PT-symmetric Solutions of Schrödinger Equation with position-dependent mass via Point Canonical Transformation

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Abstract

PT-symmetric solutions of Schrödinger equation are obtained for the Scarf and generalized harmonic oscillator potentials with the position-dependent mass. A general point canonical transformation is applied by using a free parameter. Three different forms of mass distributions are used. A set of the energy eigenvalues of the bound states and corresponding wave functions for target potentials are obtained as a function of the free parameter.
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1 Introduction

Exact solutions of the effective mass Schrödinger equation(SE) for some physical potentials have received much attention. Important applications are obtained in the fields of material science and condensed matter physics such as semiconductors [1], quantum well and quantum dots [2], ^{3}H clusters [3], quantum liquids [4], graded alloys and semiconductor heterostructures [5,6]. Recently, number of exact solutions on these topics increased [6-23]. Various solution methods are used in the calculations. The point canonical transformations (PCT) is one of these methods providing exact solution of energy eigenvalues and corresponding eigenfunctions [24-28]. It is also used for solving the Schrödinger equation with position-dependent effective mass for some potentials [8-13]. In the present work, we solve three different potentials with the three mass distributions. The point canonical transformation is taken in the more general form introducing a free parameter. This general form of the transformation will provide us a set of solutions for different values of the free parameter.

On the other hand, there has been considerable work on non-Hermitian Hamiltomians in recent years. Much attentioan has also been focused on $\mathcal{PT}$-symmetric Hamiltonians. Following the early studies of Bender *et al.* [29], the $\mathcal{PT}$-symmetry formulation has been successfully utilized by many authors [30-36]. The $\mathcal{PT}$-symmetric but non-Hermitian Hamiltonians have real spectra whether the Hamiltonians are Hermitian or not. Non-Hermitian Hamiltonians with real or complex spectra have also been analyzed by using different methods [31-34,37]. Non-Hermitian but $\mathcal{PT}$-symmetric models have applications in different fields, such as optics [38], nuclear physics [39], condensed matter [40] and population biology [41]. There are some recent works on these topics[42-44].

The contents of the paper is as follows. In section 2, we present briefly the solution of the Schrödinger equation by using point canonical transformation. In section 3, we introduce some applications for specific potentials. Results are discussed in section 4.

2 Method

We write the most general form of the Hamiltonian as [45]

$$H = \frac{1}{4}\left[m^{\eta} p m^{\lambda} p m^{\mu} + m^{\mu} p m^{\lambda} p m^{\eta} + V(x)\right] \tag{1}$$

where the parameters η, λ and μ are called the ambiguity parameters. They are constraint by the relation $\eta + \lambda + \mu = -1$. Different forms of the Hamiltonian are used in the literature depending on the choices of set of parameters. Here, we take the set such that $\eta = \mu = 0$, $\lambda = 1$ [46,47] introduced by Ben Danie-Duke. Thus, the Hamiltonian is invariant under instantaneous Galilean transformation [47]. The Schrödinger equation with $\hbar = 1$ takes the form

$$-\frac{d}{dx}\left[\frac{1}{2m(x)}\frac{d\Psi}{dx}\right] + V(x)\Psi(x) = E\Psi(x). \tag{2}$$

The wave function should be continuous at the mass discontinuity and verify the following condition

$$\left.\frac{1}{m(x)}\frac{d\Psi(x)}{dx}\right|_{-} = \left.\frac{1}{m(x)}\frac{d\Psi(x)}{dx}\right|_{+}. \tag{3}$$

On the other hand a Hamiltonian is said to be PT-symmetric if

$$[PT, H] = 0 \tag{4}$$

where P is the parity operator and T is the time reversal operator. They act on the position and momentum states as

$$P : x \to -x,\ \ P \to -P \qquad and \qquad T : x \to x,\ \ P \to -P, \qquad i \to -i. \tag{5}$$

Thus, we get the following conditions to have a PT-symmetric Hamiltonian

$$m(x) = m(-x) \qquad and \qquad V^{*}(-x) = V(x). \tag{6}$$

We shall use a general form of PCT with a free parameter to solve the Schrödinger equation for any potential $V(x)$. Defining the transformation with a free parameter β

$$\Psi(x) = m^{\beta}(x)\phi(x) \tag{7}$$

The SE takes

$$-\frac{1}{2m}\left[\phi'' + (2\beta - 1)\frac{m'}{m}\phi' + \beta(\beta - 2)\left(\frac{m'}{m}\right)^2\phi + \beta\left(\frac{m''}{m}\right)\phi\right] + V(x)\phi = E\phi. \tag{8}$$

It is solved by Roy [42] for $\beta = \frac{1}{4}$. Roy obtained this form by making the transformation

$$y = \int^x m(t)^{\frac{1}{2}}dt. \tag{9}$$

In the computations, three different position-dependent mass distributions[47] will be used. The reference potentials are PT-symmetric Scarf II[49,50] and generalized generalized harmonic oscillator[51] potentials. We will consider two different values of β.

3 Case A:$\beta = \frac{1}{2}$

For $\beta = \frac{1}{2}$, $Eq.$(8) has the following compact form

$$-\frac{1}{2}\frac{d^2\phi}{dx^2} + \Omega(x)\phi(x) = E\phi(x) \tag{10}$$

where

$$\Omega(x) = \frac{3}{8}\left(\frac{m'}{m}\right)^2 - \frac{1}{4}\left(\frac{m''}{m}\right) + mV - (m-1)E \tag{11}$$

Eq.(10) has the same form with the constant mass SE.

3.1 Mass Distribution $m(x) = \left(\frac{\alpha+x^2}{1+x^2}\right)^2$

i) The Scarf II potential is

$$\Omega(x) = -\lambda sech^2(x) - i\mu sech(x)\tanh(x) \tag{12}$$

The energy eigenvalues and corresponding wave functions are

$$E_n = -\left(n - p - 1\right)^2, \qquad n = 0, 1, 2, ... < \frac{s+t-1}{2} \tag{13}$$

and

$$\phi_n(x) = \frac{\Gamma(n - 2p + \frac{1}{4})}{n!\,\Gamma(\frac{1}{2} - 2p)}\, z^{-p}\, (z^*)^{-q}\; P_n^{(-2p-\frac{1}{2},-2q-\frac{1}{2})}\,(i\sinh(y)) \tag{14}$$

where

$$z = \frac{1 - i\sinh(x)}{2}, \tag{15}$$

$$p = -\tfrac{1}{4} \pm \tfrac{1}{2}\sqrt{\tfrac{1}{4} + \lambda + \mu},$$

$$= -\frac{1}{4} \pm \frac{t}{2}, \tag{16}$$

$$q = -\tfrac{1}{4} \pm \tfrac{1}{2}\sqrt{\tfrac{1}{4} + \lambda - \mu}$$

$$= -\frac{1}{4} \pm \frac{s}{2}. \tag{17}$$

For the Scarf II potential and the position-dependent mass case, we obtain the effective potential as

$$V(x) = \left(\frac{1+x^2}{\alpha + x^2}\right)^2 \left\{\Omega(x) - 4(1-\alpha)^2 \frac{x^2}{(1+x^2)^2)(\alpha+x^2)^2} + \frac{(1-\alpha)(1-3x^2)}{(1+x^2)^2(\alpha+x^2)} + \left[(\frac{\alpha+x^2}{1+x^2})^2 - 1\right] E\right\} \tag{18}$$

where

$$\frac{m'}{m} = 4(1-\alpha)\frac{x}{(1+x^2)(\alpha+x^2)} \tag{19}$$

and

$$\frac{m''}{m} = \frac{4(1-\alpha)}{(1+x^2)^2(\alpha+x^2)^2}\left[2(1-\alpha x^2)+(\alpha+x^2)(1-3x^2)\right]. \tag{20}$$

ii) PT-symmetric generalized harmonic oscillator

$$\Omega(x) = (x-i\varepsilon)^2 + \frac{g^2-\frac{1}{4}}{(x-i\varepsilon)^2}. \tag{21}$$

The energy eigenvalues and corresponding wave functions are

$$E_n = 4n - 2qg + 2, \qquad n = 0, 1, 2, ... \tag{22}$$

and

$$\phi_n(x) = e^{-\frac{1}{2}(x-i\varepsilon)^2}(x-i\varepsilon)^{-pg+\frac{1}{2}}L_n^{-qg}\left((x-i\varepsilon)^2\right) \tag{23}$$

where $q = \pm 1$ is called quasi-parity. Here the effective potential is

$$V(x) = \left(\frac{1+x^2}{\alpha+x^2}\right)^2\left\{\Omega(x) - \frac{3}{8}\left(\frac{m'}{m}\right)^2 + \frac{1}{4}\left(\frac{m''}{m}\right) + \left[\left(\frac{\alpha+x^2}{1+x^2}\right)^2 - 1\right]E\right\} \tag{24}$$

where $\frac{m'}{m}$ and $\frac{m''}{m}$ are given in $Eqs.(20, 21)$.

3.2 Mass Distribution $M(x) = \left(\frac{\alpha+x^2}{1+x^2}\right)^4 = m^2$

i)Scarf II potential

The form of the Scarf II potential is given in Eq. (13).

where $\frac{m'}{m}$, $\frac{m''}{m}$ and $\Omega(x)$ are given in $Eqs.(20, 21, 13)$. Solution of the SE for the Scarf II potential gives us the energy eigenvalues and corresponding wave functions as

$$E = -(n-p-1)^2 \qquad\qquad n = 0,1,2,... < \frac{s+t-1}{2} \tag{25}$$

ii) PT-symmetric generalized oscillator

It is given in $Eq.(22)$

The potential is given in $Eq.(13)$

$$V(x) = \left(\frac{1+x^2}{\alpha+x^2}\right)^4 \{\Omega(x) - (\frac{m^{'}}{m})^2 + \frac{1}{2}(\frac{m^{''}}{m}) + [(\frac{\alpha+x^2}{1+x^2})^2 - 1]E\} \tag{26}$$

$$V(x) = \left(\frac{1+x^2}{\alpha+x^2}\right)^4 \left\{\Omega(x) - \left(\frac{m^{'}}{m}\right)^2 + \frac{1}{2}\left(\frac{m^{''}}{m}\right) + \left[\left(\frac{\alpha+x^2}{1+x^2}\right)^2 - 1\right] E\right\} \tag{27}$$

where $\frac{m^{'}}{m}$, $\frac{m^{''}}{m}$ and $\Omega(x)$ are given in $Eqs.(20,21,22)$. The wave function is

$$\phi_n(y) = \frac{\Gamma(n-2p+\frac{1}{4})}{n!\,\Gamma(\frac{1}{2}-2p)}\, z^{-p}\,(z^*)^{-q}\;\; P_n^{(-2p-\frac{1}{2},-2q-\frac{1}{2})}\,(i\sinh(y)) \tag{28}$$

where z, p and q are defined in $Eqs.(16,17,18)$.

4 Case B: $\beta = \frac{2-\gamma}{4}$

Here, we define a new independent variable

$$y = \int^x m^{\frac{\gamma}{2}}(t)dt. \tag{29}$$

Then, $Eq.(3)$ takes the form

$$-\frac{1}{2m}\left[m^\gamma\phi^{''} + \left(\frac{\gamma}{2} + (\beta-1)\,m^{\frac{\gamma}{2}-1}m^{'}\right)\phi^{'} + \beta\,(\beta-1)\left(\frac{m^{'}}{m}\right)^2\phi + \beta\left(\frac{m^{''}}{m}\right)\phi\right] + V\phi = E\phi \tag{30}$$

To remove the term involving first derivative of the wave function, we impose

$$\tfrac{\gamma}{2} + 2\beta - 1 = 0.$$

This is the constraint on the parameter β to get the exact solution.

Thus, we get

$$-\frac{1}{2}\phi'' + \Omega(y)\phi = E\phi \tag{31}$$

where

$$\Omega(y) = -\frac{\beta}{2}m^{-\gamma}\left[(\beta-2)\left(\frac{m'}{m}\right)^2 + \frac{m''}{m}\right] + (V-E)\,m^{1-\gamma} + E \tag{32}$$

and also

$$V(x) = m^{\gamma-1}\left[\Omega(y) + \frac{2-\gamma}{\gamma m^{\gamma}}\left[-\frac{\gamma+6}{4}\left(\frac{m'}{m}\right)^2 + \frac{m''}{m}\right] + E\left(1-m^{1-\gamma}\right)\right]. \tag{33}$$

4.1 Mass Distribution $m(x) = \left(\frac{\alpha+x^2}{1+x^2}\right)^{\frac{2}{\gamma}}$

The new independent variable is

$$y = \left[x + (\alpha-1)\tan^{-1}(x)\right]. \tag{34}$$

i) Scarf II potential

The scarf II potential for the potential and position dependent mass has the form

$$V(x) = m^{\gamma-1}\left[\Omega(y) + \frac{2-\gamma}{8m^{\gamma}}\left[-\frac{\gamma+6}{4}\left(\frac{m'}{m}\right)^2 + \frac{m''}{m}\right] + E\left(1-m^{1-\gamma}\right)\right] \tag{35}$$

where $\frac{m'}{m}$ and $\frac{m''}{m}$ are given in $Eqs.(35, 36)$.

$$\frac{m'}{m} = \frac{4(1-\alpha)}{\gamma}\frac{x}{(1+x^2)(\alpha+x^2)} \tag{36}$$

and

$$\frac{m''}{m} = \frac{4(1-\alpha}{\gamma(1+x^2)^2\,(\alpha+x^2)^2}\left[2(k-1)(1-\alpha)x^2 + (\alpha+x^2)(1-3x^2)\right] \tag{37}$$

where $k = \frac{2}{\gamma}$

ii) PT-symmetric generalized oscillator

Solution of the SE for the PT-symmetric generalized oscillator, $Eq.(22)$, gives us energy eigenvalues and corresponding wave functions as

$$E_n = 4n - 2q\rho + 2, \qquad n = 0, 1, 2, ... \tag{38}$$

and

$$\phi_n(y) = e^{-\frac{1}{2}(y-i\varepsilon)^2}(y - i\varepsilon)^{-p\rho+\frac{1}{2}} \; L_n^{-q\rho}\left((y - i\varepsilon)^2\right) \tag{39}$$

where $q = \pm 1$ is called quasi-parity. For the potential and position dependent mass, we get

5 Conclusions

We have applied the point canonical transformation in a general form by introducing a free parameter to solve the Schrödinger equation for the Rosen-Morse and Scarf potentials with spatially dependent mass. We have obtained a set of exactly solvable target potentials by using two position-dependent mass distributions. Energy eigenvalues and corresponding wave functions for the target potentials are written in the compact form.

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