Partial Wave Analysis of the First Order Born Amplitude of a Dirac particle in an Aharonov-Bohm Potential

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Abstract

A partial wave analysis using the basis of the total angular momentum operator J_3 is carried out for the first order Born amplitude of a Dirac particle in an Aharonov-Bohm (AB) potential. It is demonstrated that the s-partial wave contributes to the scattering amplitude in contrast to the case with scalar non-relativistic particles. We suggest that this explains the fact that the first order Born amplitude of a Dirac particle coincides with the exact amplitude expanded to the same order, where it does not for a scalar particle. An interesting algebra involving the Dirac velocity operator and the angular observables is discovered and its consequences are exploited.

1 Introduction

The first attempts to calculate the Aharonov-Bohm (AB) scattering[1] amplitude for a scalar particle using perturbation theory [2, 3] revealed a discrepancy between the first order Born amplitude and the exact amplitude when expanded to the same order.Moreover, the second order Born amplitude turned out to be divergent. These results were attributed [3] to the fact that the first order Born amplitude based on the Schrödinger Hamiltonian of a scalar particle misses the contribution of the $l = 0$ partial wave, as it is of second order.

The problem manifested itself also in the field theory models of the AB effect with scalar particles, namely the Chern-Simons models[4]. It also appeared in the perturbative calculations in the many-body anyon theories near the bosonic end[5]. It was noted that introducing a contact interaction into the Hamiltonian remedies these problems[4]. Subsequently this interaction was attributed to a spin-magnetic moment interaction[6]. The first order Born amplitude for a Dirac particle was calculated in [7] and the second order in [8] where full agreement with the expansions of the exact amplitude[9] to the corresponding order was found. Non-relativistic perturbative calculations within the framework of the field theory models of the AB with spin-1/2 particles suffered no problems[10, 11].

No partial wave analysis of the first order Born amplitude for a Dirac particle where it would be interesting to investigate the behavior of the $l = 0$ partial wave was reported in the literature. The main motivation behind this work is to carry out such an analysis.

In section 2, we present a comprehensive partial wave analysis of the first order Born amplitude for non-relativistic scalar and spin 1/2 particles. In section 3, we carry out a partial wave analysis of the Born amplitude of a Dirac particle, using the cylindrical partial modes of the conserved total angular momentum operator. An interesting closed algebra involving the Dirac velocity operator and the angular observables of the theory is discovered, and its consequences pursued.

2 Partial wave Born amplitude for non-relativistic scalar and spin-1/2 particles

Before embarking on the treatment of the Dirac particle, we will first carry out the partial wave analysis for the non-relativistic scalar and spin-1/2 particles in the AB potential. While the results of this discussion are generally known and were mentioned in the literature in various contexts[6], there is no published work that we know of, which contains a systematic and complete treatment. Thus we present it here for completeness and to set the stage for the discussion of the relativistic case.

The AB potential in the cylindrical coordinates reads

$$\mathbf{A} = \frac{\Phi}{2\pi\rho}\epsilon_\varphi, \tag{1}$$

where $\rho = \sqrt{x^2 + y^2}$, ϵ_φ is the unit vector along the φ direction and Φ is the flux through the tube. The Schrödinger equation for a scalar particle in this potential, written in cylindrical coordinates, is ($\hbar = c = 1$):

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}(\frac{\partial}{\partial\varphi} + i\alpha)^2 + \frac{\partial^2}{\partial z^2} + k^2\right]\Psi(\mathbf{r}) = 0, \tag{2}$$

where $\alpha = -\frac{e\Phi}{2\pi}$. We take $0 < \alpha < 1$, as in this work we will be mainly interested in perturbative calculations.

As usual, one can separate the z-dependence of the wavefunction and neglect it all together without any loss of generality. The interaction potential in Eq. (2) can be identified as

$$U = -\frac{1}{\rho^2}(2i\alpha\frac{\partial}{\partial\varphi}) + \frac{\alpha^2}{\rho^2}. \tag{3}$$

The first order Born scattering amplitude can now be readily constructed, and reads

$$f^{(1)}(\theta) = \left(\frac{i}{2(2\pi ik)^{\frac{1}{2}}}\right)\int e^{-i\mathbf{k}'.\mathbf{x}}\left(\frac{2i\alpha}{\rho^2}\frac{\partial}{\partial\varphi}\right)e^{i\mathbf{k}.\mathbf{x}}\rho d\rho d\varphi, \tag{4}$$

where $\mathbf{k}$ and $\mathbf{k}'$ are, respectively, the wave vectors of the incident (from left) and scattered waves, with $|\mathbf{k}| = |\mathbf{k}'|$; and θ is the scattering angle. A calculation of $f^{(1)}(\theta)$ yields [2, 3]:

$$f^{(1)}(\theta) = -\alpha\left(\frac{\pi}{2ik}\right)^{\frac{1}{2}}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}, \qquad \theta \neq 0. \tag{5}$$

The exact amplitude first calculated in [1], and corrected in [13], for $0 < \alpha < 1$ reads:

$$f(\theta) = -\frac{i}{\sqrt{2\pi ik}}(\sin\pi\alpha)\frac{e^{-i\theta/2}}{\sin\frac{\theta}{2}} \tag{6}$$

For small α, one gets,

$$f(\theta) = \left(\frac{\pi}{2ik}\right)^{\frac{1}{2}}\left(-\alpha\cot\frac{\theta}{2} - i\alpha\right) + O\left(\alpha^2\right), \qquad \theta \neq 0. \tag{7}$$

$f^{(1)}(\theta)$ given in Eq. (5) clearly misses the -$i\alpha$ term of Eq. (7). This discrepancy was attributed to the fact that the first order Born amplitude misses the contribution of the s-partial wave [3]. This can be seen most transparently by looking at the partial Born amplitudes separately, to which we will now turn.

The plane waves in Eq. (4) can be expanded in terms of the conserved orbital angular momentum operator L_3 by employing the well-known expansion

$$e^{ikx\cos\alpha} = \sum_{l=-\infty}^{+\infty} i^l e^{il\alpha} J_l\left(x\right), \tag{8}$$

where $J_l\left(x\right)$ are the Bessel functions of order l. After carrying out the angular integral in Eq. (4), we get:

$$f^{(1)}(\theta) = \left(\frac{i\alpha}{(2\pi ik)^{\frac{1}{2}}}\right)\sum_l l e^{il\theta}\int\frac{d\rho}{\rho}\left[J_l\left(k\rho\right)\right]^2 \tag{9}$$

Now, it is obvious that the $l = 0$ partial wave amplitude, i.e., $f_0^{(1)}(\theta)$ vanishes. Integrating over the Bessel functions with the aid of the formula

$$\int_0^\infty dr\frac{\left[J_l\left(r\right)\right]^2}{r} = |2l|^{-1}, \qquad l \neq 0, \tag{10}$$

we get

$$f^{(1)}(\theta) = -\sum_l{}' i\alpha\pi \left(\frac{1}{2\pi ik}\right)^{\frac{1}{2}} \mathrm{sgn}(l) e^{il\theta}, \tag{11}$$

where $\mathrm{sgn}\,(l) = \frac{l}{|l|}$, and the prime denotes that the $l = 0$ term is excluded from the summation. Recalling that generally

$$f^{(1)}(\theta) = \sum_l f_l^{(1)}(\theta) e^{il\theta},$$

we get the partial amplitudes as:

$$f_l^{(1)}(\theta) = \begin{cases} \frac{-i\pi\alpha}{(2\pi ik)^{\frac{1}{2}}} \mathrm{sgn}(l) \quad , & l \neq 0 \\ \quad 0 \quad , & l = 0. \end{cases} \tag{12}$$

To compare the above partial amplitudes with the exact ones expanded in terms of α, we note that the exact phase shifts reported in [12, 13] read (when $0 < \alpha < 1$),

$$\delta_m = \begin{cases} -\frac{\pi}{2}\alpha \quad , & m \geq 0 \\ \frac{\pi}{2}\alpha \quad , & m < 0 \end{cases}$$

Therefore, the exact partial amplitudes become [12]:

$$f_l(\theta) = \left(e^{-i\mathrm{sgn}(l)\pi\alpha} - 1\right)(2\pi ik)^{-\frac{1}{2}} \qquad , \qquad l = 0, \pm 1, \pm 2, ... \tag{13}$$

which, for small α reduce to Eq. (12) when $l \neq 0$. When $l = 0$, $f_0(\theta)$ reduces to $\frac{-i\alpha\pi}{(2\pi ik)^{\frac{1}{2}}}$ for small α, while $f_0^{(1)}(\theta)$ vanishes!

We turn now to the non-relativistic spin-1/2 particles, where we will see that the $l = 0$ partial amplitude is non-vanishing. In addition to this, it will turn out that, it is this partial amplitude that leads to the modification of the exact amplitude when the spin is included.

The starting point is the Pauli equation

$$\frac{1}{2m}(\boldsymbol{\sigma} \cdot \mathbf{\Pi})^2 \Psi = E\Psi, \tag{14}$$

where $\mathbf{\Pi} = (\mathbf{p} - e\mathbf{A})$, and $\mathbf{A}$ is the AB potential given in Eq. (1), and $\sigma^i,\ i = 1, 2, 3$, are the Pauli spin matrices. Suppressing again the z degree of freedom we get,

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}(\frac{\partial}{\partial\varphi} + i\alpha)^2 - 2\pi\alpha\sigma_3\delta(\mathbf{r}) + k^2\right]\Psi(\mathbf{r}) = 0\ . \tag{15}$$

The first order Born amplitude now reads,

$$f^{(1)}(\theta) = \left(\frac{i}{2(2\pi ik)^{\frac{1}{2}}}\right) \int e^{-i\mathbf{k'}.\mathbf{x}} \chi^{\dagger(s')} \left(\frac{2i\alpha}{\rho^2}\frac{\partial}{\partial\varphi} - 2\pi\alpha\sigma_3\delta(\mathbf{r})\right) \chi^{(s)} e^{i\mathbf{k}.\mathbf{x}} \rho d\rho d\varphi, \tag{16}$$

where $\chi^{(s)}$ and $\chi^{(s')}$ are the spinors of the incident and outgoing waves, respectively. Expanding the plane waves, and carrying out the integrals as before, we get:

$$f^{(1)}(\theta) = \frac{1}{(2\pi ik)^{\frac{1}{2}}} \sum_l e^{il\theta} \chi^{\dagger(s')} \left[-i\pi\alpha \mathrm{sgn}\,(l)\,(1-\delta_{l,0}) - i\pi\alpha\delta_{l,0}\sigma_3\right] \chi^{(s)}. \quad (17)$$

Taking $\chi^{(s)}$ to be the spin state of a particle polarized along an arbitrary direction specified by a unit vector $\mathbf{n}$ with polar angle β, and considering transitions to a final state polarized along the same direction, we get the amplitude as:

$$f^{(1)}(\theta) = \sum_l (2\pi ik)^{-\frac{1}{2}} e^{il\theta} \left[-i\pi\alpha \mathrm{sgn}\,(l)\,(1-\delta_{l,0}) - i\pi\alpha\delta_{l,0}\cos\beta\right], \quad (18)$$

$$f_l^{(1)}(\theta) = \begin{cases} \frac{-i\pi\alpha}{(2\pi ik)^{\frac{1}{2}}}\mathrm{sgn}(l) & , \quad l \neq 0 \\ -\frac{i\pi\alpha}{(2\pi ik)^{\frac{1}{2}}}\cos\beta & , \quad l = 0\ . \end{cases} \quad (19)$$

The above results demonstrate that the $l = 0$ partial amplitude is non-vanishing, the reason being the spin-magnetic moment interaction term. We also note that for our choice of the spin orientations, it is only the s-wave that flips the spin, modifying the unpolarized amplitude only when the incident particle's spin has a component perpendicular to the solenoid. This result was first reported in [9] for the exact amplitude, and verified for the first-order Born amplitude in [7]. This is quite natural, as the s-wave is the only partial wave that can feel the solenoid; the other waves being banned by the centrifugal barrier.

3 Partial Wave Born Amplitudes for a Dirac Particle

The Hamiltonian for a Dirac particle in an electromagnetic potential is:

$$H = H_\circ + H_{int}, \quad (20)$$

where

$$H_\circ = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m, \quad (21)$$

and

$$H_{int} = eA_\circ - e\boldsymbol{\alpha}.\mathbf{A} \quad (22)$$

Here, $\alpha_i = \beta\gamma_i$ and $\beta = \gamma_4$. The γ's are the Dirac matrices: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$.

The first-order Born amplitude for the scattering of a Dirac particle in an electromagnetic field then reads,

$$S_{fi}^{(1)} = -i \int d^4x \overline{\psi}_f^{(s')}(\vec{x})\,(e\gamma_\mu A^\mu)\,\psi_i^{(s)}(x). \quad (23)$$

With the AB potential as given in Eq. (1), and with the choice of gauge $A_0 = 0$, and suppressing the z degree of freedom, and an energy consering δ-function, we get

$$S_{fi}^{(1)} = i\alpha \int d\rho d\varphi \overline{\psi}_f^{(s')}(\vec{x}) \left(-\sin\varphi\gamma_1 + \cos\varphi\gamma_2\right) \psi_i^{(s)}(\vec{x}). \tag{24}$$

where $p^{\perp}$ is the magnitude of the momentum perpendicular to the solenoid. For later convenience, we write $S_{fi}^{(1)}$ as :

$$S_{fi}^{(1)} = i\alpha \int d\rho d\varphi \overline{\psi}_f^{(s')}(\vec{x}) \left(D^+ + D^-\right) \psi_i^{(s)}(\vec{x}). \tag{25}$$

where the operators D$^{\pm}$ are defined by:

$$D^{\pm} = \left(\frac{\gamma_2 \pm i\gamma_1}{2}\right) e^{\pm i\varphi}. \tag{26}$$

Prior to carrying out a partial wave analysis of (24), we have to note that an expansion of the incident and outgoing waves in terms of the L_3 eigenstates will be inconclusive in this case. The reason, physically speaking, is that L_3 is not a constant of the motion in the Dirac theory, not even (as is well-known) in the free theory. The spinors, $u_i^{(s)}$ and $u_f^{(s)}$ are now functions of the angle φ. So, one has to expand the free spinors in terms of the eigenstates of the conserved total angular momentum operator $J_3 = L_3 + \frac{\Sigma_3}{2}$. We need first to find these states. These will be taken to be simultaneous eigenstates of the set of commuting operators: $H_\circ, J_3, S_3 = \beta\Sigma_3 + \frac{\xi p_3}{m}$ and p_3 (where $\xi = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$) according to:

$$\begin{array}{rc} H_\circ \Psi_{ls} = & E\Psi_{ls} \\ J_3 \Psi_{ls} = & \left(l + \frac{1}{2}\right)\Psi_{ls} \\ p_3 \Psi_{ls} = & p_3 \Psi_{ls} \\ S_3 \Psi_{ls} = & \pm s \Psi_{ls} \end{array} \tag{27}$$

Here, we are diagonalizing the spin operator S_3 along with the Hamiltonian rather than the more conventional helicity operator. S_3 is usually used when one has a magnetic field along the z-axis[14]. In the non-relativistic limit the upper components of Ψ_{ls} are eigenstates of σ_3. The eigenvalues of S_3 are

$$s = \pm\sqrt{1 + \left(\frac{p_3}{m}\right)^2}, \tag{28}$$

which reduce to $s = \pm 1$ when p_3 is set to zero. The Ψ_{ls} that solve the set of equations (27) read

$$\Psi_{ls} = \frac{e^{-i(Et - p_3 x_3 - l\varphi)}}{\sqrt{2\pi}\sqrt{2E}\sqrt{2s}} \left(\begin{array}{c} \sqrt{E+sm}\sqrt{s+1}J_l\left(p^{\perp}\rho\right) \\ ie^{i\varphi}\epsilon_3\sqrt{E-sm}\sqrt{s-1}J_{l+1}\left(p^{\perp}\rho\right) \\ \epsilon_3\sqrt{E+sm}\sqrt{s-1}J_l\left(p^{\perp}\rho\right) \\ ie^{i\varphi}\sqrt{E-sm}\sqrt{s+1}J_{l+1}\left(p^{\perp}\rho\right) \end{array}\right), \tag{29}$$

where $\epsilon_3 = \text{sgn}(s)\text{sgn}(p_3)$,$p^{\perp}$ is the magnitude of the momentum perpendicular to the solenoidand, and s assumes the values given in Eq. (28). Setting p_3 to zero one gets Ψ_{ls} modes as:

$$\Psi_{ls}(\mathbf{x}) = \frac{e^{i(l\varphi)}}{\sqrt{2\pi}\sqrt{2E}\sqrt{2s}} \begin{pmatrix} \sqrt{E+sm}\sqrt{s+1}J_l\left(p^{\perp}\rho\right) \\ ie^{i\varphi}\epsilon_3\sqrt{E-sm}\sqrt{s-1}J_{l+1}\left(p^{\perp}\rho\right) \\ \epsilon_3\sqrt{E+sm}\sqrt{s-1}J_l\left(p^{\perp}\rho\right) \\ ie^{i\varphi}\sqrt{E-sm}\sqrt{s+1}J_{l+1}\left(p^{\perp}\rho\right) \end{pmatrix}, \tag{30}$$

where $s = \pm 1$ now, and the cylindrical partial modes $\Psi_{ls}(\mathbf{x})$ are normalized as

$$\int \rho d\rho d\varphi \Psi^{\dagger}_{l's'}(\mathbf{x})\Psi_{ls}(\mathbf{x}) = \frac{1}{p^{\perp}}\delta\left(p^{\perp} - p^{\perp\prime}\right)\delta_{l,l'}\delta_{s.s'}. \tag{31}$$

The above partial modes are now the correct expansion basis that are to be used in the partial wave analysis. The incident and outgoing waves which are also eigenstates of S_3 are($p_3 = 0,\ s = \mp 1$) :

$$\Psi^{(s)}_i(\mathbf{x}) = e^{i\mathbf{p}_i\cdot\mathbf{x}}u_i = \frac{e^{ip^{\perp}\rho\cos\varphi}}{\sqrt{4\pi}\sqrt{2s}} \begin{pmatrix} \sqrt{E+sm}\sqrt{s+1} \\ \epsilon_3\sqrt{E-sm}\sqrt{s-1} \\ \epsilon_3\sqrt{E+sm}\sqrt{s-1} \\ \sqrt{E-sm}\sqrt{s+1} \end{pmatrix}, \tag{32}$$

$$\Psi^{(s)}_f(\mathbf{x}) = e^{i\mathbf{p}_f\cdot\mathbf{x}}u_f = \frac{e^{ip^{\perp}\rho\cos(\varphi-\theta)}}{\sqrt{4\pi}\sqrt{2s}} \begin{pmatrix} \sqrt{E+sm}\sqrt{s+1} \\ \epsilon_3 e^{i\theta}\sqrt{E-sm}\sqrt{s-1} \\ \epsilon_3\sqrt{E+sm}\sqrt{s-1} \\ e^{i\theta}\sqrt{E-sm}\sqrt{s+1} \end{pmatrix}, \tag{33}$$

The incident and outgoing waves given in (32) and (33) are normalized as $\int d^2x\psi^{\dagger(s')}(\vec{x})\psi^{(s)}(\vec{x}) = E\delta(\vec{p} - \vec{p'})$, which is the Lorentz-invariant normalization.

We can verify the following expansion of $\Psi_i(\mathbf{x})$ and $\Psi_f(\mathbf{x})$ in terms of the cylindrical modes $\Psi_{ls}(\mathbf{x})$:

$$\begin{array}{rl} \Psi^{(s)}_i(\mathbf{x}) = & \sqrt{E_i}\sum_l (i)^l\,\Psi_{ls}(\mathbf{x}) \\ \Psi^{(s)}_f(\mathbf{x}) = & \sqrt{E_f}\sum_l (i)^l\, e^{-il\theta}\Psi_{ls}(\mathbf{x})\ . \end{array} \tag{34}$$

The amplitude $S^{(1)}_{fi}$ now takes the form

$$S^{(1)}_{fi} = i\alpha E\sum_l (i)^l \sum_{l'} (-i)^{l'} e^{il'\theta}\mathcal{M}, \tag{35}$$

where

$$\mathcal{M} = \int d\rho d\varphi \overline{\Psi}_{l's'}(\mathbf{x})\left(D^+ + D^-\right)\Psi_{ls}(\mathbf{x}), \tag{36}$$

and $E = E_i = E_f$.

Now, the operators $D^{\pm}$, being linear combinations of the γ matrices will flip the spinors Ψ_{ls}. On the other hand, since $[J_3, D^{\pm}] = [S_3, D^{\pm}] = 0$, then

$D^{\pm}\Psi_{ls}$ should still be eigenstates of J_3 and S_3. It turns out that $D^{\pm}$ operators together with the angular observables of the theory obey an interesting algebra which leads to a fulfillment of the the above requirements. Let us first note that $D^{\pm}\Psi_{ls}$ are eigenstates of L_3 and $\frac{\Sigma_3}{2}$ as can be verified directly, though Ψ_{ls} obviously is not

$$\begin{array}{rl} \frac{\Sigma_3}{2}D^{\pm}\Psi_{ls} = & \mp\frac{1}{2}D^{\pm}\Psi_{ls} \\ L_3 D^{\pm}\Psi_{ls} = & \begin{pmatrix} l+1 \\ l \end{pmatrix} D^{\pm}\Psi_{ls}\ . \end{array} \tag{37}$$

It follows from Eq. (37)that

$$\left(L_3 + \frac{\Sigma_3}{2}\right) D^{\pm}\Psi_{ls} = \begin{pmatrix} \left(-\frac{1}{2}\right) + (l+1) \\ \left(+\frac{1}{2}\right) + (l) \end{pmatrix} D^{\pm}\Psi_{ls} = \left(l + \frac{1}{2}\right) D^{\pm}\Psi_{ls} \tag{38}$$

as should be. Therefore, the operators $D^{\pm}$ acting on the Ψ_{ls} modes, project them into eigenstates of L_3 and $\frac{\Sigma_3}{2}$ such that the sum of the eigenvalues is always equal to the J_3 eigenvalue; $l+\frac{1}{2}$. To get a further insight into the mechanism in action, we first note that the Ψ_{ls} modes can be written as linear combinations of the eigenstates of the L_3 and $\frac{\Sigma_3}{2}$ operators. Explicitly:

$$\Psi_{ls=+1} = \frac{1}{\sqrt{4\pi E}}\left[\begin{pmatrix} \sqrt{E+m}J_l\left(p^{\perp}\rho\right)e^{i(l\varphi)} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ i\sqrt{E-m}J_{l+1}\left(p^{\perp}\rho\right)e^{i(l+1)\varphi} \end{pmatrix}\right], \tag{39}$$

or in a more compact notation

$$|j_3, s=1\rangle = \left|j_3, s=1;\, l, +\frac{1}{2}\right\rangle + \left|j_3, s=1;\, l+1, -\frac{1}{2}\right\rangle, \tag{40}$$

where, the quantum numbers $(l, l+1)$ and $(\frac{1}{2}, -\frac{1}{2})$ above refer to the eigenvalues of L_3 and $\frac{\Sigma_3}{2}$, respectively, and the total orbital angular momentum's quantum number is always $j_3{=}l+\frac{1}{2}$. Similarly, for $s=-1$, we have

$$|j_3, s=-1;\rangle = \left|j_3, s=-1;\, l+1, -\frac{1}{2}\right\rangle + \left|j_3, s=-1;\, l, +\frac{1}{2}\right\rangle. \tag{41}$$

One can verify the following algebra

$$\begin{array}{rl} [L_3, D^{\pm}] = & \pm D^{\pm} \\ \left[\frac{\Sigma_3}{2}, D^{\pm}\right] = & \mp D^{\pm} \\ [D^+, D^-] = & 2\left(\frac{\Sigma_3}{2}\right)\ . \end{array} \tag{42}$$

Note also that

$$\left(D^+\right)^2 = \left(D^-\right)^2 = 0. \tag{43}$$

This algebra means that the operators $D^{\pm}$ are some sort of raising and lowering operators in the angular momentum space of the theory. Indeed, denoting the simultaneous eigenstate of L_3 and $\frac{\Sigma_3}{2}$ as $|l_3, \sigma_3\rangle$, one has:

$$\begin{array}{rl} L_3 D^{\pm}|l_3, \sigma_3\rangle = & (l_3 \pm 1)\, D^{\pm}|l_3, \sigma_3\rangle \\ \frac{\Sigma_3}{2}D^{\pm}|l_3, \sigma_3\rangle = & (\sigma_3 \mp 1)\, D^{\pm}|l_3, \sigma_3\rangle\,. \end{array} \tag{44}$$

Therefore, $D^{\pm}\left|l_3,\sigma_3\right\rangle = c_{\pm}\left|\left(l_3 \pm 1\right),\left(\sigma_3 \mp 1\right)\right\rangle$. The complex numbers $c_{\pm}$ are readily verified to be pure phases which we set to 1. Moreover, note that Eq. (43) implies

$$D^{+}\left|l_3,\sigma_3=-\frac{1}{2}\right\rangle = D^{-}\left|l_3,\sigma_3=+\frac{1}{2}\right\rangle = 0. \tag{45}$$

Thus, we have

$$D^{\pm}\left|l_3,\sigma_3\right\rangle = \left|l_3 \pm 1,\sigma_3 \mp 1\right\rangle. \tag{46}$$

Going back to our Ψ_{ls} functions given in Eqs. (40) and (41), we see now that

$$\begin{pmatrix} D^{+} \\ D^{-} \end{pmatrix}\left|j_3,s=1\right\rangle = \begin{pmatrix} \left|j_3,s=1;\, l+1,-\frac{1}{2}\right\rangle \\ \left|j_3,s=1;\, l,+\frac{1}{2}\right\rangle \end{pmatrix}, \tag{47}$$

and

$$\begin{pmatrix} D^{+} \\ D^{-} \end{pmatrix}\left|j_3,s=-1\right\rangle = \begin{pmatrix} \left|j_3,s=-1;\, l+1,-\frac{1}{2}\right\rangle \\ \left|j_3,s=-1;\, l,+\frac{1}{2}\right\rangle \end{pmatrix}. \tag{48}$$

This means that the operators $D^{\pm}$ acting on $\left|j_3,s=\pm 1\right\rangle$ projects out eigenstates of L_3 and $\frac{\Sigma_3}{2}$ such that $l_3+\sigma_3 = l+\frac{1}{2}$ only, i.e. J_3 eigenstates.

This mechanism of conserving the J_3 quantum number can be only observed upon employing the partial wave expansion of the Dirac spinors.

Going back to our amplitude; upon substituting the explicit forms of the partial modes of Eq. (30), in Eq. (36), and carrying out the φ integral we finally get,

$$\mathcal{M} = (\pi)\,\frac{\sqrt{E^2-s^2m^2}}{2E}\,(2s)\,\delta_{l,l'}\delta_{s,s'}\int J_{l+1}\left(p^{\perp}\rho\right)\,J_l\left(p^{\perp}\rho\right)d\rho. \tag{49}$$

The above expression clearly conserves J_3 and S_3 quantum numbers as it should do. Moreover, the $l=0$ partial wave contributes to the amplitude on equal footing with the other partial waves.

The Bessel function integral in Eq. (49) is tabulated for positive values of ℓ (formula 6.512-3 in [15]). For negative values of ℓ, we make use of the well-known relation valid for integral ℓ, $J_{-\ell}(x) = (-1)^{\ell}J_{\ell}(x)$, so that we convert the integral over Besssel functions of negative order to an integral over Bessel functions of positive order, getting an overall minus sign. So, we get finally for the first order amplitude,

$$S_{fi}^{(1)} = \sum_{\ell}\frac{1}{2}i\alpha sgn(\ell)e^{i\ell\theta} \tag{50}$$

The partial amplitudes are therefore,

$$S_{\ell}^{(1)} = \frac{1}{2}i\alpha sgn(\ell), \quad \ell = 0,\mp 1,\mp 2,\ldots \tag{51}$$

Note the appearance of $\mathrm{sgn}(\ell)$ which resulted from the Bessel function integral in Eq. (49). This is same as in the non-relativistic amplitude. To compare our final expression in Eq. (51) with the non-relativistic partial scattering amplitudes,

$f_\ell^{(1)}\ (\theta)$, we note that the S-matrix and the scattering amplitude are related in two dimensions via [12]:

$$(S-1)(k,\theta) = \left(\frac{ik}{2\pi}\right)^{1/2} f(k,\theta). \tag{52}$$

Expanding $S(k,\theta)$ and $f(k,\theta)$ in powers of the coupling constant, and imposing the equality for each partial wave, we get

$$f_\ell^{(1)}(k,\theta) = \sqrt{\frac{2\pi}{ik}} S_\ell^{(1)}(k,\theta) \tag{53}$$

Substituting $S_\ell^{(1)}$ given in Eq. (51), in Eq. (53), we get the partial scattering amplitudes:

$$f_\ell^{(1)}(\theta) = (2\pi ik)^{-1/2} i\alpha\pi sgn(\ell), \quad 1 = 0, \mp 1, \mp 2, \ldots \tag{54}$$

Eq. (54) compares (up to an overall minus sign) with Eq. (19). The discrepancy for the partial amplitudes $f_0^{(1)}(\theta)$ is a result of the difference in the spin orientations of the incident and outgoing particles in the two cases.

4 Conclusions

We have demonstrated through an explicit partial wave analysis, that the inclusion of spin into the Hamiltonian of a non-relativistic particle in an AB field leads to a non-vanishing $l = 0$ first order partial Born amplitude. Moreover, this particular amplitude is the one responsible for the modification of the total amplitude reported in [9] as a result of the inclusion of spin. A partial wave analysis of the first order Born amplitude for a Dirac particle shows that all the partial amplitudes, including the $l = 0$ are non-vanishing and contribute equally to the total amplitude. An interesting algebra involving the Dirac velocity operator and the angular observables of the Dirac theory was discovered, and shown to lead to a mechanism for the conservation of the total angular momentum quantum number upon transitions from the initial to the final states at the level of each partial wave.

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