# Exact solution of Effective mass Schrödinger Equation for the Hulthen potential 

Ramazan Sever ${ }^{1}$, Cevdet Tezcan ${ }^{2}$, Özlem Yeşiltaş ${ }^{3}$, Mahmut Bucurgat ${ }^{1}$<br>${ }^{1}$ Middle East Technical University,Department of Physics, 06531 Ankara, Turkey<br>${ }^{2}$ Faculty of Engineering, Başkent University, Bag̃lıca Campus, Ankara, Turkey<br>${ }^{3}$ Gazi University, Faculty of Arts and Sciences, Department of Physics, 06500, Ankara,Turkey

October 25, 2018


#### Abstract

A general form of the effective mass Schrödinger equation is solved exactly for Hulthen potential. Nikiforov-Uvarov method is used to obtain energy eigenvalues and the corresponding wave functions. A free parameter is used in the transformation of the wave function.


PACS numbers: 03.65.-w; 03.65.Ge; 12.39.Fd
Keywords: Position-dependent mass, Effective mass Schrödinger equation, Morse potential, Nikiforov-Uvarov method

## 1 Introduction

Quantum mechanical systems with position dependent effective mass (PDM) have been studied in different branches of physics by many authors $[1,2,3,4,5,6,7]$. Several authors have obtained the exact solutions of Schrödinger equation with position dependent mass [8-18]. Moreover, the Morse potential [19], one dimensional Coulomb-like potential [20], hard core potential [21], harmonic oscillator potential [22] are known as some real physical potentials that have been investigated within PDM framework.

In recent years many authors have been used Nikiforov-Uvarov (NU) approach for solving the Schrödinger equation (SE) [23,24,25,26,27,28,29].

In this work, the general form of PDEM Schrödinger equation is obtained by using a more general transformation of the wave function as $\varphi=m^{\eta}(x) \psi(x)$. NU approach is adapted to this general equation. Using an appropriate mass function, it is solved for Hulthen potential within this generalization. Energy eigenvalues and the corresponding wave functions are obtained. The contents of the paper is as follows: in section II, we introduce PDM approach and NikiforovUvarov method. The next section involves solutions of the general PDM equation. Results are discussed in section IV.

## 2 Method

We write the one-dimensional effective mass Hamiltonian of the SE as [29]

$$
\begin{equation*}
H_{e f f}=-\frac{d}{d x}\left(\frac{1}{m(x)} \frac{d}{d x}\right)+V_{e f f}(x) \tag{1}
\end{equation*}
$$

where $V_{\text {eff }}$ has the form

$$
\begin{equation*}
V_{e f f}=V(x)+\frac{1}{2}(\beta+1) \frac{m^{\prime \prime}}{m^{2}}-[\alpha(\alpha+\beta+1)+\beta+1] \frac{m^{\prime 2}}{m^{3}} \tag{2}
\end{equation*}
$$

with $\alpha, \beta$ are ambiguity parameters. Primes stand for the derivatives with respect to $x$ and we have set $\hbar=2 m_{0}=1$. Thus the SE takes the form

$$
\begin{equation*}
\left(-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{m^{\prime}}{m} \frac{d}{d x}+V_{e f f}-E\right) \varphi(x)=0 \tag{3}
\end{equation*}
$$

We apply the following transformation

$$
\begin{equation*}
\varphi=m^{\eta}(x) \psi(x) \tag{4}
\end{equation*}
$$

Hence, the SE takes the form

$$
\begin{equation*}
\left\{-\frac{d^{2}}{d x^{2}}-(2 \eta-1) \frac{m^{\prime}}{m} \frac{d}{d x}-(\eta(\eta-2)+\alpha(\alpha+\beta+1)+\beta+1) \frac{m^{\prime 2}}{m^{2}}+\left(\frac{1}{2}(\beta+1)-\eta\right) \frac{m^{\prime \prime}}{m}+m(V-E)\right\} \psi= \tag{5}
\end{equation*}
$$

Hulthen potential is given by [26]

$$
\begin{equation*}
V(x)=-V_{0} \frac{e^{-\lambda x}}{1-q e^{-\lambda x}} \tag{6}
\end{equation*}
$$

We give the following parameters including mass relation:

$$
\begin{align*}
A^{*} & =\alpha(\alpha+\beta+1)+\beta+1  \tag{7}\\
m(x) & =\left(1-q e^{-\lambda x}\right)^{-1}  \tag{8}\\
\frac{m^{\prime}}{m} & =-q \lambda \frac{e^{-\lambda x}}{1-q e^{-\lambda x}}  \tag{9}\\
\frac{m^{\prime \prime}}{m} & =q \lambda^{2} e^{-\lambda x} \frac{1+q e^{-\lambda x}}{\left(1-q e^{-\lambda x}\right)^{2}} \tag{10}
\end{align*}
$$

We introduce a variable changing in Eq.(5)given as

$$
\begin{equation*}
s=\frac{1}{1-q e^{-\lambda x}} \tag{11}
\end{equation*}
$$

and if we use Eqs.(6),(7),(8),(9),(10) and (11) in Eq.(5), it becomes,

$$
\begin{equation*}
\left(\frac{d^{2}}{d s^{2}}+\frac{2 \eta-(2 \eta+1) s}{s(1-s)} \frac{d}{d s}+\frac{1}{s^{2}(1-s)^{2}}\left(-\xi_{1} s^{2}+\xi_{2} s-\xi_{3}\right)\right) \psi=0 \tag{12}
\end{equation*}
$$

Parameters defined in Eq.(12) have the following form:

$$
\begin{align*}
-\xi_{1} & =\left(\eta(\eta-2)+A^{*}\right)-2\left(\frac{1}{2}(\beta+1)-\eta\right)+\frac{V_{0}}{q \lambda^{2}}  \tag{13}\\
\xi_{2} & =-2\left(\eta(\eta-2)+A^{*}\right)+3\left(\frac{1}{2}(\beta+1)-\eta\right)-\frac{V_{0}}{q \lambda^{2}}+\frac{E}{\lambda^{2}}  \tag{14}\\
-\xi_{3} & =\eta(\eta-2)+A^{*}-\left(\frac{1}{2}(\beta+1)-\eta\right) \tag{15}
\end{align*}
$$

where $V(s)=\frac{V_{0}}{q}(1-s)$. Now, we apply the NU method starting from its standard form

$$
\begin{equation*}
\psi_{n}^{\prime \prime}(s)+\frac{\tilde{\tau}(s)}{\sigma(s)} \psi^{\prime}(s)+\frac{\tilde{\sigma}(s)}{\sigma^{2}(s)} \psi_{n}(s)=0 . \tag{16}
\end{equation*}
$$

Comparing Eqs.(12) and (16), we obtain

$$
\begin{equation*}
\sigma=s, \tilde{\tau}(s)=3-4 \eta, \tilde{\sigma}(s)=-\xi_{1} s^{2}-\xi_{2} s+\xi_{3} \tag{17}
\end{equation*}
$$

In the NU method, the function $\pi$ and the parameter $\lambda$ are defined as [23]

$$
\begin{equation*}
\pi(s)=\frac{\sigma^{\prime}-\tau(s)}{2} \pm \sqrt{\left(\frac{\sigma^{\prime}-\tau(s)}{2}\right)^{2}-\tilde{\sigma}(s)+k \sigma(s)} \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda=k+\pi^{\prime} \tag{19}
\end{equation*}
$$

To find a physical solution, the expression in the square root must be square of a polynomial. Then, a new eigenvalue equation for the SE becomes

$$
\begin{equation*}
\lambda=\lambda_{n}=-n \tau^{\prime}-\frac{n(n-1)}{2} \sigma^{\prime \prime}(s),(n=0,1,2, \ldots) \tag{20}
\end{equation*}
$$

where

$$
\begin{equation*}
\tau(s)=\tilde{\tau}(s)+2 \pi(s) \tag{21}
\end{equation*}
$$

and it should have a negative derivative [23]. A family of particular solutions for a given $\lambda$ has hypergeometric type of degree. Thus, $\lambda=0$ will corresponds to energy eigenvalue of the ground state, i.e. $n=0$. The wave function is obtained as a multiple of two independent parts:

$$
\begin{equation*}
\psi(s)=\phi(s) y(s) \tag{22}
\end{equation*}
$$

where $y(s)$ is the hypergeometric type function written with a weight function $\rho$ as

$$
\begin{equation*}
y_{n}(s)=\frac{B_{n}}{\rho(s)} \frac{d^{n}}{d s}\left[\sigma^{n}(s) \rho(s)\right] \tag{23}
\end{equation*}
$$

where $\rho(s)$ must satisfy the condition [23]

$$
\begin{equation*}
(\sigma \rho)^{\prime}=\tau \rho \tag{24}
\end{equation*}
$$

The other part is defined as a logarithmic derivative

$$
\begin{equation*}
\frac{\phi^{\prime}(s)}{\phi(s)}=\frac{\pi(s)}{\sigma(s)} \tag{25}
\end{equation*}
$$

## 3 Solutions

If we take Eq.(12) into account, comparing with Eq.(16), it is observed that $\tilde{\tau}=2 \eta-(2 \eta+1) s$, $\sigma=s(1-s), \tilde{\sigma}=-\xi_{1} s^{2}+\xi_{2} s-\xi_{3}$. Using $z=\frac{1}{2}(1-2 \eta)$, one obtains

$$
\pi=z(1-s) \pm \begin{cases}\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}-\sqrt{\xi_{3}+z^{2}}\right) s+\sqrt{\xi_{3}+z^{2}}, & k_{1}=\xi_{2}-2 \xi_{3}+2 \zeta  \tag{26}\\ \left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right) s-\sqrt{\xi_{3}+z^{2}}, & k_{2}=\xi_{2}-2 \xi_{3}-2 \zeta\end{cases}
$$

where $\zeta=\sqrt{\xi_{3}\left(\xi_{1}-\xi_{2}+\xi_{3}+z^{2}\right)-z^{2}\left(\xi_{2}-\xi_{1}\right)}$. Now we can introduce $\tau(s)$ as given below,

$$
\tau(s)=\left\{\begin{array}{l}
1-2 s+2\left(\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}-\sqrt{\xi_{3}+z^{2}}\right) s+\sqrt{\xi_{3}+z^{2}}\right)  \tag{27}\\
1-2 s-2\left(\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}-\sqrt{\xi_{3}+z^{2}}\right) s+\sqrt{\xi_{3}+z^{2}}\right) \\
1-2 s+2\left(\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right) s-\sqrt{\xi_{3}+z^{2}}\right) \\
1-2 s-2\left(\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right) s-\sqrt{\xi_{3}+z^{2}}\right)
\end{array}\right.
$$

Derivative of $\tau(s)$ is obtained as

$$
\tau^{\prime}=\left\{\begin{array}{l}
-2+2\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}-\sqrt{\xi_{3}+z^{2}}\right)  \tag{28}\\
-2-2\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}-\sqrt{\xi_{3}+z^{2}}\right) \\
-2+2\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right) \\
-2-2\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right)
\end{array}\right.
$$

Here, first derivative of $\tau$ should be $\tau^{\prime}<0$ in order to obtain physical solutions. Thus we choose $k$ and our functions which help us to derive the energy eigenvalues and eigenfunctions:

$$
\begin{align*}
k & =\xi_{2}-2 \xi_{3}-2 \sqrt{\xi_{3}\left(\xi_{1}-\xi_{2}+\xi_{3}+z^{2}\right)-z^{2}\left(\xi_{2}-\xi_{1}\right)}  \tag{29}\\
\tau & =1-2 s-2\left[\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right) s-\sqrt{\xi_{3}+z^{2}}\right]  \tag{30}\\
\pi & =z(1-s)-\left[\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right) s-\sqrt{\xi_{3}+z^{2}}\right]  \tag{31}\\
\tau^{\prime} & =-2-2\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right) . \tag{32}
\end{align*}
$$

Using Eq.(19), the relation given below

$$
\begin{equation*}
\lambda=z^{2}-z+\xi_{1}-\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right)^{2}-\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right) \tag{33}
\end{equation*}
$$

is obtained. With the aid of Eq.(20), this equality can be written:

$$
\begin{equation*}
\lambda=\lambda_{n}=-n\left(-2-2\left(\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}\right)\right)+n(n-1) \tag{34}
\end{equation*}
$$

Substituting $\Lambda=\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}, \Lambda$ can be written

$$
\begin{equation*}
\Lambda=\frac{1}{2}(-(2 n+1) \pm \sqrt{1+4 \gamma}) \tag{35}
\end{equation*}
$$

where $\gamma=\xi_{1}+z(z-1)$. Now let us discuss two cases here depending on signs of $\Lambda$. Case 1:

$$
\begin{equation*}
\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}=\frac{1}{2}(-(2 n+1)+\sqrt{1+4 \gamma}) \tag{36}
\end{equation*}
$$

then, $\xi_{3}$ is obtained as

$$
\begin{equation*}
\xi_{3}=\left(\frac{\xi_{2}-\xi_{1}+z^{2}}{2 n+1-\sqrt{1+4 \gamma}}+\frac{1}{4}(2 n+1-\sqrt{1+4 \gamma})\right)^{2} \tag{37}
\end{equation*}
$$

Using the definitions of $\xi_{1}, \xi_{2}$ and $\xi_{3}, E_{n}$ is given by

$$
\begin{equation*}
E_{n}=-\frac{\lambda^{2}}{4}\left(2 n+1-\sqrt{1+4 \gamma}-2 \sqrt{-\eta(\eta-1)-A^{*}+\frac{\beta+1}{2}}\right)^{2}-\lambda^{2}\left(\eta-\frac{1}{2}\right)^{2} \tag{38}
\end{equation*}
$$

Case 2:

$$
\begin{equation*}
\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}+\sqrt{\xi_{3}+z^{2}}=\frac{1}{2}(-(2 n+1)-\sqrt{1+4 \gamma}) \tag{39}
\end{equation*}
$$

then, $\xi_{3}$ reads

$$
\begin{equation*}
\xi_{3}=\left(\frac{\xi_{2}-\xi_{1}+z^{2}}{2 n+1+\sqrt{1+4 \gamma}}-\frac{1}{4}(2 n+1+\sqrt{1+4 \gamma})\right)^{2} \tag{40}
\end{equation*}
$$

Energy eigenvalues can be written:

$$
\begin{equation*}
E_{n}=\frac{\lambda^{2}}{4}\left(2 n+1+\sqrt{1+4 \gamma}+2 \sqrt{-\eta(\eta-1)-A^{*}+\frac{\beta+1}{2}}\right)^{2}-\lambda^{2}\left(\eta-\frac{1}{2}\right)^{2} \tag{41}
\end{equation*}
$$

Using $\operatorname{Eqs}(24)$ and (25), $\phi$ and $\rho$ are obtained as

$$
\begin{equation*}
\phi=s^{z+\sqrt{\xi_{3}+z^{2}}}(1-s)^{\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}} \tag{42}
\end{equation*}
$$

and

$$
\begin{equation*}
\rho(s)=s^{2 \sqrt{\xi_{3}+z^{2}}}(1-s)^{2 \sqrt{\xi_{1}-\xi_{2}+\xi_{3}}} \tag{43}
\end{equation*}
$$

Solution of $y$ can be obtained from Eq.(23):

$$
\begin{equation*}
y_{n}(s)=P_{n}^{\left(2 \sqrt{\xi_{3}+z^{2}}, 2 \sqrt{\xi_{1}-\xi_{2}+\xi_{3}}\right.}(1-2 s) . \tag{44}
\end{equation*}
$$

Hence, the wave function has the following form:

$$
\begin{equation*}
\psi_{n}=s^{z+\sqrt{\xi_{3}+z^{2}}}(1-s)^{\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}} P_{n}^{\left(2 \sqrt{\xi_{3}+z^{2}}, 2 \sqrt{\xi_{1}-\xi_{2}+\xi_{3}}\right)}(1-2 s) \tag{45}
\end{equation*}
$$

If $z+\sqrt{\xi_{3}+z^{2}}<0$ and $\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}>0$, it is required that $\left|z+\sqrt{\xi_{3}+z^{2}}\right| \geq \sqrt{\xi_{1}-\xi_{2}+\xi_{3}}$ and if $\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}<0, z+\sqrt{\xi_{3}+z^{2}}>0,\left|\sqrt{\xi_{1}-\xi_{2}+\xi_{3}}\right| \geq z+\sqrt{\xi_{3}+z^{2}}$ for physical solutions.

## 4 Conclusions

NU method adapted solutions are obtained for Hulthen potential within PDEM Schrödinger equation. We have proposed a transformation of the wavefunction in a general form that leads to solutions of well-known eigenvalues and eigenfunctions of Hulthen potential. Furthermore, energy relations of the mass independent equation are obtained for two cases.

## 5 Acknowledgements

This research was partially supported by the Scientific and Technological Research Council of Turkey.

## References

[1] G Bastard, Wave Mechanics Applied to Semiconductor Heterostructures (Les Ulis: Editions de Physique), 1998.
[2] L I Serra, E Lipparini, Europhys. Lett. 401997667.
[3] M Barranco, M Pi, S M Gatica, E S Hernandez, J Navarro, Phys. Rev. B 5619978997.
[4] F de Saavedra Arias et al, Phys. Rev. B 5019944248.
[5] Von Roos O 1983 Phys. Rev. B 277547.
[6] L Dekar, L Chetouani, T F Hammann, J. Math. Phys. 39 (1998) 2551.
[7] A R Plastino, A Rigo, M Casas, F Garcias, A Plastino, Phys. Rev. A 60 (1999) 4318
[8] A Ganguly, L M Nieto, J. Phys. A, 40 (2007) 7265; Chun-Sheng Jia, Jian-Yi Liu, PingQuan Wang, Chao-Shan Che, Phys. Lett. A, 369 (2007) 274.
[9] B Bagchi, J. Phys. A: Math. Theor. 40 (2007) F1041.
[10] T Tanaka, J. Phys. A: Math. Gen. 392006 219-234.
[11] B Roy, P Roy, J. Phys. A 35 (2002) 3961.
[12] B Roy, Europhys. Lett. 722005 1-6; R Koç, M Koca, J. Phys. A: Math. Gen. 36 (2003) 81058112.
[13] C Quesne, SIGMA 3 (2007) 067.
[14] G Chen, Zi-dong Chen, Phys. Lett. A, 331(5) 2004 312-315.
[15] R De, R Dutt and U Sukhatme, J. Phys. A, 25(13) 1992 L843-L850.
[16] A D Alhaidari, Phys. Rev. A 662002042116.
[17] Shang-Wu Qian et al, New J. of Physics, 42002 13.1-13.6; Chen Gang, Chinese Phys. 14 2005 460-462.
[18] B Bagchi, P Gorain, C Quesne, R Roychoudhury, Mod. Phys. Lett. A 1920042765.
[19] J Yu, S H Dong, G H Sun, Phys. Lett. A, 322 (2004) 290.
[20] J Yu, S H Dong, Phys. Lett. A, 325 (2004) 194.
[21] S H Dong, M Lozada-Cassou, Phys. Lett. A, 337 (2005) 313.
[22] L Jiang, L Z Yi, C S Jia, Phys. Lett. A, 345 (2005) 279.
[23] A F Nikiforov and V B Uvarov, Special Functions of Mathematical Physics (Birkhauser, Bassel, 1988).
[24] H Eg̃rifes, D Demirhan, F Büyükkiliç, Phys. Lett. A, 275(4) 2000 229-237; H Eg̃rifes, D Demirhan, F Büyükkiliç, Phys. Scr. 59 No 2 (1999) 90-94.
[25] Ö Yeşiltaş M. Şimşek, R. Sever, C. Tezcan, Phys. Scr. 67 (2003) 472-475.
[26] H Eg̃rifes, R Sever, Phys. Lett. A, 344(2-4) 2005 117-126.
[27] H Eg̃rifes, R Sever, Int. J. of Theo. Phys., 46(4) 2007935.
[28] S M Ikhdair, R Sever, arXiv:quant-ph/0605045v1
[29] B. Bagchi, P.S. Gorain, C. Quesne, Mod. Phys. Lett. A 21 (2006) 2703-2708.

