Supersymmetric Solutions of PT-/non-PT-Symmetric and non-Hermitian central Potentials via Hamiltonian Hierarchy Method

Gholamreza Faridfathi, Ramazan Sever*, Metin Aktaş
Department of Physics, Middle East Technical University, 06531 Ankara, Turkey

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Abstract

The supersymmetric solutions of PT-/non-PT-symmetric and non-Hermitian deformed Morse and Pöschl-Teller potentials are obtained by solving the Schrödinger equation. The Hamiltonian hierarchy method is used to get the real energy eigenvalues and corresponding eigenfunctions.

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*Corresponding author: sever@metu.edu.tr

1 Introduction

PT-symmetric quantum systems have generated much interest in recent years [1]. About ten years ago, Besis suggested that the eigenvalue spectrum of complex-valued Hamiltonians is real and positive. Bender and Boettcher claimed that this result is due to PT-symmetry where P and T are the parity and time reversal operators respectively. It is neither a necessary nor a sufficient condition for a Hamiltonian to have real spectrum. In particular, the spectrum of the Hamiltonian is real if PT-symmetry is not spontaneously broken. Thus, the property of exactness guarantees the real eigenvalues. Recently, Mostafazadeh introduced another concept for a class of PT-invariant Hamiltonians called $\eta - pseudo - Hermiticity$ [2]. In fact, Hamiltonians of this type satisfy the transformation $\eta\ \hat{H}\ \eta^{-1} = \hat{H}^{\dagger}$ [3]. Moreover, completeness and orthonormality conditions for the eigenstates of such potentials are proposed [4]. Various techniques have been applied in the study of PT-invariant potentials such as variational methods [5], numerical approaches [6], Fourier analysis [7], semi-classical estimates [8], quantum field theory [9] and Lie group theoretical approaches [10-13]. In the applications, a generalization of the symmetry concept is encountered in the supersymmetric quantum mechanics (SUSYQM) [14]. A variety of PT-symmetric examples can be found using the SUSYQM techniques [15-20]. Furthermore, one can get more examples of the PT-symmetric and non-PT-symmetric and also non-Hermitian potential cases such as oscillator type potentials [21], flat, step and double square-well like potentials within the framework of SUSYQM [22, 23], exponential type screened potentials [24], quasi/conditionally exactly solvable ones [25], complex Hénon-Heiles potentials [26], periodic isospectral potentials [27] and some others[28, 29].

The aim of the present work is to calculate the energy eigenvalues and the corresponding eigenfunctions of the deformed Morse and Pöschl-Teller potentials using the Hamiltonian hierarchy method [30] within the framework of the PT-SUSYQM. This method is also known as the factorization method introduced by Schrödinger [31] and later developed by Infeld and Hull [32]. It is useful to obtain the energy spectra for different potentials in non-relativistic quantum mechanics [33, 34, 35].

The organization of this paper is as follows: In Sec. II we introduce a brief review of the Hamiltonian hierarchy method. In Secs. III and IV we present the supersymmetric solutions of PT-symmetric and Hermitian/non-Hermitian forms of the well-known potentials by using this method. We discuss the results in Sec. V.

2 Hamiltonian Hierarchy Method

The radial Schrödinger equation for some specific potential energies can be solved analytically only for the states with zero angular momentum [36, 37]. However, in supersymmetric quantum mechanics, one can get exact results with the hierarchy problem by using effective potentials for non-zero angular momentum states. In the framework of SUSYQM, this method provides an eigenvalue spectra for adjacent members of supersymmetric partner Hamiltonians. These Hamiltonians share the same eigenvalue spectra except for the missing ground state.

In the application of this method, we first look for an effective potential similar to the original specific potential and inspired by the SUSYQM to propose a superpotential, namely $W_{(l+1)}(x)$, as an ansatz, where $(l+1)$ denotes the partner number with $l = 0, 1, 2....$ Substituting the proposed superpotential into the Riccati equation,

$$V_{(l+1)}(x) - E^0_{(l+1)} = W^2_{(l+1)}(x) - \frac{dW_{(l+1)}(x)}{dx}, \tag{1}$$

the $(l+1)$th member of the Hamiltonian hierarchy can be obtained. Taking into account the shape invariance requirement [14], the bound-state energies can be obtained through Eq. (1), and the corresponding eigenfunctions by means of,

$$\Psi_{(l+1)}(x) = N \exp(-\int^{r} W_{(l+1)}(x')dx'). \qquad (2)$$

3 Generalized Morse Potential

The generalized Morse potential is given by [24]

$$V(x) = V_1 e^{-2\alpha x} - V_2 e^{-\alpha x}. \qquad (3)$$

To apply the Hamiltonian hierarchy method, we shall take the coefficients as $V_1 = V$, and $\frac{V_2}{V_1} = q$. Inspired by the SUSYQM, we propose an ansatz for the superpotential

$$W_{(l+1)}(x) = -\lambda e^{-\alpha x} + (\lambda q - \frac{2l+1}{2}), \qquad (4)$$

where, $\lambda^2 = \frac{2mV}{a^2\hbar^2}$, and $(2l+1)$ denotes the partner number with $l = 0, 1, 2, ...$, and the parameter m is the reduced mass of a diatomic molecule. The superpotential chosen in Eq.(4) leads to the $(l+1)th$ member of the Hamiltonian hierarchy through the Riccati equation as,

$$V_{(l+1)}(x) - E^0_{(l+1)} = W^2_{(l+1)}(x) - \frac{1}{\alpha}\frac{dW_{(l+1)}(x)}{dx}, \qquad (5)$$

which yields,

$$V_{(l+1)}(x) = \lambda^2(e^{-2\alpha x} - qe^{-\alpha x}) + 2l\lambda e^{-\alpha x}. \qquad (6)$$

Now, using the shape invariance requirement, the energy eigenvalues for any n-th state become,

$$E^n_{(l+1)} = -(\lambda q - \frac{2l+n+1}{2})^2, \qquad n = 0, 1, 2, \ldots \qquad (7)$$

The corresponding eigenfunctions are obtained through Eq. (2) as

$$\Psi^{n=0}_{(l+1)}(x) = N \exp\left[-\frac{\lambda}{\alpha}e^{-\alpha x} - (\lambda q - \frac{2l+1}{2})x\right], \tag{8}$$

where, N is the normalization constant.

3.1 Non-PT symmetric and non-Hermitian Morse Case

We define the potential parameters in Eq. (3) as $V_1 = (A+iB)^2$, $V_2 = (2C+1)(A+iB)$, and $\alpha = 1$. A, B and C are real, and $i = \sqrt{-1}$. For simplicity we define $A + iB = i\omega$, $(A+iB)^2 = -\omega^2$, $2C+1 = K$. Thus, the potential becomes

$$V(x) = -\frac{\omega^2}{K}\left[Ke^{-2x} - \frac{K^2}{i\omega}e^{-x}\right]. \tag{9}$$

To get the final compact form, we also define $\frac{\omega^2}{K} = G$, and $\frac{K^2}{\omega} = t$, and $GK = D$, and also $\frac{t}{K} = P$. As a result, we get,

$$V(x) = -D\left[e^{-2x} + iPe^{-x}\right]. \tag{10}$$

We propose an ansatz for the superpotential as,

$$W_{(l+1)}(x) = -i\ \lambda e^{-x} + (\lambda - \frac{2l+1}{2}), \tag{11}$$

where $\lambda^2 = \frac{2mD}{a^2\hbar^2}$. Consequently, according to the Hamiltonian hierarchy method, we get,

$$V_{(l+1)}(x) = -\lambda^2(e^{-2x} + 2i\ e^{-x}) + 2il\lambda e^{-x}, \tag{12}$$

By substituting this into Eq. (1), the corresponding eigenvalues and eigenfunctions are obtained as

$$E^n_{(l+1)} = -(\lambda - \frac{n+2l+1}{2})^2, \qquad n = 0, 1, 2, \ldots, \tag{13}$$

and

$$\Psi^{n=0}_{(l+1)}(x) = N \exp\left[-i\lambda e^{-x} - (\lambda - \frac{2l+1}{2})x\right], \tag{14}$$

where, N is a normalization constant.

3.2 The first type of PT-symmetric and non-Hermitian Morse case

We take the coefficients of the generalized Morse potential as $V_1 = (A + iB)^2$, $V_2 = (2C + 1)(A + iB)$, and $\alpha = i$. Following the same procedure, we propose an ansatz for the superpotential

$$W_{(l+1)}(x) = -\lambda e^{-ix} + (\lambda - \frac{2l+1}{2}). \tag{15}$$

The Hamiltonian hierarchy method yields,

$$V_{(l+1)}(x) = \lambda^2(e^{-2ix} - e^{-ix}) + 2l\lambda e^{-ix}. \tag{16}$$

This form of potential gives the same eigenvalues as in Eq. (13).

3.3 The second type of PT-symmetric and non-Hermitian Morse case

Now, we take the parameters as $V_1 = -\omega^2$, and $V_2 = D$, and $\alpha = i\alpha$ in Eq. (3), where V_1 and V_2 are real. For $V_1 \Longrightarrow 0$, we get no real spectra for this kind of PT-symmetric Morse potentials. The superpotential can be proposed as,

$$W_{(l+1)}(x) = -e^{-i\alpha x} + (2l + 1 + \frac{D}{2\omega}). \tag{17}$$

By applying the Hamiltonian hierarchy method, we get the potential

$$V_{(l+1)}(x) = e^{-2i\alpha x} - 2\left[(2l+1) + \frac{D}{2\omega} + \frac{i\alpha}{2}\right] e^{-i\alpha x}, \tag{18}$$

and the corresponding eigenvalues for any n-th state are,

$$E^{n}_{(l+1)} = -(2l + n + 1 + \frac{D}{2\omega})^2. \tag{19}$$

4 Pöschl-Teller Potential

The Pöschl-Teller potential is given as,

$$V(x) = -4V_0 \frac{e^{-2\alpha x}}{(1 + qe^{-2\alpha x})^2}. \tag{20}$$

In the framework of the SUSYQM, the corresponding superpotential can be proposed as,

$$W_{(l+1)}(x) = -\frac{\hbar}{\sqrt{2m}} \frac{(l+1)\, e^{-2\alpha x}}{(1 + qe^{-2\alpha x})^2} + \sqrt{\frac{m}{2}} \frac{e^2}{\hbar}\left[\frac{1}{(l+1)} - \frac{(l+1)}{2}\beta\right], \tag{21}$$

where, $\beta = \frac{\hbar^2}{me^2}$, and $l = 0, 1, 2, ...$ By applying the Hamiltonian hierarchy method we get,

$$V_{(l+1)}(x) = \frac{\hbar^2}{2m} \frac{e^{-4\alpha x}}{(1 + qe^{-2\alpha x})^4} l(l+1) - e^2 \frac{e^{-2\alpha x}}{(1 + qe^{-2\alpha x})^2}\left[1 - l\,(l+1)\,\frac{\beta}{2}\right]. \tag{22}$$

As a result, the corresponding eigenvalues of this potential for any n-th state are,

$$E^{n}_{(l+1)} = -\frac{q^2 m e^4}{2\hbar^2}\left[\frac{1}{(n+l+1)} - \frac{(n+l+1)}{2}\beta\right]^2. \tag{23}$$

The corresponding eigenfunctions are,

$$\Psi_{(l+1)}^{n=0}(x) = N\left(1+qe^{-2\alpha x}\right)^{l+1}\exp\left\{-\frac{me^2}{\hbar^2}\left[\frac{1}{l+1}-\frac{(l+1)}{2}\beta\right]x\right\}, \tag{24}$$

where, N is a normalization constant.

4.1 Non-PT symmetric and non-Hermitian Pöschl-Teller cases

Here, V_0 and q are complex parameters: $V_0 = V_{0R} + iV_{0I}$ and $q = q_R + iq_I$, but α is a real parameter. Although the potential is complex and the corresponding Hamiltonian is non-Hermitian and also non-PT symmetric, there may be real spectra if and only if $V_{0I}\, q_R = V_{0R}\, q_I$. When both parameters V_0, and q are taken pure imaginary, the potential turns out to be,

$$V(x) = -4V_0\frac{2qe^{-4\alpha x}+i(1-q^2e^{-4\alpha x})}{(1+q^2e^{-4\alpha x})^2}. \tag{25}$$

For simplicity, we use the notation V_0 and q instead of V_{0I} and q_I. To obtain the energy eigenvalues, we propose the superpotential

$$W_{(l+1)}(x) = -\frac{\hbar}{\sqrt{2m}}\frac{(l+1)\,qe^{-4\alpha x}}{(1+q^2e^{-4\alpha x})^2}+\sqrt{\frac{m}{2}}\frac{e^2}{\hbar}\left[\frac{1}{(l+1)}-\frac{(l+1)}{2}\beta\right]. \tag{26}$$

Therefore, substituting this equation into the Riccati equation, we get the same potential as in Eq. (22), and also the same energy eigenvalues as in Eq. (23).

4.2 PT symmetric and non-Hermitian Pöschl-Teller case

We choose the parameters V_0 and q as real, and also $\alpha \Longrightarrow i\alpha$ in Eq. (25). Here, we propose the superpotential similar to this potential as,

$$W_{(l+1)}(x) = -\frac{\hbar}{\sqrt{2m}} \frac{(l+1)\, q e^{-4i\alpha x}}{\left(1+q^2 e^{-4i\alpha x}\right)^2} + \sqrt{\frac{m}{2}} \frac{e^2}{\hbar} \left[\frac{1}{(l+1)} - \frac{(l+1)}{2} \beta \right]. \qquad (27)$$

By applying the same procedure, we get the energy eigenvalues as in Eq.(23).

5 Conclusions

We have applied the PT-symmetric formulation to solve the Schrödinger equation for more general Morse and Pöschl-Teller potentials. The Hamiltonian hierarchy method within the framework of the SUSYQM is used. We have obtained the energy eigenvalues and the corresponding eigenfunctions for different forms of these potentials. The energy spectrum of the PT-invariant complex-valued non-Hermitian potentials may be real or complex depending on the parameter values. We have imposed some restrictions on the potential parameters to get the real spectra in PT-symmetric, or more generally, in non-Hermitian cases. It is also pointed out that the superpotentials, and partners must satisfy the PT-symmetry condition. Finally, we have pointed out that our exact results of complexified general Morse and Pöschl-Teller potentials may increase the number of applications in the study of different quantum systems.

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