Gamma(n+2\epsilon+1)$$

$$\times\sum_{p=0}^{n}\frac{(-1)^p q^{n-p}}{p!(n-p)!\Gamma(p+2\epsilon+1)\Gamma(n+2b-p+1)}s^{n-p}(1 - s)^p, \tag{35}$$

$$P_n^{(2b,2\epsilon)}(1 - 2s) = \frac{\Gamma(n+2b+1)}{\Gamma(n+2b+2\epsilon+1)}\sum_{r=0}^{n}\frac{(-1)^r q^r\Gamma(n+2b+2\epsilon+r+1)}{r!(n-r)!\Gamma(2b+r+1)}s^r. \tag{36}$$

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$$1 = N_{nq}^2(-1)^n \frac{\Gamma(n+2\epsilon+1)\Gamma(n+2b+1)^2}{\Gamma(n+2\epsilon+2b+1)} \left\{ \sum_{p=0}^{n} \frac{(-1)^p q^{n-p}}{p!(n-p)!\Gamma(p+2\epsilon+1)\Gamma(n+2b-p+1)} \right\}$$

$$\times \left\{ \sum_{r=0}^{n} \frac{(-1)^r q^r \Gamma(n+2\epsilon+2b+r+1)}{r!(n-r)!\Gamma(2b+r+1)} \right\} I_{nq}(p,r), \tag{37}$$

where

$$I_{nq}(p,r) = \int_0^1 s^{n+2b+r-p}(1-s)^{p+2\epsilon} ds. \tag{38}$$

Using the following integral representation of the hypergeometric function [49]

$$\int_0^1 s^{\alpha_0-1}(1-s)^{\gamma_0-\alpha_0-1}(1-qs)^{-\beta_0} ds =_2 F_1(\alpha_0,\beta_0:\gamma_0;q) \frac{\Gamma(\alpha_0)\Gamma(\gamma_0-\alpha_0)}{\Gamma(\gamma_0)},$$

$$[\text{Re}(\gamma_0) > \text{Re}(\alpha_0) > 0, \quad |\arg(1-q)| < \pi] \tag{39}$$

which gives

$$_2F_1(\alpha_0,\beta_0:\alpha_0+1;q)/\alpha_0 = \int_0^1 s^{\alpha_0-1}(1-qs)^{-\beta_0} ds, \tag{40}$$

$$_2F_1(\alpha_0,\beta_0:\gamma_0;q) = \frac{\Gamma(\gamma_0)\Gamma(\gamma_0-\alpha_0-\beta_0)}{\Gamma(\gamma_0-\alpha_0)\Gamma(\gamma_0-\beta_0)},$$

$$[\text{Re}(\gamma_0-\alpha_0-\beta_0) > 0, \quad \text{Re}(\gamma_0) > \text{Re}(\beta_0) > 0, \tag{41}$$

for $q = 1$. Setting $\alpha_0 = n+2b+r-p+1$, $\beta_0 = -p-2\epsilon$, and $\gamma_0 = \alpha_0+1$, one gets

$$I_{nq}(p,r) = \frac{_2F_1(\alpha_0,\beta_0:\gamma_0;q)}{\alpha_0} = \frac{(n+2b+r-p+1)!(p+2\epsilon)!}{(n+2b+r-p+1)(n+2\epsilon+r+2b+1)!} \tag{42}$$

A. Real Potentials

Consider the parameters V_0, q, and α given in Eq.(13) are all real, then

(i) For any given α the spectrum consists of real eigenstate spectra $E_n(V_0, q, \alpha)$ depending on q. The sign of V_0 does not affect the bound states. For positive shape parameter q, it

is clear that while $V_0 \to 0$, $E_n = m\sqrt{1 - \left(\frac{\alpha n}{2m}\right)^2}$ tends to the value m for the ground state (i.e., $n = 0$) and $0.866m$ for the first excited state (i.e., $n = 1$), etc. In these calculations, we have used $\lambda_c = \frac{\hbar}{mc} = \frac{1}{m} = \frac{1}{\alpha}$ which is the compton wavelength of the KG particles.

(ii) There exist bound states (real solution) in case if the condition $4V_0^2 \leq q^2\alpha^2$ is achieved, otherwise there are no bound-states.

(iii) There exist bound states in case if the condition $4V_0^2 + \xi^2 \leq 16q^2m^2$ is achieved, otherwise there are no bound-states.

Moreover, this condition which gives the critical coupling value turns to be

$$n \leq \frac{1}{q\alpha}\left(\sqrt{4q^2m^2 - V_0^2} + \sqrt{\frac{q^2\alpha^2}{4} - V_0^2}\right) - \frac{1}{2}, \tag{43}$$

i.e., there are only finitely many eigenstates. In order that at least one level might exist $(n = 0)$, its necessary that the inequality

$$q\alpha - \sqrt{q^2\alpha^2 - 4V_0^2} \leq 2\sqrt{4q^2m^2 - V_0^2}, \tag{44}$$

is fulfilled

B. Complex Potentials

Under $\mathcal{P}$, the spatial coordinates (x, y, z) are replaced by $(-x, -y, -z)$ but r is replaced by r and not $-r$, in the radial wave equation (4). Thus, the s-wave differential equation is not $\mathcal{PT}-$symmetric. The radial Schrödinger wave equation becomes a different differential equation under the action of the $\mathcal{PT}-$operator and does not go into itself. This means that we must solve the problem in $1D$, on the full plane, say x-direction and not in the radial direction r.

1. Non-Hermitian $\mathcal{PT}-$Symmetric Generalized Woods-Saxon Potential

Let us consider the case where at least one of the potential parameters be complex:

If α is a complex parameter $(\alpha \to i\alpha)$, the potential (13) becomes

$$V_q(x) = -\frac{V_0}{q^2 + 2q\cos(\alpha x) + 1}\left[q + \cos(\alpha x) - i\sin(\alpha x)\right] = V_q^*(-x), \tag{45}$$

which is a $\mathcal{PT}$-symmetric but non-Hermitian. It has real spectrum given by

$$E_{nq} = -\frac{V_0}{2q} + \left(\sqrt{q^2\alpha^2 + 4V_0^2} - \alpha q(2n+1)\right)\sqrt{\frac{1}{16q^2} - \frac{m^2}{4V_0^2 - \left(\sqrt{q^2\alpha^2 + 4V_0^2} - q\alpha(2n+1)\right)^2}}, \tag{46}$$

if and only if $16q^2m^2 \leq 4V_0^2 - \left(\sqrt{q^2\alpha^2 + 4V_0^2} - q\alpha(2n+1)\right)^2$ which gives $\frac{8qm^2}{\alpha(2n+1)} + \frac{q\alpha}{2(2n+1)} + \frac{q\alpha}{2}(2n+1) \leq \sqrt{q^2\alpha^2 + 4V_0^2}$.

Figs. 1(a) and (b) show the variation of the ground-state (i.e., $n = 0$) as a function of the coupling constant V_0 for different positive and negative q, and $a = \lambda_c$. Obviously, in Fig. 1(a), the non-Hermitian $\mathcal{PT}$-symmetric generalized WS potential generates real and negative bound-states for $q > 0$, it generates real and positive bound-states for the same value of α when $q < 0$ (Fig. 1(b)). Further, Figs. 2(a) and (b) show the variation of the first three energy eigenstates as a function of α for (a) $q = 1.0, V_0 = 6m$ and (b) $q = -1.0, V_0 = 2m$. Obviously, for the given V_0, as seen from Figs. 2(a) and (b) all possible eigenstates have negative (positive) eigenenergies if the parameter q is positive (negative). It is almost notable that there are some crossing points of the relativistic energy eigenvalues for some V_0 values.

The corresponding radial wave function $\psi_{nq}(s)$ for the s-wave could be determined as

$$\psi_{nq}(s) = N_{nq}s^{ib}(1 - qs)^{i\epsilon}P_n^{(2ib,2i\epsilon)}(1 - 2s), \tag{47}$$

where $s = (1 + qe^{-i\alpha x})^{-1}$.

For the sake of comparing the relativistic and non-relativistic binding energies, we need to solve the $1D$ Schrödinger equation for the complex form of the generalized WS potential given by Eq.(45). We employ the convenient transformation $s(x) = (1 + qe^{-i\alpha x})^{-1}, 0 \leq r \leq \infty \to 0 \leq s \leq 1$, to obtain

$$\psi_{nq}''(s) + \frac{1 - 2s}{(s - s^2)}\psi_{nq}'(s) + \frac{[-\beta^2 s + \beta^2 - \epsilon^2]}{(s - s^2)^2}\psi_{nq}(s) = 0, \tag{48}$$

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for which

$$\tilde{\tau}(s) = 1 - 2s, \quad \sigma(s) = s - s^2, \quad \tilde{\sigma}(s) = -\beta^2 s + \beta^2 - \epsilon^2,$$

$$\epsilon^2 = \frac{2m}{\hbar^2 \alpha^2} E_n \quad (E_n < 0), \quad \beta^2 = -\frac{2m}{\hbar^2 \alpha^2 q} V_0 \quad (\beta^2 > 0). \tag{49}$$

Moreover, it could be obtained

$$\tau(s) = -2(1 + c + \epsilon)s + (1 + 2c), \quad c = \sqrt{\epsilon^2 - \beta^2}, \tag{50}$$

if $\pi(s) = -(c + \epsilon)s + c$ is chosen for $k_- = -(c + \epsilon)^2$. We also find the eigenvalues

$$\lambda = -(c + \epsilon)(c + \epsilon + 1), \quad \lambda_n = 2n(c + \epsilon + 1) + n(n - 1). \tag{51}$$

Finally, setting $\lambda = \lambda_n$ and solving for ϵ, then the energy eigenvalues of the system under consideration could be found as

$$E_n(V_0, q, i\alpha) = \frac{\hbar^2 \alpha^2}{2m} \left[\frac{n+1}{2} - \frac{\gamma}{(n+1)} \right]^2, \quad \gamma = \frac{mV_0}{\hbar^2 q\alpha^2}, \quad 0 \le n < \infty. \tag{52}$$

On the other hand, the radial wave function in the present case becomes

$$\psi_{nq}(s) = N_{nq} s^c (1 - s)^\epsilon P_n^{(2c,2\epsilon)}(1 - 2s), \tag{53}$$

with $s(x) = (1 + e^{-i\alpha x})^{-1}$ and N_{nq} is a new normalization constant determine by

$$1 = N_{nq}^2 (-1)^n \frac{(n + 2\epsilon)! \Gamma(n + 2c + 1)^2}{\Gamma(n + 2c + 2\epsilon + 1)} \left\{ \sum_{p=0}^n \frac{(-1)^p q^{n-p}}{p!(n-p)!(2\epsilon + p)!\Gamma(n + 2c - p + 1)} \right\}$$

$$\times \left\{ \sum_{r=0}^n \frac{(-1)^r q^r \Gamma(n + 2c + r + 2\epsilon + 1)}{r!(n-r)!\Gamma(2c + r + 1)} \right\} \int_0^1 s^{n+2c+r-p}(1 - qs)^{p+2\epsilon} ds, \tag{54}$$

where

$$\int_0^1 s^{n+2c+r-p}(1 - qs)^{p+2\epsilon} ds = {}_2F_1(n + 2c + r - p + 1, -p - 2\epsilon : n + 2c + r - p + 2 : 1)B(n + 2c + r - p + 1,$$

$$\tag{55}$$

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2. Non-Hermitian non-$\mathcal{PT}$−Symmetric Generalized Woods-Saxon Potential

Let two parameters; namely, V_0 and q be complex parameters (i.e., $V_0 \rightarrow iV_0$, $q \rightarrow iq$), then we obtain the potential as

$$V_q(x) = V_0 \frac{\left[2\cosh^2(\alpha x) - \sinh(2\alpha x) - 1\right] - i\left[\cosh(\alpha x) - \sinh(\alpha x)\right]}{1 + q^2\left[2\cosh^2(\alpha x) - \sinh(2\alpha x) - 1\right]}. \tag{56}$$

This potential is a non $\mathcal{PT}$-symmetric but non-Hermitian possesses exact real spectra

$$E_{nq} = -\frac{V_0}{2q} + \left(\sqrt{q^2\alpha^2 - 4V_0^2} - q\alpha(2n+1)\right)\sqrt{\frac{m^2}{4V_0^2 + \left(\sqrt{q^2\alpha^2 - 4V_0^2} - q\alpha(2n+1)\right)^2} - \frac{1}{16q^2}}, \tag{57}$$

if and only if $16q^2m^2 \geq 4V_0^2 + \left(\sqrt{q^2\alpha^2 - 4V_0^2} - q\alpha(2n+1)\right)^2$ which gives $-\frac{8qm^2}{\alpha(2n+1)} + \frac{q\alpha}{2(2n+1)} + \frac{q\alpha}{2}(2n+1) \leq \sqrt{q^2\alpha^2 - 4V_0^2}$.

On the other hand, the corresponding radial wave functions $\psi_{nq}(s)$ for the s-wave could be determined as

$$\psi_{nq}(s) = N_{nq}s^b(1-s)^\epsilon P_n^{(2b,2\epsilon)}(1-2s), \tag{58}$$

where $s = (1 + iqe^{-\alpha x})^{-1}$. The integral $I_{nq}(p,r) = \int_0^1 s^{n+2b+r-p}(1-s)^{p+2\epsilon}ds$ is given by

$$I_{nq}(p,r) =_2 F_1(n + 2b + r - p + 1, -p - 2\epsilon : n + 2b + r - p + 2; i)B(n + 2b + r - p + 1, 1). \tag{59}$$

3. Pseudo-Hermiticity and $\mathcal{PT}$−Symmetric Generalized Woods-Saxon Potential

When all the parameters V_0, α and q are complex parameters (i.e., $V_0 \rightarrow iV_0$, $\alpha \rightarrow i\alpha$, $q \rightarrow iq$), we obtain

$$V_q(x) = -\frac{V_0}{q^2 + 2q\sin(\alpha x) + 1}[q + \sin(\alpha x) + i\cos(\alpha x)] = V_q^*(\frac{\pi}{2} - x). \tag{60}$$

This potential is a pseudo-Hermitian potential [27,48] having a $\pi/2$ phase difference with respect to the potential (I), $\eta = P$-pseudo-Hermitian (i.e., $PTV_q(x)(PT)^{-1} = V_q(x)$, with

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$P = \eta : x \to \frac{\pi}{2\alpha} - x$ and $T : i \to -i$), it is also a non $\mathcal{PT}$-symmetric but non-Hermitian having exact real spectrum given by

$$E_{nq} = -\frac{V_0}{2q} + \left(\sqrt{q^2\alpha^2 + 4V_0^2} + q\alpha(2n+1)\right)\sqrt{\frac{1}{16q^2} - \frac{m^2}{4V_0^2 - \left(\sqrt{q^2\alpha^2 + 4V_0^2} + q\alpha(2n+1)\right)^2}},$$

$$(61)$$

if and only if $16q^2m^2 \le 4V_0^2 - \left(\sqrt{q^2\alpha^2 + 4V_0^2} + q\alpha(2n+1)\right)^2$ which gives $-\frac{8qm^2}{\alpha(2n+1)} - \frac{q\alpha}{2(2n+1)} - \frac{q\alpha}{2}(2n+1) \ge \sqrt{q^2\alpha^2 + 4V_0^2}$

Figs. 3(a) and (b) show the variation of the ground-state (i.e., $n = 0$) as a function of the coupling constant V_0 for different positive q with $a = 10$ and negative q with $\alpha = 1$. Obviously, in Fig. 3(a), the P-pseudo-Hermitian non-$\mathcal{PT}$-symmetric generalized WS potential generates real and negative bound-states for $q > 0$, it also generates real and positive bound-states for the same value of α when $q < 0$ (Fig. 3(b)). Further, Figs. 4(a) and (b) show the variation of the first three energy eigenstates as a function of α for (a) $q = 1.0, V_0 = 2m$ and (b) $q = -1.0, V_0 = 4m$. Obviously, for the given V_0, as seen from Figs.4(a) and (b) all possible eigenstates have negative (positive) eigenenergies if the shape parameter q is positive (negative). There are some limitation on the number of bound states. It is seen that the range parameter α has limits for any given state.

On the other hand, the corresponding radial wave functions $\psi_{nq}(s)$ for the s-wave could be determined as

$$\psi_{nq}(s) = N_{nq}s^{ib}(1-s)^{i\epsilon}P_n^{(2ib,2i\epsilon)}(1-2s),\qquad(62)$$

with $s = (1 + iqe^{-i\alpha x})^{-1}$. The integral $I_{nq}(p,r) = \int_0^1 s^{n+2ib+r-p}(1-s)^{p+2i\epsilon}ds$ is given by

$$I_{nq}(p,r) =_2 F_1(n + 2ib + r - p + 1, -p - 2i\epsilon : n + 2ib + r - p + 2; i)B(n + 2ib + r - p + 1, 1).$$

$$(63)$$

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IV. RESULTS AND CONCLUSIONS

We have seen that the s-wave KG equation for the generalized WS potential can be solved exactly. The relativistic bound-state energy spectrum and the corresponding wave functions for the generalized WS potential have been obtained by the NU method. Some interesting results including the $\mathcal{PT}$-symmetric and pseudo-Hermitian versions of the generalized WS potential have also been discussed for the real bound-states. In addition, we have discussed the relation between the non-relativistic and relativistic solutions and the possibility of existence of bound states for complex parameters. We have also shown the possibility to obtain relativistic bound-states of complex quantum mechanical formulations. Finally, the relativistic model provides real solution for the complex exponential potential where this solution is not available in the nonrelativistic model.

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FIGURES

FIG. 1. The variation of the ground-state ($n = 0$) energy with $\alpha = 1$, in a non-Hermitian $\mathcal{PT}$-symmetric potential given by Eq.(45), as a function of V_0 for three different (a) positive and (b) negative q.

FIG. 2. The variation of the first three energy eigenstates, in a non-Hermitian $\mathcal{PT}$-symmetric potential given by Eq.(45), as a function of α for (a) $q = 1.0, V_0 = 6m$ and (b) $q = -1.0, V_0 = 2m$.

FIG. 3. The variation of the ground-state ($n = 0$) energy, in a P-pseudo Hermitian non-$\mathcal{PT}$-symmetric potential given by Eq.(60), as a function of V_0 for three different (a) positive q with $a = 10$ and (b) negative q with $\alpha = 1$.

FIG. 4. The variation of the first three energy eigenstates, in a P-pseudo -Hermitian non-$\mathcal{PT}$-symmetric potential given by Eq.(60), as a function of α for (a) $q = 1.0, V_0 = 2m$ and (b) $q = -1.0, V_0 = 4m$.