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Implications of the index of a fixed point subgroup

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Abstract – Let G be a finite group and $A \leq \mathrm{Aut}(G)$. The index $|G : C_G(A)|$ is called the *index of A in G* and is denoted by $\mathrm{Ind}_G(A)$. In this paper, we study the influence of $\mathrm{Ind}_G(A)$ on the structure of G and prove that $[G, A]$ is solvable in case where A is cyclic, $\mathrm{Ind}_G(A)$ is squarefree and the orders of G and A are coprime. Moreover, for arbitrary $A \leq \mathrm{Aut}(G)$ whose order is coprime to the order of G, we show that when $[G, A]$ is solvable, the Fitting height of $[G, A]$ is bounded above by the number of primes (counted with multiplicities) dividing $\mathrm{Ind}_G(A)$ and this bound is best possible.

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1. Introduction

Throughout this paper, we consider only finite groups. To introduce the notation we use in this paper, let G be a group and $x \in G$. The conjugacy class of G containing x is denoted by x^G and its length is denoted by $\mathrm{Ind}_G(x)$ which is the index $|G : C_G(x)|$. The product of solvable normal subgroups of G which is the maximal solvable normal subgroup of G in finite case is denoted by $S(G)$. The Fitting subgroup of G is denoted by $F(G)$, the Fitting height of a solvable group K by $h(K)$ and the set of prime divisors of order of G by $\pi(G)$.

Arithmetical conditions on the length of conjugacy classes of G influence nonsimplicity, solvability, supersolvability, and nilpotency of G. There are many results in this problem that can be seen in the historical order in [1].

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Let A be a subgroup of the automorphism group of G and let the fixed point subgroup $\{g \in G | \alpha(g) = g$ for all $\alpha \in A\}$ of A in G be denoted by $C_G(A)$. The index of $C_G(A)$ in G, denoted by $\mathrm{Ind}_G(A)$ influences solvability of G and $[G, A]$. There are many results for the case where $C_G(A)$ is small. One of the most famous paper related to this type of problems is probably Higman's result [2]. There are relatively less papers for the case $C_G(A)$ is large.

One of the papers joining the first type of problems with the latter case of the second type is Kazarin's work [3]. In 1990, he studied the case where $A = \langle\alpha\rangle$ and $\mathrm{Ind}_G(\alpha)$ is a prime power. Namely, he proved the following:

Theorem 1.1 ([3], Corollary 1). *Let G be a finite group and ϕ one of its automorphisms. If $C_G(\phi)$ contains a Sylow r-subgroup for all $r \in \pi(G) \setminus \{p\}$ then ϕ induces the identity automorphism on $G/S(G)$.*

This result led us to investigate the structure of $[G, \alpha]$ when $\mathrm{Ind}_G(\alpha)$ is divisible by at least two distinct primes, starting with the case $\mathrm{Ind}_G(\alpha)$ is squarefree. Although the orders of G and α are not necessarily coprime in Kazarin's result, the following example shows the indispensability of the coprimeness condition $(|G|, |\alpha|) = 1$, in our case.

Example 1.2. Let $G = A_5$ and α be the inner automorphism of S_5 induced by the transposition $(1, 2)$. Then $\mathrm{Ind}_G(\alpha) = 10$ but $[A_5, \alpha] = A_5$ is nonabelian simple.

We prove the following theorem as a result in my thesis work ([9], Theorem 2):

Theorem 1.3. *Let G be a finite group and α be an automorphism of G such that $(|G|, |\alpha|) = 1$. If $\mathrm{Ind}_G(\alpha)$ is squarefree then $[G, \alpha]$ is solvable.*

In the proof of Theorem 1.3, we use the classification of the finite simple groups (CFSG) to show non-simplicity of $[G, \alpha]$.

One may ask if it is possible to replace the assumption that "$\mathrm{Ind}_G(\alpha)$ is squarefree" with assumption "$\mathrm{Ind}_G(\alpha)$ is not divisible by 4." The following example shows that this is not possible:

Example 1.4. Let $G = \mathrm{PSL}(3, \mathbb{F}_{3^5})$ and let σ be a field automorphism of order 5. Since G is a simple group, we have $[G, \sigma] = G$. Since

$$|G| = 2^4.3^{15}.11^4.13.61.4561,$$

we have $(|G|, |\sigma|) = 1$. It can be seen that $C_G(\sigma) = \mathrm{PSL}(3, \mathbb{F}_3)$ and hence $|C_G(\sigma)| = 2^4.3^3.13$. It follows that,

$$\mathrm{Ind}_G(\sigma) = 3^{12}.11^4.61.4561$$

is odd but $[G, \sigma]$ is nonabelian simple.

Another work studying consequences of arithmetical properties of $\mathrm{Ind}_G(A)$ for given pair G, A with $A \leq \mathrm{Aut}(G)$ is due to Parker and Quick [4]. They proved the following:

THEOREM 1.5 ([4], Theorem A). *Let G be a finite group and let A be a group of automorphisms of G such that the orders of G and A are coprime. If $|G{:}\, C_G(A)| \leq n$ then $|[G, A]| \leq n^{\log_2(n+1)}$.*

Motivated by this result we investigate the influence of $\mathrm{Ind}_G(A)$ on the nilpotent height of $[G, A]$ when $[G, A]$ is a solvable group and $(|G|, |A|) = 1$. Namely, we obtain the following as a result in my thesis work ([9], Theorem 1):

THEOREM 1.6. *Let G be a group and $A \leq \mathrm{Aut}(G)$ such that $(|G|, |A|) = 1$ and $\mathrm{Ind}_G(A) = m$. If $[G, A]$ is solvable then the Fitting height of $[G, A]$ is bounded above by $\ell(m)$ where $\ell(m)$ is the number of primes dividing m, counted with multiplicities.*

The Classification of Finite Simple Groups is not needed in the proof of this theorem. The bound given by Theorem 1.6 is best possible because of the example below:

EXAMPLE 1.7. Let G be the group

$$\langle a, b, c, d \mid a^3 = b^7 = c^3 = d^7 = a^{-1}a^c = a^{-1}a^d = b^{-1}b^c = b^{-1}b^d = b^{-2}b^a = d^{-2}d^c = 1\rangle$$

and let α be the involutory automorphism of G given by $\alpha(a) = cd^5$, $\alpha(b) = d^2$, $\alpha(c) = ab$ and $\alpha(d) = b^4$. We observe by [10] that $|G| = 441$, $|[G, \alpha]| = 147$, $F([G, \alpha]) = 49$, $F_2([G, \alpha]) = [G, \alpha]$, $C_G(\alpha) = \langle abc, b^{a^{-1}}d\rangle$ and $|G : C_G(\alpha)| = 21$. Now, $(|G|, |A|) = (441, 2) = 1$, $\ell([G : C_G(\alpha)]) = \ell(21) = 2$ and Fitting height of $[G, \alpha]$ is 2.

2. The proof of Theorem 1.3

Proof. We use induction on the order of the semidirect product $G\langle\alpha\rangle$.

Suppose G is a group and $\alpha \in \mathrm{Aut}(G)$ so that the semidirect product $G\langle\alpha\rangle$ has the smallest order among all the pairs (G, α) that satisfies the hypothesis of Theorem 1.3 but $[G, \alpha]$ is not solvable.

We deduce a contradiction over a series of steps.

Let p be a prime divisor of order of α. Then there is a positive integer k so that $|\alpha| = kp$.

Suppose $k > 1$. Since $C_G(\alpha) \le C_G(\alpha^k)$, we have $\mathrm{Ind}_G(\alpha^k)$ divides $\mathrm{Ind}_G(\alpha)$. Hence, $\mathrm{Ind}_G(\alpha^k)$ is squarefree. As $|G\langle\alpha^k\rangle| < |G\langle\alpha\rangle|$, we have $[G, \alpha^k]$ is solvable.

$[G, \alpha^k]$ is an α-invariant normal subgroup of G and so α induces an automorphism by $G/[G, \alpha^k]$. Clearly, $\mathrm{Ind}_{G/[G,\alpha^k]}(\alpha)$ divides $\mathrm{Ind}_G(\alpha)$. So $\mathrm{Ind}_{G/[G,\alpha^k]}(\alpha)$ is squarefree. It follows by induction assumption that $[G/[G, \alpha^k], \alpha] = [G, \alpha]/[G, \alpha^k]$ is solvable. Therefore, $[G, \alpha]$ is solvable which contradicts to our assumption.

Hence, $k = 1$ and α is of prime order p.

Let N be a proper normal subgroup of $G\langle\alpha\rangle$.

Suppose $\alpha \in N$. Then $N = N_1\langle\alpha\rangle$ where $N_1 = N \cap G$ which is α-invariant. Since $\mathrm{Ind}_{N_1}(\alpha)$ divides $\mathrm{Ind}_G(\alpha)$, we get $\mathrm{Ind}_{N_1}(\alpha)$ is squarefree. So by induction assumption $[N_1, \alpha] = [N, \alpha]$ is solvable. Now, $\langle\alpha^N\rangle = [N, \alpha]\langle\alpha\rangle$ is solvable. It follows that, $\alpha \in S(N)$ and as $S(N) \le S(G\langle\alpha\rangle)$ we get $\alpha \in S(G\langle\alpha\rangle)$. So, $\langle\alpha^{G\langle\alpha\rangle}\rangle \le S(G\langle\alpha\rangle)$. As $[G, \alpha] \le \langle\alpha^{G\langle\alpha\rangle}\rangle = \langle\alpha^G\rangle = [G, \alpha]\langle\alpha\rangle$, we have $[G, \alpha]$ is solvable, which is a contradiction.

Hence, α is not contained in a proper normal subgroup of $G\langle\alpha\rangle$.

Consider the quotient group $G\langle\alpha\rangle/N$.

$\mathrm{Ind}_{G\langle\alpha\rangle/N}(\alpha N)$ is squarefree since it is a divisor of $\mathrm{Ind}_G(\alpha)$. By induction, $[G\langle\alpha\rangle/N, \alpha N]$ is solvable. It follows that,

$$\langle(\alpha N)^{G\langle\alpha\rangle/N}\rangle = [G\langle\alpha\rangle/N, \alpha N]\langle\alpha N\rangle$$

is solvable and hence $(\alpha N)^{G\langle\alpha\rangle/N} \in S(G\langle\alpha\rangle/N)$.

Now, $S(G\langle\alpha\rangle/N) = X/N$ for some normal subgroup X of $G\langle\alpha\rangle$ and $\alpha \in X$. It follows that, $X = G\langle\alpha\rangle$.

Therefore, for any proper normal subgroup N of $G\langle\alpha\rangle$, we have $G\langle\alpha\rangle/N$ is solvable.

Suppose $S(G\langle\alpha\rangle) \ne 1$. Since $S(G\langle\alpha\rangle)$ is a proper normal subgroup of $G\langle\alpha\rangle$, we get $G\langle\alpha\rangle/S(G\langle\alpha\rangle)$ is solvable. Hence, $G\langle\alpha\rangle$ is solvable, a contradiction. Therefore, $S(G\langle\alpha\rangle) = 1$.

Let K be a minimal normal subgroup of $G\langle\alpha\rangle$. If $K \not\leq G$, then $K \cap G = 1$. Hence, $|K| = p$ is prime, which leads the contradiction $K \leq S(G\langle\alpha\rangle) = 1$. Thus, $K \leq G$.

Suppose $K \neq G$. Then $\mathrm{Ind}_K(\alpha)$ is squarefree since it divides $\mathrm{Ind}_G(\alpha)$. Hence, by induction $[K, \alpha]$ is solvable and so is $\langle\alpha^K\rangle = [K, \alpha]\langle\alpha\rangle$. Then we get $\alpha \in \langle\alpha^K\rangle = \langle\alpha^{K\langle\alpha\rangle}\rangle \leq S(K\langle\alpha\rangle)$. Now,

$$[K, \alpha] \leq S(K\langle\alpha\rangle) \cap K \leq S(K) \leq S(G\langle\alpha\rangle) = 1.$$

It follows that, $\alpha \in C_{G\langle\alpha\rangle}(K) \trianglelefteq G\langle\alpha\rangle$. So $C_{G\langle\alpha\rangle}(K) = G\langle\alpha\rangle$. Then we get the contradiction $K \leq Z(G\langle\alpha\rangle) \leq S(G\langle\alpha\rangle) = 1$.

Therefore, G is the unique minimal normal subgroup of $G\langle\alpha\rangle$, G is characteristically simple and $(G\langle\alpha\rangle)' = G$. Hence, G is a product of isomorphic copies of a simple group say, $E \leq G$. As G is not solvable, E is nonabelian.

Suppose $G \neq E$. Consider the family $\{E^{\alpha^k} \mid k = 0, 1, 2, \ldots, p-1\}$ of subgroups of G. The subgroup $M = E \times E^{\alpha} \times \ldots \times E_{\alpha^{p-1}}$ is an α-invariant normal subgroup of G. So $M \trianglelefteq G\langle\alpha\rangle$. Since G is the unique minimal normal subgroup of $G\langle\alpha\rangle$ we get $G = M$. It follows that $C_G(\alpha) = \{xx^{\alpha}x^{\alpha^2} \ldots x^{\alpha^{p-1}} \mid x \in E\}$. Hence $|C_G(\alpha)| = |E|$ and $\mathrm{Ind}_G(\alpha) = |E|^{p-1}$. Since 2 is a divisor of $|E|$ and $(|G|, |\alpha|) = 1$, we have $p > 2$ and so $\mathrm{Ind}_G(\alpha)$ is divisible by $|E|^2$, a contradiction.

Therefore, $G = E$ is a nonabelian simple group.

From Atlas of Finite Groups [8], we observe that G is not a sporadic simple group as they have no coprime automorphism. Since alternating groups has no coprime automorphism, we get G is not an alternating group. Thus, G is a simple group of Lie type. It follows that α is a field automorphism up to conjugation since $(|\alpha|, |G|) = 1$.

Let r be a prime number. Let n_r denote the largest power of r that divides n and $L(q)$ denote a simple group of Lie type over the finite field of order q. By Proposition 4.9.1 in [5], if $q = r^{ps}$ for some integer s and $G = L(q)$ and α is a field automorphism of order p, then $C_G(\alpha) \cong L(r^s)$.

Let $G = A_m(r^{ps})$ for $m \geq 2$ then $C_G(\alpha) = A_m(r^s)$. It follows that,

$$\mathrm{Ind}_G(\alpha)_r = (r^{ps})^{m(m+1)/2}/(r^s)^{m(m+1)/2} = r^{sm(m+1)(p-1)/2}$$

For each other family of simple groups of Lie type the argument is the same. In all cases, r^2 divides $\mathrm{Ind}_G(\alpha)$. This contradiction completes the proof. □

3. The proof of Theorem 1.6

Proof. We use induction on the order of G. Let G be a minimal counter example to Theorem 1.6 and $A \leq \mathrm{Aut}(G)$ as in Hypothesis of Theorem 1.6.

Suppose that $[G, A]$ is properly contained in G. Since $(|G|, |A|) = 1$, we have $G = [G, A]C_G(A)$ by Lemma 8.2.7 in [6]. It follows that

$$\begin{aligned}
|[G, A] : C_{[G,A]}(A)| &= |[G, A] : ([G, A] \cap C_G(A))| \\
&= |[G, A]C_G(A) : C_G(A)| \\
&= |G : C_G(A)| \\
&= m.
\end{aligned}$$

As $|[G, A]| < |G|$, by minimality of G, we get $h([G, A, A]) \leq \ell(m)$.

By Lemma 8.2.7 in [6], we know $[G, A, A] = [G, A]$ since $(|G|, |A|) = 1$. This leads to the contradiction $h([G, A]) \leq \ell(m)$.

Hence, $[G, A] = G$.

If G is nilpotent then $h(G) = 1 \leq \ell(m)$. Thus, we may assume that $F(G) \lneqq G$.

Next, suppose that $F(G)$ is a subgroup of $C_G(A)$. As $[F(G), G] \leq F(G)$, we have $[F(G), G, A] = 1$ and $[A, F(G), G] = 1$. It follows by the Three Subgroup Lemma (2.2 Theorem 2.3 in [7]) that $[G, A, F(G)] = [G, F(G)] = 1$.

Since $C_G(F(G)) \subseteq F(G)$ by 6.1 Theorem 1.3 in [7], we get $G = F(G)$, which is not the case. Hence, $F(G) \nleq C_G(A)$.

Now, $C_G(A)F(G) \neq C_G(A)$ and hence,

$$\ell(|G : C_G(A)F(G)|) \lneqq \ell(m).$$

As $|G/F(G)| < |G|$, we have

$$\begin{aligned}
h(G) - 1 &= h(G/F(G)) \\
&\leq \ell(|G/F(G) : C_{G/F(G)}(A)|) \\
&\leq \ell(|G/F(G) : C_G(A)F(G)/F(G)|) \\
&\leq \ell(|G : C_G(A)F(G)|) \\
&< \ell(m).
\end{aligned}$$

Consequently, $h([G, A]) - 1 = h(G) - 1 \leq \ell(m) - 1$ and hence $h([G, A]) \leq \ell(m)$, completing the proof. □

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