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Stability and asymmetric vibrations of pressurized compressible hyperelastic cylindrical shells

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Cylindrical shells of arbitrary wall thickness subjected to uniform radial tensile or compressive dead-load traction are investigated. The material of the shell is assumed to be homogeneous, isotropic, compressible and hyperelastic. The stability of the finitely deformed state and small, free, radial vibrations about this state are investigated using the theory of small deformations superposed on large elastic deformations. The governing equations are solved numerically using both the multiple shooting method and the finite element method. For the finite element method the commercial program ABAQUS is used.¹ The loss of stability occurs when the motions cease to be periodic. The effects of several geometric and material properties on the stress and the deformation fields are investigated.

1. Introduction

The research on compressible, isotropic, hyperelastic solids is relatively new and the scope of these works is more restricted due to the fact that the only controllable deformations for compressible, isotropic, hyperelastic solids are homogeneous deformations as was proved by Ericksen [1] in 1955. The earlier works in this area were limited to determining a suitable model representing compressible, isotropic, hyperelastic behavior [2–4], and to analyzing deformations and stresses in bodies of different geometries which undergo finite elastic deformations for various boundary conditions [5–8].

In 1982, Ball [9] investigated discontinuous equilibrium solutions and cavitation in non-linear elasticity by modeling the appearance of a cavity in the interior of a solid, homogeneous, isotropic, hyperelastic body once a critical load is reached. Following Ball's work, a class of problems concerning the void nucleation and growth in such bodies has been the subject of extensive research [10–13]. Horgan and Abeyaratne [14], and

Horgan [15] investigated, respectively, the bifurcation of a solid cylinder made of generalized Blatz–Ko material (foam rubber) and a solid sphere made of compressible Varga material, both subjected to a uniform radial tension on their outer surfaces. They concluded that “the bifurcation problem may be viewed as providing an idealized model describing the growth of a pre-existing micro-void”. A unified derivation of the necessary and sufficient conditions for ellipticity of the three-dimensional displacement equations was recently provided by Horgan [16] for the Blatz–Ko material. In [16], it was shown that when the material constant f is equal to unity, i.e. the strain energy density function is independent of the second invariant, the equilibrium equations are globally elliptic while ellipticity is always lost when $0 \leq f < 1.0$.

One of the most important contributions to finite elasticity of compressible hyperelastic materials is due to Carroll [17]. Emphasis in [17] was placed on determining the forms of admissible deformation fields for John’s material [2] and two new classes introduced by the author himself. Recently, Murphy [18] examined the most general compressible, hyperelastic material for which plane strain cylindrical inflation is possible. He concluded that, although the finite inflation and compaction of compressible cylindrical shell is possible for harmonic material proposed by John, and Carroll material, the most general form is the so-called generalized Varga material introduced by Haughton.

Fewer results are available for the stability and vibrations of finitely deformed, compressible, hyperelastic solids. Roxborough and Ogden [19] analyzed the stability and vibrations of a compressible, hyperelastic, rectangular plate subjected to finite, pure homogeneous deformations. Vandyke and Wineman [20] studied small amplitude sinusoidal vibrations of a compressible circular cylinder which undergoes a circumferential shear deformation only. In [20] it is concluded that, the prestress solution has a significant effect on the response in the radial direction while it has a small effect on the response in the circumferential direction. In a recent work, Akyuz and Ertepinar [21] studied the breathing motion and loss of stability of infinitely long, circular cylindrical shells of arbitrary wall thickness subjected to finite radial deformations. In [21] the effect of material constants and the shell thickness on the frequency and the critical outer stretch ratio were analyzed numerically. It was observed in [21] that, for the polynomial hyperelastic material, the system is always stable when f is equal to unity, while failure occurs without bound at a critical value of the external tensile load when $0 \leq f < 1.0$.

The present work deals with asymmetric vibrations and the loss of stability of infinitely long, circular cylindrical shells of arbitrary wall thickness subjected to finite radial inward and outward deformations. The material of the body is assumed to be a polynomial material [4] which is homogeneous, isotropic, compressible, hyperelastic and reduces to Blatz–Ko material when the material is a foam rubber. The shell is first subjected to a uniform, radial tensile or compressive dead load traction on its exterior surface. The stress and the displacement fields of this finitely deformed state are expressed using the theory of finite elasticity [22]. The resulting highly non-linear differential system of this state is solved numerically by using a multiple shooting method [23]. The shell is then assumed to undergo small, free asymmetric vibrations about the prestressed state. The formulation of this state is based on the theory of small deformations superposed on large elastic deformations [24]. The associated boundary conditions of this state are obtained from the requirement that the secondary surface tractions vanish for free vibrations. The homogeneous, linear differential system governing the secondary state is solved numerically using the method of complementary functions. For a non-trivial solution of the problem, it is required that the characteristic determinant of the system vanishes. This determinant contains parameters pertaining to the finitely deformed state, the frequencies of small, free vibrations about this state, harmonic mode number, material constants and the initial geometry of the shell. Hence, the solution of this equation yields the frequencies numerically. The loss of stability occurs when the motions about the finitely deformed state cease to be periodic, i.e. when the frequency of vibrations equals zero.

Numerical results are obtained to investigate the effects of several geometric and material properties on the frequencies and the critical stretch of the outer surface when instability occurs. The problem is also solved by

using the commercial non-linear finite element code ABAQUS [25] for some geometric and material properties.

2. Formulation and the analysis of the problem

Consider a long circular cylindrical shell made of a homogeneous, isotropic, compressible and hyperelastic material. Let the inner and outer radii of the shell be denoted, respectively, by r_1 and r_2 . The shell is subjected to a uniform radial tensile or compressive dead-load traction q on its exterior surface. The coordinates of a material point in the undeformed and the deformed states are, respectively, given by²

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z \quad (2.1)$$

and

$$X_1 = R(r) \cos \theta, \quad X_2 = R(r) \sin \theta, \quad X_3 = z. \quad (2.2)$$

The non-zero contravariant components of the metric tensors of the undeformed and deformed states, g^{ij} and G^{ij} , are

$$(g^{11}, g^{22}, g^{33}) = (R'^2, 1/r^2, 1), \quad (G^{11}, G^{22}, G^{33}) = (1, 1/R^2, 1), \quad (2.3)$$

where a prime denotes derivative with respect to r . The three strain invariants of the deformation field are given by

$$I_1 = R'^2 + \frac{R^2}{r^2} + 1, \quad I_2 = R'^2 + \frac{R^2}{r^2} + \frac{R^2 R'^2}{r^2}, \quad I_3 = \frac{R^2 R'^2}{r^2}. \quad (2.4)$$

For bodies made of a homogeneous, isotropic, hyperelastic material it is assumed that the strain energy density function W can be expressed in terms of three strain invariants as

$$W = W(I_1, I_2, I_3). \quad (2.5)$$

The components of the stress tensor τ^{ij} of the strained body are given by

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + p G^{ij}, \quad (2.6)$$

where

$$\begin{aligned} \Phi &= \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1}, \quad \Psi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2}, \quad p = 2\sqrt{I_3} \frac{\partial W}{\partial I_3}, \\ B^{ij} &= I_1 g^{ij} - g^{ir} g^{js} G_{rs}. \end{aligned} \quad (2.7)$$

It is now further assumed that the strain energy density function W has the form

$$W = \frac{\mu}{2} \left[f(I_1 - 3) + (1 - f) \left(\frac{I_2}{I_3} - 3 \right) + 2(1 - 2f)(\sqrt{I_3} - 1) + (2f + \beta)(\sqrt{I_3} - 1)^2 \right], \quad (2.8)$$

which has been proposed by Levinson and Burgess [4] and has been named ‘‘polynomial compressible material’’ by the authors. In Eq. (2.8), μ is the shear modulus of the material for vanishingly small strains, f is

²The details of the formulation can be found in the text by Green and Zerna [22].

a material constant whose value lies between zero and unity, and β is expressed as

$$\beta = \frac{4\nu - 1}{1 - 2\nu}, \quad (2.9)$$

where ν is the Poisson's ratio for the material as the deformations become vanishingly small. It is noted that for highly elastic rubbers and rubber-like materials $f = 0$ while for solid natural and synthetic rubbers $f = 1$. When $\nu \rightarrow \frac{1}{2}$ the expression for the strain energy density function reduces to that of a neo-Hookean material. It is also noted that Levinson–Burgess and Blatz–Ko models are identical for $\nu = 0.25$ and $f = 0$.

The only non-zero equation of equilibrium for this finitely deformed state is the one in the radial direction which is given by

$$\frac{\partial \tau^{11}}{\partial R} + \frac{\tau^{11} - R^2 \tau^{22}}{R} = 0. \quad (2.10)$$

Using Eqs. (2.3), (2.4), (2.6)–(2.8), Eq. (2.10) reduces to

$$f \left(\frac{R'}{R} - \frac{1}{r} + \frac{rR''}{R} \right) + (1-f) \left(\frac{r^3}{R^4} - \frac{1}{RR'^3} + \frac{3rR''}{RR'^4} \right) + (2f + \beta) \left(\frac{R'^2}{r} - \frac{RR'}{r^2} + \frac{RR''}{r} \right) = 0. \quad (2.11)$$

The associated boundary conditions for the shell which is assumed to be free of tractions on its inner surface and subjected to a uniform radial tensile or compressive dead-load traction on its exterior surface, are

$$\tau^{11}(R_1) = 0, \quad \tau^{11}(R_2) = \mp q \left(\frac{r_2}{R_2} \right)^2. \quad (2.12)$$

The shell is now exposed to a secondary dynamic displacement field described by

$$w_1 = u(R(r), \theta, t), \quad w_2 = Rv(R(r), \theta, t), \quad w_3 = 0 \quad (2.13)$$

to investigate the existence of small, free, radial vibrations about the finitely deformed state.

The formulation of this state is based on the theory of small deformations superposed on large elastic deformations³ [24]. The incremental metric tensors and the incremental stresses are given by

$$G_{ij}^* = w_{i,j} + w_{j,i} - 2\Gamma_{ij}^r w_r, \quad G^{*ij} = -G^{ir} G^{js} G_{rs}^*, \quad (2.14)$$

$$\tau^{*ij} = g^{ij} \Phi^* + B^{ij} \Psi^* + B^{*ij} \Psi + G^{*ij} p + G^{ij} p^*, \quad (2.15)$$

where

$$\Phi^* = \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1^2} I_1^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_2} I_2^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_3} I_3^* - \frac{\Phi}{2I_3} I_3^*,$$

$$\Psi^* = \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_2} I_1^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_2^2} I_2^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_2 \partial I_3} I_3^* - \frac{\Psi}{2I_3} I_3^*,$$

$$p^* = I_3 \left(\frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_3} I_1^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_2 \partial I_3} I_2^* + \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_3^2} I_3^* \right) + \frac{p}{2I_3} I_3^*,$$

$$B^{*ij} = (g^{ij} g^{rs} - g^{ir} g^{js}) G_{rs}^* \quad (2.16)$$

³A detailed discussion of the theory has been given in Ref. [22].

and Γ_{ij}^r are the Christoffel symbols of the second kind, and a comma denotes differentiation with respect to the following subscript. The incremental strain invariants I_1^* , I_2^* , and I_3^* are given by

$$\begin{aligned} I_1^* &= 2R'^2 w_{1,R} + \frac{2}{r^2} (w_{2,\theta} + R w_1), \\ I_2^* &= \left(2R'^2 + \frac{2R'^2 R^2}{r^2} \right) w_{1,R} + \left(\frac{2}{r^2} + \frac{2R'^2}{r^2} \right) (w_{2,\theta} + R w_1), \\ I_3^* &= 2 \frac{R^2 R'^2}{r^2} \left(w_{1,R} + \frac{1}{R^2} w_{2,\theta} + \frac{1}{R} w_1 \right). \end{aligned} \quad (2.17)$$

The incremental equations of motion are

$$\begin{aligned} \frac{\partial \tau^{*11}}{\partial R} + \frac{\partial \tau^{*21}}{\partial \theta} + \frac{1}{R} (\tau^{*11} - R^2 \tau^{*22}) + \left(2w_{1,RR} + \frac{1}{R^2} w_{2,R\theta} \right) \tau^{11} + w_{1,\theta\theta} \tau^{22} \\ + \frac{1}{R^2} \left(R w_{1,R} - \frac{2}{R} w_{2,\theta} - w_1 \right) (\tau^{11} + R^2 \tau^{22}) = \rho \ddot{w}_1 \end{aligned} \quad (2.18a)$$

in radial direction, and

$$\begin{aligned} \frac{\partial \tau^{*12}}{\partial R} + \frac{\partial \tau^{*22}}{\partial \theta} + \frac{3}{R} \tau^{*12} + \frac{1}{R^2} \left(w_{2,RR} + \frac{2}{R^2} w_2 - \frac{2}{R} w_{2,R} \right) \tau^{11} \\ + \left(w_{1,R\theta} - \frac{2}{R^2} w_{2,\theta\theta} + \frac{3}{R} w_{1,\theta} + \frac{1}{R} w_{2,R} - \frac{2}{R^2} w_2 \right) \tau^{22} = \frac{\rho}{R^2} \ddot{w}_2 \end{aligned} \quad (2.18b)$$

in circumferential direction. The equation of motion in axial direction is trivially satisfied. Upon substitution of Eqs. (2.6)–(2.8) and (2.13)–(2.17) equations of motion reduce to

$$\begin{aligned} u_{,rr} \left\{ f \frac{r}{RR'} + (1-f) \frac{3r}{RR'^5} + (2f + \beta) \frac{R}{rR'} \right\} \\ + u_{,r} \left\{ f \left(\frac{1}{rR'^2} - \frac{rR''}{RR'^2} \right) + (1-f) \left(\frac{4}{RR'^5} - \frac{15rR''}{RR'^6} - \frac{r^3}{R^4 R'^2} \right) + (2f + \beta) \left(\frac{1}{r} - \frac{RR''}{rR'^2} \right) \right\} \\ + u \left\{ f \left(\frac{1}{rRR'} - \frac{2rR''}{R^2 R'} - \frac{2}{R^2} \right) + (1-f) \left(\frac{2}{R^2 R'^4} - \frac{6rR''}{R^2 R'^5} - \frac{5r^3}{R^5 R'} \right) - (2f + \beta) \frac{R'}{rR} \right\} \\ + v_{,\theta} \left\{ -2f \left(\frac{1}{R^2} + \frac{rR''}{R^2 R'} \right) + (1-f) \left(\frac{2}{R^2 R'^4} - \frac{r}{R^3 R'^3} - \frac{6rR''}{R^2 R'^5} - \frac{5r^3}{R^5 R'} \right) - (2f + \beta) \frac{R'}{rR} \right\} \\ + u_{,\theta\theta} \left\{ \frac{1}{rRR'} + (1-f) \frac{r}{R^3 R'^3} \right\} + v_{,r\theta} \left\{ (1-f) \left(\frac{r}{R^2 R'^4} + \frac{r^3}{R^4 R'^2} \right) + (2f + \beta) \frac{1}{r} \right\} = \frac{\rho}{\mu} \ddot{u}, \end{aligned} \quad (2.19a)$$

$$\begin{aligned}
& v_{,rr} \left\{ f \frac{r}{R^2 R'} + (1-f) \frac{r^3}{R^4 R'^3} \right\} + v_{,r} \left\{ f \frac{1}{R^2 R'} + (1-f) \left(\frac{3r^2}{R^4 R'^3} - \frac{2r^3 R''}{R^4 R'^4} - \frac{2r^3}{R^5 R'^2} \right) \right\} \\
& + v \left\{ -f \left(\frac{1}{R^3} + \frac{r R''}{R^3 R'} \right) + (1-f) \left(\frac{2r^3}{R^6 R'} + \frac{r^3 R''}{R^5 R'^3} - \frac{3r^2}{R^5 R'^2} \right) \right\} \\
& + v_{,\theta\theta} \left\{ f \frac{1}{r R^2 R'} + (1-f) \frac{3r^3}{R^6 R'} + (2f + \beta) \frac{R'}{r R^2} \right\} \\
& + u_{,\theta} \left\{ f \left(\frac{1}{R^3} + \frac{1}{r R^2 R'} + \frac{r R''}{R^3 R'} \right) + (1-f) \left(\frac{r^3}{R^6 R'} - \frac{r^3 R''}{R^5 R'^3} + \frac{3r^2}{R^5 R'^2} \right) + (2f + \beta) \frac{R'}{r R^2} \right\} \\
& + u_{,r\theta} \left\{ (1-f) \left(\frac{r}{R^3 R'^4} + \frac{r^3}{R^5 R'^2} \right) + (2f + \beta) \frac{1}{r R} \right\} = \frac{\rho}{\mu} \ddot{v}, \tag{2.19b}
\end{aligned}$$

where ρ is the mass density of finitely deformed body and a dot denotes differentiation with respect to time. The current mass density ρ is related to the mass density ρ_0 of the natural state by

$$\rho = \frac{\sqrt{g}}{\sqrt{G}} \rho_0, \tag{2.20}$$

using the principle of conversation of mass. In Eq. (2.20), g and G denote, respectively, the determinants $|g_{ij}|$ and $|G_{ij}|$. For this secondary state, the boundary conditions, which are obtained from the requirement that the secondary surface tractions vanish, are

$$\begin{aligned}
\tau^{*11} - G^{*11} \tau^{11} &= 0 \quad \text{at } R = R_1 \quad \text{and} \quad R = R_2, \\
\tau^{*12} - G^{*12} \tau^{11} &= 0 \quad \text{at } R = R_1 \quad \text{and} \quad R = R_2.
\end{aligned} \tag{2.21}$$

The solution of Eqs. (2.19a) and (2.19b) may be assumed to be of the form

$$u = \sum_{n=0}^{\infty} R_{1n}^*(R) \cos(n\theta) e^{i\omega t}, \quad v = \sum_{n=0}^{\infty} R_{2n}^*(R) \sin(n\theta) e^{i\omega t}, \tag{2.22}$$

where ω is the frequency of vibrations about the finitely deformed state, n is the circumferential mode number, and R_{1n}^* and R_{2n}^* are unknown functions of R which, in turn, is a function of r . The case $n=0$ corresponds to pure radial vibrations about the prestressed state which was studied in [21]. The case $n=1$ includes rigid body motions and is left out of discussion with the understanding that the rigid body motions of the shell are prevented. The higher modes corresponding to $n \geq 2$ are considered in what follows.

The governing equations of both states are now non-dimensionalized by introducing

$$\begin{aligned}
\bar{r} &= \frac{r}{r_2}, \quad \bar{R} = \frac{R}{r_2}, \quad \bar{R}_{1n}^* = \frac{\bar{R}_{1n}^*}{r_2}, \quad \bar{R}_{2n}^* = \frac{R_{2n}^*}{r_2}, \\
\bar{\tau}^{11} &= \frac{\tau^{11}}{\mu}, \quad \bar{q} = \frac{q}{\mu}, \quad \bar{u} = \frac{u}{r_2}, \quad \bar{\omega} = \omega \sqrt{\frac{\rho_0 r_2^2}{\mu}}.
\end{aligned} \tag{2.23}$$

Hence, the non-dimensional form of the equation of equilibrium (2.11) of the finitely deformed state is

$$f\left(\frac{\bar{R}'}{\bar{R}} - \frac{1}{\bar{r}} + \frac{\bar{r}\bar{R}''}{\bar{R}}\right) + (1-f)\left(\frac{\bar{r}^3}{\bar{R}^4} - \frac{1}{\bar{R}\bar{R}^3} + \frac{3\bar{r}\bar{R}''}{\bar{R}\bar{R}'^4}\right) + (2f+\beta)\left(\frac{\bar{R}'^2}{\bar{r}} - \frac{\bar{R}\bar{R}'}{\bar{r}^2} + \frac{\bar{R}\bar{R}''}{\bar{r}}\right) = 0, \quad (2.24)$$

while the associated boundary conditions given by Eq. (2.12) reduce to

$$\begin{aligned} f\frac{\bar{r}_1\bar{R}'_1}{\bar{R}_1} - (1-f)\frac{\bar{r}}{\bar{R}_1\bar{R}'_1} + (2f+\beta)\frac{\bar{R}_1\bar{R}'_1}{\bar{r}_1} + (1-4f-\beta) &= 0, \\ f\frac{\bar{R}'_2}{\bar{R}_2} - (1-f)\frac{1}{\bar{R}_2\bar{R}'_2} + (2f+\beta)\bar{R}_2\bar{R}'_2 + (1-4f-\beta) &= \bar{\eta}\bar{q}\left(\frac{1}{\bar{R}_2}\right)^2. \end{aligned} \quad (2.25)$$

The non-dimensional governing equations for the secondary dynamical state is obtained by substituting Eqs. (2.22) and (2.23) into Eqs. (2.19a), (2.19b), and (2.20) as

$$\begin{aligned} A\bar{R}_{1n,rr}^* + B\bar{R}_{1n,r}^* + C\bar{R}_{1n}^* + D\bar{R}_{2n,r}^* + E\bar{R}_{2n}^* &= 0, \\ F\bar{R}_{2n,rr}^* + G\bar{R}_{2n,r}^* + H\bar{R}_{2n}^* + I\bar{R}_{1n,r}^* + J\bar{R}_{1n}^* &= 0, \end{aligned} \quad (2.26)$$

where

$$\begin{aligned} A &= f + (1-f)\frac{3}{\bar{R}'^4} + (2f+\beta)\frac{\bar{R}^2}{\bar{r}^2}, \\ B &= f\left(\frac{\bar{R}}{\bar{r}^2\bar{R}'} - \frac{\bar{R}''}{\bar{R}'}\right) + (1-f)\left(\frac{4}{\bar{r}\bar{R}'^4} - \frac{15\bar{R}''}{\bar{R}'^5} - \frac{\bar{r}^2}{\bar{R}^3\bar{R}'}\right) + (2f+\beta)\left(\frac{\bar{R}\bar{R}'}{\bar{r}^2} - \frac{\bar{R}^2\bar{R}''}{\bar{r}^2\bar{R}'}\right), \\ C &= f\left(\frac{1-n^2}{\bar{r}^2} - \frac{2\bar{R}''}{\bar{R}} - \frac{2\bar{R}'}{\bar{r}\bar{R}}\right) + (1-f)\left(\frac{2}{\bar{r}\bar{R}\bar{R}'^3} - \frac{6\bar{R}''}{\bar{R}\bar{R}'^4} - \frac{5\bar{r}^2}{\bar{R}^4} - \frac{n^2}{\bar{R}^2\bar{R}'^2}\right) - (2f+\beta)\frac{\bar{R}'^2}{\bar{r}^2} + \bar{\omega}^2, \\ D &= n\left\{(1-f)\left(\frac{1}{\bar{R}\bar{R}'^3} + \frac{\bar{r}^2}{\bar{R}^3\bar{R}'}\right) + (2f+\beta)\frac{\bar{R}\bar{R}'}{\bar{r}^2}\right\}, \\ E &= n\left\{-2f\left(\frac{\bar{R}'}{\bar{r}\bar{R}} + \frac{\bar{R}''}{\bar{R}}\right) + (1-f)\left(\frac{2}{\bar{r}\bar{R}\bar{R}'^3} - \frac{1}{\bar{R}^2\bar{R}'^2} - \frac{6\bar{R}''}{\bar{R}\bar{R}'^4} - \frac{5\bar{r}^2}{\bar{R}^4}\right) - (2f+\beta)\frac{\bar{R}'^2}{\bar{r}^2}\right\}, \end{aligned} \quad (2.27a)$$

$$F = f + (1-f)\frac{\bar{r}^2}{\bar{R}^2\bar{R}'^2},$$

$$G = f\frac{1}{\bar{r}} + (1-f)\left(\frac{3\bar{r}}{\bar{R}^2\bar{R}'^2} - \frac{2\bar{r}^2\bar{R}''}{\bar{R}^2\bar{R}'^3} - \frac{2\bar{r}^2}{\bar{R}^3\bar{R}'}\right),$$

$$H = -f\left(\frac{n^2}{\bar{r}^2} + \frac{\bar{R}'}{\bar{r}\bar{R}} + \frac{\bar{R}''}{\bar{R}}\right) + (1-f)\left(\frac{\bar{r}^2}{\bar{R}^4}(2-3n^2) + \frac{\bar{r}^2\bar{R}''}{\bar{R}^3\bar{R}'^2} - \frac{3\bar{r}}{\bar{R}^3\bar{R}'}\right) + \bar{\omega}^2,$$

$$I = n\left\{(1-f)\left(\frac{1}{\bar{R}\bar{R}'^3} + \frac{\bar{r}^2}{\bar{R}^3\bar{R}'}\right) + (2f+\beta)\frac{\bar{R}\bar{R}'}{\bar{r}^2}\right\},$$

$$J = n\left\{f\left(\frac{\bar{R}'}{\bar{r}\bar{R}} + \frac{1}{\bar{r}^2} + \frac{\bar{R}''}{\bar{R}}\right) + (1-f)\left(\frac{\bar{r}^2}{\bar{R}^4} - \frac{\bar{r}^2\bar{R}''}{\bar{R}^3\bar{R}'^2} + \frac{3\bar{r}}{\bar{R}^3\bar{R}'}\right) + (2f+\beta)\frac{\bar{R}'^2}{\bar{r}^2}\right\}. \quad (2.27b)$$

The boundary conditions, Eq. (2.21), of this state reduce to

$$\begin{aligned} M\bar{R}_{1n,r}^* + N(\bar{R}_{1n}^* + n\bar{R}_{2n}^*) &= 0 \quad \text{at } \bar{R} = \bar{R}_1 \quad \text{and} \quad \bar{R} = \bar{R}_2, \\ \frac{\bar{R}}{\bar{R}'} \bar{R}_{2n,r}^* - (\bar{R}_{2n}^* + n\bar{R}_{1n}^*) &= 0 \quad \text{at } \bar{R} = \bar{R}_1 \quad \text{and} \quad \bar{R} = \bar{R}_2, \end{aligned} \quad (2.28)$$

where

$$\begin{aligned} M &= f \frac{\bar{r}}{\bar{R}} + (1-f) \frac{3\bar{r}}{\bar{R}\bar{R}'^4} + (2f+\beta) \frac{\bar{R}}{\bar{r}}, \\ N &= -f \frac{\bar{r}\bar{R}'}{\bar{R}^2} + (1-f) \frac{\bar{r}}{\bar{R}^2\bar{R}'^3} + (2f+\beta) \frac{\bar{R}'}{\bar{r}}. \end{aligned} \quad (2.29)$$

For the finitely deformed state, no closed-form solution of the highly non-linear system of equations. Eqs. (2.24) and (2.25), seems possible. To solve these equations numerically, the boundary-value problem is converted to an initial-value problem by using the multiple shooting method [23]. For this purpose, Eq. (2.24) is first converted to a set of two first-order equations of the form

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= \frac{f(1/x - y_2/y_1) + (1-f)(1/y_1y_2^3 - x^3/y_1^4) + (2f+\beta)(y_1y_2/x^2 - y_2^2/x)}{fx/y_1 + (1-f)3x/y_1y_2^4 + (2f+\beta)y_1/x} \end{aligned} \quad (2.30)$$

and the associated boundary conditions are now expressed as

$$\begin{aligned} f \frac{xy_2}{y_1} - (1-f) \frac{x}{y_1y_2^3} + (2f+\beta) \frac{y_1y_2}{x} + (1-4f-\beta) &= 0 \quad \text{at } \bar{R}_1, \\ f \frac{y_2}{y_1} - (1-f) \frac{1}{y_1y_2^3} + (2f+\beta)y_1y_2 + (1-4f-\beta) &= \bar{q} \left(\frac{1}{y_1} \right)^2 \quad \text{at } \bar{R}_2, \end{aligned} \quad (2.31)$$

where

$$x = \bar{r}, \quad y_1 = \bar{R}, \quad y_2 = \bar{R}'. \quad (2.32)$$

The solution of Eq. (2.30) together with Eq. (2.31) is obtained by using a FORTRAN code called BVPSOL and developed by Deuffhard and Bader [23]. This subroutine is a “boundary value problem solver for highly non-linear two point boundary-value problems using local linear solver (condensing algorithm) for the solution of the arising linear subproblems by multiple shooting approach”. In this study 1991 version of BVPSOL is used. For a nearly solid shell 5001 equally spaced nodal points along the radial direction are used to reach the desired accuracy.

Eq. (2.26) governing the secondary state are coupled linear ordinary differential equations. Since no closed-form solution is available for the finitely deformed state, the coefficients $A, B, C, D, E, F, G, H, I,$ and J in Eq. (2.26) contain constants known pointwise and the unknown frequency, and, hence, a numerical approach must be used to obtain the solution of this state. The method of complementary functions is used to transform the linear boundary-value problem to an initial-value problem. For this purpose, the equation of motion (2.28) are converted to four first-order ordinary differential equations as

$$z_1' = z_3, \quad z_2' = z_4, \quad z_3' = -\frac{Bz_3 + Cz_1 + Dz_4 + Ez_2}{A}, \quad z_4' = -\frac{Gz_4 + Hz_2 + Iz_3 + Jz_1}{F}, \quad (2.33)$$

where

$$z_1 = \bar{R}_{1n}^*, \quad z_2 = \bar{R}_{2n}^*, \quad z_3 = \bar{R}_{1n,r}^*, \quad z_4 = \bar{R}_{2n,r}^*. \quad (2.34)$$

The general solution of the unknown functions z_i , $i = 1, \dots, 4$ is assumed to be

$$z_i = \sum_{j=1}^4 b_j z_i^j, \quad i = 1, \dots, 4, \quad (2.35)$$

where b_j s are constants, and z_i^j s are the homogeneous solutions. To determine z_i^j at nodal points, Eq. (2.33) are integrated using Runge–Kutta method of order four with initial conditions $z_i^j = \delta_{ij}$, $i, j = 1, \dots, 4$ where δ_{ij} is the Kronecker delta. Hence, Eq. (2.28) now reduce to, in matrix form,

$$\begin{bmatrix} (Mz_3^1 + Nz_1^1 + nz_2^1)_{R_1} & (Mz_3^2 + Nz_1^2 + nz_2^2)_{R_1} & (Mz_3^3 + Nz_1^3 + nz_2^3)_{R_1} & (Mz_3^4 + Nz_1^4 + nz_2^4)_{R_1} \\ \left(\frac{\bar{R}}{\bar{R}'} z_4^1 - z_2^1 - nz_1^1\right)_{R_1} & \left(\frac{\bar{R}}{\bar{R}'} z_4^2 - z_2^2 - nz_1^2\right)_{R_1} & \left(\frac{\bar{R}}{\bar{R}'} z_4^3 - z_2^3 - nz_1^3\right)_{R_1} & \left(\frac{\bar{R}}{\bar{R}'} z_4^4 - z_2^4 - nz_1^4\right)_{R_1} \\ (Mz_3^1 + Nz_1^1 + nz_2^1)_{R_2} & (Mz_3^2 + Nz_1^2 + nz_2^2)_{R_2} & (Mz_3^3 + Nz_1^3 + nz_2^3)_{R_2} & (Mz_3^4 + Nz_1^4 + nz_2^4)_{R_2} \\ \left(\frac{\bar{R}}{\bar{R}'} z_4^1 - z_2^1 - nz_1^1\right)_{R_2} & \left(\frac{\bar{R}}{\bar{R}'} z_4^2 - z_2^2 - nz_1^2\right)_{R_2} & \left(\frac{\bar{R}}{\bar{R}'} z_4^3 - z_2^3 - nz_1^3\right)_{R_2} & \left(\frac{\bar{R}}{\bar{R}'} z_4^4 - z_2^4 - nz_1^4\right)_{R_2} \end{bmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (2.36)$$

where the subscript after the parenthesis denotes the boundary condition at that position. The characteristic determinant, which must vanish for a non-trivial solution, contains parameters pertaining to the finitely deformed state, material properties and initial geometry of the shell, and the unknown frequencies of small, free vibration about the finitely deformed state. When the frequency of vibrations ceases to be real valued, the corresponding finitely deformed state becomes unstable.

For the sake of comparison, the problem is also solved by using a finite element method. Recently, the radial compaction of a circular hyperelastic tube was examined by Dragoni [26] using a finite element method for nearly incompressible materials. By using the finite element code ABAQUS, he concluded that, for nearly incompressible materials ($\nu > 0.4999$), hybrid elements give better results than standard displacement field elements. In this study, the calculations are performed on a Sun Sparc 10 workstation using commercial non-linear finite element code ABAQUS version 5.6. A user subroutine for ABAQUS has been developed to model the displacement dependencies of mass density. In ABAQUS, the usage of complete circular geometry without any constraints leads to rigid body motion. In order to analyze the system with a reasonable accuracy separate constrained models were used for different modes with regard to the symmetry of the mode shapes. A quarter of a circular cylinder was created with 1000 elements with $v = 0$ at $\theta = \theta^\circ$, and $u = 0$ at $\theta = 90^\circ$ constraints for the second and fourth modes. For mode three, one-third of a circular cylinder is used ($30^\circ \leq \theta \leq 150^\circ$) and the solid body is divided into 1500 elements with $v = 0$ at $\theta = 30^\circ$ and $\theta = 150^\circ$, and $u = 0$ at $\theta = 90^\circ$ constraints. In all models four-node bilinear plane strain quadrilateral element CPE4 is used. The comparisons are made for some geometric and material properties.

3. Discussion of the results

Numerical examples are produced to compare the results obtained by the multiple shooting and the finite element methods and to investigate the effects of the material constants and the thickness ratio of the shell ($\chi = \bar{r}_1/\bar{r}_2$) on the frequencies of vibration and the critical stretch ratio. In particular, the behaviors of the foam rubber and the slightly compressible materials are investigated. When $f = 1.0$ and $\nu = 0.5$, the results obtained reduce to those obtained by Wang and Ertepinar [27].

For all the numerical examples considered, the results of the multiple shooting and the finite element methods almost coincide for all loading history except for high tension zone (see Fig. 1). This is mainly due to the discretization in the finite element method and improvement of results can be achieved using finer mesh sizes for high tension zone.

In Fig. 1, the variation of the frequency of vibrations as a function of the outer normal stress is displayed for a thickness ratio $\chi = 0.80$ and made of foam rubber with $f = 0.0$ and $\nu = 0.25$. For all modes of vibration, the shell exhibits a softening behavior for increasing compressive loading and a hardening behavior for increasing tensile loading. When the compressive surface traction reaches a critical value, the frequency of vibrations approaches zero which is equivalent to stating that the shell buckles in the corresponding mode. For all cases under investigation, it is observed that the fundamental mode of buckling corresponds to $n = 2$ and, hence, further analysis is restricted to the case $n = 2$.

The response of the shell made of a nearly incompressible polynomial material with $f = 1.0$, $\nu = 0.49$ and $\chi = 0.80$ is given in Fig. 2. A comparison of Figs. 1 and 2 show that shells made of these materials behave, qualitatively, in a similar manner. A quantitative comparison indicates that, for foam rubber, the vibrational

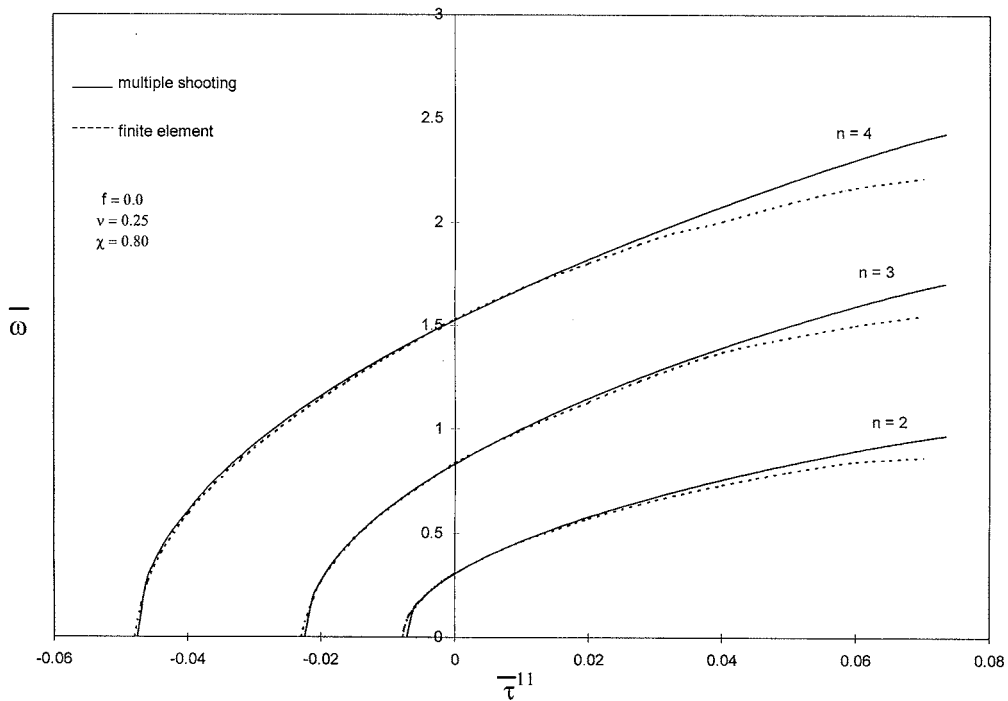


Fig. 1. The variation of the frequency as a function of the outer normal stress-foam rubber.

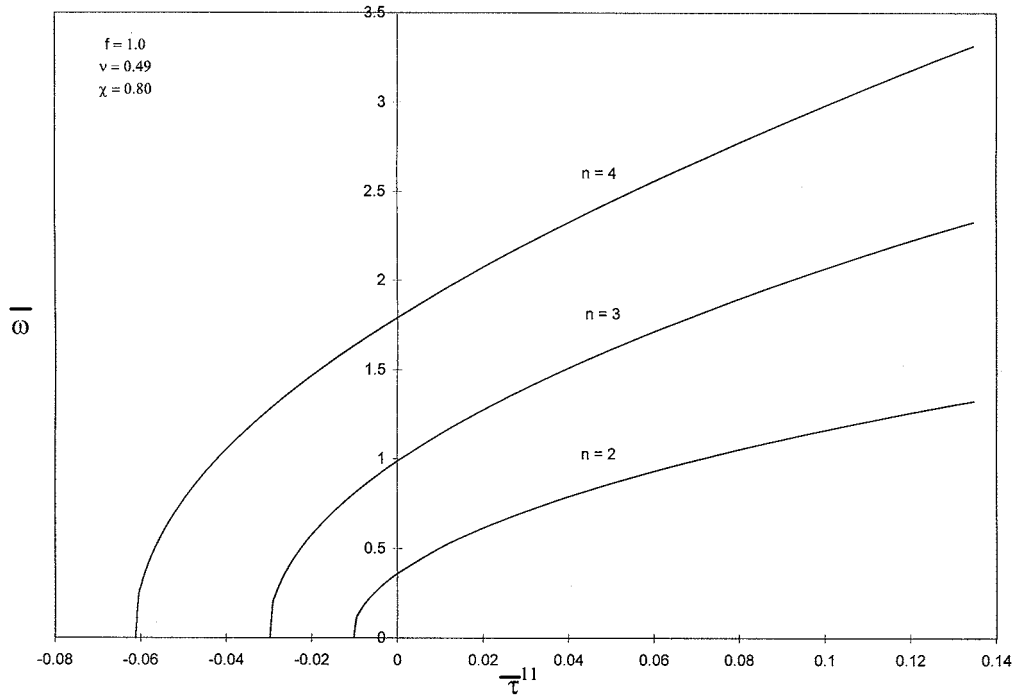


Fig. 2. The variation of frequency as a function of the outer normal stress-polynomial compressible material.

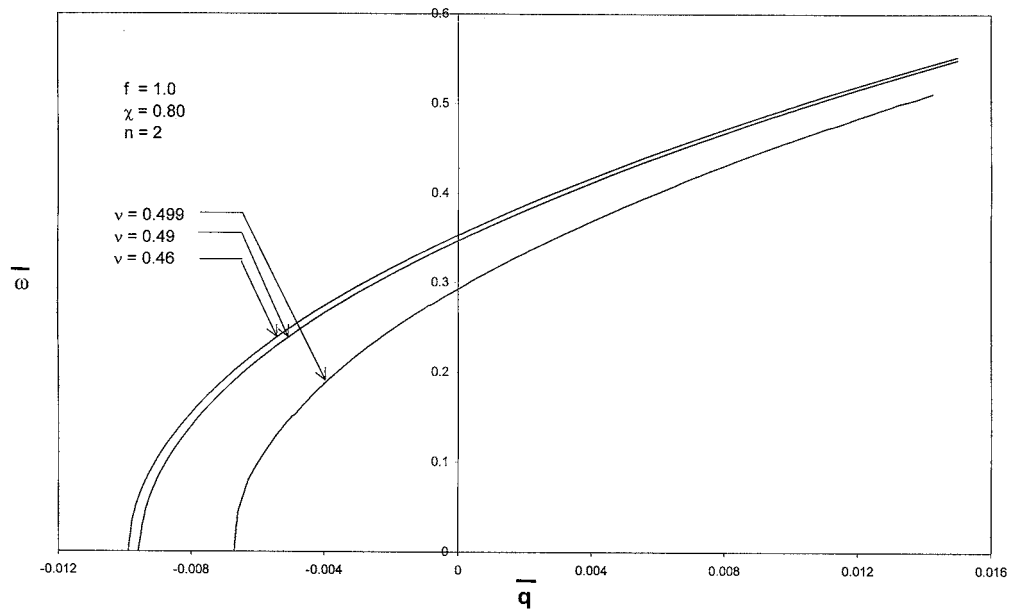


Fig. 3. The effect of Poisson's ratio on the surface traction versus frequency curves.

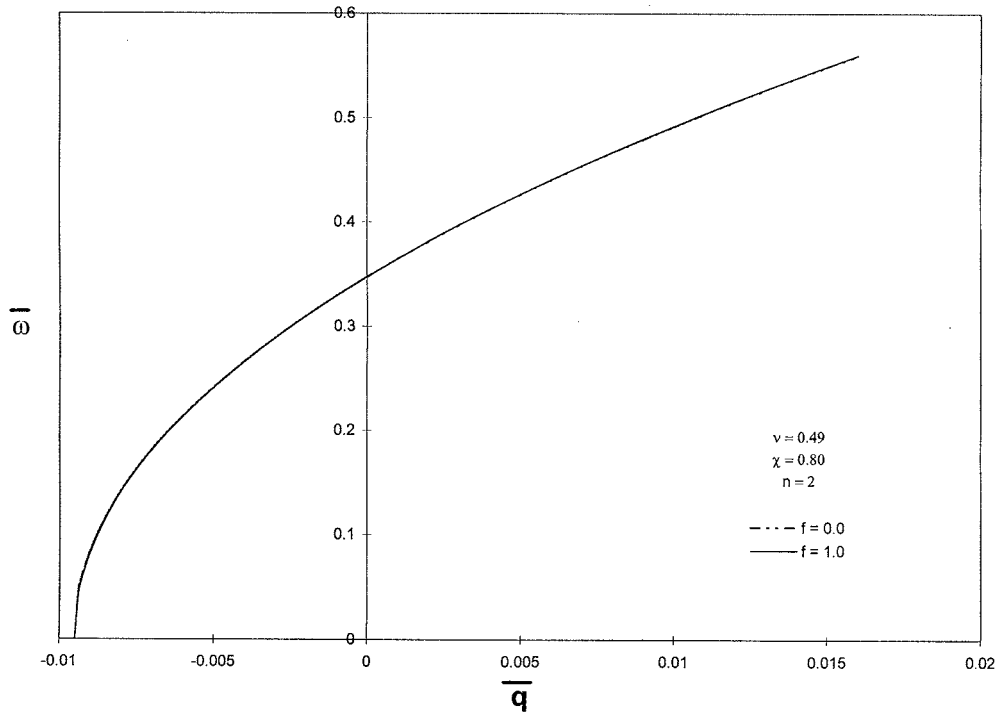


Fig. 4. The effect of material constant f on the surface traction versus frequency curves.

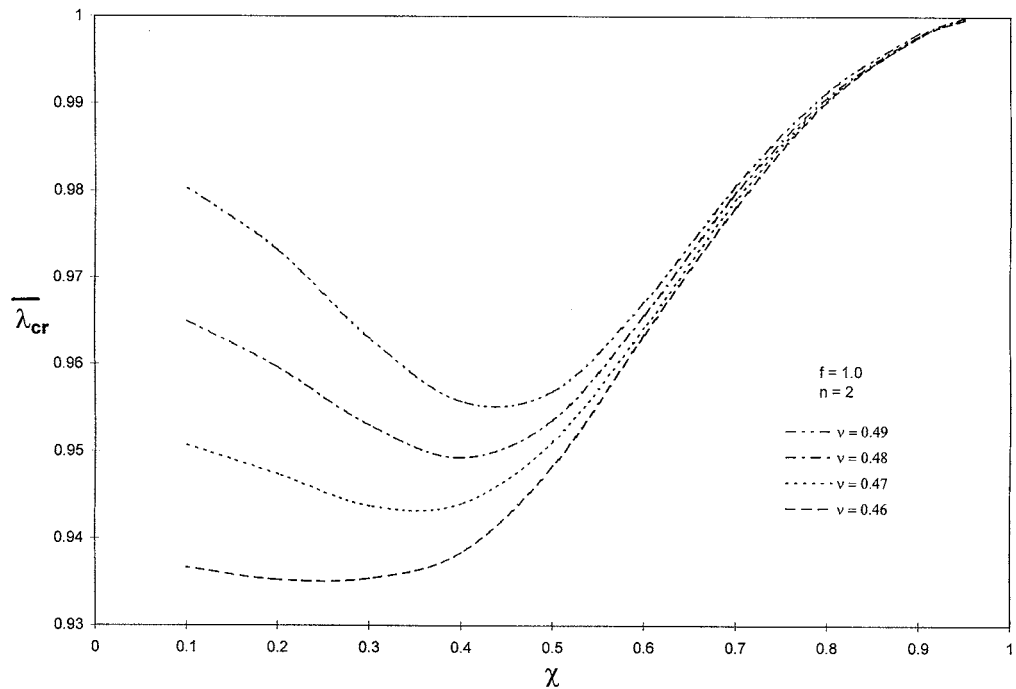


Fig. 5. The effect of Poisson's ratio on critical outer stretch ratio versus shell thickness curves-polynomial compressible material.

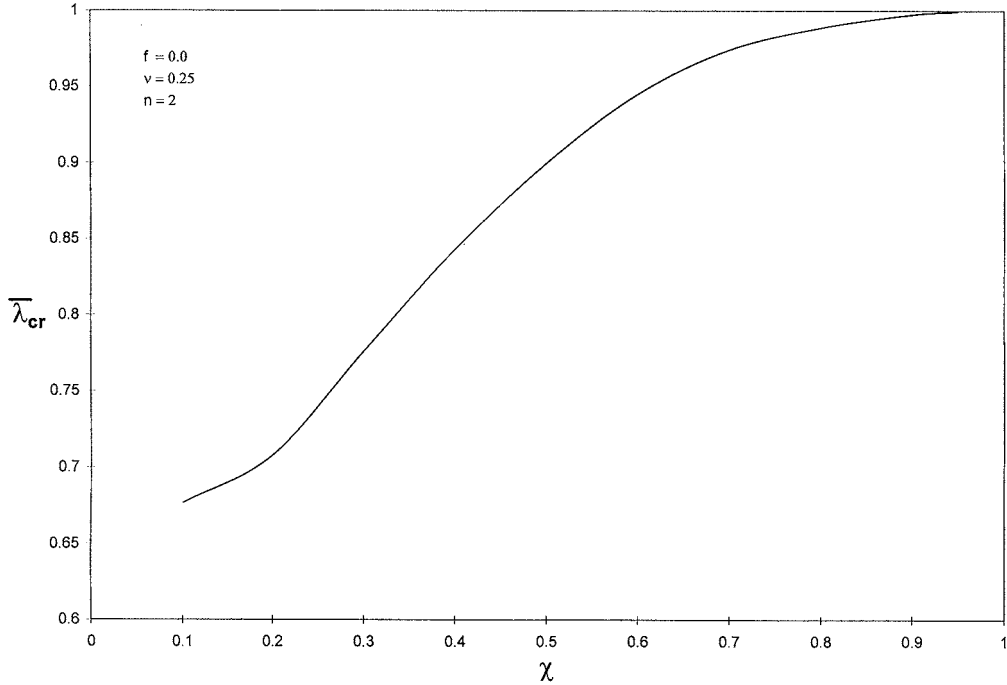


Fig. 6. The effect of ν on critical outer stretch ratio versus shell thickness curves-foam rubber.

frequencies and critical pressures are smaller and the difference becomes more pronounced for higher modes of vibration.

Fig. 3 shows that the effect of Poisson's ratio ν on the surface traction versus frequency curves is significant in the range $0.49 \leq \nu \leq 0.499$, and rather insignificant in the range $0.46 \leq \nu \leq 0.49$. According to the experimental studies done by Blatz and Ko [3] $\nu = 0.25$ gains a meaning only if f is set equal to zero. Similarly, Fig. 4 shows that surface traction versus frequency curves are insensitive to a change in the material constant f for moderately thick, nearly incompressible shells.

The change of the critical stretch ratio at the outer surface of the shell as a function of the thickness ratio is investigated in Figs. 5 and 6 for shells made of a polynomial material and foam rubber, respectively. As the Poisson's ratio of the material decreases, the critical stretch ratio of the shell also decreases. For thick shells of polynomial compressible material (χ less than nearly 0.45) the critical stretch ratio decreases with increasing thickness ratio. As the shell becomes thinner, the critical stretch ratio increases with increasing thickness ratio. This behavior is in accordance with the findings of Wang and Ertepinar [27] for incompressible shells. For foam rubber, the critical stretch ratio increases with increasing thickness ratio. No results could be generated for shells with a thickness ratio less than 0.1 due to numerical sensitivity.

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