Scattering of a spinless particle by an asymmetric Hulthén potential within the effective mass formalism

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Abstract

Effective mass Klein-Gordon equation for the asymmetric Hulthén potential is solved in terms of hypergeometric functions. Results are obtained for the scattering and bound states with the position dependent mass and constant mass, as a special case. In both cases, we derive a condition for the existence of transmission resonance ($T = 1$). We also study how the transmission resonance depends on the particle energy and the shape of the external potential.

Keywords: Asymmetric Hulthén potential, Klein-Gordon equation, transmission and reflection coefficients, bound-states, transmission resonances, position-dependent mass.

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I. INTRODUCTION

Solution of the Schrödinger equation for an external potential for the bound and scattering states [1] is a fundamental problem. In the low-momentum limit $k \to 0$ ($E \to 0$), the transmission and reflection coefficients for a physical potential are well behaved at infinity in one dimension going to 0 and 1, respectively, unless the external potential supports a bound state for this limit [2]. In that case, the zero energy resonance (half-bound state) described by the not square integrable wave function is finite at infinity [1, 3]. So transmission coefficient goes to 1 (*unity*) while reflection coefficient goes to 0. This phenomenon is called as *transmission resonance* [4]. On the other hand, a condition for the existence of the transmission resonance in view of asymmetric potentials has been recently investigated in Refs. [5, 6] for the non-relativistic particles. The transmission resonance concept has been recently generalized to the relativistic case [7, 8, 9, 10]. Dombey and Kennedy [8] showed that in the low-momentum limit $k \to 0$, Dirac particles scattered by an external potential have half-bound states at $E = \pm m$ in contrast to non-relativistic particles where half-bound state occurs only at zero energy. Thus, we should speak of zero momentum resonances in the relativistic case rather than zero energy resonances [8]. Afterwards, the scattering and bound state solutions of the Dirac equation for the Woods-Saxon potential have been obtained in the low-momentum limit. Conditions for a zero momentum resonance (transmission resonance: $T = 1$) and supercriticality (as the particle bound state at $E = -m$) have been derived in Ref. [9]. After these pioneering studies, the transmission resonance and supercriticality for the relativistic/non-relativistic particles in an external potentials have been extensively discussed [3, 11, 12, 13, 14, 15, 16, 17, 18]. In one of these studies [13], the authors showed that the transmission coefficient obtained for the Klein-Gordon particle displays a behavior similar to that of the one obtained for the Dirac particle [8].

Recently, solving the non-relativistic/relativistic wave equations with external potential and obtaining the bound states have been widely studied in view of the position-dependent mass formalism. It has extensive applications in condensed matter physics and material science such as electronic properties of the semi-conductors [19], quantum dots [20], and quantum liquids [21]. Besides, the scattering problem in the position-dependent mass framework has been recently received much attentions [22, 23, 24, 25, 26] . The effective mass Dirac equation for the Coulomb field has been investigated in Ref. [22]. Dutra *et al.* have obtained exact solution of the Dirac equation for the inversely linear potential in the presence of the position-dependent mass [24].

In the present work, we investigate the transmission resonances for the Klein-Gordon particle scattered by the asymmetric Hulthén potential within the position-dependent mass formalism. The Hulthén potential [27] is one of the most significant short-range potential in physics and has been used in atomic physics, condensed matter, nuclear and particle physics, and chemical physics [28, 29, 30, 31]. The generalized Hulthén potential is a potential containing several potential forms such as usual Hulthén potential, Woods-Saxon potential, Cusp potential and Coulomb potential. Recently, Sogut [32] have obtained the exact solution of the one-dimensional Duffin-Kemmer-Petiau equation for the asymmetric Hulthén potential and investigated the bound and scattering states of vector bosons.

The asymmetric Hulthén potential is given in the following form [32]

$$V_{AHP} = V_0 \left[\frac{\Theta(-x)}{e^{-ax} - q} + \frac{\Theta(x)}{e^{bx} - \tilde{q}} \right] , \tag{1}$$

where V_0 is the strength of the potential, a, b, q (< 1) and $\tilde{q}$ (< 1) are the positive parameters related to the shape of potential. $\Theta(x)$ is the Heaviside step function. It is worth talking about that the asymmetric Hulthén potential transforms to the usual Hulthén potential for $a = b$ and $q = \tilde{q}$ and the asymmetric Cusp potential for $q = \tilde{q} = 0$. The form of the asymmetric Hulthén potential is shown in Fig. 1.

We take mass distribution as

$$m(x) = m_0 + m_1 \, f(x) \, , \tag{2}$$

where m_0 and m_1 are positive parameters and the function is given as $f(x) = \frac{\Theta(-x)}{e^{-ax} - q} + \frac{\Theta(x)}{e^{bx} - \tilde{q}}$. This form of the mass function makes it possible to solve the problem analytically.

The paper is organized as follows: In the next section, we find the exact solution of the Klein-Gordon equation in terms of hypergeometric functions. In section 3, transmission and reflection coefficients are obtained by using asymptotic behavior of the hypergeometric functions. In section 4, we investigate the bound states. Section 5 is devoted to discussions. Finally, we summarize the results in the last section.

II. EFFECTIVE MASS KLEIN-GORDON EQUATION FOR THE ASYMMETRIC HULTHÉN POTENTIAL

In $(1+1)$ dimensions, the time-independent Klein-Gordon equation with scalar $S(x)$ and vector $V(x)$ potentials in the presence of the effective mass can be written as [24]

$$\frac{d^2\Psi(x)}{dx^2} + \left\{[E - V(x)]^2 - [m(x) + S(x)]^2\right\} \Psi(x) = 0 \tag{3}$$

where E is the energy of the relativistic particle. Here, we take $\hbar = c = 1$ for the simplicity. We investigate the scattering and bound state solutions of the Eq. (3) by using the mass distribution given in Eq. (2) together with the following scalar $S(x)$ and vector $V(x)$ potentials

$$S(x) \;=\; S_0\, f(x)\,, \tag{4}$$

$$V(x) \;=\; V_0\, f(x)\,, \tag{5}$$

where S_0 and V_0 are positive parameters.

In order to find the scattering of a Klein-Gordon particle from the asymmetric Hultén potential in the presence of the effective mass, we first seek the solution of the Klein-Gordon eqution for $x < 0$. In that case, Eq. (3) becomes

$$\frac{d^2\Psi_L(x)}{dx^2} + \left\{\left[E - \frac{V_0}{e^{-ax} - q}\right]^2 - \left[m_0 + \frac{m_1 + S_0}{e^{-ax} - q}\right]^2\right\} \Psi_L(x) = 0. \tag{6}$$

Changing the variable $e^{-ax} = q/y$, Eq. (6) takes the following form

$$\begin{aligned} y(1-y)^2 \left(y\frac{d^2\Psi_L(y)}{dy^2} + \frac{d\Psi_L(y)}{dy}\right) + \Bigg\{\left[\left(E + \frac{V_0}{q}\right)^2 - \left(m_0 - \frac{m_1 + S_0}{q}\right)^2\right] \\ \times\frac{y^2}{a^2} - 2\left(E^2 - m_0^2 + \frac{EV_0}{q} + \frac{m_0(m_1 + S_0)}{q}\right)\frac{y}{a^2} + \frac{(E^2 - m_0^2)}{a^2}\Bigg\} \Psi_L(y) = 0. \end{aligned} \tag{7}$$

By setting $\Psi_L(y) = y^{\mu}(1-y)^{\nu}H(y)$ and substituting it into the above equation, one gets the hypergeometric equation [33]

$$y(1-y)\frac{d^2H(y)}{dy^2} + [2\mu + 1 - (2\mu + 2\nu + 1)y]\frac{dH(y)}{dy} - (\mu + \nu - \gamma)(\mu + \nu + \gamma)H(y) = 0 \tag{8}$$

where

$$\mu = \frac{ik}{a} \quad \text{with} \quad k = \sqrt{E^2 - m_0^2} \tag{9}$$

$$\nu = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{a^2 q^2}[V_0^2 - (m_1 + S_0)^2]} \tag{10}$$

$$\gamma = \frac{1}{a}\sqrt{\left(m_0 - \frac{m_1 + S_0}{q}\right)^2 - \left(E + \frac{V_0}{q}\right)^2}. \tag{11}$$

It is worth noting that $|E| > m$ ensures that k is real and V_0 is real and positive for the scattering states [9, 10]. One can write the solution of the Eq. (8) in terms of hypergeometric functions [33]

$$\begin{aligned} H(y) &= A_1 \; F(\mu + \nu - \gamma, \mu + \nu + \gamma, 1 + 2\mu, y) \\ &+ A_2 y^{-2\mu} \; F(-\mu + \nu - \gamma, -\mu + \nu + \gamma, 1 - 2\mu, y) \,. \end{aligned} \tag{12}$$

Thus, the left solution is obtained as follows

$$\begin{aligned} \Psi_L(y) &= A_1 y^{\mu}(1-y)^{\nu} \; F(\mu + \nu - \gamma, \mu + \nu + \gamma, 1 + 2\mu, y) \\ &+ A_2 y^{-\mu}(1-y)^{\nu} \; F(-\mu + \nu - \gamma, -\mu + \nu + \gamma, 1 - 2\mu, y). \end{aligned} \tag{13}$$

Let us investigate the scattering solution for $x > 0$. In that case, Eq. (3) becomes

$$\frac{d^2\Psi_R(x)}{dx^2} + \left\{ \left[E - \frac{V_0}{e^{bx} - \tilde{q}}\right]^2 - \left[m_0 + \frac{m_1 + S_0}{e^{bx} - \tilde{q}}\right]^2 \right\} \Psi_R(x) = 0. \tag{14}$$

Changing the variable $e^{bx} = \tilde{q}/z$ and setting $\Psi_R(z) = z^{\delta}(1-z)^{-\alpha}G(z)$, Eq. (14) yields

$$z(1-z)\frac{d^2G(z)}{dz^2} + [2\delta + 1 - (2\delta - 2\alpha + 1)z]\frac{dG(z)}{dz} - (\delta - \alpha - \beta)(\delta - \alpha + \beta)G(z) = 0 \tag{15}$$

where

$$\delta = \frac{ik}{b} \tag{16}$$

$$\alpha = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{b^2 \tilde{q}^2}[V_0^2 - (m_1 + S_0)^2]} \tag{17}$$

$$\beta = \frac{1}{b}\sqrt{\left(m_0 - \frac{m_1 + S_0}{\tilde{q}}\right)^2 - \left(E + \frac{V_0}{\tilde{q}}\right)^2}. \tag{18}$$

Eq. (15) is the hypergeometric equation and its general solution is given as [33]

$$\begin{aligned} G(z) &= A_3 \; F(\delta - \alpha - \beta, \delta - \alpha + \beta, 1 + 2\delta, z) \\ &+ A_4 z^{-2\delta} \; F(-\delta - \alpha - \beta, -\delta - \alpha + \beta, 1 - 2\delta, z). \end{aligned} \tag{19}$$

Finally, the right solution can be written as

$$\begin{aligned}\Psi_R(z) \;&=\; A_3 z^{\delta}(1-z)^{-\alpha}\; F(\delta-\alpha-\beta,\delta-\alpha+\beta,1+2\delta,z) \\ &+\; A_4 z^{-\delta}(1-z)^{-\alpha}\; F(-\delta-\alpha-\beta,-\delta-\alpha+\beta,1-2\delta,z).\end{aligned} \tag{20}$$

III. TRANSMISSION AND REFLECTION COEFFICIENTS

In order to obtain the transmission and reflection coefficients, we have to investigate the asymptotic behavior of the left and right solutions. As $x\to-\infty$, then $y\to 0$, $(1-y)^{\nu}\to 1$ and $y^{\pm\mu}\to q^{\pm\mu}e^{\pm a\mu x}$ which leads to

$$\Psi_L(x\to-\infty)\sim A_1 q^{\mu}e^{ikx}+A_2 q^{-\mu}e^{-ikx} \tag{21}$$

where following property of the hypergeometric functions is used: ${}_2F_1(a,b;c;0)=1$ [33]. Asymptotic behavior of the left solution can be written in terms of incident Ψ_{inc} and reflected Ψ_{ref} waves in the limit $x\to-\infty$. Then, it is seen from Eq. (21) that Ψ_{inc} and Ψ_{ref} behave like a plane wave travelling to the right and left, respectively.

On the other hand, as $x\to\infty$, then $z\to 0$, $(1-z)^{-\alpha}\to 1$ and $z^{\pm\delta}\to\tilde{q}^{\pm\delta}e^{\mp b\delta x}$ which leads to

$$\Psi_R(x\to\infty)\sim A_3 q^{\delta}e^{-b\delta x}+A_4 q^{-\delta}e^{b\delta x}. \tag{22}$$

To obtain the transmitted wave traveling from left to right, we set $A_3=0$. So, we get

$$\Psi_R(x\to\infty)\sim A_4 q^{-\delta}e^{ikx}. \tag{23}$$

Thus, from now on, the following right solution is used:

$$\Psi_R(z)=A_4 z^{-\delta}(1-z)^{-\alpha}\; F(-\delta-\alpha-\beta,-\delta-\alpha+\beta,1-2\delta,z). \tag{24}$$

Then, the transmission and reflection coefficients can be expressed as [11]

$$T=\left|\frac{\Psi_{trans}}{\Psi_{inc}}\right|^2=\left|\frac{A_4}{A_1}\right|^2 \tag{25}$$

$$R=\left|\frac{\Psi_{ref}}{\Psi_{inc}}\right|^2=\left|\frac{A_2}{A_1}\right|^2. \tag{26}$$

Let's use the following continuity conditions on the wave functions and their first derivatives at $x=0$ to obtain explicit expressions for R and T

$$\Psi_R(x=0)\;=\;\Psi_L(x=0) \tag{27}$$

$$\frac{d}{dx}\Psi_R(x=0)\;=\;\frac{d}{dx}\Psi_L(x=0). \tag{28}$$

Using Eqs. (27) and (28) with Eqs. (13) and (20), one can obtain

$$A_4 C_1 F_1 = A_1 C_2 F_2 + A_2 C_3 F_3, \tag{29}$$

$$A_4 C_1 b(C_4 F_4 - C_5 F_1) = A_1 C_2 a(C_6 F_2 + C_7 F_5) - A_2 C_3 a(C_8 F_3 - C_9 F_6) \tag{30}$$

where we have used the following property: $\frac{d}{dx} F(a,b;c;x) = \frac{ab}{c} F(a+1,b+1;c+1;x)$ [33]. Here, some new coefficients and definitions are used to shorten the Eqs. (29) and (30) and they are listed in Table 1. Consequently, transmission and reflection coefficients are, respectively, calculated as follows

$$T = \left|\frac{A_4}{A_1}\right|^2 = \left|\frac{aC_2[F_2(C_8F_3 - C_9F_6) + F_3(C_6F_2 + C_7F_5)]}{C_1[aF_1(C_8F_3 - C_9F_6) + bF_3(C_4F_4 - C_5F_1)]}\right|^2 \tag{31}$$

$$R = \left|\frac{A_2}{A_1}\right|^2 = \left|\frac{C_2[aF_1(C_6F_2 + C_7F_5) - bF_2(C_4F_4 - C_5F_1)]}{C_3[aF_1(C_8F_3 - C_9F_6) + bF_3(C_4F_4 - C_5F_1)]}\right|^2. \tag{32}$$

It is significant to discuss the condition for existence of the transmission resonances in the scattering states. The transmission resonance condition can be derived by setting $R = 0$ (or $T = 1$) which means that there is no reflected wave. Then, considering Eq. (32), the condition for the transmission resonances is obtained as

$$aF_1\left[C_6F_2 + C_7F_5\right] - bF_2\left[C_4F_4 - C_5F_1\right] = 0. \tag{33}$$

IV. BOUND STATES

In this section, we investigate the bound state solution of the effective mass Klein-Gordon equation for the asymmetric Hulthén potential. The scalar (4) and vector (5) asymmetric Hulthén potentials give a bound state $|E| < m$ as they become attractive ($V_0 \to -V_0$ and $S_0 \to -S_0$) [34]. Then, in the bound state case, Eq. (3) yields the following second-order differential equations for $x < 0$ and $x > 0$, respectively

$$\frac{d^2\Phi_L(x)}{dx^2} + \left\{\left[E + \frac{V_0}{e^{-ax} - q}\right]^2 - \left[m_0 + \frac{m_1 - S_0}{e^{-ax} - q}\right]^2\right\}\Phi_L(x) = 0 \tag{34}$$

$$\frac{d^2\Phi_R(x)}{dx^2} + \left\{\left[E + \frac{V_0}{e^{bx} - \tilde{q}}\right]^2 - \left[m_0 + \frac{m_1 - S_0}{e^{bx} - \tilde{q}}\right]^2\right\}\Phi_R(x) = 0. \tag{35}$$

Following the same procedure as above, we obtain the solutions of the Klein-Gordon equation for both $x < 0$ and $x > 0$ in terms of hypergeometric functions as follow

$$\Phi_L(y) = B_1 y^{\tilde{\mu}}(1-y)^{\tilde{\nu}} F(\tilde{\mu} + \tilde{\nu} - \tilde{\gamma}, \tilde{\mu} + \tilde{\nu} + \tilde{\gamma}, 1 + 2\tilde{\mu}, y)$$
$$+ B_2 y^{-\tilde{\mu}}(1-y)^{\tilde{\nu}} F(-\tilde{\mu} + \tilde{\nu} - \tilde{\gamma}, -\tilde{\mu} + \tilde{\nu} + \tilde{\gamma}, 1 - 2\tilde{\mu}, y) \tag{36}$$

$$\Phi_R(z) = B_3 z^{\tilde{\delta}}(1-z)^{-\tilde{\alpha}}\ F(\tilde{\delta}-\tilde{\alpha}-\tilde{\beta}, \tilde{\delta}-\tilde{\alpha}+\tilde{\beta}, 1+2\tilde{\delta}, z) + B_4 z^{-\tilde{\delta}}(1-z)^{-\tilde{\alpha}}\ F(-\tilde{\delta}-\tilde{\alpha}-\tilde{\beta}, -\tilde{\delta}-\tilde{\alpha}+\tilde{\beta}, 1-2\tilde{\delta}, z) \quad (37)$$

where

$$\tilde{\mu} = \frac{1}{a}\sqrt{m_0^2 - E^2}, \quad \tilde{\delta} = \frac{1}{b}\sqrt{m_0^2 - E^2} \quad (38)$$

$$\tilde{\nu} = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{a^2 q^2}[V_0^2 - (m_1 - S_0)^2]}\right), \ \tilde{\alpha} = \frac{1}{2}\left(-1 + \sqrt{1 - \frac{4}{b^2 \tilde{q}^2}[V_0^2 - (m_1 - S_0)^2]}\right) \quad (39)$$

$$\tilde{\gamma} = \frac{1}{a}\sqrt{\left(m_0 - \frac{m_1 - S_0}{q}\right)^2 - \left(E - \frac{V_0}{q}\right)^2}, \ \tilde{\beta} = \frac{1}{b}\sqrt{\left(m_0 - \frac{m_1 - S_0}{\tilde{q}}\right)^2 - \left(E - \frac{V_0}{\tilde{q}}\right)^2} \quad (40)$$

In order to obtain the bound state solutions requiring the vanishing of the wave functions (36) and (37) at $\pm\infty$, the following regular solutions are chosen for $\Phi_L(y)$ and $\Phi_R(z)$, respectively

$$\Phi_L(y) = B_1 y^{\tilde{\mu}}(1-y)^{\tilde{\nu}}\ F(\tilde{\mu}+\tilde{\nu}-\tilde{\gamma}, \tilde{\mu}+\tilde{\nu}+\tilde{\gamma}, 1+2\tilde{\mu}, y) \quad (41)$$

$$\Phi_R(z) = B_3 z^{\tilde{\delta}}(1-z)^{-\tilde{\alpha}}\ F(\tilde{\delta}-\tilde{\alpha}-\tilde{\beta}, \tilde{\delta}-\tilde{\alpha}+\tilde{\beta}, 1+2\tilde{\delta}, z). \quad (42)$$

Imposing the continuity conditions on $\Phi_L(y)$ and $\Phi_R(z)$ at $x = 0$, we obtain

$$B_1\tilde{C}_1\tilde{F}_1 - B_3\tilde{C}_2\tilde{F}_2 = 0 \quad (43)$$

$$B_1 a\tilde{C}_3(\tilde{C}_4\tilde{F}_1 + \tilde{C}_5\tilde{F}_3) - B_3 b\tilde{C}_6(\tilde{C}_7\tilde{F}_2 + \tilde{C}_8\tilde{F}_4) = 0. \quad (44)$$

Here, we use some abbreviations given in Table 2. From the last two equations, one can calculate the energy eigenvalue equation as follows

$$a\tilde{C}_2\tilde{C}_3\tilde{F}_2(\tilde{C}_4\tilde{F}_1 + \tilde{C}_5\tilde{F}_3) - b\tilde{C}_1\tilde{C}_6\tilde{F}_1(\tilde{C}_7\tilde{F}_2 + \tilde{C}_8\tilde{F}_4) = 0. \quad (45)$$

The energy eigenvalues of the bound states can be obtained in terms of potential strength V_0 by solving the Eq. (45) numerically. The dependence of the energy eigenvalues on V_0 is given in Fig. 2. From Fig. 2, one can observe that bound state energy of the Klein-Gordon particle decreases with increasing potential strength V_0. Finally, this energy takes the value of negative mass of the particle which means that the bound state joins the negative-energy continuum [35, 14]. This result agrees with previous ones [11, 14]. On the other hand, the potential is called critical when a discrete energy level crosses the value $E = -m$ and joins the negative energy continuum.

V. DISCUSSIONS

A. Low-momentum limit

In the low-momentum limit ($E \to -m$) which leads to $\mu = \delta = 0$, transmission resonance condition (33) becomes

$$\begin{aligned} & a\ F(-\alpha-\beta,-\alpha+\beta,1,\tilde{q}) \times \\ & \left[q(\nu^2-\gamma^2)\ F(\nu-\gamma+1,\nu+\gamma+1,2,q) - \frac{q\nu}{1-q}\ F(\nu-\gamma,\nu+\gamma,1,q)\right] - \\ & b\ F(\nu-\gamma,\nu+\gamma,1,q) \times \\ & \left[\tilde{q}(\beta^2-\alpha^2)\ F(1-\alpha-\beta,1-\alpha+\beta,2,\tilde{q}) - \frac{\tilde{q}\alpha}{1-\tilde{q}}\ F(-\alpha-\beta,-\alpha+\beta,1,\tilde{q})\right] = 0. \end{aligned} \quad (46)$$

On the other hand, bound state energy equation (45) turns into the following form in the low-momentum limit ($\tilde{\mu} = \tilde{\delta} = 0$):

$$\begin{aligned} & a\ F(-\tilde{\alpha}-\tilde{\beta},-\tilde{\alpha}+\tilde{\beta},1,\tilde{q}) \times \\ & \left[q(\tilde{\nu}^2-\tilde{\gamma}^2)\ F(\tilde{\nu}-\tilde{\gamma}+1,\tilde{\nu}+\tilde{\gamma}+1,2,q) - \frac{q\tilde{\nu}}{1-q}\ F(\tilde{\nu}-\tilde{\gamma},\tilde{\nu}+\tilde{\gamma},1,q)\right] - \\ & b\ F(\tilde{\nu}-\tilde{\gamma},\tilde{\nu}+\tilde{\gamma},1,q) \times \\ & \left[\tilde{q}(\tilde{\beta}^2-\tilde{\alpha}^2)\ F(1-\tilde{\alpha}-\tilde{\beta},1-\tilde{\alpha}+\tilde{\beta},2,\tilde{q}) - \frac{\tilde{q}\tilde{\alpha}}{1-\tilde{q}}\ F(-\tilde{\alpha}-\tilde{\beta},-\tilde{\alpha}+\tilde{\beta},1,\tilde{q})\right] = 0. \end{aligned} \quad (47)$$

Comparing Eq. (46) with Eq. (47) and considering Eqs. (10), (11), (17) and (18), it is not difficult to see that transmission resonance condition is reduced to the bound-energy condition after the transformations $V_0 \to -V_0$ and $S_0 \to -S_0$ in the low-momentum limit which means that the asymmetric Hulthén potential supports a zero-momentum (half-bound) state in the presence of the effective mass and also constant mass as well.

Eq. (47) can be used to calculate the value of the critical potential ($E = -m$). This value is found to be $V_c = 1.89014$ for $a = 1$, $b = 0.8$, $q = 0.5$, $\tilde{q} = 0.4$, $m = 1$, $m_1 = 0$, $S = 0$ and $V_c = 2.0512$ for $a = 1$, $b = 0.8$, $q = 0.5$, $\tilde{q} = 0.4$, $m = 1$, $m_1 = 0.2$, $S = 0$. From this point of view, we can say that the value of the critical potential increases in the presence of the effective mass compared to the constant mass case.

B. Unitary condition

Fig. 3 shows the existence of the unitary condition for the asymmetric Hulthén potential within the effective mass formalism. From Fig. 3, we can see that unitary condition ($T+R=1$) is valid for both position dependent (right plot) and constant (left plot) mass cases.

C. Transmission resonances

Figs. 4-9 display the transmission coefficients for the asymmetric Hulthén potential. All of the figures show that transmission resonances for the asymmetric Hulthén potential exist for both of the effective mass and constant mass cases. Fig. 4 presents the behavior of the transmission coefficient versus the scalar particles energy. In Fig. 4, solid line represents the constant mass case for the usual Hulthén potential displayed in Fig. 3 in Ref. [18]. We can readily see from Fig. 4 that the width of the transmission resonances is sensitive to the effective mass parameter m_1. The peaks of the resonances become narrower when m_1 increases. Besides, in the case of the effective mass, the first resonance peak ($T=1$) appears at lower energy compared to the constant mass case.

Fig. 5 displays the transmission coefficients as a function of potential strength V_0. In Ref. [13], it is shown that transmission resonances for the Klein-Gordon particles in the presence of the Woods-Saxon potential vanishes for $E-m<V_0<E+m$ and they appear for $V_0>E+m$. However, from Fig. 5, we can see that transmission resonances appear all range of the asymmetric Hulthén potential which means that there is no V_0 values making the asymmetric Hulthén potential entirely impenetrable. The reason is that there is no way to reduce the asymmetric Hulthén potential to square well. Thus, asymmetric Hulthén potential for the Klein-Gordon particle is completely penetrable as in the vector particle case [32]. The result given in Fig. 5 with solid line ($m_1=0$ constant mass) also agrees with the one presented in Fig. 4 in Ref. [18]. Effects of the position-dependent mass on the transmission coefficients are represented with the other three lines. From these three lines, it is easy to conclude that transmission resonances also appear in all range of the asymmetric Hulthén potential in the presence of the effective mass and resonance peaks become narrower and shorter with increasing the m_1.

D. Effect of the potential parameters on the transmission resonances

From the Figs. 6-8, it is clear that intensities of the transmission resonances as well as the width of the resonance peaks depend on the shape of the external potential. From the Fig. 6, one can observe that the dependence of the transmission coefficient on the energy of the Klein-Gordon particle is the same for both $a > b$ and $b > a$. The reason can be found by considering the Fig. 1. Based on the left plot of the Fig. 1, one can readily notice that the height and the width of the potential barrier remains the same whether $a > b$ or $b > a$. However, comparing the Fig. 6 with the Fig. 4, we can conclude that the intensity of the resonance peaks increases with decreasing a (or b) for $q = \tilde{q}$. On the other hand, the first transmission resonance peak appears at smaller values of the Klein-Gordon particles energy in the presence of the position-dependent mass.

Relationship between potential parameters q and $\tilde{q}$ that define the shape of the potential (1) and transmission coefficient is given in the Fig. 7. From the Fig. 7, we can see that existence of the transmission resonances depends on q and $\tilde{q}$. However, it should be noticed that the form of the transmission resonances remain the same in both $q > \tilde{q}$ and $q < \tilde{q}$. This can be explained by considering the change in the hight and the width of asymmetric Hulthén potential as shown in middle plot of the Fig. 1. From this plot, we can conclude that magnitudes of the hight and the width of the asymmetric Hulthén potential remains the same for $q > \tilde{q}$ and $q < \tilde{q}$. However, by considering the Figs. 7 and 8, it is seen that the number of the transmission resonances increases with increasing the q and $\tilde{q}$.

We also investigate the energy dependence of the transmission resonances for both $a > b > q > \tilde{q}$ and $a < b < q < \tilde{q}$. The results are given in the Fig. 8. From the left plot of the Fig. 8, it is concluded that the intensities and widths of the resonance peaks decrease as the potential parameters a and b are bigger than the q and $\tilde{q}$. The reason can be found by considering the right plot of the Fig. 1. This plot gives that the hight and the width of the asymmetric Hulthén potential (1) decrease when $a > b > q > \tilde{q}$. On the other hand, the Fig. 8 shows that if the a and b are smaller than the q and $\tilde{q}$, then the number of transmission resonance peaks increases. Based on the Figs. 5-8, we can conclude that existence of the transmission resonances as well as the intensity and width of the resonance peaks depend on shape of the asymmetric Hulthén potential. Besides, we can also see from the Figs. 5-8 that the Klein-Gordon equation within the position-dependent mass formalism exhibits transmission resonance in the presence of the asymmetric Hulthén potential.

E. Effect of the unequal scalar and vector potentials on the transmission resonances

The Fig. 9 displays the effect of the unequal scalar and vector potentials on the transmission resonances. From the Fig. 9, one can observe that transmission resonance peaks disappear when $S_0 \neq 0$ and it does not matter whether $V_0 = S_0$ or $V_0 < S_0$.

VI. CONCLUSIONS

In the present study, exact solution of the one-dimensional effective mass Klein-Gordon equation for the asymmetric Hulthén potential has been found in terms of hypergeometric functions. Considering the asymptotic behavior of the hypergeometric functions and using the continuity of the wave functions, we have obtained the scattering and bound states of the Klein-Gordon particle. Then, the condition for the existence of the transmission resonances $(T = 1, R = 0)$ is derived for the position-dependent mass and constant mass, as a special case. From the Figs. 4-8, it has been observed that the intensity and width of transmission resonance peaks depend on the shape and the strength V_0 of the external potential. Based on the Figs. 3-8, we have concluded that asymmetric and symmetric Hulthén potentials are entirely penetrable for all values of potential strength V_0 since asymmetric and symmetric Hulthén potentials cannot be reduced to square well. In the low-momentum limit, it has been shown that asymmetric Hulthén potential supports zero-momentum (half-bound) states. On the other hand, our results show that the Klein-Gordon equation exhibits the transmission resonances in the presence of the position-dependent mass and form of the resonances depends on the effective mass parameter m_1 as well as the shape and the strength of the external potential. Furthermore, asymmetric Hulthén potential has a general form and reduces to the well-known potential such as usual Hulthén and Cusp potentials. Thus, our results contain the scattering and bound state solutions of the scalar particles for the Cusp and usual Hulthén potentials. Finally, it should be mentioned that these results can be useful for understanding the behavior of elementary particles and nuclei in view of effective mass.

VII. ACKNOWLEDGMENTS

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TABLE I: Table for coefficients and definitions used to calculate the T and R.

C_1	$\tilde{q}^{-\delta}(1-\tilde{q})^{-\alpha}$	C_6	$\mu - \frac{\nu q}{1-q}$
C_2	$q^{\mu}(1-q)^{\nu}$	C_7	$q\frac{(\mu+\nu-\gamma)(\mu+\nu+\gamma)}{1+2\mu}$
C_3	$q^{-\mu}(1-q)^{\nu}$	C_8	$\mu + \frac{\nu q}{1-q}$
C_4	$\tilde{q}\frac{(\delta+\alpha+\beta)(\beta-\delta-\alpha)}{1-2\delta}$	C_9	$q\frac{(\nu-\mu-\gamma)(\nu-\mu+\gamma)}{1-2\mu}$
C_5	$\frac{\alpha\tilde{q}}{1-\tilde{q}} - \delta$		
F_1	$F(-\delta-\alpha-\beta, -\delta-\alpha+\beta; 1-2\delta; \tilde{q})$	F_4	$F(1-\delta-\alpha-\beta, 1-\delta-\alpha+\beta; 2-2\delta; \tilde{q})$
F_2	$F(\mu+\nu-\gamma, \mu+\nu+\gamma; 1+2\mu; q)$	F_5	$F(\mu+\nu-\gamma+1, \mu+\nu+\gamma+1; 2+2\mu; q)$
F_3	$F(-\mu+\nu-\gamma, -\mu+\nu+\gamma; 1-2\mu; q)$	F_6	$F(1-\mu+\nu-\gamma, 1-\mu+\nu+\gamma; 2-2\mu; q)$

TABLE II: Table for coefficients and definitions used to calculate the bound-state energy.

$\tilde{C}_1$	$q^{\tilde{\mu}}(1-q)^{\tilde{\nu}}$	$\tilde{C}_5$	$\frac{(\tilde{\mu}+\tilde{\nu}-\tilde{\gamma})(\tilde{\mu}+\tilde{\nu}+\tilde{\gamma})}{1+2\tilde{\mu}}$
$\tilde{C}_2$	$\tilde{q}^{\tilde{\delta}}(1-\tilde{q})^{-\tilde{\alpha}}$	$\tilde{C}_6$	$-\tilde{q}^{\tilde{\delta}+1}(1-\tilde{q})^{-\tilde{\alpha}}$
$\tilde{C}_3$	$q^{\tilde{\mu}+1}(1-q)^{\tilde{\nu}}$	$\tilde{C}_7$	$\frac{\tilde{\alpha}}{1-\tilde{q}} + \frac{\tilde{\delta}}{\tilde{q}}$
$\tilde{C}_4$	$\frac{\tilde{\mu}}{q} - \frac{\tilde{\nu}}{1-q}$	$\tilde{C}_8$	$\frac{(\tilde{\delta}-\tilde{\alpha}-\tilde{\beta})(\tilde{\beta}+\tilde{\delta}-\tilde{\alpha})}{1+2\tilde{\delta}}$
$\tilde{F}_1$	$F(\tilde{\mu}+\tilde{\nu}-\tilde{\gamma}, \tilde{\mu}+\tilde{\nu}+\tilde{\gamma}; 1+2\tilde{\mu}; q)$	$\tilde{F}_3$	$F(\tilde{\mu}+\tilde{\nu}-\tilde{\gamma}+1, \tilde{\mu}+\tilde{\nu}+\tilde{\gamma}+1; 2+2\tilde{\mu}; q)$
$\tilde{F}_2$	$F(\tilde{\delta}-\tilde{\alpha}-\tilde{\beta}, \tilde{\delta}-\tilde{\alpha}+\tilde{\beta}; 1+2\tilde{\delta}; \tilde{q})$	$\tilde{F}_4$	$F(1+\tilde{\delta}-\tilde{\alpha}-\tilde{\beta}, 1+\tilde{\delta}-\tilde{\alpha}+\tilde{\beta}; 2+2\tilde{\delta}; \tilde{q})$

FIG. 1: The form of the asymmetric Hulthén potential with $V_0 = 1$.

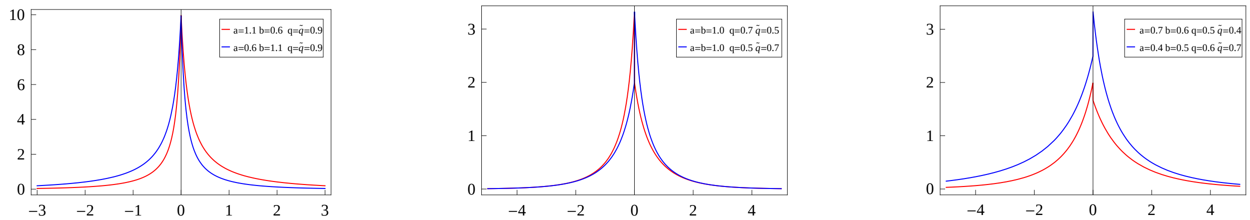
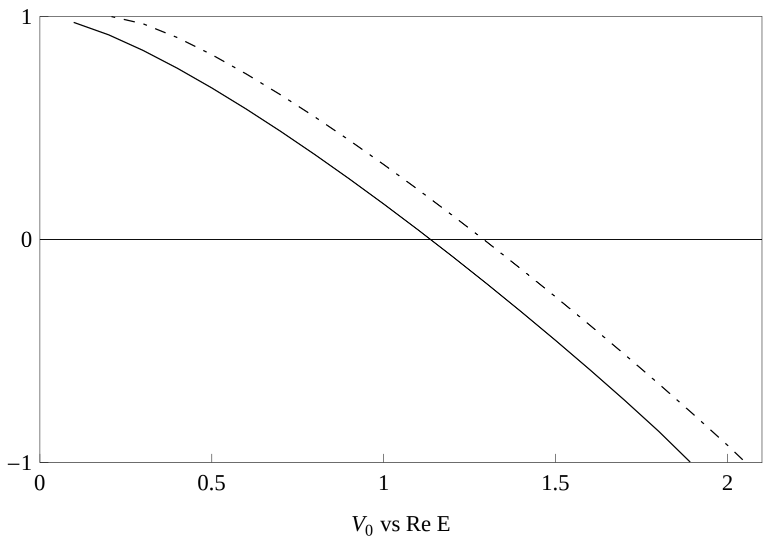


FIG. 2: Energy of the lowest bound-state vs potential strength in the presence of the position-independent(solid line)/dependent (dashed line) mass cases.



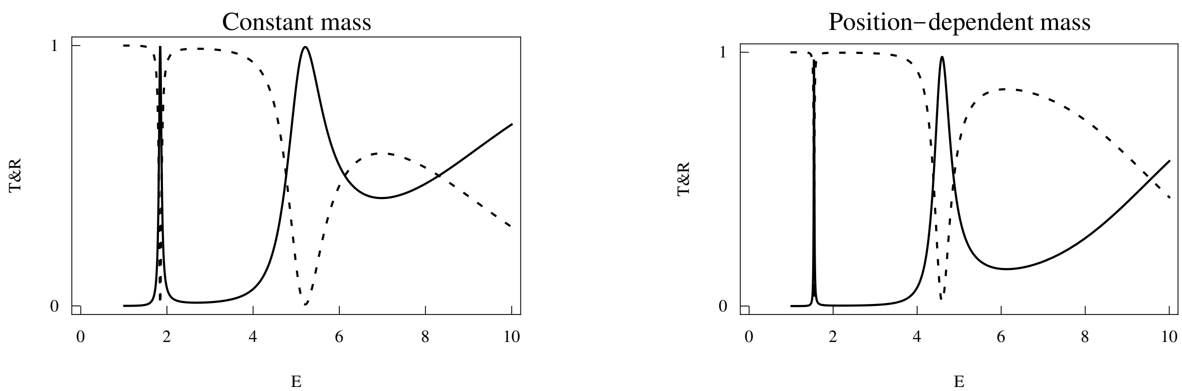


FIG. 3: T (solid line) and R (dashed line) coefficients for $a = 0.8$, $q = 0.5$, $b = 0.9$, $\tilde{q} = 0.6$, $m_0 = 1$, $S_0 = 0$ and $V_0 = 4$ where $m_1 = 0.0$ and $m_1 = 0.5$ for the left and right plots, respectively.

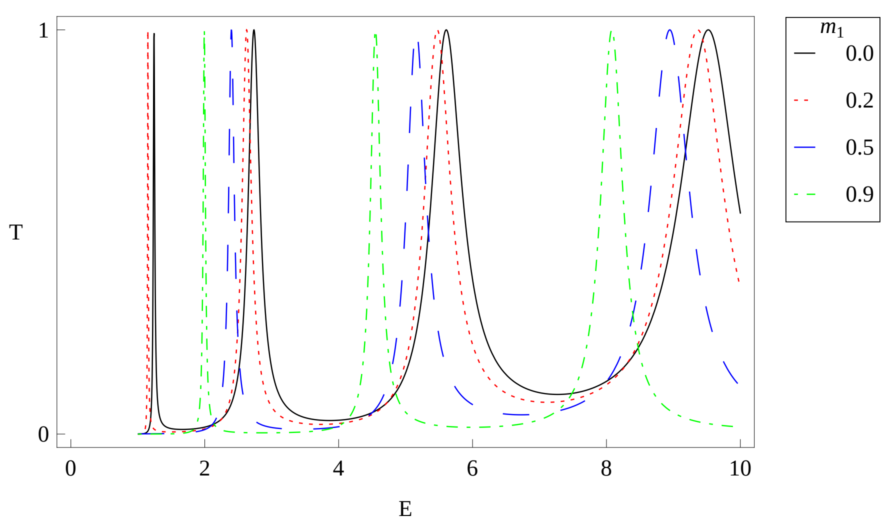


FIG. 4: Transmission coefficients for constant mass (solid line) and position-dependent mass (dashed line) with $q = 0.9$, $\tilde{q} = 0.9$, $m_0 = 1$, $m_1 = 0.5$, $S_0 = 0$ and $V_0 = 4$. We also take $a = 1.1$ and $b = 0.6$ for the left plot and $a = 0.6$ and $b = 1.1$ for the right plot.

FIG. 5: Transmission coefficient vs energy for $a = 1$, $b = 1$, $q = 0.9$, $\tilde{q} = 0.9$, $m_0 = 1$, $S_0 = 0$, $V_0 = 4$.

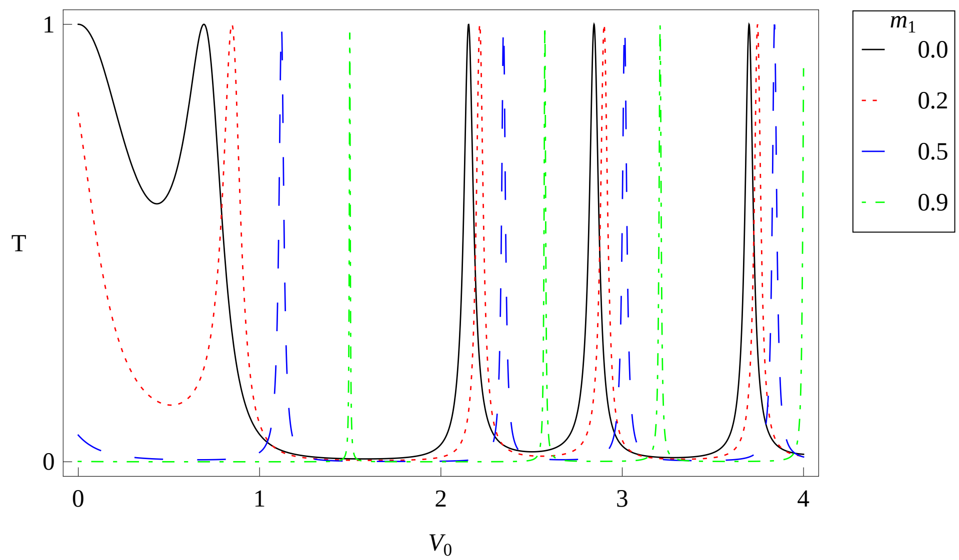


FIG. 6: Transmission coefficient vs potential strength for $a = 1$, $b = 1$, $q = 0.9$, $\tilde{q} = 0.9$, $m_0 = 1$, $S_0 = 0$, $E = 2m_0$.

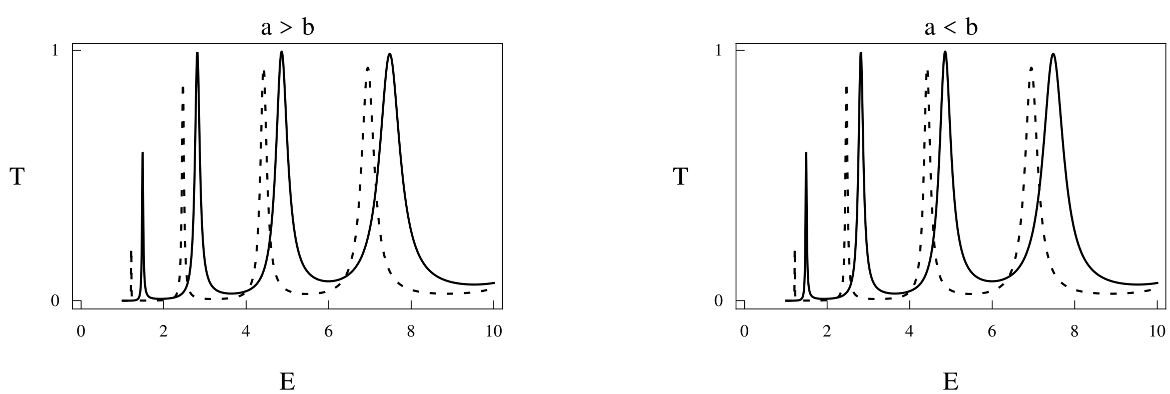


FIG. 7: Transmission coefficients for constant mass (solid line) and position-dependent mass (dashed line) with $a = 1$, $b = 1$, $m_0 = 1$, $m_1 = 0.5$, $S_0 = 0$ and $V_0 = 4$. We also take $q = 0.6$ and $\tilde{q} = 0.5$ for the left plot and $q = 0.5$ and $\tilde{q} = 0.6$ for the right plot.

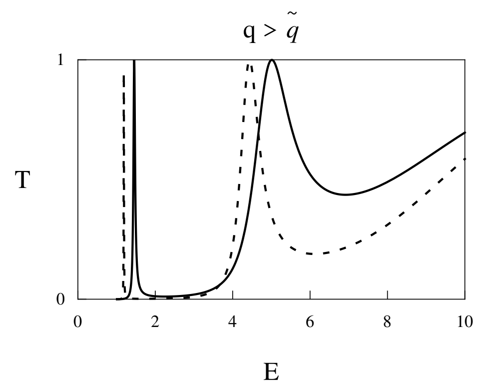


FIG. 8: Transmission coefficients for constant mass (solid line) and position-dependent mass (dashed line) with $a = 0.7$, $b = 0.6$, $q = 0.5$, $\tilde{q} = 0.4$, $m_0 = 1$, $m_1 = 0.5$, $S_0 = 0$ and $V_0 = 4$ for the left plot and $a = 0.4$, $b = 0.5$, $q = 0.6$, $\tilde{q} = 0.7$, $m_0 = 1$, $m_1 = 0.5$, $S_0 = 0$ and $V_0 = 4$ for the right plot.

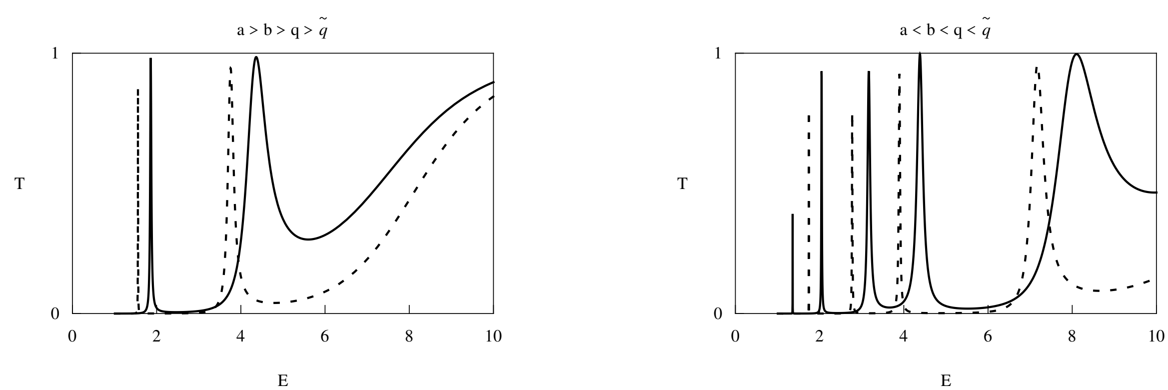


FIG. 9: Transmission coefficients in the constant mass (solid line) case and position-dependent mass (dashed line) case with $a = b = q = \tilde{q} = 0.5$, $m_0 = 1$, $m_1 = 0.5$ for $S_0 = V_0 = 0.5$ (the left plot) and for $S_0 = 0.4$, $V_0 = 0.6$ (the right plot).