Effective Mass Quantum Systems with Displacement Operator: Inverse Square plus Coulomb-like Potential

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Abstract

The Schrödinger-like equation written in terms of the displacement operator is solved analytically for a inverse square plus Coulomb-like potential. Starting from the new Hamiltonian, the effects of the spatially dependent mass on the bound states and normalized wave functions of the "usual" inverse square plus Coulomb interaction are discussed.

Keywords: position-dependent mass, translation operator, inverse square potential, Coulomb potential, Schrödinger equation

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I. INTRODUCTION

It has been argued that the divergences appearing in field theories can be removed with the idea of noncommutativity and this can be done by using a universal invariant length parameter. This approach has became a central idea in the physical and mathematical points of view [1, 2]. Within the quantum mechanics, the noncommutative coordinates written by the terms of minimum length scale lead to some modifications in position-momentum commutators [3, 4]. This yields an extended Hamiltonian having a position-dependent mass term in kinetic part with some ambiguity parameters α, β, δ satisfying $\alpha + \beta + \delta = 1$, and a Schrödinger equation with effective mass [5-7].

Recently, Filho and co-workers have analysed a quantum system with position-dependent mass by using a different approach where they suggest a displacement operator given by

$$T_\gamma(dx)|x >= |x + dx + \gamma x dx > , \tag{1}$$

where γ is a real constant describing the mixing between the displacement and the original position state. This operator transforms a well-localized state around x to another well-localized state around $x + (1 + \gamma x)dx$ while all other physical properties remain unchanged [8]. This operator is written explicitly as

$$T_\gamma(dx) = I - \frac{i}{\hbar}\hat{p}_\gamma dx \, , \tag{2}$$

where $\hat{p}_\gamma$ corresponds to the generalized linear momentum operator. The commutator between $\hat{p}_\gamma$ and $\hat{x}$ operator is written as by $[\hat{x}, \hat{p}_\gamma] = (1 + \gamma x)i\hbar$ which gives a generalized uncertainty relation

$$\Delta x \Delta p_\gamma \geq (1 + \gamma < x >)\,\frac{\hbar}{2} \, , \tag{3}$$

The generalized momentum operator can be given as [8]

$$\hat{p}_\gamma|\alpha >= -i\hbar(1 + \gamma x)\frac{d}{dx}|\alpha > , \tag{4}$$

and the corresponding deformed derivative is written as $D_\gamma = (1+\gamma x)\,\frac{d}{dx}$ where $\hat{p}_\gamma = -i\hbar D_\gamma$. In Ref. [9], the generalized momentum operator is written in Hermitian form which enables us to write the Hamiltonian of the system as a Hermitian operator.

The time-dependent form of the equation for a particle moving in a potential field $V(x)$ is written as stated by Filho and co-workers [10]

$$\left\{i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}(1+\gamma x)\left[(1+\gamma x)\frac{\partial^2}{\partial x^2}+\gamma\frac{\partial}{\partial x}\right]-V(x)\right\}\Psi(x,t)=0\,, \tag{5}$$

If we consider the time-independent Hamiltonian operator to be $H=\hat{p}_\gamma^2/2m+V(x)$, we assume the wave function as $\Psi(x,t)=\phi(x)e^{-iEt/\hbar}$, and obtain the following Schrödinger-like equation for a single particle

$$\left[-\frac{\hbar^2}{2m}D_\gamma^2-E+V(x)\right]\phi(x)=0\,, \tag{6}$$

or

$$\left[\frac{2}{m(x)}\frac{d^2}{dx^2}+\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}+\frac{4}{\hbar^2}[E-V(x)]\right]\phi(x)=0\,, \tag{7}$$

with $m(x)=m(1+\gamma x)^{-2}$. In Refs. [8, 9], the authors have tested their ideas for a free particle, and a particle moving in a one-dimensional infinite well of length L. They have obtained analytical solutions for the above systems. They have discussed the expectation values of the position, and the normalization of the wave functions. In Ref. [8], the authors have also studied the dependence of the transmission and tunnelling probability on parameter γ for a particle subjected to a potential barrier with height $V_0>0$. In Ref. [11], the general bound state solutions and the corresponding normalized wave functions have been discussed for a potential function including a quartic and a quadratic term. In the present work, starting from the Schrödinger-like equation given in Eq. (7), we will search the analytical solutions for a particle moving in an inverse square plus Coulomb-like potential of the form

$$V(x)=\frac{A}{x^2}-\frac{B}{x}\,,$$

Our aim is to find the bound states and to see the effect of the parameter γ on the energy eigenvalues. We will also find the wave functions with their normalization constants.

II. ANALYTICAL SOLUTIONS

In order to study the effects of displacament operator on the results of the present problem, we change the variable to $z=1+\gamma x$, and write the above potential function into Eq. (7) giving

$$\frac{d^2\phi(z)}{dz^2}+\frac{1}{z}\frac{d\phi(z)}{dz}+\left[\frac{a_1}{z^2}+\frac{a_2}{z(1-z)}+\frac{a_3}{(1-z)^2}\right]\phi(z)=0\,, \tag{8}$$

where

$$a_1 = M[E - \gamma(A\gamma + B)]\,, \tag{9a}$$

$$a_2 = -\gamma M(2A\gamma + B)\,, \tag{9b}$$

$$a_3 = -AM\gamma^2\,, \tag{9c}$$

with $M = 2m/\gamma^2\hbar^2$.

The transformation on the wave function such as $\phi(z) = z^p(1-z)^q\psi(z)$ gives a second order differential equation

$$z(1-z)\frac{d^2\psi(z)}{dz^2} + [1 + 2p - (1 + 2p + 2q)z]\frac{d\psi(z)}{dz} + (a_2 - 2pq - q)\psi(z) = 0\,, \tag{10}$$

which could be a hypergeometric differential equation if the parameters used in the equation satisfy [12-14]

$$p^2 = M[\gamma(A\gamma + B) - E]\ ;\ q = \frac{1}{2}\left[1 \pm \sqrt{1 + 4AM\gamma^2}\right]\,, \tag{11}$$

Comparing with the hypergeometric differential equation as following

$$y(1-y)\frac{d^2\omega}{dy^2} + [c - (a + b + 1)y]\frac{d\omega}{dy} - ab\omega = 0\,, \tag{12}$$

we obtain the solution of Eq. (9) as

$$\psi(z) \sim {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}\,, \tag{13}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function and $(a)_k$ is Pochhammer symbol [12-14]. The new parameters in ${}_2F_1(a, b; c; z)$ should be

$$a = p + q - i\sqrt{ME}\ ;\ b = p + q + i\sqrt{ME}\ ;\ c = 1 + 2\sqrt{M[\gamma(A\gamma + B) - E]}\,, \tag{14}$$

The mathematical solutions of Eq. (8) is written as

$$\phi(z) = Nz^p(1-z)^q\,{}_2F_1(a, b; c; z)\,. \tag{15}$$

where the normalization constant N is obtained below. We now obtain the energy eigenvalues of the system in the next section.

A. Energy Spectrum

In order to obtain a physical solution for the wave functions, the parameter a in ${}_2F_1(a,b;c;z)$ should be $a=-n$ $(n=0,1,2,\ldots)$ which is the quantization rule of the system and gives us the energy eigenvalues as

$$E(n,\gamma,A,B)=-\frac{1}{4}\left[\frac{\gamma\hbar}{\sqrt{2m}}\left(n+\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8Am}{\hbar^2}}\right)-\frac{\sqrt{2m}}{\hbar}\frac{A\gamma+B}{n+\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8Am}{\hbar^2}}}\right]^2, \tag{16}$$

Firstly, we want to compare our results analytically for the case of "constant" mass. To achieve this aim, we introduce the principal quantum number as $N=n+1$. In this case, the energy eigenvalues are written as

$$E(N,\gamma,A,B)=-\frac{1}{4}\left[\frac{\gamma\hbar}{2\sqrt{2m}}\left(2N-1+\sqrt{1+\frac{8Am}{\hbar^2}}\right)-\frac{2\sqrt{2m}}{\hbar}\frac{A\gamma+B}{2N-1+\sqrt{1+\frac{8Am}{\hbar^2}}}\right]^2, \tag{17}$$

We obtain the following from the last equation

$$E(N,0,A,B)=-\frac{1}{4}\left[\frac{2\sqrt{2m}}{\hbar}\frac{B}{2N-1+\sqrt{1+\frac{8Am}{\hbar^2}}}\right]^2, \tag{18}$$

which is exactly the same with Eq. (15) given in Ref. [15]. Eq. (17) gives the result for "usual" Coulomb-like potential for the case of "constant" mass

$$E(N,0,0,B)=-\frac{2m}{\hbar^2}\frac{B^2}{4N^2}, \tag{19}$$

which is the same with Eq. (27) obtained in Ref. [15]. It is suitable now to give the result for the Coulomb-like potential for the case where the mass depends on spatially coordinate

$$E(N,\gamma,0,B)=\frac{1}{2}\gamma B-\frac{\gamma^2\hbar^2}{32m}n'^2-\frac{2m}{\hbar^2}\frac{B^2}{n'^2}. \tag{20}$$

with $n'=2N-1$.

Secondly, we summarize the numerical results in Table I. In general, the numerical analyse for such potentials are computed for diatomic molecules. So, we give the bound state energies for a diatomic molecule (CO molecule) in eV where the parameter values used here are taken from Ref. [16] inserting a new quantity $E_0 = \hbar^2/mr_0^2$ and, by comparing, we set the potential parameters as $A = D_e r_e^2$ and $B = 2D_e r_e$ (D_e is the dissociation energy and r_e is the equilibrium distance). By using Eq. (16), the binding energies are calculated four different values of parameter γ to see the effect of position-dependent mass on energy levels of the inverse square plus Coulomb-like potential. It is also seen in Table I that we give some numerical results for the potential for the constant mass case ($\gamma = 0$). There is an increasingly contribution of the parameter γ on the energy levels of the potential. Finally, we summarize some numerical energy eigenvalues obtaining from Eq. (20) for the Coulomb-like potential ($A = 0$) for both two cases of $\gamma \neq 0$ and $\gamma = 0$, respectively, in Table II. Here, our aim is just to giving an idea about the effect of parameter γ on the energy levels for the Coulomb problem, the results are obtained in atomic units. It is seen that the contribution of parameter γ increases while it's value increases. In Tables, we use only the values for γ falling into the range $[0, 1]$. The dependency of the energy eigenvalues on the displacement operator for the case where the γ-values greater than one are given in Figure I. We plot the variation of energy eigenvalue only for ground states for the inverse square plus Coulomb-like potential, and Coulomb-like potential, respectively, because the shape for the upper energy levels are similar. It is observed that the results in Fig. I are consistent with the ones given in Tables.

B. Normalization

The wave functions should be satisfy

$$\int_{-\infty}^{+\infty} |\phi(z)|^2 dz = 1\,, \tag{21}$$

By using the following identity for the hypergeometric functions for $|z| \to \infty$ [14]

$${}_2F_1(a, b; c; z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}\,(-z)^{-a} + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}\,(-z)^{-b}\,, \tag{22}$$

Eq. (21) is written as

$$2|N|^2 \int_0^{\infty} (z)^{2p}(1-z)^{2q} \left(\Gamma_1 z^n + \Gamma_2 z^{-(n+2p+2q)}\right)^2 dz = 1\,, \tag{23}$$

where

$$\Gamma_1 = (-1)^n\,\frac{\Gamma(2n+2p+2q)\Gamma(1+2p)}{\Gamma(n+2p+2q)\Gamma(1+n+2p)}\;;\;\Gamma_2 = (-1)^{-(n+2p+2q)}\,\frac{\Gamma(-2n-2p-2q)\Gamma(1+2p)}{\Gamma(-n)\Gamma(1-n-2q)}\,, \tag{24}$$

By defining a new variable such as $y = z/(1+z)$ we can use the integral equation [13, 14]

$$\int_0^1 t^{r-1}(1-t)^{r'-1}(1-tx)^{-r-r'}dt = B(r,r')\,{}_2F_1(r+r',r;r+r';x)\,, \tag{25}$$

where $B(a',b')$ is the Beta integral [12-14]. Using Eq. (25) gives us the normalization constant as

$$N = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{I_1+I_2+I_3}}\,, \tag{26}$$

where

$$\begin{aligned}
I_1 &= \Gamma_1^2 B(1+2n+2p,-1-2n-2p-2q)\,{}_2F_1(-2q,1+2n+2p;-2q;2)\,,\\
I_2 &= 2\Gamma_1\Gamma_2 B(1-2q,-1)\,{}_2F_1(-2q,1-2q;-2q;2)\,,\\
I_3 &= \Gamma_2^2 B(1-2n-2p-4q,-1-2n-2p-2q)\,{}_2F_1(-2q,1-2n-2p-2q;-2q;2)
\end{aligned} \tag{27}$$

III. CONCLUSION

Starting from the Schrödinger-like equation written in terms of the translation operator, we have analysed the changes of the bound states of a inverse square plus Coulomb-like potential. We have computed the corresponding normalized wave functions analytically. We have also given two tables, and a figure to see the variation of the bound states according to the parameter γ. We have found that our analytical results obtained for the bound states are in agreement with the ones obtained for the case where the mass is constant as $\gamma \to 0$.

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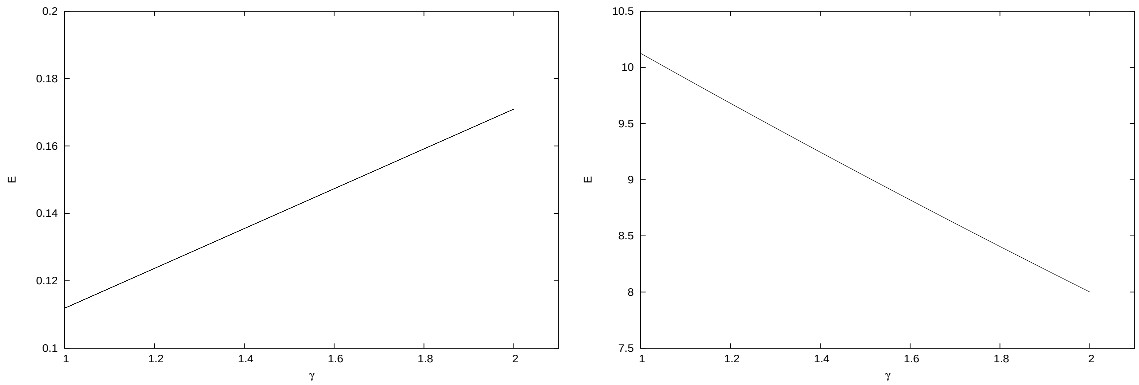
TABLE I: Energy eigenvalues for the inverse square plus Coulomb-like potential.

n	$\gamma = 0$	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 1.0$
0	0.051710	0.058521	0.082237	0.111846
1	0.153947	0.172279	0.241986	0.328809
2	0.254787	0.284476	0.399400	0.542203
3	0.354256	0.395141	0.554523	0.752090
4	0.452378	0.504302	0.707395	0.958530
5	0.549178	0.611985	0.858054	0.958530

TABLE II: Energy eigenvalues ($-E$) for the Coulomb-like potential ($B = 5$).

n	$\gamma = 0$	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 1.0$
0	12.5000	12.2513	11.2813	10.1250
1	3.12500	2.88000	2.00000	1.12500
2	1.38889	1.15014	0.42014	0.01389
3	0.78125	0.55125	0.03125	0.28125
4	0.50000	0.28125	0.03125	—
5	0.34722	0.14222	0.22222	—



(a) energy for inverse square plus Coulomb-like potential.

(b) energy for Coulomb-like potential ($-E$).

FIG. 1: The dependencies of energy eigenvalues for the inverse square plus Coulomb-like potential (left panel), and the Coulomb-like potential (right panel) on mixing parameter γ.