

Analytical Solutions of Klein-Gordon Equation with Position-Dependent Mass for q -Parameter Pöschl-Teller potential

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Abstract

The energy eigenvalues and the corresponding eigenfunctions of the one-dimensional Klein-Gordon equation with q -parameter Pöschl-Teller potential are analytically obtained within the position-dependent mass formalism. The parametric generalization of the Nikiforov-Uvarov method is used in the calculations by choosing a mass distribution.

Keywords: Pöschl-Teller potential, Klein-Gordon equation, Position-Dependent Mass, Nikiforov-Uvarov Method

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I. INTRODUCTION

In last few years, the formalism constructed on varying of the mass with coordinates has been received much attentions because of finding many applications in different areas. The position-dependent mass formalism has been used to describe the electronic properties of semiconductors and quantum dots [1-5]. It also gives interesting results in the study of ^3He clusters [6], and quantum liquids [7].

The position-dependent mass formalism brings some basic problems in the investigation of physical systems, such as the possible discontinuities of choosen mass functions, the invariance of theory under the Galileo transformations, and ordering-ambiguity between momentum, and mass operators in kinetic energy term [8, 9].

Recently, the solutions of non-relativistic wave equation with constat mass have been extended to the position-dependent mass (PDM) case [10-25]. In Ref. [26], a general formalism giving both of energy spectra and also wave functions were found in non-relativistic problems by mapping between reference and target systems by using point canonical transformation. The Schrödinger equation with position-dependent mass is studied by using parameter algebra, and in curved spaces has been investigated in the Coulomb example [27]. In Ref. [28], the shape-invariance technique has been applied to solvable Hamiltonians. Bagchi, and co-workers have generated a formalism to obtain PT -symmetric Hamiltonians in the PDM case [29]. In Re. [30] has been made a generalization of the point canonical transformation to solve the Schrödinger equation in d -dimension. The authors have also investigated the η -pseudo-hermicity of non-Hermitian Hamiltonians with position-dependent mass [31]. The investigation of a non-Hermitian Hamiltonian for an oscillator defined by Swanson's model has been made in the context of N-fold supersymmetry [32].

Another area received much attentions is the extension of the non-relativistic solutions to the relativistic solutions in the view of position-dependent mass formalism. The energy spectra and corresponding wave functions of Klein-Gordon and Dirac equations have been found for different potentials by using different methods, such as complexified Lorentz scalar interactions [33], Coulomb potential [34], hyperbolic potentials [35], non-Hermitian complexified potentials [36, 37], PT -symmetric trigonometric potential [38], PT -symmetric harmonic oscillator-like, PT -symmetric inversely linear plus linear, and PT -symmetric kink-like potentials [39], inversely linear plus linear potential, and Scarf II potential [40].

Quantum deformation [41] has received much attentions because of its relation with applications in nuclei [42-45], statistical-quantum theory, string/brane theory and conformal field theory [46-49]. Recently, some authors have been introduced some potentials in terms of hyperbolic functions in the view of q -deformation [50]. One of the potential from this family can be written as

$$V(x; q) = -\frac{V_0}{\cosh_q^2(\alpha x)}, \quad (1)$$

which is the q -parameter form of the usual Pöschl-Teller potential [51]. Here, q is the deformation parameter and used to denote a mapping from a c -number N to a q -number $[N]_p$ by relation

$$[N]_p = \frac{(e^{\kappa N})^p - (e^{\kappa N})^{-p}}{(e^{\kappa})^p - (e^{\kappa})^{-p}} \rightarrow N, \quad (2)$$

in the $q \rightarrow 1$ limit, κ is a real parameter, and $e^{\kappa} = q$ in the above equation [41]. The q -parameter hypergeometric functions in Eq. (1) are defined

$$\sinh_q(z) = \frac{1}{2}(e^z - qe^{-z}) ; \cosh_q(z) = \frac{1}{2}(e^z + qe^{-z}), \quad (3)$$

$$\tanh_q(z) = \frac{e^z - qe^{-z}}{e^z + qe^{-z}} ; \operatorname{sech}_q(z) = \frac{2}{e^z + qe^{-z}}. \quad (4)$$

In the present work, we intend to solve analytically the one-dimensional effective Klein-Gordon equation for q -parameter Pöschl-Teller potential to investigate the effect of the PDM to the energy spectra and corresponding wave functions. We choose an exponentially mass distribution function which makes it possible to analytically solve the Klein-Gordon equation. We use the parametric generalization of the Nikiforov-Uvarov (NU) method [35] to find the energy eigenvalues, and corresponding wave functions [53]. The NU method describes the way to solve a Schrödinger-like equation by turning it into a hypergeometric type equation [52].

The organization of this work is as follows. In Section II, we solve the one-dimensional Klein-Gordon equation for q -parameter Pöschl-Teller potential, and give the energy spectrum, and the corresponding wave functions in the case of position-dependent mass. We summarize our conclusion in Section III.

II. BOUND-STATES OF KLEIN-GORDON EQUATION

The one-dimensional Klein-Gordon equation for a particle with mass m subject to scalar, $V_s(x)$, and vector, $V_v(x)$, potentials reads ($\hbar = c = 1$)

$$\frac{d^2\psi(x)}{dx^2} + \left[[E - V_v(x)]^2 - [m - V_s(x)]^2 \right] \psi(x) = 0, \quad (5)$$

where E is the relativistic energy of particle.

We prefer to use the mass function equal to the vector part of the potential as

$$m(x) = m_0 + 4V_0 \frac{e^{-2\alpha x}}{(1 + qe^{-2\alpha x})^2}, \quad (6)$$

to obtain an exactly solvable Schrödinger-like equation from Eq. (5) in the absence of scalar potential. The mass function should also be a physically distribution, so we restrict ourself in the range $0 \leq x \leq \infty$, which gives the following finite mass values

$$m(x) = \begin{cases} m_0 + 2V_0 & (\text{for } q \rightarrow 1), \quad x \rightarrow 0, \\ m_0, & x \rightarrow \infty. \end{cases}$$

Indeed, this distribution corresponds to shifted scalar potential function in the problem. Substituting Eq. (1), and Eq. (6) into Eq. (5), we get

$$\frac{d^2\psi(x)}{dx^2} + \left\{ (E - m_0 + 8V_0 \frac{e^{-2\alpha x}}{(1 + qe^{-2\alpha x})^2})(E + m_0) \right\} \psi(x) = 0, \quad (7)$$

By using the new variable $s = e^{-2\alpha x}$ ($0 < s < 1$), we have

$$\begin{aligned} \frac{d^2\psi(s)}{ds^2} + \frac{1 + qs}{s(1 + qs)} \frac{d\psi(s)}{ds} + \frac{1}{[s(1 + qs)]^2} \\ \times \left\{ \eta^2 q^2 (E^2 - m_0^2) s^2 + [2\eta^2 q (E^2 - m_0^2) + 8\eta^2 V_0 (E - m_0)] s \right. \\ \left. + \eta^2 (E^2 - m_0^2) \right\} \psi(s) = 0, \end{aligned} \quad (8)$$

where $\eta^2 = 1/4\alpha^2$. Following Ref. [36], we obtain the parameter set as

$$\begin{aligned}
\alpha_1 &= 1, & -\xi_1 &= \eta^2 q^2 (E^2 - m_0^2) \\
\alpha_2 &= -q, & \xi_2 &= 2\eta^2 q (E^2 - m_0^2) + 8\eta^2 V_0 (E - m_0) \\
\alpha_3 &= -q, & -\xi_3 &= \eta^2 (E^2 - m_0^2) \\
\alpha_4 &= 0, & \alpha_5 &= \frac{q}{2} \\
\alpha_6 &= \xi_1 + \frac{q^2}{4}, & \alpha_7 &= -\xi_2 \\
\alpha_8 &= \xi_3, & \alpha_9 &= \xi_1 + q\xi_2 + q^2\xi_3 + \frac{1}{4}q^2 \\
\alpha_{10} &= 1 + 2\sqrt{\xi_3}, & \alpha_{11} &= -2q + 2\left(\sqrt{\xi_1 + q\xi_2 + q^2\xi_3 + \frac{1}{4}q^2} - q\sqrt{\xi_3}\right) \\
\alpha_{12} &= \sqrt{\xi_3}, & \alpha_{13} &= \frac{q}{2} - \left(\sqrt{\xi_1 + q\xi_2 + q^2\xi_3 + \frac{1}{4}q^2} - q\sqrt{\xi_3}\right)
\end{aligned} \tag{9}$$

and we deduce the parameters required for the method [36]

$$\begin{aligned}
\pi(s) &= \frac{1}{2}q \pm \left\{ \left[\frac{1}{4}q^2 - 2q\sqrt{\eta^2(m_0^2 - E^2)[8V_0\eta^2q(E - m_0) + \frac{1}{4}q^2]} \right] s^2 \right. \\
&\quad - \left[2\eta^2q(E^2 - m_0^2) - 2\sqrt{\eta^2(m_0^2 - E^2)[8V_0\eta^2q(E - m_0) + \frac{1}{4}q^2]} \right] s \\
&\quad \left. - \eta^2(E^2 - m_0^2) \right\}^{1/2},
\end{aligned} \tag{10}$$

and

$$k = -2\sqrt{\eta^2(m_0^2 - E^2)[8V_0\eta^2q(E - m_0) + \frac{1}{4}q^2]}, \tag{11}$$

$$\begin{aligned}
\tau(s) &= 1 - 2qs - 2\left\{ \left[\sqrt{8V_0\eta^2q(E - m_0) + \frac{1}{4}q^2} - q\sqrt{\eta^2(m_0^2 - E^2)} \right] s \right. \\
&\quad \left. - \sqrt{\eta^2(m_0^2 - E^2)} \right\}.
\end{aligned} \tag{12}$$

We need to know λ and λ_n [36]

$$\lambda = -2\sqrt{\eta^2(m_0^2 - E^2)[8V_0\eta^2q(E - m_0) + \frac{1}{4}q^2]} + \pi'(s), \tag{13}$$

$$\lambda_n = -n \left[-2q - 2\sqrt{8V_0\eta^2q(E - m_0) + \frac{1}{4}q^2} + 2q\sqrt{\eta^2(m_0^2 - E^2)} \right] - qn(n-1), \tag{14}$$

to get the eigenvalue equation for the energy spectra of the q -parameter Pöschl-Teller potential as

$$\frac{1}{\alpha} \sqrt{m_0^2 - E^2} + \sqrt{\frac{1}{4} + \frac{2V_0}{q\alpha^2}(E - m_0)} = n + \frac{1}{2}, \tag{15}$$

which gives two independent solutions corresponding to particle and antiparticle states. This equation gives $E = \pm m_0$ for the constant mass case when the potential vanishes.

The corresponding unnormalized eigenfunctions are obtained in terms of following functions [36]

$$\rho(s) = s^{2\sqrt{\eta^2(m_0^2-E^2)}} (1+qs)^{-2\sqrt{\frac{\frac{8}{q}V_0\eta^2(E-m_0)+\frac{1}{4}}{q}}}, \quad (16)$$

$$y_n(s) = P_n^{(2\sqrt{\eta^2(m_0^2-E^2)}, -2\sqrt{\frac{\frac{8}{q}V_0\eta^2(E-m_0)+\frac{1}{4}}{q}})} (1+2qs), \quad (17)$$

$$(18)$$

and

$$\phi(s) = s^{\sqrt{\eta^2(m_0^2-E^2)}} (1+qs)^{-\frac{1}{2}-\sqrt{\frac{\frac{8}{q}V_0\eta^2(E-m_0)+\frac{1}{4}}{q}}}, \quad (19)$$

as

$$\begin{aligned} \psi_n(s) &= a_n s^{\sqrt{\eta^2(m_0^2-E^2)}} (1+qs)^{-\frac{1}{2}-\sqrt{\frac{\frac{8}{q}V_0\eta^2(E-m_0)+\frac{1}{4}}{q}}} \\ &\times P_n^{(2\sqrt{\eta^2(m_0^2-E^2)}, -2\sqrt{\frac{\frac{8}{q}V_0\eta^2(E-m_0)+\frac{1}{4}}{q}})} (1+2qs). \end{aligned} \quad (20)$$

where a_n is a normalization constant. The eigenfunctions are dependent on the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. The asymptotic behavior of the wave function could be given in the limit $s \rightarrow 0$ as

$$\psi_n(s) \rightarrow 0, \quad (21)$$

because $P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(1+\alpha)}$ [54], and we write the wave function for $s \rightarrow 1$ as

$$\psi_n(s) \sim P_n^{(\alpha,\beta)}(1+2q), \quad (22)$$

where a physical solution can be obtained only for $-1 < q \leq 1$.

III. CONCLUSION

We have solved analytically the Klein-Gordon equation for the q -parameter Pöschl-Teller potential in one-dimension in the case of position-dependent mass function. We obtain a

energy eigenfunction, which gives particle and antiparticle states, by using the parametric generalization of the NU-method, and get the corresponding eigenfunctions in terms of Jacobi polynomials. We also study the energy spectra for the case where mass is constant and potential vanishes. Finally, we give the behavior of the corresponding wave function at zero.

IV. ACKNOWLEDGMENTS

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