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AN OBSERVATION ON REALCOMPACT SPACES

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ABSTRACT. We give a characterization of realcompact spaces in terms of nets. By using the technique of this characterization we give easy proofs of the Tychonoff Theorem and the Alaoglu Theorem.

Let X be Tychonoff space and let (x_i) be a net in X. It is clear that (x_i) converges if $lim f(x_i) = f(x)$ for each real-valued continuous function f on X for some $x \in X$. This motivates the following question: Does (x_i) converge if $lim f(x_i)$ exists for each continuous real-valued function f on X? We will show that this characterizes realcompact spaces. We believe that this characterization may be of interest because of its simplicity and accessibility.

Throughout this paper we suppose that the topological spaces are Hausdorff. As usual $C(X)$ stands for the algebra of real-valued continuous functions on a topological space X under the pointwise operations; that is, for each $f, g, h \in C(X)$ and $\alpha \in \mathbb{R}$,

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x).$$

A map $\varphi : C(X) \longrightarrow \mathbb{R}$ is called an *(algebra) homomorphism* if

$$\varphi(fg + \alpha h) = \varphi(f)\varphi(g) + \alpha\varphi(h)$$

for each $f, g, h \in C(X)$ and $\alpha \in \mathbb{R}$. In [8] it is shown that a map $\varphi : C(X) \longrightarrow \mathbb{R}$ is a homomorphism if and only if it is a *ring homomorphism*; that is,

$$\varphi(fg + h) = \varphi(f)\varphi(g) + \varphi(h)$$

for each $f, g, h \in C(X)$. The subalgebra of bounded functions in $C(X)$ is denoted by $C_b(X)$. We refer to [7] for more details on $C(X)$ and $C_b(X)$ algebras.

Recall that a topological space X is called *Tychonoff* if for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a bounded function f in $C(X)$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$. A topological space X is called a *realcompact space* if it is homeomorphic to a closed subspace of the product space $\prod_{i \in I} \mathbb{R}$ for some index set I. It is well known that a Tychonoff space X is a realcompact space if and only if for each non-zero homomorphism $\pi : C(X) \longrightarrow \mathbb{R}$ there exists $x_\pi \in X$ such that $\pi(f) = f(x_\pi)$ for each $f \in C(X)$ (an elementary proof of this without using of Axiom of Choice can be found in a recent paper [6]). Each Tychonoff space X can be embedded into a unique realcompact space υX as a dense subspace such that $C(X)$ and $C(\upsilon X)$ are algebraic isomorphic (see [5, p. 218]). For each $f \in C(X)$, $f^r \in C(\upsilon X)$ denotes the image of f. If $\pi : C(X) \longrightarrow \mathbb{R}$,

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then $\pi^r : C(\upsilon X) \longrightarrow \mathbb{R}$ is defined by $\pi^r(g) = \pi(h)$, where h is the restriction of g to X. This observation leads to the following lemma.

Lemma 1. *Let X be a Tychonoff space and let $\pi : C(X) \longrightarrow \mathbb{R}$ be a nonzero homomorphism. Then there exists a net (x_i) in X such that $\pi(f) = lim f(x_i)$ for each $f \in C(X)$.*

Proof. Let $\pi : C(X) \longrightarrow \mathbb{R}$ be a nonzero homomorphism. As the algebra $C(X)$ can be identified with $C(\upsilon X)$, where υX is defined above, there exists $a \in \upsilon X$ and a net $(x_i) \in X$ with $x_i \longrightarrow a$ such that

$$\pi(f) = \pi^r(f^r) = f^r(a) = lim f^r(x_i) = lim f(x_i).$$

This completes the proof. □

Now we are ready to present the main result as follows.

Theorem 2. *Let X be a Tychonoff space. Then the following are equivalent.*

i. *X is a realcompact space.*

ii. *A net (x_i) in X converges if and only if $lim f(x_i)$ exists for each $f \in C(X)$.*

Proof. Suppose that **i** holds and (x_i) is a net in X such that $lim f(x_i)$ exists for each $f \in C(X)$. Let

$$\pi : C(X) \longrightarrow \mathbb{R}, \quad \pi(f) = lim f(x_i).$$

Then π is a nonzero homomorphism. As X is a realcompact space there exists an $x \in X$ such that $\pi(f) = f(x)$. Since X is a Tychonoff space and $lim f(x_i) = f(x)$ for each $f \in C(X)$, (x_i) converges to x. Now suppose that **ii** holds and $\pi : C(X) \longrightarrow \mathbb{R}$ is a nonzero homomorphism. From the above lemma there exists a net (x_i) in X such that $\pi(f) = lim f(x_i)$ for each $f \in C(X)$. From the hypothesis, $x_i \rightarrow x$, so $\pi(f) = f(x)$ for each $f \in C(X)$. Hence X is a realcompact space. □

In [10], a net (x_i) of points of a topological space X is called a $C(X)$-net in X if for every real-valued continuous function f on X the net $(f(x_i))$ is convergent. A topological space X is C-complete whenever every $C(X)$-net in X has a cluster point in X. In [10] it is observed that every realcompact space is C-complete. For a Tychonoff space X, from the above theorem, the proof of the corollary below is easy.

Corollary 3. *Let X be a Tychonoff space. Then the following are equivalent.*

i. *X is a realcompact space.*

ii. *X is C-complete.*

iii. *Each $C(X)$-net is convergent.*

In general the construction of the Stone-Čech compactification of a Tychonoff space is given by using the Tychonoff Theorem. There is another way of doing this construction, that is as follows: Let X be a Tychonoff space and βX be the set of homomorphisms $\pi : C_b(X) \longrightarrow \mathbb{R}$ with $\pi\mathbf{1} = 1$. Then by using elementary arguments it is not difficult to show that (without using the Tychonoff Theorem) βX is a compact Hausdorff space such that $\pi_\alpha \longrightarrow \pi$ in βX if and only if $\pi_\alpha(f) \longrightarrow \pi(f)$ for each $f \in C_b(X)$ (see [4]). $\{\{\pi : \pi(f) \neq 0\} : f \in C_b(X)\}$ is a base for the topological space βX. Under the map $x \longrightarrow \pi_x$, $\pi_x(f) = f(x)$, X is dense in βX, and $C_b(X)$ and $C(\beta X)$ are algebraic isomorphic under the operator $T : C_b(X) \longrightarrow C(\beta X)$, $T(f)(\pi) = \pi(f)$. By using an argument similar to the above theorem and the construction of βX as in the above, for a Tychonoff space X the proof of the next theorem is clear.

Theorem 4. *Let X be a Tychonoff space. Then the following are equivalent.*
i. *X is compact.*
ii. *A net (x_i) in X converges if and only if $lim f(x_i)$ exists for each $f \in C_b(X)$.*

One of the important theorems of functional analysis is the Alaoglu Theorem (see [1] for a proof). The known proof of this theorem depends on the Tychonoff Theorem. By using the above theorem we can give an alternative and easy proof (depending on the Stone-Čech compactification and without using the Tychonoff Theorem) of this theorem as follows.

Theorem 5 (Alaoglu, [1])**.** *Let X be a locally convex space and let V be a neighborhood of zero. Then the polar V° is $\sigma(X^{'}, X)$-compact.*

Proof. It is clear that V° is a $\sigma(X^{'}, X)$-Tychonoff space. Let (f_α) be a net in V° such that $\lim \pi(f_\alpha)$ exists for each real-valued bounded continuous function on V°. Then in particular $\lim \pi_x(f_\alpha)$ exists for each $x \in X$, where $\pi_x : V^\circ \longrightarrow \mathbb{R}$, $\pi_x(f) = f(x)$. Let $f : X \longrightarrow \mathbb{R}$ be defined by $f(x) = lim \pi_x(f_\alpha)$. Now it is clear that $f \in V^\circ$ and $f_\alpha \longrightarrow f$ in $\sigma(X^{'}, X)$. Now from Theorem 4, V° is $\sigma(X^{'}, X)$-compact. □

The classical Tychonoff Theorem states that the product space X of compact Hausdorff spaces (X_α) is also compact. In the literature there are many different proofs of this theorem (see [3], [9]). By using Theorem 3 we can give an alternative and easy proof of this as follows:

Theorem 6 (Tychonoff Theorem)**.** *Let $\{X_\alpha : \alpha \in \Gamma\}$ be a set of compact Hausdorff spaces. Then the product space $X = \prod_\alpha X_\alpha$ is compact.*

Proof. Clearly X is a Tychonoff space. For each $\alpha_0 \in \Gamma$, let $P_{\alpha_0} : X \longrightarrow X_{\alpha_0}$ be defined by $P_{\alpha_0}((x_\alpha)) = x_{\alpha_0}$. Let $x = ((x_i^\alpha)_\alpha)_i$ be a net in the product space X such that $(F((x_i^\alpha)_\alpha))_i$ is convergent for each real-valued bounded continuous function F on X. Then for each α_0 and each bounded function $f \in C(X_{\alpha_0})$, $(f(x_i^{\alpha_0})) = (f \circ P_{\alpha_0}(x))$ is convergent. As X_{α_0} is compact, from Theorem 4, $x_i^{\alpha_0} \longrightarrow x^{\alpha_0}$ for some $x^{\alpha_0} \in X_{\alpha_0}$. Let $x_\infty = (x^\alpha)$. Then x is convergent to x_∞ in X. Again from Theorem 4, X is compact. □

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