DIFFERENT APPROACHES FOR MØLLER'S ENERGY IN THE KASNER-TYPE SPACE-TIME‡

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Abstract. Considering the Møller energy definition in both Einstein's theory of general relativity and tele-parallel theory of gravity, we find the energy of the universe based on viscous Kasner-type metrics. The energy distribution which includes both the matter and gravitational field is found to be zero in both of these different gravitation theories and this result agrees with previous works of Cooperstock and Israelit, Rosen, Johri *et al.*, Banerjee-Sen, Vargas who investigated the problem of the energy in Friedmann-Robertson-Walker universe in Einstein's theory of general relativity and Aydogdu-Saltı who considered the same problem in tele-parallel gravity. In all of these works, they found that the energy of the Friedmann-Robertson-Walker space-time is zero. Our result is the same as obtained in the studies of Saltı and Havare. They used the viscous Kasner-type metric and found total energy and momentum by using Bergmann-Thomson energy-momentum formulation in both general relativity and tele-parallel gravity. The result that the total energy and momentum components of the universe is zero supports the viewpoints of Albrow and Tryon.

Keywords: Total energy; Kasner-type metric; Møller's prescription.
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1. Introduction

The conserved quantities such as energy and momentum play a crucial role as they provide the first integrals of equations of motions, helping one to solve otherwise intractable problems[1]. Furthermore the energy content in a sphere of radius R in a given space-time gives a taste of the effective gravitational mass that a test particle situated at the same distance from the gravitating object experiences. A large number of researchers have devoted considerable attention to the problem of finding the energy as well as momentum and angular momentum associated with various space-times.

The problem of obtaining the energy is considered for Einstein's theory of general relativity and also tele-parallel theory gravity. From the advents of these different gravitation theories various methods have been proposed to deduce the conservation laws that characterize the gravitational systems. The first of such attempts was made

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by Einstein who proposed an expression for the energy-momentum distribution of the gravitational field. There have many attempts to resolve the energy-momentum problem[2, 3, 4, 5, 6, 7, 8, 9, 10]. There exists an opinion that the energy-momentum definitions are not useful to get finite and meaningful results in a given geometry. Virbhadra and his collaborators re-opened the problem of the energy and momentum by using the energy-momentum complexes. The Einstein energy-momentum complex, used for calculating the energy in general relativistic systems, was followed by many complexes: e.g. Tolman, Papapetrou, Bergmann-Thomson, Møller, Landau-Liftshitz, Weinberg, Qadir-Sharif and the tele-parallel gravity analogs of the Einstein, Landau-Lifshitz, Bergmann-Thomson and Møller's. The energy-momentum complexes give meaningful results when we transform the line element in quasi-Cartesian coordinates. The energy and momentum complex of Møller gives the possibility to perform the calculations in any coordinate system[11]. To this end Virbhadra and his collaborators have considered many space-time models and have shown that several energy-momentum complexes give the same and acceptable results for a given space-time[12, 13, 14, 15, 16, 17]. In Phys. Rev. D60-104041 (1999), Virbhadra, using the energy and momentum complexes of Einstein, Landau-Lifshitz, Papapetrou and Weinberg for a general non-static spherically symmetric metric of the Kerr-Schild class, showed that all of these energy-momentum formulations give the same energy distribution as in the Penrose energy-momentum formulation.

Albrow[18] and in the following work, Tryon[19] suggested that in our universe, all conserved quantities have to vanish. Tryon's big bang model predicted a homogenous, isotropic and closed universe including matter and anti-matter equally. They argue that any closed universe has zero energy. The subject of the energy-momentum distributions of closed and open universes was initiated by an interesting work of Cooperstock and Israelit[20]. They found the zero value of energy for any homogenous isotropic universe described by a Friedmann-Robertson-Walker metric in the context of general relativity. This interesting result influenced some general relativists, for example: Rosen[21], Johri *et al.*[22], Banerjee and Sen[23]. Johri *et al.* using the Landau-Liftshitz's energy-momentum complex, found that the total energy of an Friedmann-Robertson-Walker spatially closed universe is zero at all times. Banerjee and Sen who investigated the problem of total energy of the Bianchi-I type space-times using the Einstein complex, obtained that the total energy is zero. This result agrees with the studies of Johri *et al.* since the line element of the Bianchi-I type space-time reduces to the spatially flat Friedmann-Robertson-Walker line element in a special case. Vargas[10] using the definitions of Einstein and Landau-Lifshitz in tele-parallel gravity, found that the total energy is zero in Friedmann-Robertson-Walker space-times. This results agree with the works of Rosen and Johri. Saltı and his collaborators considered different space-times for various definitions in tele-parallel gravity and obtained the energy-momentum distributions in a given model. Firstly, Saltı and Havare[24] considered Bergmann-Thomson's complex in both general relativity and tele-parallel gravity for the Viscous Kasner-type metric and in another work, Saltı[25] using the Einstein and Landau-

Lifshitz's complexes associated with the same metric in tele-parallel gravity, found that total energy and momentum are zero. At the last, Aydogdu and Saltı[26] used Møller's definition in tele-parallel gravity for Bianchi-I type metric and found that the total energy is zero.

The paper is organized as follows: In the next section, we introduce viscous Kasner-type space-times. In section III, by using Møller's energy-momentum complex we calculate the total energy in the viscous Kasner-type space-times in general relativity. Section IV gives us, the total energy distribution for the same metric in tele-parallel gravity. At the last, we summarize and discuss our results. Throughout this paper, Latin indices (i, j, ...) represent the vector number, and Greek indices (μ, ν,...) represent the vector components. All indices run from 0 to 3 and we use the convention that $G = 1$, $c = 1$ units.

2. The Viscous Kasner-type Space-time

The Friedmann-Robertson-Walker cosmological model has attracted considerable attention in the relativistic cosmology literature. Maybe one of the most important properties of this model is, as predicted by inflation[27, 28, 29], the flatness, which agrees with the observed cosmic microwave background radiation.

In the early universe the sorts of the matter fields are uncertain. The existence of anisotropy at early times is a very natural phenomenon to investigate, as an attempt to clarify among other things, the local anisotropies that we observe today in galaxies, clusters and super-clusters. So, at the early time, it appears appropriate to suppose a geometry that is more general than just the isotropic and homogenous Friedmann-Robertson-Walker geometry. Even though the universe, on a large scale, appears homogenous and isotropic at the present time, there are no observational data that guarantee this in an epoch prior to the recombination. The anisotropies defined above have many possible sources; they could be associated with cosmological magnetic or electric fields, long-wave length gravitational waves, Yang-Mills fields[30].

The line element of the viscous Kasner-type universe[31] is given as,

$$ds^2 = -dt^2 + t^{2a}dx^2 + t^{2b}dy^2 + t^{2c}dz^2 \tag{1}$$

where a, b and c are three parameters that we shall require to be constants. The expansion factors t^{2a}, t^{2b} and t^{2c} could be determined via Einstein's field equations.

The Kasner universe, in Einstein's theory(with cosmological constant $\Lambda = 0$), refers to a vacuum cosmological model given by (1) where the numbers a, b and c satisfy the constraints

$$a + b + c = a^2 + b^2 + c^2 = 1 \tag{2}$$

An anisotropic Kasner-type universe can be considered to be filled with an ideal(non-viscous) fluid which has an equation of state $p = \rho$(stiff matter-the velocity of sound coincides with the speed of light), where ρ is the energy density and p is the isotropic pressure.

For the line element (1), $g_{\mu\nu}$ and $g^{\mu\nu}$ are defined by

$$g_{\mu\nu} = -\delta^0_\mu\delta^0_\nu + t^{2a}\delta^1_\mu\delta^1_\nu + t^{2b}\delta^2_\mu\delta^2_\nu + t^{2c}\delta^3_\mu\delta^3_\nu \tag{3}$$

$$g^{\mu\nu} = -\delta^\mu_0\delta^\nu_0 + t^{-2a}\delta^\mu_1\delta^\nu_1 + t^{-2b}\delta^\mu_2\delta^\nu_2 + t^{-2c}\delta^\mu_3\delta^\nu_3 \tag{4}$$

The non-trivial tetrad field induces a tele-parallel structure on space-time which is directly related to the presence of the gravitational field, and the Riemannian metric arises as

$$g_{\mu\nu} = \eta_{ab}h^a_\mu h^b_\nu \tag{5}$$

Using this relation, we obtain the tetrad components:

$$h^a_\mu = \delta^a_0\delta^0_\mu + t^a\delta^a_1\delta^1_\mu + t^b\delta^a_2\delta^2_\mu + t^c\delta^a_3\delta^3_\mu \tag{6}$$

and its inverse is

$$h^\mu_a = \delta^0_a\delta^\mu_0 + t^{-a}\delta^1_a\delta^\mu_1 + t^{-b}\delta^2_a\delta^\mu_2 + t^{-c}\delta^3_a\delta^\mu_3 \tag{7}$$

From the Christoffel symbols which are defined by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \tag{8}$$

we obtain following non-vanishing components:

$$\begin{aligned} &\Gamma^0_{11} = at^{2a-1}, \qquad \Gamma^0_{22} = bt^{2b-1}, \qquad \Gamma^0_{33} = ct^{2c-1} \\ &\Gamma^1_{01} = \Gamma^1_{10} = \frac{a}{t}, \qquad \Gamma^2_{02} = \Gamma^2_{20} = \frac{b}{t}, \qquad \Gamma^3_{03} = \Gamma^3_{30} = \frac{c}{t} \end{aligned} \tag{9}$$

The metric given by (1) reduces to the spatially flat Friedmann-Robertson-Walker metric in a special case. Defining $t^a = R(t)$ with $a = b = c$, and transforming the line element (1) to t, x, y, z coordinates according to $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ gives

$$ds^2 = -dt^2 + R^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{10}$$

which describes the well-known spatially flat Friedmann-Robertson-Walker space-time.

3. Mϕller's Energy in General Relativity

In this section considering Mϕller's energy-momentum complex in general relativity we calculate total energy associated with the viscous Kasner-type space-time.

In general theory of relativity the energy-momentum complex of Mϕller[5] is given by

$$M^\nu_\mu = \frac{1}{8\pi}\chi^{\nu\alpha}_{\mu,\alpha} \tag{11}$$

satisfying the local conservation laws:

$$\frac{\partial M^\nu_\mu}{\partial x^\nu} = 0 \tag{12}$$

where the antisymmetric super-potential $\chi^{\nu\alpha}_\mu$ is

$$\chi^{\nu\alpha}_\mu = \sqrt{-g}[g_{\mu\beta,\gamma} - g_{\mu\gamma,\beta}]g^{\nu\gamma}g^{\alpha\beta}. \tag{13}$$

The locally conserved energy-momentum complex M^{ν}_{μ} contains contributions from the matter, non-gravitational and gravitational fields. M^0_0 is the energy density and M^0_a are the momentum density components. The energy-momentum components are given by

$$P_{\mu} = \int \int \int M^0_{\mu} dxdydz. \tag{14}$$

Using Gauss's theorem, the energy and momentum are

$$P_{\mu} = \frac{1}{8\pi} \int \int \chi^{\nu\alpha}_{\mu} \mu_{\alpha} dS. \tag{15}$$

where μ_{α} is the outward unit normal vector over the infinitesimal surface element dS. P_i give momentum components P_1, P_2, P_3 and P_0(say E_{GR}) gives the energy. We wish to find the total energy in the space-time which is described by the line element (1). Using equation (13) we found the components of $\chi^{\nu\alpha}_{\mu}$ are zero. From this point of view using equation (11) we obtain

$$M^0_0 = M^0_a = 0 \tag{16}$$

However, we easily see that in the viscous Kasner-type universe Mϕller's energy is found as

$$E_{GR} = 0 \tag{17}$$

4. Mϕller's Energy in Tele-Parallel Gravity

The tele-parallel gravity is an alternative approach to gravitation and corresponds to a gauge theory for the translation group based on Weitzenböck geometry[32]. In the theory of the tele-parallel gravity, gravitation is attributed to torsion[33], which plays the role of a force[34], and the curvature tensor vanishes identically. The essential field is acted by a nontrivial tetrad field, which gives rise to the metric as a by-product. The translational gauge potentials appear as the nontrivial item of the tetrad field, so induces on space-time a tele-parallel structure which is directly related to the presence of the gravitational field. The interesting place of tele-parallel gravity is that, due to its gauge structure, it can reveal a more appropriate approach to consider some specific problem. This is the situation, for example, in the energy and momentum problem, which becomes more transparent.

Mϕller modified general relativity by constructing a new field theory in tele-parallel space[35]. The aim of this theory was to overcome the problem of the energy-momentum complex that appears in Riemannian Space[36]. The field equations in this new theory were derived from a Lagrangian which is not invariant under local tetrad rotation. Saez[37] generalized Mϕller theory into a scalar tetrad theory of gravitation. Meyer[38] showed that Mϕller theory is a special case of Poincare gauge theory[39, 40].

The super-potential of Mϕller in tele-parallel gravity is given by Mikhail *et al.*[9] as

$$U^{\nu\beta}_{\mu} = \frac{(-g)^{1/2}}{2\kappa} P^{\tau\nu\beta}_{\chi\rho\sigma} \left[\Phi^{\rho} g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} \gamma^{\chi\rho\sigma} - (1-2\lambda) g_{\tau\mu} \gamma^{\sigma\rho\chi}\right] \tag{18}$$

where $P^{\tau\nu\beta}_{\chi\rho\sigma}$ is

$$P^{\tau\nu\beta}_{\chi\rho\sigma} = \delta^{\tau}_{\chi} g^{\nu\beta}_{\rho\sigma} + \delta^{\tau}_{\rho} g^{\nu\beta}_{\sigma\chi} - \delta^{\tau}_{\sigma} g^{\nu\beta}_{\chi\sigma} \tag{19}$$

with $g^{\nu\beta}_{\rho\sigma}$ being a tensor defined by

$$g^{\nu\beta}_{\rho\sigma} = \delta^{\nu}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\nu}_{\sigma}\delta^{\beta}_{\rho} \tag{20}$$

and $\gamma_{\mu\nu\beta}$ is the con-torsion tensor given by

$$\gamma_{\mu\nu\beta} = h_{i\mu} h^{i}_{\nu;\rho} \tag{21}$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. g is the determinant of the $g_{\mu\nu}$, Φ_{μ} is the basic vector field defined by

$$\Phi_{\mu} = \gamma^{\rho}_{\mu\rho} \tag{22}$$

κ is the Einstein constant and λ is the free dimension-less parameter. The energy may be expressed by the surface integral[5]

$$E_{TG} = \lim_{r\to\infty} \int_{r=constant} U_{0}^{0\alpha} n_{\alpha} dS \tag{23}$$

here n_{α} is the unit three-vector normal to the surface element dS. Taking the results which are given by (6) and (7) into equation (21) we get the non-vanishing components of $\gamma_{\mu\nu\beta}$ as:

$$\begin{aligned}
\gamma_{011} &= -\gamma_{101} = at^{2a-1} \\
\gamma_{022} &= -\gamma_{202} = bt^{2b-1} \\
\gamma_{033} &= -\gamma_{303} = ct^{2c-1}
\end{aligned} \tag{24}$$

Using this result, we find following non-vanishing component of basic vector field:

$$\Phi^{0} = \frac{1}{t} \tag{25}$$

From equation (18) with the results which are given in equation (24) and (25) we find the required components of Møller's super-potential are vanishing. Substituting these values into equation (23) we get

$$E_{TG} = 0 \tag{26}$$

This is the same result as which is obtained by general relativity version of Møller's energy-momentum complex.

5. Discussions

The definition of energy-momentum localization in both the general theory of relativity and tele-parallel gravity has been very exciting and interesting; however, it has been associated with some debate.

Through this paper, to compute the total energy of the universe based on the viscous Kasner-type metric, we have considered two approaches of Møller's energy-momentum complex: the versions in both the general theory of relativity and tele-parallel gravity. We found the total energy in these different gravitation theories as:

$$E_{GR} = E_{TG} = 0 \tag{27}$$

which agree with the result that obtained by Cooperstock and Israelit, Rosen, Johri *et al.*, Banerjee-Sen, Vargas, Saltı *et al.*. Moreover; the result that the total energy of the universe in the viscous Kasner-type space-time is zero supports the viewpoints of Albrow and Tyron. It is also independent of the tele-parallel dimensionless coupling constant, which means that it is valid in any tele-parallel model. The Moller approach is one among others to compute total energy, and the Kasner-type metrics should be analyzed and get the same results in some of these different ways: e.g. Papapetrou, Landau-Lifshitz, Qadir-Sharif, Weinberg, Tolman and Einstein.

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