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A CONCRETE DESCRIPTION OF $CD_0(K)$-SPACES AS $C(X)$-SPACES AND ITS APPLICATIONS

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ABSTRACT. We prove that for a compact Hausdorff space K without isolated points, $CD_0(K)$ and $C(K \times \{0,1\})$ are isometrically Riesz isomorphic spaces under a certain topology on $K \times \{0,1\}$. Moreover, K is a closed subspace of $K \times \{0,1\}$. This provides concrete examples of compact Hausdorff spaces X such that the Dedekind completion of $C(X)$ is $B(S)$ (= the set of all bounded real-valued functions on S) since the Dedekind completion of $CD_0(K)$ is $B(K)$ ($CD_0(K,E)$ and $CD_w(K,E)$ spaces as Banach lattices).

For a nonempty set X, $c_0(X)$ denotes the set of all real-valued bounded functions $d : X \to \mathbf{R}$ such that the set $\{x \in X : \epsilon \leq |d(x)|\}$ is finite for each $0 < \epsilon$. As usual, for a topological space K, $C(K)$ is the set of all real-valued continuous functions. Let K be a compact Hausdorff space without isolated points. Then

$$CD_0(K) = C(K) \oplus c_0(K)$$

is an AM-space with order unit $\mathbf{1}$ under pointwise operations. The order and norm properties of $CD_0(K)$ have been studied in [1], [3] and [5]. We refer to [2] for some important applications of the space $CD_0(K)$. The well-known Krein-Kakutani representation theorem allows that for each compact Hausdorff space K without isolated points there exists a compact Hausdorff space M such that $CD_0(K)$ and $C(M)$ are isometrically Riesz isomorphic spaces. In this paper we show that M is homeomorphic to $K \times \{0,1\}$ under the topology which is defined below.

For $f \in CD_0(K)$, f_c is the continuous part of f and f_d denotes the other part. That is, $f = f_c + f_d$ with $f_c \in C(K)$ and $f_d \in c_0(K)$. $\mathcal{X}_k$ denotes the characteristic function of $k \in K$.

Let K be a compact Hausdorff space without isolated points. It is easy to verify that $K \times \{0,1\}$ is a Hausdorff topological space under the convergence

$$(k_\alpha, r_\alpha) \to (k,r) \quad \text{if and only if} \quad f_c(k_\alpha) + r_\alpha f_d(k_\alpha) \to f_c(k) + r f_d(k)$$

for each $f \in CD_0(K)$. Indeed, let

$$\mathcal{T} = \{A \subset K \times \{0,1\} : (k_\alpha, r_\alpha) \in A, (k_\alpha, r_\alpha) \longrightarrow (k,r) \Longrightarrow (k,r) \in A\}.$$

It is obvious that $\mathcal{T}$ is closed under arbitrary intersection. Let $A, B \in \mathcal{T}$ and let $((k_\alpha, r_\alpha))$ be a net in $A \cup B$ with $(k_\alpha, r_\alpha) \longrightarrow (k,r)$. Then there exists a subnet $((k_{\alpha_\beta}, r_{\alpha_\beta}))$ in A (or in B) of $((k_\alpha, r_\alpha))$ with $(k_{\alpha_\beta}, r_{\alpha_\beta}) \longrightarrow (k,r)$, so that $(k,r) \in A$

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(or $(k,r) \in B$); therefore, $(k,r) \in A \cup B$. This shows that $\mathcal{T}$ is closed under finite union. Hence $K \times \{0,1\}$ is a topological space. Let $((k_\alpha, r_\alpha))$ be a net in $K \times \{0,1\}$ with $(k_\alpha, r_\alpha) \longrightarrow (k_0, r_0)$ and $(k_\alpha, r_\alpha) \longrightarrow (k_1, r_1)$. Then $f(k_\alpha) \longrightarrow f(k_0)$ and $f(k_\alpha) \longrightarrow f(k_1)$ for each $f \in C(K)$, so that $k_\alpha \longrightarrow k_0 = k_1$. Since

$$r_1 = r_1 \mathcal{X}_{k_1}(k_1) \longleftarrow r_\alpha \mathcal{X}_{k_1}(k_\alpha) = r_\alpha \mathcal{X}_{k_0}(k_\alpha) \longrightarrow r_0 \mathcal{X}_{k_0}(k_0) = r_0,$$

we have $(k_0, r_0) = (k_1, r_1)$. This shows that the limit of a net in this topological space is unique. So the topological space $K \times \{0,1\}$ is Hausdorff.

Throughout the paper $K \times \{0,1\}$ is topologized with the preceding topology.

Lemma 1. *Let K be a compact Hausdorff space without isolated points. Then $K \times \{0,1\}$ is a compact Hausdorff space.*

Proof. By the definition it is clear that $k_\alpha \longrightarrow k$ and $(k_{\alpha_\beta}, r_{\alpha_\beta}) \to (k,r)$ for each subnet $((k_{\alpha_\beta}, r_{\alpha_\beta}))$ of a net $((k_\alpha, r_\alpha))$ in $K \times \{0,1\}$ whenever $(k_\alpha, r_\alpha) \to (k,r)$.

Claim 1. $(k_\alpha, r_\alpha) \to (k,1)$ if and only if there exists α_0 such that $(k_\alpha, r_\alpha) = (k,1)$ for each $\alpha_0 \leq \alpha$. Indeed, otherwise, there exists a subnet (k_{α_β}) of (k_α) such that $k_{\alpha_\beta} \neq k$ for each β. Then

$$0 = r_{\alpha_\beta} \mathcal{X}_k(k_{\alpha_\beta}) \to \mathcal{X}_k(k) = 1,$$

So $k_\alpha = k$ for each $\alpha_0 \leq \alpha$ for some α_0. If there exists a subnet (r_{α_β}) of (r_α) with $r_{\alpha_\beta} = 0$, then

$$0 = r_{\alpha_\beta} \mathcal{X}_k(k) \to \mathcal{X}_k(k) = 1, \quad \alpha_\beta \geq \alpha_0.$$

This contradiction proves the claim.

Claim 2. $(k_\alpha, 1) \to (k,0)$ if and only if $k_\alpha \to k$ and there exists α_0 such that $k_\alpha \neq k$ for each $\alpha_0 \leq \alpha$. Indeed, if $(k_\alpha, 1) \to (k,0)$, then $k_\alpha \to k$ and $\mathcal{X}_k(k_\alpha) \to 0\mathcal{X}_k(k) = 0$. So there exists α_0 such that $k_\alpha \neq k$ for each $\alpha_0 \leq \alpha$. Conversely, let $k_\alpha \neq k$ for each $\alpha_0 \leq \alpha$ for some α_0 and $k_\alpha \to k$. Then (k_α) has infinite range. Suppose that $(k_\alpha, 1) \not\to (k,0)$. Since $k_\alpha \to k$, there exists $d \in C_0(K)$ such that $d(k_\alpha) \not\to 0d(k) = 0$. Then there exists a subnet (k_{α_β}) of (k_α) and $0 < \epsilon$ such that $\epsilon < d(k_{\alpha_\beta})$ for each β; so (k_{α_β}) has finite range. This implies that there exists a subnet $(k_{\alpha_{\beta_\gamma}})$ of (k_{α_β}) such that $k_{\alpha_{\beta_\gamma}} = k_{\alpha_{\beta_0}}$ for each γ and for some β_0. Since $k_\alpha \to k$, we have $k_{\alpha_{\beta_0}} = k$. This is a contradiction.

Claim 3. $(k_\alpha, 0) \to (k,0)$ if and only if $k_\alpha \to k$. This is clear.

From the above claims it is easy to check that under the described convergence, $K \times \{0,1\}$ is a compact Hausdorff space. □

Now we can prove the following theorem.

Theorem 2. *Let K be a compact Hausdorff topological space without isolated points. Then $CD_0(K)$ and $C(K \times \{0,1\})$ are isometrically Riesz isomorphic spaces.*

Proof. $K \times \{0,1\}$ is a compact Hausdorff space under the convergence given in the above lemma. Define

$$T : CD_0(K) \to C(K \times \{0,1\}), \quad T(f)(k,r) = f_c(k) + r f_d(k).$$

It is easy to check that T is well defined and clearly it is a one-to-one operator. It is also clear that $0 \leq T(f)$ if and only if $0 \leq f$ and $T(CD_0(K))$ is a Riesz subspace of $C(K \times \{0,1\})$. From [5] for each $f \in CD_0(K)$, $||T(f)|| = ||f||$. Then $T(CD_0(K))$ is also a closed subspace of $C(X \times \{0,1\})$. Let $(k_1, r_1) \neq (k_2, r_2)$ in $K \times \{0,1\}$. If

$k_1 \neq k_2$, then choose $f \in C(K)$ with $f(k_1) \neq f(k_2)$ and $T(f)(k_1, r_1) \neq T(f)(k_2, r_2)$. If $k = k_1 = k_2$ and $r_1 \neq r_2$, then $T(\mathcal{X}_k)(k_1, r_1) \neq T(\mathcal{X}_k)(k_2, r_2)$. This shows that $T(CD_0(K))$ separates the points of $K \times \{0, 1\}$. By the Stone-Weierstrass theorem, $T(CD_0(K))$ is dense in $C(K \times \{0, 1\})$. This completes the proof since $T(CD_0(K))$ is closed. □

Remark. Let K and M be compact Hausdorff spaces without isolated points. It was proved in [3] that K and M are homeomorphic whenever $CD_0(K)$ and $CD_0(M)$ are isometric. We can prove this in our context as follows: If $CD_0(K)$ and $CD_0(M)$ are isometric, then $C(K \times \{0, 1\})$ and $C(K \times \{0, 1\})$ are isometric; so $K \times \{0, 1\}$ and $M \times \{0, 1\}$ are homeomorphic. Now it is clear that $K \times \{0\}$ and $M \times \{0\}$ are homeomorphic since the spaces are without isolated points, $\{(k, 1)\}$ is open in $K \times \{0, 1\}$, and $\{(m, 1)\}$ is open in $M \times \{0, 1\}$ for each $k \in K, m \in M$. Thus K and M are homeomorphic from claim 3 of the above lemma.

It should be remarked that in the above theorem if K is a compact metric space without isolated points, for each $g \in C(K \times \{0, 1\})$, we define the maps on K by

$$g_f(k) = g(k, 0) \quad \text{and} \quad g_d(k) = g(k, 1) - g(k, 0).$$

Then $g_f \in C(K)$, $g_d \in c_0(K)$ and $T(g_f + g_d) = g$. This observation is used to prove the next theorem. For a Banach lattice E and a compact Hausdorff space K without isolated points, the space

$$CD_0(K, E) = C(K, E) \oplus c_0(K, E)$$

is defined similarly to $CD_0(K)$ (see [5]). The proof of the following theorem is similar to the above theorem and follows from the above observation.

Theorem 3. *Let K be a compact metric space without isolated points and E a Banach lattice. Then $CD_0(K, E)$ and $C(K \times \{0, 1\}, E)$ are isometrically Riesz isomorphic spaces.*

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