

Quantum Mechanical Treatment of the Problem of Constraints in Nonextensive Formalism Revisited

G. B. Bağcı^a, Altuğ Arda^b, Ramazan Sever^{c*}

^a *Department of Physics, University of North Texas,
P.O. Box 311427, Denton, TX 76203-1427, USA*

^b *Physics Education, Hacettepe University, 06532,
Ankara, Turkey*

^c *Department of Physics,
Middle East Technical University, 06531,
Ankara, Turkey*

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The purity of Werner state in nonextensive formalism associated with two different constraints has been calculated in a previous paper by G. B. Bağcı et al. [G. B. Bağcı et al. , Int. J. Mod. Phys. **20**, 2085 (2006)]. Two different results have been obtained corresponding to ordinary probability and escort probability. The former has been shown to result in negative values thereby leading authors to deduce the advantage of escort probabilities over ordinary probabilities. However, these results have been only derived for a limited interval of q values which lie between 0 and 1. In this paper, we solve the same problem for all values of nonextensive index q by using a perturbative approach and show that the simultaneous use of both types of constraints is necessary in order to obtain the solution for whole spectrum of nonextensive index. In this sense, the existence of these different constraints in nonextensive formalism must not be seen as a deficiency in the formalism but rather must be welcomed as a means of providing solutions for all values of parameter q .

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I. INTRODUCTION

A nonextensive generalization of the standard Boltzmann-Gibbs (BG) entropy has been proposed by C. Tsallis in 1988¹⁻⁴. This new definition of entropy is given by

$$S_q = k \frac{1 - \sum_{i=1}^W p_i^q}{q - 1}, \quad (1)$$

where k is a positive constant which becomes the usual Boltzmann constant in the limit $q \rightarrow 1$, p_i is the probability of the system in the i th microstate, W is the total number of the configurations of the system. The entropic index q is a real number, which characterizes the degree of nonextensivity as can be seen from the following pseudo-additivity rule:

$$S_q(A + B)/k = [S_q(A)/k] + [S_q(B)/k] + (1 - q)[S_q(A)/k][S_q(B)/k], \quad (2)$$

where A and B are two independent systems i.e., $p_{ij}(A+B) = p_i(A)p_j(B)$. As $q \rightarrow 1$, the nonextensive entropy definition in Eq. (1) becomes

$$S_{q \rightarrow 1} = -k_B \sum_{i=1}^W p_i \ln p_i, \quad (3)$$

*Corresponding Author
; Electronic address: sever@newton.physics.metu.edu.tr

which is the usual BG entropy. This means that the definition of nonextensive entropy contains BG statistics as a special case. The cases $q < 1$, $q > 1$ and $q = 1$ correspond to superextensivity, subextensivity and extensivity, respectively.

The nonextensive formalism has been used successfully to investigate earthquakes⁵, models of fracture roughness⁶, entropy production⁷, voltage-gated ion channels⁸ and climatological models⁹.

The outline of the paper is as follows: In Section II, we use quantum divergences in the nonextensive formalism in order to calculate the purity of Werner state within perturbative approach and analyze the role of different intervals of the values of nonextensive parameter q . We show that each of them provides us with a different answer and is complementary of one another. We summarize the results in Section III.

II. BREGMAN DIVERGENCE VERSUS CSISZÁR DIVERGENCE

The physical meaning of divergence which is also called relative entropy in the literature is the free energy difference in ordinary (extensive) case.

$$K(\rho \parallel \sigma) = Tr[\rho(\ln \rho - \ln \sigma)]. \quad (4)$$

Quantum divergence has the same physical meaning in nonextensive formalism as in the ordinary case explained above. Additionally, the nonextensive divergence of Csiszár type¹⁰ has been used by Abe and Rajagopal to show an expected violation of the second law if the nonextensive index q is not in the range $(0, 2]$ ¹¹.

$$I_q(\rho \parallel \sigma) = \frac{1}{q-1} [Tr(\rho^q \sigma^{1-q}) - 1]. \quad (5)$$

Recently, an alternative definition of relative entropy of Bregman type¹² has been provided by J. Naudts¹³ and T. D. Frank¹⁴ which reads

$$D_q(\rho \parallel \sigma) = \frac{1}{q-1} [Tr(\rho^q) - Tr(\rho\sigma^{q-1})] - [Tr(\rho\sigma^{q-1}) - Tr(\sigma^q)]. \quad (6)$$

The quantum divergence is also used to calculate purity of states in quantum information theory. Unfortunately, The Kullback-Leibler divergence given by Eq. (4) works only when the contribution of ρ is smaller than σ . If one chooses σ to be a pure state, then the contribution of σ is smaller than the contribution of ρ . Therefore, the Kullback-Leibler divergence cannot be used for this particular case. In a previous study, S. Abe¹⁵ made use of Eq. (5) in order to calculate the purity of Werner state in the nonextensive framework. For this purpose, he used Werner state¹⁶ which is given by the density matrix

$$\rho_W = F |\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3} (|\Psi^+\rangle\langle\Psi^+| + |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|), \quad \frac{1}{4} \leq F \leq 1 \quad (7)$$

where

$$|\Psi^\pm\rangle \equiv \frac{1}{\sqrt{2}} (|+-\rangle \pm |--\rangle), \quad (8)$$

and

$$|\Phi^\pm\rangle \equiv \frac{1}{\sqrt{2}} (|++\rangle \pm |--\rangle). \quad (9)$$

F is the fidelity of ρ_W with respect to the pure reference state $\sigma = |\Psi^-\rangle\langle\Psi^-|$. For the time being, we restrict ourselves to the interval $q \in (0, 1)$ since Eq. (5) should not be too sensitive to small eigenvalues of the matrices as in Ref. [15]. If we substitute Werner state in order to find the degree of purity with respect to σ in Eq.(5), we obtain

$$I_q(\rho_W \parallel |\Psi^-\rangle\langle\Psi^-|) = \frac{1}{1-q}(1-F^q). \quad (10)$$

This is the result already obtained by Abe in Ref. [15]. If we consider the alternative definition of quantum divergence proposed by Naudts and T. D. Frank in Eq. (6) and redo the above calculation by substituting σ , as defined earlier, we obtain

$$D_q(\rho \parallel \sigma) = \frac{1}{q-1} [F^q + 3(\frac{1-F}{3})^q - F] - (F-1). \quad (11)$$

Obviously, Eq. (11) is different from Eq. (10). It leads to negative values for $q \in (0, 1)$ and F smaller than 1. This result has been obtained in Ref. [17] where it has been deduced that the relative entropy of Bregman type is not suitable to handle these kind of situations, since it is not positive definite for $q \in (0, 1)$. Now, let us look closer at these two different relative entropy expressions i.e., Eqs. (5) and (6) by trying to solve them within a perturbative approach. The reason for this is to ensure that all eigenvalues of σ are different than zero. From now on, we do not have to restrict ourselves to any particular interval of q as long as it is not equal to 1. In order to do this, let us rewrite σ as

$$\sigma = (1-\epsilon) |\Psi^-\rangle\langle\Psi^-| + \frac{\epsilon}{3}(1 - |\Psi^-\rangle\langle\Psi^-|). \quad (12)$$

This definition of σ corresponds to our earlier definition when we set ϵ equal to zero. If we recalculate Eqs. (5) and (6) now, we obtain

$$I_q(\rho_W \parallel \sigma) = \frac{1}{q-1} [(1-\epsilon)^q F^q + \epsilon^{1-q}(1-F)^q - 1], \quad (13)$$

whereas for Frank-Naudts version, we have

$$D_q(\rho_W \parallel \sigma) = \frac{1}{q-1} [F^q + 3^{1-q}(1-F)^q - (1-\epsilon)^{q-1}F - (\frac{\epsilon}{3})^{q-1}(1-F)] - (1-\epsilon)^{q-1}F - (\frac{\epsilon}{3})^{q-1}(1-F) + (1-\epsilon)^q + 3^{1-q}\epsilon^q. \quad (14)$$

Now let us consider the limit $\epsilon \rightarrow 0$ for these two distinct expressions of divergence. Then, for $q \in (0, 1)$, we obtain

$$I_q(\rho_W \parallel \sigma) = \frac{1}{1-q}(1-F^q) \quad (15)$$

and

$$D_q(\rho_W \parallel \sigma) = +\infty. \quad (16)$$

On the other hand, for q values greater than 1, we have

$$I_q(\rho_W \parallel \sigma) = +\infty \quad (17)$$

and

$$D_q(\rho_W \parallel \sigma) = \frac{1}{q-1} [F^q + 3(\frac{1-F}{3})^q - F] - (F-1). \quad (18)$$

It is important to note that both type of quantum divergences share the same feature of Kullback-Leibler measure in resulting divergence. The Bregman type diverges for q values which lie between 0 and 1 whereas the Csiszàr type diverges for q values greater than 1. But, we have non-diverging results due to the existence of entropic index q which is an advantage of nonextensive formalism. Note that the physical meaning of Bregman divergence is the same as Eq. (4) i.e. difference of free energies if one employs ordinary constraint¹⁸ whereas the same meaning can be preserved for Csiszàr type divergence by employing escort probability¹⁹ (see Ref. [20] for details).

III. RESULTS AND DISCUSSIONS

We have studied two definitions of divergence in current use in the nonextensive formalism in order to calculate the degree of purification of Werner state by adopting a perturbative method. This in turn enabled us to generalize the results of our previous work¹⁷ for all values of the nonextensive index q . In other words, the existence of two types of divergence associated with two types of constraints makes the calculation possible for all values of entropic index, rendering the problem of purity of state fully solvable. In this sense, one can interpret the existence of multiple constraints as an advantage of nonextensive formalism, not a defect which is to be eliminated.

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