# A categorical approach to the maximum theorem 

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#### Abstract

Berge's maximum theorem gives conditions ensuring the continuity of an optimised function as a parameter changes. In this paper we state and prove the maximum theorem in terms of the theory of monoidal topology and the theory of double categories.

This approach allows us to generalise (the main assertion of) the maximum theorem, which is classically stated for topological spaces, to pseudotopological spaces and pretopological spaces, as well as to closure spaces, approach spaces and probabilistic approach spaces, amongst others. As a part of this we prove a generalisation of the extreme value theorem.


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## 0. Introduction

Berge's maximum theorem [3], which is used in mathematical economics for instance, concerns a relation $J: A \rightarrow B$ between topological spaces, which we regard as a subset $J \subseteq A \times B$, as well as a continuous map $d: A \rightarrow[-\infty, \infty]$ into the extended real line, as depicted in the following diagram.


We may "extend $d$ along $J$ " by "optimising $d$ for each $y \in B$ ", thus obtaining a map $l: B \rightarrow[-\infty, \infty]$ given by the suprema

$$
\begin{equation*}
l(y)=\sup _{x \in J^{\circ} y} d(x), \tag{1}
\end{equation*}
$$

where $J^{\circ} y=\{x \in A \mid(x, y) \in J\}$ denotes the preimage of $y$ under $J$. The main assertion of the maximum theorem states that the optimised function $l$ is continuous as soon as the relation $J$ is 'hemi-continuous' and $J^{\circ} y \neq \emptyset$ for each $y \in B$. Among the conditions included in hemi-continuity is the compactness of the preimages $J^{\circ} y$ so that, by the extreme value theorem, hemi-continuity of $J$ implies that the suprema defining $l$ are attained as maxima-a consequence that is used in the classical proof of the maximum theorem.

[^0]Regarding the ordered set $[-\infty, \infty]$ as a category allows us to think of the suprema in (1) as being limits. In fact, we may consider the full optimised function $l$ as the 'left Kan extension of $d$ along $J^{\prime}$, a construction that is fundamental to category theory. Recently it has been shown that, mostly in purely categorical settings, structure on a 'morphism' $D: \mathcal{A} \rightarrow \mathcal{M}$ carries over to Kan extensions of $D$ under certain conditions - the monoidal structure on a functor for instance, see [26], [20] and [33]. The maximum theorem can be thought of as fitting in the same scheme of results: it shows that the continuity of the map $d$ carries over to its left Kan extension $l$. In view of this, one might hope to discover a purely categorical result that, in the topological setting, recovers the classical maximum theorem while, when considered in other settings, allows us to obtain generalisations of the maximum theorem. This paper realises this hope to a large extent.

Besides recognising optimised functions as Kan extensions, the second ingredient of our categorical approach to the maximum theorem is to regard topological structures as algebraic structures - a point of view that forms the basis of the study of 'monoidal topology' [17]. In the fundamental example for instance, one regards topologies on a set $A$ as closure operations, i.e. relations $c: P A \rightarrow A$ between the powerset $P A$ of $A$ and $A$ itself: one defines $(S, x) \in c$ precisely if $x \in \bar{S}$, the closure of $S \subseteq A$. The axioms for a topology on $A$ then translate to three axioms on the 'closure relation' $c$ and, by weakening or removing some of these axioms, generalisations of the notion of topological space are recovered, such as that of pretopological space [4] and closure space.

The closure relation $c: P A \rightarrow A$ above can be equivalently thought of as a map $c: P A \times A \rightarrow\{\perp, \top\}$ taking values in the set $\{\perp, \top\}$ of truth values. A second way of generalising the notion of topological space, which is fundamental to monoidal topology, is to replace the set of truth values by a different set of values $\mathcal{V}$. In this way for instance, by considering $[0, \infty]$-valued closure relations $\delta: P A \times A \rightarrow[0, \infty]$, one recovers the notion of approach space [25], consisting of a set $A$ equipped with a point-set distance $\delta(S, x) \in[0, \infty]$ for each subset $S \subseteq A$ and point $x \in A$. Likewise, by allowing closure relations to take 'distance distribution functions' $\phi:[0, \infty] \rightarrow[0,1]$ as values, one obtains the notion of probabilistic approach space [21].

Besides closure operations, the notion of topology can be described algebraically in terms of ultrafilter convergence as well [2]: topologies on a set $A$ correspond precisely to convergence relations $\alpha: U A \rightarrow A$ satisfying certain axioms, where $U A$ denotes the set of ultrafilters on $A$. As with closure operations, by weakening these axioms, or by considering $\mathcal{V}$-valued convergence relations $\alpha: U A \times A \rightarrow \mathcal{V}$, one recovers generalisations of the notion of topological space, such as the notions of pretopological space and (probabilistic) approach space, as well as that of pseudotopological space [4], amongst others. In our approach to the maximal theorem we will consider both closure relations and ultrafilter convergence relations, as well as the relationship between them. In our study of the latter we closely follow [22].

The language allowing us to naturally describe the relations between the two ingredients of our approach-Kan extensions and algebraic descriptions of topological structures - is that of double categories, in the sense of e.g. [13]. The notion of double category extends that of category by considering two types of morphisms instead of the usual single type: e.g. between sets we will consider both functions $f: A \rightarrow C$ as well as $\mathcal{V}$-valued relations $J: A \times B \rightarrow \mathcal{V}$. Throughout this paper the language of double categories will lead us in the right direction. At the start for instance, when we consider approach spaces (equipped with $[0, \infty]$-valued closure relations), it naturally leads us to consider Kan extensions that are 'weighted' by $[0, \infty]$-valued relations $J: A \times B \rightarrow[0, \infty]$, instead of Kan extensions along ordinary relations $J: A \rightarrow B$ as described above. Later it naturally leads to the generalisation of the notion of hemi-continuous relation, as well as to the proper generalisation
of Kan extensions "whose suprema are attained by maxima". Finally, the language of double categories leads us to the generalisations of the maximum theorem and of the extreme value theorem themselves.

In closing this introduction we remark on some present restrictions of our approach. In recent work on the maximum theorem (e.g. [10]), as well as in recent textbooks (e.g. [1]), the name 'maximum theorem' is designated to a result that extends and generalises in two ways the main assertion described above:

- more generally, it concerns optimisations $k: B \rightarrow[-\infty, \infty]$ of the form

$$
k(y)=\sup _{x \in J^{\circ} y} e(x, y)
$$

for each $y \in B$, where $J: A \rightarrow B$ is a hemi-continuous relation with graph $\operatorname{Gr} J \subseteq A \times B$ and $e: \operatorname{Gr} J \rightarrow[-\infty, \infty]$ is a continuous function;

- besides continuity of the optimised function $k$, it also proves the 'upper hemi-continuity' of the 'solution relation' $J^{*}: A \rightarrow B$ that is defined by

$$
(x, y) \in J^{*} \quad: \Leftrightarrow \quad(x, y) \in J \quad \text { and } \quad e(x, y)=k y .
$$

Investigating ways of incorporating these generalisations in the categorical approach presented here have to be left as a further study.

## Outline

We start in Section 1 by recalling the language and basic theory of double categories, mostly from [12] and [13]. Guided by the classical setting of functions $f: A \rightarrow C$ and relations $J: A \rightarrow B$ between sets, we restrict to double categories whose cells, which describe the relations between the two types of morphism, are uniquely determined by their boundaries, and in which every 'vertical morphism' $f: A \rightarrow C$ induces two corresponding 'horizontal morphisms' $f_{*}: A \rightarrow C$ and $f^{*}: C \rightarrow A$. Such double categories we will call 'thin equipments'. Our main examples are the thin equipments $\mathcal{V}$-Rel, of relations $J: A \times B \rightarrow \mathcal{V}$ taking values in a 'quantale' $\mathcal{V}$ : loosely speaking, any ordered set $\mathcal{V}$ with "enough structure to replace the set of truth values". After recalling some examples of quantales, such as the quantale $\Delta$ of distance distribution functions, we recall the notion of 'monoid' in a thin equipment. Monoids in $\{\perp, \top\}$-Rel are ordered sets while monoids in $[0, \infty]$-Rel and $\Delta$-Rel respectively recover the notions of generalised metric space [23] and probabilistic metric space [27].

In Section 2 the double categorical notion of Kan extension, introduced in [19], is considered in thin equipments. After describing Kan extensions between monoids in $\mathcal{V}$-Rel, we consider the classical situation of a Kan extension into $[-\infty, \infty]$ whose suprema are attained as maxima, and generalise it in terms of a 'Beck-Chevalley condition' for Kan extensions. The main result of this section shows that, in a thin equipment, Kan extensions satisfying the Beck-Chevalley condition are precisely the 'absolute Kan extensions' of [14].

Given a 'monad' $T$ on a thin equipment, we start Section 3 by recalling from e.g. [17] the notions of ' $T$-graph' and ' $T$-category', as well as some related notions. For the ultrafilter monad $U$ extended to ordinary relations these notions recover those of pseudotopological space and topological space, as well as that of pretopological space. Extending $U$ to $\mathcal{V}$-valued relations recovers to the notion of $\mathcal{V}$-valued topological space [21] which, by taking $\mathcal{V}=[0, \infty]$ and $\mathcal{V}=\Delta$, includes the notions of approach space and probabilistic approach space respectively. Likewise, by taking the powerset monad extended to $\mathcal{V}$-valued relations we obtain the notion of $\mathcal{V}$-valued (pre-)closure space and several of its generalisations. As a variant
on the main theorem of [22], which shows that $\mathcal{V}$-valued topological spaces correspond precisely to $\mathcal{V}$-valued closure spaces whose closure relations $c: P A \times A \rightarrow \mathcal{V}$ 'preserve finite joins', the main result of this section establishes a similar correspondence between $\mathcal{V}$-valued pretopological spaces and 'finite-join-preserving' $\mathcal{V}$-valued preclosure spaces.

In Section 4 we consider objects in a thin equipment that are equipped with compatible monoid and $T$-graph structures. Following [31] we call such objects 'modular $T$-graphs'. We prove that the correspondences described in Section 3 lift to give correspondences between modular $\mathcal{V}$-valued (pre-)topological spaces and finite-join-preserving modular $\mathcal{V}$-valued (pre-)closure spaces.

The generalisations of the maximum theorem given in Section 7 apply to Kan extensions $l: B \rightarrow M$, between $T$-graphs, that satisfy one of the following conditions. Either $l$ satisfies the Beck-Chevalley condition, in the sense of Section 2, or the $T$-graph $M$ is ' $T$-cocomplete', as described in Section 5. The latter condition extends the $T$-cocompleteness property considered in [17]. Loosely speaking, the $T$-graph structure of a $T$-cocomplete modular $T$-graph is completely determined by its 'generic points': a modular approach space $A$ for example, equipped with both a generalised metric $A(x, y) \in[0, \infty]$, where $x, y \in A$, and a $[0, \infty]$-valued ultrafilter convergence $\alpha: U A \rightarrow A$, is $U$-cocomplete whenever for every ultrafilter $\mathfrak{x}$ on $A$ a generic point $x_{0} \in A$ is chosen such that

$$
\alpha(\mathfrak{x}, y)=A\left(x_{0}, y\right)
$$

for all $y \in A$. We will describe how every 'completely distributive' quantale $\mathcal{V}$ itself admits two $U$-cocomplete modular $\mathcal{V}$-valued topological space structures.

In Section 6 the notions of lower and upper hemi-continuity, for ordinary relations between topological spaces, are generalised to the notions of ' $T$-open' and ' $T$-closed' horizontal morphism $J: A \rightarrow B$ between $T$-graphs $A$ and $B$. Restricting ourselves to the extensions $P$ and $U$ of the powerset and ultrafilter monads to $\mathcal{V}$-relations, we describe the relationship between $P$-openness and $U$-openness, as well as that between $P$-closedness and $U$-closedness, in terms of the correspondences between $\mathcal{V}$-valued (pre-)closure spaces and $\mathcal{V}$-valued (pre-)topological spaces given in Section 3 .

Finally in Section 7 we state and prove four generalisations of the classical maximum theorem, in terms of Kan extensions between $T$-graphs. These come in pairs, one pair for 'left' Kan extensions and the other for 'right' Kan extensions: each pair either assumes that the Kan extension satisfies the Beck-Chevalley condition, in the sense of Section 2, or has a $T$-cocomplete target, in the sense of Section 5 . Besides showing how to recover the classical maximum theorem we describe its generalisations to preclosure spaces, approach spaces and probabilistic approach spaces.

Generalising the classical extreme value theorem, in the last section we obtain conditions ensuring the Beck-Chevalley condition for Kan extensions between modular $\mathcal{V}$-valued pseudotopological spaces.

## 1. Thin equipments

In this preliminary section we consider the notion of thin equipment, which forms the main setting for this paper. As this notion is modeled to describe the interaction between functions $f: A \rightarrow C$ and relations $J: A \rightarrow B$ between sets, we start by briefly setting out some notation for relations. We think of a relation $J: A \rightarrow B$ as a subset $J \subseteq A \times B$, and shorten $(x, y) \in J$ to $x J y$. We will write

$$
J S=\{y \in B \mid \exists x \in S: x J y\}
$$

for the $J$-image of $S \subseteq A$; also we write $J x:=J\{x\}$ for all $x \in A$. The reverse $J^{\circ}: B \rightarrow A$ of $J$ is defined by

$$
y J^{\circ} x \quad: \Leftrightarrow \quad x J y,
$$

allowing us to denote by $J^{\circ} T$ the $J$-preimage of $T \subseteq B$.
Relations between $A$ and $B$ are ordered by inclusion; in fact, to describe the interplay between functions and relations it is useful to depict by a cell

the property that $(f x) K(g y)$ for every $x J y$. For example these cells allow us to formalise, in the definition of thin equipment below, the relation between a function $f: A \rightarrow C$ and the two relations $f_{*}: A \rightarrow C$ and $f^{*}: C \rightarrow A$ that it induces, that are defined by

$$
x\left(f_{*}\right) y \quad: \Leftrightarrow \quad f x=y \quad \Leftrightarrow: \quad y\left(f^{*}\right) x .
$$

Notice that cells like the one above can be composed both vertically and horizontally: any two vertically adjacent cells combine as on the left below while any two horizontally adjacent cells combine as on the right. Here $\odot$ denotes the usual composition of relations: $x(J \odot H) z$ precisely if $x J y$ and $y H z$ for some $y \in B$.


The preceding describes the prototypical thin equipment Rel, of functions and relations between sets. It naturally gives rise to the following general definition of thin equipment.

Definition 1.1. A thin equipment $\mathcal{K}$ consists of a pair of categories $\mathcal{K}_{\mathrm{v}}=\left(\mathcal{K}_{\mathrm{v}}, \circ\right.$, id $)$ and $\mathcal{K}_{\mathrm{h}}=\left(\mathcal{K}_{\mathrm{h}}, \odot, 1\right)$, on the same collection of objects, that is equipped with a collection $\mathcal{K}_{\mathrm{c}}$ of square-shaped cells
each of which is uniquely determined by its boundary morphisms $f, g \in \mathcal{K}_{\mathrm{v}}$ and $J, K \in \mathcal{K}_{\mathrm{h}}$. This data is required to satisfy the following axioms:

- $\mathcal{K}_{\mathrm{c}}$ is closed under vertical and horizontal composition as depicted in (2) above;
- $\mathcal{K}_{\mathrm{c}}$ contains identity cells as shown below, one for each $f \in \mathcal{K}_{\mathrm{v}}$ and one for each $J \in \mathcal{K}_{\mathrm{h}}$;


- the ordering on morphisms in $\mathcal{K}_{\mathrm{h}}$ that is induced by $\mathcal{K}_{\mathrm{c}}$ is separated, that is the existence of both cells below implies $J=K$;

- for each morphism $f: A \rightarrow C$ in $\mathcal{K}_{\mathrm{v}}$ there are two morphisms $f_{*}: A \rightarrow C$ and $f^{*}: C \rightarrow A$ in $\mathcal{K}_{\mathrm{h}}$ such that the cells below exist.





We call the morphisms of $\mathcal{K}_{\mathrm{v}}$ the vertical morphisms of $\mathcal{K}$ and those of $\mathcal{K}_{\mathrm{h}}$ the horizontal morphisms. Cells with identities as vertical morphisms, such as in (4), are called horizontal cells; if either cell in (4) exists then we write $J \leq K$ or $K \leq J$ respectively.

For $f: A \rightarrow C$ in $\mathcal{K}_{\mathrm{v}}$ we call the horizontal morphism $f_{*}: A \rightarrow C$ the companion of $f$ and $f^{*}: C \rightarrow A$ the conjoint of $f$. Notice that the companion and conjoint of $f$ are uniquely determined by the existence of the four cells above, as a consequence of the separated ordering on horizontal morphisms. It follows that $(g \circ f)_{*}=f_{*} \odot g_{*}$ and $(g \circ f)^{*}=g^{*} \odot f^{*}$ for composable morphisms $f$ and $g$, while $\left(\mathrm{id}_{A}\right)_{*}=1_{A}=\left(\mathrm{id}_{A}\right)^{*}$; in short the assignments $f \mapsto f_{*}$ and $f \mapsto f^{*}$ are functorial. Given morphisms $f: A \rightarrow C, K: C \rightarrow D$ and $g: B \rightarrow D$ we write

$$
K(f, g):=f_{*} \odot K \odot g^{*}
$$

and call $K(f, g)$ the restriction of $K$ along $f$ and $g$; notice that $1_{C}(f, \mathrm{id})=f_{*}$ and $1_{C}(\mathrm{id}, f)=f^{*}$. In terms of restrictions the functoriality of companions and conjoints means that $K(f, g)(h, k)=K(h \circ f, k \circ g)$ and $K(\mathrm{id}, \mathrm{id})=K$.

When drawing cells we will often depict identity morphisms by the equal sign $(=)$. Although the cells of a thin equipment are uniquely determined by their boundaries, often it will be useful to give them names. In those cases we will use greek letters $\phi, \psi, \ldots$, as well as denoting vertical and horizontal composition of cells by $\circ$ and $\odot$, while vertical and horizontal unit cells will be denoted by $1_{f}$ and $\mathrm{id}_{J}$ respectively.

Besides the direct definition given above, a thin equipment can equivalently be defined as a flat strict double category, in the sense of Section 1 of [12], whose horizontal bicategory is locally skeletal and in which every vertical morphism has both a companion and conjoint (also called horizontal adjoint), the latter in the sense of Section 1 of [13]. The term 'equipment' originates from the term 'proarrow equipment' used by Wood in 35] for structures closely related to "double categories $\mathcal{K}$ with all companions and conjoints": one can think of such $\mathcal{K}$ as equipping their underlying vertical bicategories with the 'proarrows' of their underlying horizontal bicategories.
Example 1.2. Instead of the classical relations between sets $A$ and $B$ we will also consider metric relations $J: A \rightarrow B$, given by functions $J: A \times B \rightarrow[0, \infty]$. Composition of metric relations $J: A \rightarrow B$ and $H: B \rightarrow E$ is given by "shortest path distance"

$$
(J \odot H)(x, z)=\inf _{y \in B} J(x, y)+H(y, z)
$$

Together with functions between sets, metric relations form a thin equipment MetRel in which a cell as in (3) exists precisely if $J(x, y) \geq K(f x, g y)$ for all $x \in A$ and $y \in B$.
Example 1.3. Generalising the previous example, relations between sets can take values in any 'quantale' as follows. A quantale $\mathcal{V}=(\mathcal{V}, \otimes, k)$ is a complete lattice $\mathcal{V}$ equipped with a (not necessarily commutative) monoid structure $\otimes$ with unit $k$, such that $\otimes$ preserves suprema on both sides. Given a quantale $\mathcal{V}$, a $\mathcal{V}$-relation $J: A \rightarrow B$ between sets $A$ and $B$ is a function $J: A \times B \rightarrow \mathcal{V}$. The composite of $\mathcal{V}$-relations $J: A \rightarrow B$ and $H: B \rightarrow E$ is given by "matrix multiplication"

$$
(J \odot H)(x, z)=\sup _{y \in B} J(x, y) \otimes H(y, z)
$$

the identity $\mathcal{V}$-relations $1_{A}: A \rightarrow A$ for this composition are given by $1_{A}(x, y)=k$ if $x=y$ and $1_{A}(x, y)=\perp$ if $x \neq y$, where $\perp=\sup \emptyset$ is the bottom element of $\mathcal{V}$. Functions and $\mathcal{V}$-relations between sets combine to form a thin equipment $\mathcal{V}$-Rel, in which a cell as in (3) exists precisely if $J(x, y) \leq K(f x, g y)$ for all $x \in A$ and $y \in B$. Since the ordering on $\mathcal{V}$ is separated the ordering on parallel $\mathcal{V}$-relations is separated as well. The companion $f_{*}: A \rightarrow C$ and conjoint $f^{*}: C \rightarrow A$ of a function $f: A \rightarrow C$ are the $\mathcal{V}$-relations given by $f_{*}(x, y)=k=f^{*}(y, x)$ if $f x=y$ and $f_{*}(x, y)=\perp=f^{*}(y, x)$ if $f x \neq y$. The restriction $K(f, g)$ of a $\mathcal{V}$-relation $K: C \rightarrow D$ along functions $f: A \rightarrow C$ and $g: B \rightarrow D$ is indeed given by restriction: $K(f, g)(x, y)=K(f x, g y)$ for all $x \in A$ and $y \in B$.

If $\mathcal{V}$ is the two-chain $2=\{\perp \leq T\}$ of truth values, equipped with the monoid structure $(\wedge, \top)$ given by conjunction, then 2-Rel is isomorphic to the thin equipment Rel of relations, under the identification of ordinary relations $J \subseteq A \times B$ with 2-relations $J: A \times B \rightarrow 2$. If $\mathcal{V}$ is the Lawvere quantale $[0, \infty]$, equipped with the opposite order $\geq$ and the monoid structure $(+, 0)$, then $[0, \infty]$-Rel coincides with the thin equipment MetRel of the previous example. Similarly the completion $[-\infty, \infty]$ of $\mathbb{R}$, either with the natural order $\leq$ or with the reversed order $\geq$, forms a quantale under addition. As is customary, when referring to infima and suprema in $([0, \infty], \geq)$ and $([-\infty, \infty], \geq)$ we will always consider the natural order $\leq$.

In the same vein the unit interval $[0,1]$, with its natural order, admits several monoid structures \& that make it into a quantale: one can take the usual multiplication $\&=\times$ of real numbers, the frame operation $p \& q=\min \{p, q\}$ or the Łukasiewicz operation $p \& q=\max \{p+q-1,0\}$. Notice that, besides preserving suprema in both variables, each of these multiplications is commutative and has unit $k=1$ : monoid structures on $[0,1]$ with these properties are known as left-continuous t-norms.
Example 1.4. A distance distribution function is a function $\phi:[0, \infty] \rightarrow[0,1]$ satifying the left-continuity condition $\sup _{s<t} \phi(s)=\phi(t)$ for all $t \in[0, \infty]$. As a consequence $\phi$ preserves the natural order $\leq$, while $\phi(0)=0$. Any left-continuous t-norm \& on $[0,1]$ induces a quantale structure on the set $\Delta$ of all distance distribution functions: $\Delta$ inherits a pointwise ordering from $[0,1]$ while its multiplication is given by the convolution product

$$
(\phi \otimes \psi)(t)=\sup _{r+s \leq t} \phi(r) \& \psi(s)
$$

for all $t \in[0, \infty]$. The resulting quantales $\Delta_{\&}$ share their unit $k$, which is given by $k(t)=1$ for $t>0$ and $k(0)=0$, while their orderings fail to be linear.
Example 1.5. Any frame $\mathcal{V}$, that is a lattice such that $v \mapsto \min \{v, w\}$ preserves suprema for all $w \in \mathcal{V}$, can be regarded as a quantale with $v \otimes w=\min \{v, w\}$.

By using companions and conjoints any general cell in a thin equipment corresponds to a couple of horizontal cells as follows.

Lemma 1.6. In a thin equipment consider morphisms as in the boundary of the cell below. The cell below exists if and only if $J \odot g_{*} \leq K(f, \mathrm{id})$ if and only if $f^{*} \odot J \leq K(\mathrm{id}, g)$.


Proof. By composing the cell above with the cells defining the companions and conjoints of $f$ and $g$ we obtain the horizontal cells that exhibit the inequalities. In the same way the cell can be recovered from either horizontal cell that exhibits one of the inequalities.

As in any double category (see e.g. Section 11 of [30]) one can consider monoids in a thin equipment, as follows.

Definition 1.7. Let $\mathcal{K}$ be a thin equipment.

- A monoid $A=(A, \bar{A})$ in $\mathcal{K}$ is an object $A$ equipped with a horizontal morphism $\bar{A}: A \rightarrow A$ (which we will often denote by $A$ ) satisfying the associativity and unit axioms $\bar{A} \odot \bar{A} \leq \bar{A}$ and $1_{A} \leq \bar{A}$.
- A vertical morphism $f: A \rightarrow C$ between monoids is called a monoid homomorphism if the cell on the left below exists.

- A horizontal morphism $J: A \rightarrow B$ between monoids is called a bimodule if $\bar{A} \odot J \odot \bar{B} \leq J$.
- A cell between monoid homomorphisms and bimodules, as on the right above, is simply a cell in $\mathcal{K}$ between the underlying vertical and horizontal morphisms.

The structure on $\mathcal{K}$ lifts to make monoids, their homomorphisms and bimodules, as well as the cells between those, into a thin equipment $\operatorname{Mod}(\mathcal{K})$. The unit bimodule of a monoid $A$ is its structure morphism $\bar{A}: A \rightarrow A$, while the companion and conjoint of a monoid morphism $f: A \rightarrow C$ are the bimodules given by the restrictions $f_{*}=$ $\bar{C}(f, \mathrm{id})$ and $f^{*}=\bar{C}(\mathrm{id}, f)$. The restriction $K(f, g)$ of a bimodule $K: C \rightarrow D$ along homomorphisms $f: A \rightarrow C$ and $g: B \rightarrow D$ coincides with the restriction $K(f, g)$ of the underlying horizontal morphism $K$ in $\mathcal{K}$, along the vertical morphisms underlying $f$ and $g$.

Example 1.8. Being an ordered set we may regard any quantale $\mathcal{V}=(\mathcal{V}, \otimes, k)$ as a category; the monoid structure $(\otimes, k)$ then makes $\mathcal{V}$ into a monoidal category. In these terms monoids in $\mathcal{V}$-Rel are precisely $\mathcal{V}$-enriched categories, in the usual sense of e.g. [18], while their homomorphisms are $\mathcal{V}$-functors. A bimodule $J: A \rightarrow B$ is a $\mathcal{V}$-bimodule in the sense of Section 3 of [23]: a $\mathcal{V}$-relation $J: A \rightarrow B$ such that

$$
A\left(x_{1}, x_{2}\right) \otimes J\left(x_{2}, y_{1}\right) \otimes B\left(y_{1}, y_{2}\right) \leq J\left(x_{1}, y_{2}\right)
$$

for all $x_{1}, x_{2} \in A$ and $y_{1}, y_{2} \in B$. Also called $\mathcal{V}$-distributors, we will call such bimodules $\mathcal{V}$-profunctors. We write $\mathcal{V}$-Prof $:=\operatorname{Mod}(\mathcal{V}$-Rel $)$.

We remark that $\mathcal{V}$, as a monoidal category, is biclosed: the suprema preserving maps $x \otimes-$ and $-\otimes y$, where $x, y \in \mathcal{V}$, have right adjoints $x \multimap-$ and $-\circ-y$ defined by

$$
\begin{equation*}
y \leq x \multimap z \quad \Leftrightarrow \quad x \otimes y \leq z \quad \Leftrightarrow \quad x \leq z \circ-y \tag{5}
\end{equation*}
$$

for all $x, y, z \in \mathcal{V}$ or, equivalently,

$$
x \multimap z=\sup \{v \in \mathcal{V} \mid x \otimes v \leq z\} \quad \text { and } \quad z \circ y=\sup \{v \in \mathcal{V} \mid v \otimes y \leq z\}
$$

Both $\multimap$ and $\circ$ can be used to enrich $\mathcal{V}$ over itself, resulting in two $\mathcal{V}$-categories $\mathcal{V}_{-}$ and $\mathcal{V}_{0-}$ with hom-objects $\mathcal{V}_{-\circ}(x, y)=x \multimap y$ and $\mathcal{V}_{\circ-}(x, y)=x \circ-y$ respectively. If the monoid structure on $\mathcal{V}$ is commutative then the right adjoints $x \multimap-$ and $-a-x$ coincide.

We will use the fact that the adjoints $x \multimap-$ and $-0-y$ induce right adjoints to the sup-maps $J \odot-$ and $-\odot H$, for any $\mathcal{V}$-relations $J: A \rightarrow B$ and $H: B \rightarrow E$. Denoting these adjoints by $J \triangleleft-$ and $-\triangleright H$ respectively, they are defined by

$$
H \leq J \triangleleft K \quad \Leftrightarrow \quad J \odot H \leq K \quad \Leftrightarrow \quad J \leq K \triangleright H
$$

for all $J: A \rightarrow B, H: B \rightarrow E$ and $K: A \rightarrow E$ or, equivalently,
and

$$
\begin{aligned}
(J \triangleleft K)(y, z) & =\inf _{x \in A} J(x, y) \multimap K(x, z) \\
(K \triangleright H)(x, y) & =\inf _{z \in E} K(x, z) \circ H(y, z),
\end{aligned}
$$

for all $x \in A, y \in B$ and $z \in E$.
Example 1.9. Monoids $A$ in 2-Rel, that is categories enriched in the set 2 of truth values, can be identified with ordered sets, whose order relations $\bar{A}: A \rightarrow A$ are transitive and reflexive, while homomorphisms of such monoids are order preserving maps. A 2-profunctor $J: A \rightarrow B$ between ordered sets is a modular relation satisfying

$$
x_{1} \leq x_{2}, \quad x_{2} J y_{1} \quad \text { and } \quad y_{1} \leq y_{2} \quad \Rightarrow \quad x_{1} J y_{2},
$$

for all $x_{1}, x_{2} \in A$ and $y_{1}, y_{2} \in B$. The value of $y \multimap z$ in 2 is the Boolean truth value of the implication $y \rightarrow z$, so that the two ways of enriching 2 over itself simply recover the natural and reversed orderings of 2.
Example 1.10. If $\mathcal{V}$ is the Lawvere quantale $[0, \infty]$ then a monoid $A$ in $\mathcal{V}$-Rel, that is a $[0, \infty]$-category, is a generalised metric space in Lawvere's sense 23], whose distance function $A: A \times A \rightarrow[0, \infty]$ satisfies

$$
A(x, y)+A(y, z) \geq A(x, z) \quad \text { and } \quad A(x, x)=0
$$

for all $x, y$ and $z \in A$, but which need not be symmetric. A [0, $\infty$ ]-functor $f: A \rightarrow C$ is a non-expansive map, that satisfies $A(x, y) \geq C(f x, f y)$ for all $x, y \in A$, while a $[0, \infty]$-profunctor $J: A \rightarrow B$ is a modular metric relation $J: A \times B \rightarrow[0, \infty]$, satisfying

$$
A\left(x_{1}, x_{2}\right)+J\left(x_{2}, y_{1}\right)+B\left(y_{1}, y_{2}\right) \geq J\left(x_{1}, y_{2}\right)
$$

for all $x_{1}, x_{2} \in A$ and $y_{1}, y_{2} \in B$. In $\mathcal{V}=[0, \infty]$ the number $z \circ-y$ is the truncated difference $z \ominus y=\max \{z-y, 0\}$, so that the two ways of enriching $[0, \infty]$ over itself equip it with the (non-symmetric) metrics $[0, \infty]_{\odot}(x, y)=y \ominus x$ and $[0, \infty]_{\circ-}(x, y)=$ $x \ominus y$.
Example 1.11. The notion of metric space is further generalised by enriching over the extended real numbers $([-\infty, \infty], \geq)$ instead, thus allowing negative distances as well. Willerton in [34] uses such $[-\infty, \infty]$-categories in giving a category theoretic perspective of the Legendre-Fenchel transform, while Lawvere in [24] takes a categorical approach to entropy using categories enriched over $([-\infty, \infty], \leq)$.

Example 1.12. Analogous to the relation between metric spaces and $[0, \infty]$-categories, Flagg notes in [11] (or see Section III.2.1 of [17]) that probabilistic metric spaces, originally introduced by Menger in [27], can be regarded as categories enriched in the quantales $\Delta_{\&}$ (Example 1.4) of distance distribution functions. Instead of real-valued distances, any pair $(x, y)$ of points in a probabilistic metric space $A$ is equipped with a distance distribution function $A(x, y) \in \Delta$. For each $t \in[0, \infty]$, the value $A(x, y)(t) \in[0,1]$ is to be thought of as the "probability that the distance between $x$ and $y$ is less than $t$ ".

We close this section by restricting to the setting of thin equipments the notions of lax functor between double categories and (vertical) transformation of such functors, both introduced in Section 7 of [12].

Definition 1.13. A lax functor $F: \mathcal{K} \rightarrow \mathcal{L}$ between thin equipments $\mathcal{K}$ and $\mathcal{L}$ consists of a functor $F_{\mathrm{v}}: \mathcal{K}_{\mathrm{v}} \rightarrow \mathcal{L}_{\mathrm{v}}$ (which will be denoted $F$ ) as well as an assignment of horizontal morphisms

$$
J: A \rightarrow B \quad \mapsto \quad F J: F A \rightarrow F B
$$

that preserves horizontal composition laxly, that is

$$
1_{F A} \leq F 1_{A} \quad \text { and } \quad F J \odot F H \leq F(J \odot H)
$$

for any object $A$ and composable morphisms $J$ and $H$ in $\mathcal{K}_{\mathrm{h}}$, such that the existence of any cell in $\mathcal{K}$ as on the left below implies the existence of the middle cell in $\mathcal{L}$.


$$
\begin{gathered}
F A \stackrel{F J}{\mapsto} F B \\
\xi_{A} \downarrow|\wedge| \xi_{B} \\
G A \underset{G J}{\rightarrow} G B
\end{gathered}
$$

A transformation $\xi: F \Rightarrow G$ of lax functors $F$ and $G: \mathcal{K} \rightarrow \mathcal{L}$ is a natural transformation $\xi_{\mathrm{v}}: F_{\mathrm{v}} \Rightarrow G_{\mathrm{v}}$ (which will be denoted $\xi$ ) such that for every horizontal morphism $J: A \rightarrow B$ in $\mathcal{K}$ the naturality cell on the right above exists in $\mathcal{L}$.

Thin equipments, lax functors and their transformations form a 2-category that we will denote ThinEquip ${ }_{1}$; it is a full sub-2-category of the 2-category $\mathrm{Db}_{1}$ of double categories, lax functors and their transformations.

A lax functor $F: \mathcal{K} \rightarrow \mathcal{L}$ is called normal if it preserves horizontal units strictly, that is $F 1_{A}=1_{F A}$ for all $A \in \mathcal{K}$; a strict functor $F: \mathcal{K} \rightarrow \mathcal{L}$ is a lax functor that preserves both units and horizontal compositions strictly. Notice that a lax functor $F$ is normal if and only if it preserves companions and conjoints, in the sense that $F\left(f_{*}\right)=(F f)_{*}$ and $F\left(f^{*}\right)=(F f)^{*}$ for all $f: A \rightarrow C$ in $\mathcal{K}$. On the other hand any lax functor preserves restrictions, as the following restriction of Proposition 6.8 of [30] to thin equipments shows.

Proposition 1.14 (Shulman). For any lax functor $F: \mathcal{K} \rightarrow \mathcal{L}$ and morphisms $f: A \rightarrow C, K: C \rightarrow D$ and $g: B \rightarrow D$ in $\mathcal{K}$ we have $F(K(f, g))=(F K)(F f, F g)$.
Proof. To obtain $F(K(f, g)) \leq(F K)(F f, F g)$ we apply $F$ to the composite of cells on the left below and compose the result with the appropriate cells among those that define $(F f)_{*}$ and $(F g)^{*}$.



The inverse $(F K)(F f, F g) \leq F(K(f, g))$ is obtained by composing the composite on the right above, whose leftmost and rightmost cells are ' $F$-images' of cells defining $f_{*}$ and $g^{*}$ respectively, with the lax structure cells $1_{F A} \leq F 1_{A}, 1_{F B} \leq F 1_{B}$ and $F\left(f_{*}\right) \odot F K \odot F\left(g^{*}\right) \leq F\left(f_{*} \odot K \odot g^{*}\right)$.

We write ThinEquip ${ }_{\text {str }} \subset$ ThinEquip $_{n} \subset$ ThinEquip $_{1}$ for the locally full sub-2-categories generated by the strict and normal functors respectively. The following is Proposition 11.12 of [30] restricted to thin equipments.

Proposition 1.15 (Shulman). The assignment $\mathcal{K} \mapsto \operatorname{Mod}(\mathcal{K})$ of Definition 1.7 extends to a strict 2 -functor Mod: ThinEquip ${ }_{1} \rightarrow$ ThinEquip $_{\mathrm{n}}$, which restricts to a 2 -functor ThinEquip $_{\text {str }} \rightarrow$ ThinEquip $_{\text {str }}$.
Sketch of the proof. The image

$$
\operatorname{Mod} F: \operatorname{Mod}(\mathcal{K}) \rightarrow \operatorname{Mod}(\mathcal{L})
$$

of a lax functor $F: \mathcal{K} \rightarrow \mathcal{L}$ between thin equipments simply applies $F$ indexwise; e.g. it maps a monoid $A=(A, \bar{A})$ in $\mathcal{K}$ to the monoid $(\operatorname{Mod} F)(A):=(F A, F \bar{A})$ in $\mathcal{L}$. Notice that $\operatorname{Mod} F$ is normal, while it is strict whenever $F$ is. The naturality cells of a transformation $\xi: F \Rightarrow G$ ensure that, for every monoid $A$ in $\mathcal{K}$, the component $\xi_{A}: F A \rightarrow G A$ is a homomorphism of monoids, so that these components combine to form a transformation $\operatorname{Mod} \xi: \operatorname{Mod} F \Rightarrow \operatorname{Mod} G$.

## 2. Kan extensions in thin equipments

Using thin equipments as environment, in this section we describe the first ingredient of our categorical approach to the maximum theorem: the notion of Kan extension, which generalises that of optimised function. In the definition below we start by restricting the notion of left Kan extension in a general double category, that was introduced in [19] under the name 'pointwise left Kan extension', to thin equipments. Afterwards we will describe and study a type of Kan extension that generalises those optimised functions given by suprema that are attained as maxima, as described in the Introduction.

Definition 2.1. Let $d: A \rightarrow M$ and $J: A \rightarrow B$ be morphisms in a thin equipment $\mathcal{K}$. The cell $\eta$ in the right-hand side below defines $l: B \rightarrow M$ as the left Kan extension of $d$ along $J$ if every cell in $\mathcal{K}$, of the form as on the left-hand side, factors through $\eta$ as shown.


Horizontally dual, the cell $\varepsilon$ in the right-hand side below defines $r: A \rightarrow M$ as the right Kan extension of $e: B \rightarrow M$ along $J: A \rightarrow B$ if every cell in $\mathcal{K}$, of the form as on the left-hand side, factors through $\varepsilon$ as shown.


For a quantale $\mathcal{V}$ and a $\mathcal{V}$-functor $j: A \rightarrow B$ the following proposition implies that, in the thin equipment $\mathcal{V}$-Prof of $\mathcal{V}$-profunctors, left Kan extensions along the companion $j_{*}: A \rightarrow B$ coincide with $\mathcal{V}$-enriched left Kan extensions along $j$ in the usual sense (see e.g. Section 4 of [18]). The same holds for right Kan extensions along the conjoint $j^{*}: B \rightarrow A$.

Proposition 2.2. Let $\mathcal{V}$ be a quantale, let $d: A \rightarrow M$ and $l: B \rightarrow M$ be $\mathcal{V}$-functors and let $J: A \rightarrow B$ be a $\mathcal{V}$-profunctor. The $\mathcal{V}$-functor $l$ is the left Kan extension of d along $J$ in $\mathcal{V}$-Prof precisely if

$$
\begin{equation*}
M(l y, z)=\inf _{x \in A} J(x, y) \multimap M(d x, z) \tag{7}
\end{equation*}
$$

for all $y \in B$ and $z \in M$. In particular if $M=\mathcal{V}_{-}$(Example 1.8) then $l$ is given by

$$
l y=\sup _{x \in A} d x \otimes J(x, y)
$$

while if $M=\mathcal{V}_{0}-$ and $\mathcal{V}$ is commutative then $l$ is given by

$$
l y=\inf _{x \in A} d x \circ-J(x, y)
$$

Dually a $\mathcal{V}$-functor $r: A \rightarrow M$ is the right Kan extension of a $\mathcal{V}$-functor $e: B \rightarrow M$ along $J: A \rightarrow B$ precisely if

$$
M(z, r x)=\inf _{y \in B} M(z, e y) \circ J(x, y)
$$

for all $x \in A$ and $z \in M$. If $M=\mathcal{V}_{0-}$ then $r$ is given by

$$
r x=\sup _{y \in B} J(x, y) \otimes e y
$$

if $M=\mathcal{V}_{-}$and $\mathcal{V}$ is commutative then $r$ is given by

$$
r x=\inf _{y \in B} J(x, y) \multimap e y
$$

Sketch of the proof. We sketch the proof for the left Kan extension $l: B \rightarrow M$ of $d: A \rightarrow M$ along $J: A \rightarrow B$; the proof for right Kan extensions is analogous. For the 'if'-part first notice that the existence of the cell $\eta$ follows from the fact that

$$
k \leq M(l y, l y)=\inf _{x \in A} J(x, y) \multimap M(d x, l y) \leq J(x, y) \multimap M(d x, l y)
$$

for all $x \in A$ and $y \in B$, combined with the definition of - , see (5). To see that it satisfies the univeral property (6) notice that, by the definitions of $\odot$ and - , the cell on the left-hand side of (6) exists precisely if $H(y, z) \leq J(x, y) \multimap M(d x, g z)$ for all $x \in A, y \in B$ and $z \in C$, so that $H(y, z) \leq M(l y, g z)$ follows.

For the 'precisely'-part assume that $l$ satisfies the universal property (6), let $y \in B$ and $z \in M$, and let $v \in \mathcal{V}$ be such that $v \leq J(x, y) \multimap M(d x, z)$ for all $x \in A$. To show that $v \leq M(l y, z)$ consider on the left-hand side of (6) the cell with $C=*$, the unit $\mathcal{V}$-category with single object $*$ and hom-object $*(*, *)=k, g: C \rightarrow M$ given by $g(*)=z$ and $H: B \rightarrow C$ given by $H(s, *)=v$ if $s=y$ and $H(s, *)=\perp$ otherwise. That this cell exists follows from the assumption on $v$, while factorising it through $\eta$ gives $v \leq M(l y, z)$.

In the case that $M=\mathcal{V}_{\bigcirc}$ the equation defining $l$ reduces to

$$
\begin{aligned}
& l y \multimap z=\inf _{x \in A}(J(x, y) \multimap(d x \multimap z)) \\
&=\inf _{x \in A}((d x \otimes J(x, y)) \multimap z)=\left(\sup _{x \in A} d x \otimes J(x, y)\right) \multimap z
\end{aligned}
$$

for all $y \in B$ and $z \in M$, where we use that $-\multimap z$ transforms suprema into infima. Using the fact that $\mathcal{V}$ is separated we conclude that $l y$ and $\sup _{x \in A} d x \otimes J(x, y)$ coincide for all $y \in B$.

Finally if $M=\mathcal{V}_{0-}$ and $\mathcal{V}$ is commutative then the equation defining $l$ reduces to

$$
\begin{aligned}
& l y \circ-z=\inf _{x \in A}((d x \circ-z) \circ J(x, y)) \\
&=\inf _{x \in A}((d x \circ-J(x, y)) \circ-z)=\left(\inf _{x \in A} d x \circ-J(x, y)\right) \circ-z
\end{aligned}
$$

for all $y \in B$ and $z \in M$, where we use that $-0-z$ preserves infima. As before $l y=\inf _{x \in A} d x \circ-J(x, y)$ follows.

Example 2.3. Let $A, B$ and $M$ be ordered sets, $d: A \rightarrow M$ a monotone map and $J: A \rightarrow B$ a modular relation (Example 1.9). Taking $\mathcal{V}=2$ in the previous proposition, the defining equation (7) of the left Kan extension $l: B \rightarrow M$ of $d$ along $J$ reduces to

$$
l y=\sup _{x \in J^{\circ} y} d x \quad \text { where } \quad J^{\circ} y=\{x \in A \mid x J y\},
$$

so that $l$ exists whenever these suprema exist. Dually the right Kan extension $r: A \rightarrow M$ of a monotone map $e: B \rightarrow M$ along $J: A \rightarrow B$ is given by the infima

$$
r x=\inf _{y \in J x} e y \quad \text { where } \quad J x=\{y \in B \mid x J y\} .
$$

For general quantales $\mathcal{V}$ notice that, if $d$ is a $\mathcal{V}$-functor $A \rightarrow \mathcal{V}_{-}$, then by the previous proposition $l: B \rightarrow M$ above can be regarded as a $\mathcal{V}$-enriched Kan extension as well, along the $\mathcal{V}$-profunctor $J_{\mathcal{V}}: A \rightarrow B$ that is given by $J_{\mathcal{V}}(x, y)=k$ if $x J y$ and $J_{\mathcal{V}}(x, y)=\perp$ otherwise. Likewise if $\mathcal{V}$ is commutative and $e$ is a $\mathcal{V}$-functor $B \rightarrow \mathcal{V}_{-\circ}$ then $r: A \rightarrow M$ above is the left Kan extension, in $\mathcal{V}$-Prof, of $e$ along the $\mathcal{V}$-profunctor $\left(J^{\circ}\right) \mathcal{V}: B \rightarrow A$.

Having introduced Kan extensions we now consider the notion of exact cell. The corresponding notion for general double categories, that was introduced in [19] under the name 'pointwise exact cell', generalises the classical notion of 'exact square' of functors, as studied by Guitart in [15].

Definition 2.4. The cell $\phi$ on the left below is called left exact if, for any cell $\eta$ as in the middle that defines $l$ as a left Kan extension, the vertical composite $\eta \circ \phi$ defines $l \circ g$ as a left Kan extension. Dually $\phi$ is called right exact if, for any cell $\varepsilon$ below that defines $r$ as a right Kan extension, the composite $\varepsilon \circ \phi$ defines $r \circ f$ as a right Kan extension.


Notice that if the cell $\eta$ above is itself left exact then it defines $l$ as the absolute left Kan extension of $d$ along $K$ : for any morphism $k: M \rightarrow N$ the composite $1_{k} \circ \eta$ defines $k \circ l$ as a left Kan extension. Likewise if $\varepsilon$ above is right exact then it defines $r$ as an absolute right Kan extension.

The following result restates Corollary 4.5 of [19] in the setting of thin equipments. For each cell $\phi$ as on the left above we will call the hypotheses of the parts (a) and (b) below the left and right Beck-Chevalley conditions for $\phi$ respectively.

Proposition 2.5. For a cell $\phi$ in a thin equipment, as on the left above, the following hold; compare Lemma 1.6.
(a) If $f^{*} \odot J=K(\mathrm{id}, g)$ then $\phi$ is left exact.
(b) If $J \odot g_{*}=K(f, \mathrm{id})$ then $\phi$ is right exact.

As we shall see shortly, the following theorem describes a type of left Kan extension that generalises those optimised functions given by suprema that are attained by maxima. We will call its hypothesis the Beck-Chevalley condition for left Kan extensions. Horizontally dual, we say that a right Kan extension satisfies the Beck-Chevalley condition whenever its defining cell $\varepsilon$ satisfies the right Beck-Chevalley condition or, equivalently, $\varepsilon$ satisfies the universal property that is horizontally dual to that described in the theorem below. In a general double category, right Kan extensions defined by cells satisfying such a universal property were introduced by Grandis and Paré in [14], where they were called 'absolute right Kan extensions'.

Theorem 2.6. In a thin equipment consider a cell $\eta$ as in the right-hand side below. It satisfies the left Beck-Chevalley condition if and only if any cell as on the left-hand side factors through $\eta$ as shown.


In particular, in this case $l$ is the absolute left Kan extension of d along $J$.
Proof. The 'only if'-part. Suppose that $\eta$ satisfies the left Beck-Chevalley condition, that is $d^{*} \odot J=l^{*}$. We have to show that any cell as on the left-hand side above factors through $\eta$ as shown. By Lemma 1.6 we may equivalently prove that $l^{*} \odot H \leq$ $K(\mathrm{id}, g)$. This is shown below, where the identity is the assumption on $\eta$ and the inequality follows from applying Lemma 1.6 to the cell on the left-hand side above.

$$
l^{*} \odot H=d^{*} \odot J \odot H \leq K(\mathrm{id}, g)
$$

The 'if'-part. Assuming that $\eta$ satisfies the universal property above, we have to show that $d^{*} \odot J=l^{*}$. Applying Lemma 1.6 to $\eta$ gives $d^{*} \odot J \leq l^{*}$. For the reverse inequality apply the same lemma to the factorisation through $\eta$ in

which exists by the assumption on $\eta$.
Example 2.7. Given ordered sets $A, B$ and $M$, let $d: A \rightarrow M$ be a monotone map and $J: A \rightarrow B$ a modular relation. If the left Kan extension $l: B \rightarrow M$ of $d$ along $J$ exists then, as we saw in Example 2.3, $l$ is given by

$$
l y=\sup _{x \in J^{\circ} y} d x
$$

It is easily checked that the Beck-Chevalley condition for $l$ states that for every $y \in B$ there is $x \in J^{\circ} y$ with $d x=l y$, that is the suprema above are attained as maxima.

Example 2.8. Given generalised metric spaces $A, B$ and $M$, let $d: A \rightarrow M$ be a non-expanding map and let $J: A \rightarrow B$ be a modular metric relation (Example 1.10). By the proposition below the left Kan extension $l: B \rightarrow M$ of $d$ along $J$, if it exists, satisfies the Beck-Chevalley condition precisely when

$$
\inf _{x \in A} M(l y, d x)+J(x, y)=0
$$

for all $y \in B$.
If the map $d$ in Example 2.7 above is a continuous map $d: A \rightarrow[-\infty, \infty]$, then the Beck-Chevalley condition holds whenever the pre-images $J^{\circ} y$, for each $y \in B$, are non-empty and compact in $A$ : this is a direct consequence of Weierstraß' extreme value theorem, see e.g. Corollary 2.35 of [1] or Theorem 8.1 below. In Section 8 we will generalise the extreme value theorem to left Kan extensions of morphisms $d: A \rightarrow \mathcal{V}_{-}$of ' $\mathcal{V}$-valued topological spaces', a notion that is recalled in the next section.

Proposition 2.9. Let $\mathcal{V}$ be a quantale, let $d: A \rightarrow M$ be a $\mathcal{V}$-functor and $J: A \rightarrow B$ $a \mathcal{V}$-profunctor. The left Kan extension $l: B \rightarrow M$ of $d$ along $J$, if it exists, satisfies the Beck-Chevalley condition precisely when

$$
k \leq \sup _{x \in A} M(l y, d x) \otimes J(x, y)
$$

for all $y \in B$.
Proof. For the 'when'-part we have to show that $d^{*} \odot J=l^{*}$ follows from the inequality above. Firstly $d^{*} \odot J \leq l^{*}$ is obtained by applying Lemma 1.6 to the universal cell defining $l$. That the reverse inequality follows from the inequality above is shown by

$$
\begin{aligned}
l^{*}(z, y) & =M(z, l y)=M(z, l y) \otimes k \leq M(z, l y) \otimes\left(\sup _{x \in A} M(l y, d x) \otimes J(x, y)\right) \\
& =\sup _{x \in A}(M(z, l y) \otimes M(l y, d x) \otimes J(x, y)) \\
& \leq \sup _{x \in A} M(z, d x) \otimes J(x, y)=\left(d^{*} \odot J\right)(z, y)
\end{aligned}
$$

where $y \in B$ and $z \in M$. The 'precisely when'-part follows easily from evaluating $l^{*}=d^{*} \odot J$ at $(l y, y)$, where $y \in B$, and using that $k \leq M(l y, l y)$ by the unit axiom for $M$.

Kan extensions satisfying the Beck-Chevalley condition are preserved by any strict functor, as follows.

Proposition 2.10. Let $F: \mathcal{K} \rightarrow \mathcal{L}$ be a normal lax functor between thin equipments. Given morphisms $d: A \rightarrow M$ and $J: A \rightarrow B$ in $\mathcal{K}$ suppose that their left Kan extension $l: B \rightarrow M$ exists, and that it satisfies the Beck-Chevalley condition. The image Fl is the left Kan extension of Fd along FJ, and it satisfies the Beck-Chevalley condition, precisely if $F\left(d^{*}\right) \odot F J=F\left(d^{*} \odot J\right)$.
Proof. Since normal lax functors preserve companions we have

$$
(F d)^{*} \odot F J=F\left(d^{*}\right) \odot F J \leq F\left(d^{*} \odot J\right)=F\left(l^{*}\right)=(F l)^{*},
$$

where the second equality is the $F$-image of the Beck-Chevalley condition for $l$. The result follows directly from noticing that the Beck-Chevalley condition for Fl means that $(F d)^{*} \odot F J=(F l)^{*}$.

## 3. T-graphs

Here we describe the second ingredient of our categorical approach to the maximum theorem: expressing topological structures as algebraic structures. More precisely, taking the view-point of the study of monoidal topology [17], we will regard topological structures as 'graphs' or 'categories' over a monad on a thin equipment.

Definition 3.1. A lax monad $T$ on a thin equipment $\mathcal{K}$ is simply a monad $T=$ $(T, \mu, \iota)$ on $\mathcal{K}$ in the 2-category ThinEquip ${ }_{1}$, consisting of a lax endofunctor $T: \mathcal{K} \rightarrow \mathcal{K}$ equipped with multiplication and unit transformations $\mu: T^{2} \Rightarrow T$ and $\iota: \operatorname{id}_{\mathcal{K}} \Rightarrow T$ that satisfy the usual associativity and unit axioms. We call $T$ normal or strict whenever its underlying endofunctor is normal or strict.

Notice that any lax monad $T$ on a thin equipment $\mathcal{K}$ restricts to a monad $T_{\mathrm{v}}$ on the vertical category $\mathcal{K}_{\mathrm{v}}$. In particular, in the case $\mathcal{K}=\mathcal{V}$-Rel for some quantale $\mathcal{V}$, the lax monad $T$ can be thought of as being a "lax extension" of the Set-monad $T_{\mathrm{v}}$ to the thin 2-category of $\mathcal{V}$-relations. The latter is the traditional view-point taken in monoidal topology; that such lax extensions are equivalent to lax monads on $\mathcal{V}$-Rel, in our sense, is shown in Section III.1.13 of [17].

For general constructions of lax extensions of Set-monads to $\mathcal{V}$-relations we refer to [6] (or see Section IV.2.4 of [17]) and [29]. Here we restrict ourselves to the extensions of the powerset monad and the ultrafilter monad, which are recalled in the examples below.
Example 3.2. We denote by $P A=\{S \subseteq A\}$ the powerset of a set $A$. The assignment $A \mapsto P A$ extends to the powerset monad on Set that is given by

$$
P f: P A \rightarrow P C: S \mapsto f S ; \quad \mu_{A}: P^{2} A \rightarrow P A: \Sigma \mapsto \bigcup \Sigma ; \quad \iota_{A}: A \rightarrow P A: x \mapsto\{x\},
$$

where $f: A \rightarrow C$ is any function. It was shown by Clementino and Hofmann (Section 6.3 of $[6]$ ) that $P$ extends to a lax monad on $\mathcal{V}$-Rel, by mapping a $\mathcal{V}$-relation $J: A \rightarrow B$ to

$$
(P J)(S, T)=\inf _{t \in T} \sup _{s \in S} J(s, t)
$$

for any $S \in P A$ and $T \in P B$. In case $\mathcal{V}=2$, so that we can regard $J$ and $P J$ as ordinary relations, this reduces to

$$
S(P J) T \quad \Leftrightarrow \quad T \subseteq J S
$$

Notice that $P$ is not normal.
The following example describes the lax extensions of the ultrafilter monad $U$. To be able to extend the ultrafilter monad $U$ to $\mathcal{V}$-relations the quantale $\mathcal{V}$ needs to be 'completely distributive', as follows. Writing Dn $\mathcal{V}$ for the set of downsets $S \subseteq \mathcal{V}$, satisfying

$$
u \leq v \quad \text { and } \quad v \in S \quad \Rightarrow \quad u \in S
$$

for all $u, v \in \mathcal{V}$, the quantale $\mathcal{V}$ is called completely distributive if sup: $\operatorname{Dn} \mathcal{V} \rightarrow \mathcal{V}$ has a left adjoint $\Downarrow$. In that case let the totally below relation $\ll$ on $\mathcal{V}$ be defined by $u \ll v: \Leftrightarrow u \in \Downarrow v$; equivalently

$$
u \ll v \quad \Leftrightarrow \quad \underset{S \subseteq \mathcal{V}}{\forall}(v \leq \sup S \Rightarrow \exists s \in S: u \leq s) .
$$

Writing $\downarrow: \mathcal{V} \rightarrow \operatorname{Dn} \mathcal{V}$ for the map that sends $v \in \mathcal{V}$ to the principal downset $\downarrow v=\{u \in \mathcal{V} \mid u \leq v\}$, it follows from the chain of adjunctions $\Downarrow \dashv \sup \dashv \downarrow$ that $v=\sup \{u \in \mathcal{V} \mid u \ll v\}$ for all $v \in \mathcal{V}$; for details see e.g. Section II.1.11 of [17].

The two-chain quantale $2=\{\perp \leq \top\}$ is completely distributive, with the totally below relation given by $u \ll v \Leftrightarrow v=\top$, and so are the quantales $([0, \infty], \geq)$ and
$([-\infty, \infty], \geq)$, both with $u \ll v \Leftrightarrow u>v$. That the quantales $\Delta_{\&}$ of distance distribution functions (Example 1.4) are completely distributive is shown in Section 2.1 of 16 .
Example 3.3. For a set $A$ we denote by $U A$ the set of ultrafilters on $A$; see e.g. Section II.1.13 of [17]. The assignment $A \mapsto U A$ extends to the ultrafilter monad $U=(U, \mu, \iota)$ on Set defined by

$$
T \in(U f)(\mathfrak{x}): \Leftrightarrow f^{-1} T \in \mathfrak{x} ; \quad S \in \mu_{A}(\mathfrak{X}): \Leftrightarrow S^{\sharp} \in \mathfrak{X} ; \quad S \in \iota_{A}(x): \Leftrightarrow x \in S,
$$

where $f: A \rightarrow C, \mathfrak{x} \in U A, T \subseteq C, \mathfrak{X} \in U^{2} A, S \subseteq A$ and $x \in A$; here $S^{\sharp}$ is the set of all ultrafilters on $S \subseteq A$ :

$$
\mathfrak{x} \in S^{\sharp} \quad: \Leftrightarrow \quad S \in \mathfrak{x} .
$$

In Section 8 of [8] Clementino and Tholen show that $U$ extends to a lax monad on $\mathcal{V}$-Rel provided that $\mathcal{V}$ is completely distributive, by mapping a $\mathcal{V}$-relation $J: A \rightarrow B$ to

$$
(U J)(\mathfrak{x}, \mathfrak{y})=\inf _{\substack{S \in \mathfrak{x} \\
T \in \mathfrak{y} \\
\sup _{\begin{subarray}{c}{ } S }}^{t \in T}}\end{subarray}} J(s, t),
$$

for all $\mathfrak{x} \in U A$ and $\mathfrak{y} \in U B$; see [22] for an alternative proof. In case $\mathcal{V}=2$, so that we can regard $J$ and $U J$ as ordinary relations, the definition of $U J$ reduces to

$$
\mathfrak{x}(U J) \mathfrak{y} \quad \Leftrightarrow \quad \underset{S \in \mathfrak{x}}{\forall} J S \in \mathfrak{y} \quad \Leftrightarrow \quad \forall \quad \forall{ }_{T \in \mathfrak{y}} J^{\circ} T \in \mathfrak{x},
$$

which recovers Barr's original extension of the ultrafilter monad [2]. Returning to general $\mathcal{V}$, it was shown in Section 6.4 of [6] that $U J$ can be equivalently given by

$$
\begin{equation*}
(U J)(\mathfrak{x}, \mathfrak{y})=\sup \left\{v \in \mathcal{V} \mid \mathfrak{x}\left(U J_{v}\right) \mathfrak{y}\right\} \tag{8}
\end{equation*}
$$

where $J_{v}: A \rightarrow B$ is the (ordinary) relation defined by

$$
x J_{v} y \quad: \Leftrightarrow \quad v \leq J(x, y)
$$

for all $x \in A$ and $y \in B$.
It is easily checked that the above described extension $U$ of the ultrafilter monad to $\mathcal{V}$-relations is normal. Moreover $U$ is a strict monad in the cases $\mathcal{V}=2$ (see Sections III.1.11-12 of [17]) and $\mathcal{V}=([0, \infty], \geq)$ (see Proposition III.2.4.3 of [17]). In Section 6.4 of $[6]$ it is shown that $U$ is not strict when $\mathcal{V}=([-\infty, \infty], \geq)$; unfortunately I do not know whether $U$ is a strict monad for any of the quantales $\Delta_{\&}$ of distance distribution functions (Example 1.4).

Having described the main examples of monads $T$ on thin equipments, in the definition below we recall, from e.g. Sections III.1. 6 and III.4.1 of [17], the notions of 'graph' and 'category' over such a monad. The examples that follow describe how these notions allow us to regard topological structures as algebraic structures.

Definition 3.4. Let $T=(T, \mu, \iota)$ be a lax monad on a thin equipment $\mathcal{K}$.

- A $T$-graph $A=(A, \alpha)$ consists of an object $A$ equipped with a horizontal morphism $\alpha: T A \rightarrow A$, such that the unitor cell on the left below exists.

- A $T$-graph $A$ is called left unitary if the middle cell above exists and right unitary if the cell on the right exists; it is called unitary if both cells exist.
- A $T$-category is a $T$-graph $A=(A, \alpha)$ such that the associator cell on the left below exists.


- A morphism $f: A \rightarrow C$ between $T$-graphs is called a $T$-morphism (or again simply morphism) if the cell on the right above exists.

Notice that any $T$-category $A$ is a unitary $T$-graph: the required cells are obtained by composing the associator cell of $A$ both with the " $T$-image" of the unitor cell and with the horizontal cell $\iota_{A}^{*} \leq \alpha$ that corresponds to the unitor cell, under Lemma 1.6. Moreover any $T$-graph is left unitary when $T$ is normal: in that case we have $T 1_{A} \odot \alpha=1_{T A} \odot \alpha=\alpha$.

Together with the morphisms between them $T$-graphs form a category which we denote by $T$-Gph. Left unitary $T$-graphs, unitary $T$-graphs and $T$-categories generate full subcategories

$$
\begin{equation*}
T \text {-Cat } \hookrightarrow T \text {-UGph } \hookrightarrow T \text {-LGph } \hookrightarrow T \text {-Gph. } \tag{9}
\end{equation*}
$$

For a lax extension of a Set-monad $T$ to $\mathcal{V}$-Rel, where $\mathcal{V}$ is a quantale, $T$-graphs are traditionally called $(T, \mathcal{V})$-graphs and their morphisms $(T, \mathcal{V})$-functors; in this case we shall write $(T, \mathcal{V})-\mathrm{Gph}:=T$-Gph and $(T, \mathcal{V})$-UGph $:=T$-UGph. Likewise categories over such monads $T$ are called $(T, \mathcal{V})$-categories, and we write $(T, \mathcal{V})$-Cat : $=T$-Cat.
Example 3.5. Let $\mathcal{V}$ be a quantale and let $P$ be the powerset monad extended to $\mathcal{V}$-relations, see Example 3.2, A $(P, \mathcal{V})$-graph is a set $A$ equipped with a $\mathcal{V}$-relation $\delta: P A \times A \rightarrow \mathcal{V}$ satisfying the reflexivity axiom
(R) $k \leq \delta(\{x\}, x)$
for all $x \in A$. It is easily checked that $A$ is unitary precisely if it is left unitary, which in turn is equivalent to the extensionality axiom
(E) $S \subseteq T \quad \Rightarrow \quad \delta(S, x) \leq \delta(T, x)$
for all $S, T \in P A$.
Seal proved in Section 5.4 of [29] that a $\mathcal{V}$-relation $\delta: P A \rightarrow A$ equips the set $A$ with a $(P, \mathcal{V})$-category structure precisely if it satisfies the two axioms above as well as the transitivity axiom
(T) $v \otimes \delta\left(S^{(v)}, x\right) \leq \delta(S, x)$,
for all $x \in A, S \in P A$ and $v \in \mathcal{V}$; here $S^{(v)}:=\{y \in A \mid v \leq \delta(S, y)\}$. Called closeness spaces by Seal, we follow $\lfloor 22]$ and call $(P, \mathcal{V})$-categories $\mathcal{V}$-valued closure spaces; we write $\mathcal{V}$-Cls $:=(P, \mathcal{V})$-Cat. Similarly we shall call $(P, \mathcal{V})$-graphs $\mathcal{V}$-valued pseudoclosure spaces and unitary $(P, \mathcal{V})$-graphs $\mathcal{V}$-valued preclosure spaces, and write $\mathcal{V}$-PsCls $:=(P, \mathcal{V})$-Gph and $\mathcal{V}$-PreCls $:=(P, \mathcal{V})$-UGph. In each of these categories a morphism $f:(A, \delta) \rightarrow(C, \zeta)$ is a continuous map, satisfying
(C) $\delta(S, x) \leq \zeta(f S, f x)$
for all $S \in P A$ and $x \in A$.
Of course 2-valued closure spaces $A$ can be identified with ordinary closure spaces $(A, S \mapsto \bar{S})$ via $x \in \bar{S} \Leftrightarrow \delta(S, x)=\top$, while morphisms $f: A \rightarrow C$ are continuous in the usual sense: $f \bar{S} \subseteq \overline{f S}$ for all $S \in P A$. In Exercise III.2.G of $[17][0, \infty]$-valued closure spaces are called metric closure spaces.
Example 3.6. Let $\mathcal{V}$ be a completely distributive quantale and let $U$ be the ultrafilter monad extended to $\mathcal{V}$-relations, see Example 3.3 A $(U, \mathcal{V})$-graph is a set $A$ equipped with a $\mathcal{V}$-valued convergence relation $\alpha: U A \times A \rightarrow \mathcal{V}$, which is required to satisfy the reflexivity axiom

$$
\text { (R) } k \leq \alpha(\iota x, x)
$$

for all $x \in A$; here $\iota x$ is the principal ultrafilter generated by $x$. In the case $\mathcal{V}=2$ this recovers the notion of pseudotopological space introduced by Choquet [4]; we shall call $(U, \mathcal{V})$-graphs $\mathcal{V}$-valued pseudotopological spaces and write $\mathcal{V}$-PsTop $:=$ $(U, \mathcal{V})$-Gph. A morphism $f:(A, \alpha) \rightarrow(C, \gamma)$ of $\mathcal{V}$-valued pseudotopological spaces is a continuous map satisfying
(C) $\alpha(\mathfrak{x}, x) \leq \gamma((U f)(\mathfrak{x}), f x)$
for all $\mathfrak{x} \in U A$ and $x \in A$.
While unitary $(U, \mathcal{V})$-graphs and $(U, \mathcal{V})$-categories can be described directly in terms of ultrafilter convergence, we shall follow the approach taken by Lai and Tholen in [22] and instead describe them in terms of $\mathcal{V}$-valued preclosure spaces and $\mathcal{V}$-valued closure spaces respectively. These descriptions generalise the classical description of topological spaces in terms of closure operations. We will use this approach throughout: for instance in Section 6 we will describe "horizontal $U$-morphisms" $J: A \rightarrow B$ between $(U, \mathcal{V})$-categories in terms of the corresponding $\mathcal{V}$-valued closure space structures on $A$ and $B$.

The functor $(U, V)$-Cat $\rightarrow \mathcal{V}$-Cls that allows us to regard $(U, \mathcal{V})$-categories as $\mathcal{V}$-valued closure spaces is induced by an 'algebraic morphism' $\varepsilon: P \rightarrow U$ between the powerset monad $P$ and the ultrafilter monad $U$, in the sense of Section 7 of [32], as follows. The first assertion of the proposition below is the first assertion of Proposition 3.4 of [22].

Proposition 3.7. Let $\mathcal{V}$ be a completely distributive quantale and let $P$ and $U$ be the extensions of the powerset and ultrafilter monads to the thin equipment $\mathcal{V}$-Rel of $\mathcal{V}$-relations. The family of $\mathcal{V}$-relations $\varepsilon_{A}: P A \rightarrow U A$, where $A$ ranges over all sets, that is defined by

$$
\varepsilon_{A}(S, \mathfrak{x})= \begin{cases}k & \text { if } S \in \mathfrak{x} \\ \perp & \text { otherwise }\end{cases}
$$

for all $S \in P A$ and $\mathfrak{x} \in U A$, forms an algebraic morphism $\varepsilon: P \rightarrow U$. This means that the cells


$$
\begin{gathered}
A=A \\
\iota_{A}^{P} \downarrow \mid \wedge \downarrow \iota_{A}^{U} \\
P A
\end{gathered}
$$

$$
\begin{gathered}
P^{2} A \xrightarrow{P \varepsilon_{A}} P U A \xrightarrow{\varepsilon_{U A}} U^{2} A \\
\mu_{A}^{P} \mid \underset{\varepsilon_{A}}{\mid \wedge} \downarrow^{\downarrow} \mu_{A}^{U} \\
P A \xrightarrow{\mid} U A
\end{gathered}
$$

exist in $\mathcal{V}$-Rel, where $A$ is any set and $f: A \rightarrow C$ is any function, while

$$
P J \odot \varepsilon_{B} \leq \varepsilon_{A} \odot U J \quad \text { and } \quad P \varepsilon_{A} \odot P \alpha=P\left(\varepsilon_{A} \odot \alpha\right)
$$

for all $\mathcal{V}$-relations $J: A \rightarrow B$ and $\alpha: U A \rightarrow A$.
Furthermore the following identities hold:
(a) $\varepsilon_{A}\left(\iota_{A}^{P}\right.$, id $)=\iota_{A *}^{U}$ for all $A$;
(b) $P J \odot \varepsilon_{B}=\varepsilon_{A} \odot U J$ for all $J: A \rightarrow B$;
(c) $\varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot H\right)=H$ for all $H: U A \rightarrow B$ with $(U H)\left(\mathrm{id}, \iota_{B}^{U}\right) \leq H\left(\mu_{A}^{U}, \mathrm{id}\right)$.

Since the proof is somewhat long and technical we defer it to the end of this section. The first assertion of the following proposition is the second assertion of Proposition 3.4 of [22].

Proposition 3.8. The assignment $(A, \alpha: U A \rightarrow A) \mapsto\left(A, \varepsilon_{A} \odot \alpha\right)$, where

$$
\varepsilon_{A} \odot \alpha: P A \rightarrow A:(S, x) \mapsto \sup _{\mathfrak{x} \in S^{\sharp}} \alpha(\mathfrak{x}, x),
$$

extends to algebraic functors $\tilde{\varepsilon}:(U, \mathcal{V})$-Cat $\rightarrow \mathcal{V}$-Cls and $\tilde{\varepsilon}:(U, \mathcal{V})$-Gph $\rightarrow \mathcal{V}$-PreCls that leave morphisms unchanged.
Sketch of the proof. A routine calculation. To show that a $(U, \mathcal{V})$-graph $(A, \alpha)$ is mapped to a $\mathcal{V}$-valued preclosure space, that is $\varepsilon_{A} \odot \alpha$ forms a left unitary $(P, \mathcal{V})$-graph structure on $A$ (see Example 3.5), remember that the extension $U$ of the ultrafilter monad is normal, so that $P 1_{A} \odot \varepsilon_{A}=\varepsilon_{A}$ by the previous proposition.

We now follow [21] in calling a $\mathcal{V}$-valued closure space $(A, \delta)$ (Example 3.5) a $\mathcal{V}$-valued topological space whenever its structure relation $\delta: P A \times A \rightarrow \mathcal{V}$ preserves finite joins:

$$
\begin{equation*}
\delta(\emptyset, x)=\perp \quad \text { and } \quad \delta(S \cup T, x)=\sup \{\delta(S, x), \delta(T, x)\} \tag{10}
\end{equation*}
$$

for all $x \in A$ and $S, T \in P A$. In particular 2-valued topological spaces can be identified with topological spaces, while $[0, \infty]$-valued topological spaces coincide with Lowen's original approach spaces [25], consisting of sets $A$ equipped with a point-set distance $\delta: P A \times A \rightarrow[0, \infty]$. Taking $\mathcal{V}=\Delta_{\&}$, the quantale of distance distribution functions (Example 1.4), $\Delta_{\&}$-valued topological spaces are called \&-probabilistic approach spaces in [21].

Writing $\mathcal{V}$-Top for the full subcategory of $\mathcal{V}$-Cls generated by $\mathcal{V}$-valued topological spaces, the main result of [22] is as follows.

Theorem 3.9 (Lai and Tholen). Let $\mathcal{V}$ be a completely distributive quantale. The algebraic functor $\tilde{\varepsilon}:(U, \mathcal{V})$-Cat $\rightarrow \mathcal{V}$-Cls embeds $(U, \mathcal{V})$-Cat into $\mathcal{V}$-Cls as a full coreflective subcategory, which is precisely the category $\mathcal{V}$-Top of $\mathcal{V}$-valued topological spaces. Its right adjoint $R: \mathcal{V}$-Cls $\rightarrow(U, \mathcal{V})$-Cat is given on objects by $R(A, \delta)=\left(A, \varepsilon_{A} \triangleleft \delta\right)$, where

$$
\varepsilon_{A} \triangleleft \delta: U A \rightarrow A:(\mathfrak{x}, x) \mapsto \inf _{S \in \mathfrak{x}} \delta(S, x)
$$

while it leaves morphisms unchanged.
Example 3.10. Taking $\mathcal{V}=2$ in the theorem above recovers Barr's presentation [2]

$$
(U, 2)-\mathrm{Cat} \cong \mathrm{Top}
$$

of topological spaces in terms of ultrafilter convergence. Instead of closure operations, in terms of topologies this isomorphism is induced by the correspondence of $(U, 2)$-category structures $\alpha: U A \rightarrow A$ and topologies $\tau$ on a set $A$, given by the inverse assignments $\alpha \mapsto \tau$ and $\tau \mapsto \alpha$ that are defined by

$$
S \in \tau \quad: \Leftrightarrow \quad \underset{\mathfrak{x} \alpha x}{\forall}(x \in S \Rightarrow S \in \mathfrak{x}) \quad \text { and } \quad \mathfrak{x} \alpha x \quad: \Leftrightarrow \quad \underset{S \in \tau}{\forall}(x \in S \Rightarrow S \in \mathfrak{x}),
$$

where $S \subseteq A, \mathfrak{x} \in U A$ and $x \in A$.
Taking $\mathcal{V}=[0, \infty]$ in the theorem above recovers Clementino and Hofmann's presentation (Section 3.2 of [5])

$$
(U,[0, \infty])-\mathrm{Cat} \cong \mathrm{App}
$$

of Lowen's approach spaces in terms of metric ultrafilter convergence.
Turning to $(U, \mathcal{V})$-graphs, we follow [21] in calling a $\mathcal{V}$-valued pseudoclosure space $A=(A, \delta)($ Example 3.5) a $\mathcal{V}$-valued pretopological space whenever its structure relation $\delta: P A \times A \rightarrow \mathcal{V}$ preserves finite joins; see (10). Notice that this implies that $A$ is unitary, i.e. $A$ is a $\mathcal{V}$-valued preclosure space (see Example 3.5). Choosing $\mathcal{V}=2$ recovers the classical notion of pretopological space [4]; see Example III.4.1.3(2) of [17].

Writing $\mathcal{V}$-PreTop for the subcategory of $\mathcal{V}$-PreCls consisting of $\mathcal{V}$-valued pretopological spaces, the following theorem is a variation on Theorem 3.9.

Theorem 3.11. For a completely distributive quantale $\mathcal{V}$ consider the restriction

$$
(U, \mathcal{V})-U G p h \hookrightarrow(U, \mathcal{V})-\mathrm{Gph} \xrightarrow{\tilde{\varepsilon}} \mathcal{V} \text {-PreCls }
$$

of the functor $\tilde{\varepsilon}$ that is given in Proposition 3.8. Again denoted by $\tilde{\varepsilon}$, it embeds $(U, \mathcal{V})-U G p h$ into $\mathcal{V}$-PreCls as a full coreflective subcategory, which is precisely the category $\mathcal{V}$-PreTop of $\mathcal{V}$-valued pretopological spaces. As in Theorem 3.9 the right adjoint $R$ to $\tilde{\varepsilon}$ is given by $R(A, \delta)=\left(A, \varepsilon_{A} \triangleleft \delta\right)$.
Proof. That the composite $\tilde{\varepsilon}$ maps into $\mathcal{V}$-PreTop follows directly from the fact that, for an ultrafilter $\mathfrak{x}$ on a set $A$, we have $\emptyset \notin \mathfrak{x}$ while $S \cup T \in \mathfrak{x}$ precisely if $S \in \mathfrak{x}$ or $T \in \mathfrak{x}$. We start by checking that the assignment $R A=\left(A, \varepsilon_{A} \triangleleft \delta\right)$, for $\mathcal{V}$-valued preclosure spaces $A=(A, \delta)$, induces a functor $R: \mathcal{V}$-PreCls $\rightarrow(U, \mathcal{V})$-UGph.

Writing

$$
\alpha:=\varepsilon_{A} \triangleleft \delta: U A \rightarrow A:(\mathfrak{x}, x) \mapsto \inf _{S \in \mathfrak{x}} \delta(S, x),
$$

we have to show that $R A=(A, \alpha)$ is a unitary $(U, \mathcal{V})$-graph. That $\alpha$ is reflexive, that is $\alpha\left(\iota_{A}^{U} x, x\right)=\inf _{S \in \iota_{A}^{U} x} \delta(S, x) \geq k$ for all $x \in A$, follows easily from the fact that $A$ is reflexive and unitary; hence $R A$ is a $(U, \mathcal{V})$-graph. That $R A$ is left unitary is immediate from the fact that the ultrafilter monad $U$ is normal, so that only right unitariness remains: we have to show that

$$
(U \alpha)\left(\mathfrak{X}, \iota_{A}^{U}(x)\right) \leq \alpha\left(\mu_{A}^{U}(\mathfrak{X}), x\right)=\inf _{T \in \mu_{A}^{U}(\mathfrak{X})} \delta(T, x)
$$

for all $\mathfrak{X} \in U^{2} A$ and $x \in A$. To see this notice that for every $T \in \mu_{A}^{U}(\mathfrak{X})$, that is $T^{\sharp} \in \mathfrak{X}$, we have

$$
\begin{aligned}
&(U \alpha)\left(\mathfrak{X}, \iota_{A}^{U}(x)\right)=\inf _{\substack{\sigma \in \mathfrak{X} \\
R \in \iota_{A}^{U}(x)}} \sup _{\substack{\mathfrak{y} \in \sigma \\
r \in R}} \inf _{P \in \mathfrak{y}} \delta(P, r) \\
& \leq \sup _{\substack{\mathfrak{y} \in T^{\sharp} \\
r \in\{x\}}} \inf _{P \in \mathfrak{y}} \delta(P, r)=\sup _{\mathfrak{y} \in T^{\sharp}} \inf _{P \in \mathfrak{y}} \delta(P, x) \leq \delta(T, x) .
\end{aligned}
$$

To see that $A \mapsto R A$ extends to morphisms consider a continuous map $f: A \rightarrow C$ between $\mathcal{V}$-valued preclosure spaces $A=(A, \delta)$ and $C=(C, \zeta)$. Then

$$
\alpha(\mathfrak{x}, x)=\inf _{S \in \mathfrak{x}} \delta(S, x) \leq \inf _{S \in \mathfrak{x}} \zeta(f S, f x)=\inf _{T \in(U f)(\mathfrak{x})} \zeta(T, f x)=\left(\varepsilon_{C} \triangleleft \zeta\right)((U f)(\mathfrak{x}), f x)
$$

for all $\mathfrak{x} \in U A$ and $x \in A$, showing that $f$ forms a morphism $R A \rightarrow R C$ of unitary $(U, \mathcal{V})$-graphs.

The arguments proving that $R$ is a right adjoint of $\tilde{\varepsilon}$, as well as that $(\tilde{\varepsilon} \circ R)(A)=$ $A$ for any $\mathcal{V}$-valued pretopological space $A$, are identical to the ones given in the proof of Theorem 3.6 of [22| (whose statement is reproduced above as Theorem 3.9). To complete the proof it thus suffices to prove that $(R \circ \tilde{\varepsilon})(A)=A$ for any unitary $(U, \mathcal{V})$-graph $A=(A, \alpha)$. But this follows immediately from Proposition 3.7(c).

We close this section with the proof of Proposition 3.7.
Proof of Proposition 3.7. That the family of $\mathcal{V}$-relations $\varepsilon_{A}: P A \rightarrow U A$, whose definition is recalled below, forms an algebraic morphism $\varepsilon: P \rightarrow U$ is proved in Proposition 3.4 of [22]. Thus it remains to prove parts (a), (b) and (c).

$$
\varepsilon_{A}(S, \mathfrak{x})= \begin{cases}k & \text { if } S \in \mathfrak{x} \\ \perp & \text { otherwise }\end{cases}
$$

Part (a): $\varepsilon_{A}\left(\iota_{A}^{P}, \mathrm{id}\right)=\iota_{A *}^{U}$ for all $A$. This is a direct consequence of the fact that, for any $x \in A$ and ultrafilter $\mathfrak{x}$ on $A$, one has $\{x\} \in \mathfrak{x}$ if and only if $\mathfrak{x}=\iota_{A}^{U}(x)$, the principal ultrafilter on $x$.

Part (b): PJ $\odot \varepsilon_{B}=\varepsilon_{A} \odot U J$ for all $J: A \rightarrow B$. The inequality $\leq$ is a consequence of $\varepsilon: P \rightarrow U$ being an algebraic morphism; we claim that the converse inequality holds as well. Indeed for any $R \in P A$ and $\mathfrak{y} \in U B$ we have

$$
\begin{aligned}
\left(\varepsilon_{A} \odot U J\right)(R, \mathfrak{y}) & =\sup _{\mathfrak{x} \in U A}\left(\varepsilon_{A}(R, \mathfrak{x}) \otimes \inf _{\substack{S \in \mathfrak{x} \\
T \in \mathfrak{y} \\
s \in S \\
\sup _{t \in T}}} J(s, t)\right)=\sup _{\mathfrak{x} \in R^{\sharp}} \inf _{\substack{S \in \mathfrak{r} \\
T \in \mathfrak{y}}} \sup _{s \in S} J(s, t) \\
& \leq \inf _{T \in \mathfrak{y}} \sup _{\substack{ \\
s \in R \\
t \in T}} J(s, t) \stackrel{(\mathrm{i})}{=} \sup _{T \in \mathfrak{y}} \inf _{t \in T} \sup _{\substack{ } R} J(s, t) \\
& =\sup _{T \in P B}\left(\left(\inf _{t \in T} \sup _{s \in R} J(s, t)\right) \otimes \varepsilon_{B}(T, \mathfrak{y})\right)=\left(P J \odot \varepsilon_{B}\right)(R, \mathfrak{y}),
\end{aligned}
$$

where the equality denoted (i) follows from the lemma below. We conclude that $P J \odot \varepsilon_{B}=\varepsilon_{A} \odot U J$ for all $J: A \rightarrow B$.

Part $(\mathrm{c}): \varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot H\right)=H$ for all $H: U A \rightarrow B$ with $(U H)\left(\mathrm{id}, \iota_{B}^{U}\right) \leq H\left(\mu_{A}^{U}, \mathrm{id}\right)$. In Theorem 3.6 of [22] this was proved in case that $H=\alpha: U A \rightarrow A$ is the $\mathcal{V}$-valued convergence relation of a $(U, \mathcal{V})$-category $A$. We will modify the proof given there slightly so that it generalises to any $\mathcal{V}$-relation $H: U A \rightarrow B$ satisfying $(U H)\left(\mathrm{id}, \iota_{B}^{U}\right) \leq H\left(\mu_{A}^{U}, \mathrm{id}\right)$. First notice that $H \leq \varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot H\right)$ follows from the adjunction $\varepsilon_{A} \odot-\dashv \varepsilon_{A} \triangleleft-$. For the reverse inequality let $\mathfrak{x} \in U A$ and $y \in B$; to prove that

$$
\left(\varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot H\right)\right)(\mathfrak{x}, y)=\inf _{S \in \mathfrak{x}} \sup _{\mathfrak{y} \in S^{\sharp}} H(\mathfrak{y}, y) \leq H(\mathfrak{x}, y)
$$

it suffices to show that $v \ll \inf _{S \in \mathfrak{x}} \sup _{\mathfrak{y} \in S^{\sharp}} H(\mathfrak{y}, y)$ implies $v \leq H(\mathfrak{x}, y)$ for all $v \in \mathcal{V}$. The hypothesis here means that for all $S \in \mathfrak{x}$ there is $\mathfrak{y} \in S^{\sharp}$ with $v \leq H(\mathfrak{y}, y)$. As a consequence the sets

$$
X_{S}=\left\{\mathfrak{y} \in S^{\sharp} \mid v \leq H(\mathfrak{y}, y)\right\},
$$

where $S$ ranges over $\mathfrak{x}$, form a proper filter base and we can choose an ultrafilter $\mathfrak{X} \in U^{2} A$ containing all of them. As $S^{\sharp} \supseteq X_{S} \in \mathfrak{X}$ for all $S \in \mathfrak{x}$ it follows that $\mu_{A}^{U}(\mathfrak{X})=\mathfrak{x}$. Moreover
where the inequality follows from the fact that every $R \in \mathfrak{X}$ intersects $X_{A}$. We conclude that

$$
H(\mathfrak{x}, y)=H\left(\mu_{A}^{U}(\mathfrak{X}), y\right) \geq(U H)\left(\mathfrak{X}, \iota_{B}^{U}(y)\right) \geq v
$$

where the first inequality follows from the assumption on $H$. This completes the proof.

The following lemma was used in the proof above; it is a straightforward generalisation of Proposition 1.8.29 of [25].

Lemma 3.12. Let $\mathcal{V}$ be a completely distributive lattice, and let $f: A \rightarrow \mathcal{V}$ be any function. If $\mathfrak{x}$ is an ultrafilter on $A$ then

$$
\sup _{S \in \mathfrak{x}} \inf _{s \in S} f s=\inf _{S \in \mathfrak{r}} \sup _{s \in S} f s
$$

Proof. The inequality $\leq$ follows directly from the fact that $\inf _{s \in S} f s \leq \sup _{t \in T} f t$ for all $S, T \in \mathfrak{x}$. To show $\geq$ assume that $v \ll \inf _{S \in \mathfrak{r}} \sup _{s \in S} f s$ for some $v \in \mathcal{V}$. Hence for all $S \in \mathfrak{x}$ there is $s \in S$ with $v \leq f s$. Writing $T:=\{t \in A \mid v \leq f t\}$ we thus have $S \cap T \neq \emptyset$ for all $S \in \mathfrak{x}$. We conclude $T \in \mathfrak{x}$, from which $v \leq \sup _{S \in \mathfrak{x}} \inf _{s \in S} f s$ follows.

## 4. Modular T-graphs

Let $\mathcal{V}$ be a quantale and let $T$ be a lax monad on the thin equipment $\mathcal{V}$-Rel of $\mathcal{V}$-relations. In this section we consider $\mathcal{V}$-categories equipped with compatible $(T, \mathcal{V})$-graph structures as follows. Applying the 2 -functor Mod (Proposition 1.15) to $T$ we obtain a normal lax monad $\operatorname{Mod} T$ on the thin equipment $\mathcal{V}$-Prof $=\operatorname{Mod}(\mathcal{V}$-Rel $)$ of $\mathcal{V}$-profunctors between $\mathcal{V}$-categories (Example 1.8). Following Section 4 of 31$]$ we call $(\operatorname{Mod} T)$-graphs modular $(T, \mathcal{V})$-graphs and write $(T, \mathcal{V})$-ModGph $:=(\operatorname{Mod} T)$-Gph. A modular $(T, \mathcal{V})$-graph $A$ is a $\mathcal{V}$-category $A=$ $(A, \bar{A})$ equipped with a $T$-graph structure $\alpha: T A \rightarrow A$ that is a $\mathcal{V}$-profunctor, see the lemma below. A morphism of modular $(T, \mathcal{V})$-graphs $f: A \rightarrow C$ is simultaneously a $\mathcal{V}$-functor $f:(A, \bar{A}) \rightarrow(C, \bar{C})$ of $\mathcal{V}$-categories as well as a morphism $f:(A, \alpha) \rightarrow(C, \gamma)$ of $(T, \mathcal{V})$-graphs. As in the non-modular case (9) we have subcategories

$$
(T, \mathcal{V}) \text {-ModCat } \hookrightarrow(T, \mathcal{V}) \text {-ModUGph } \hookrightarrow(T, \mathcal{V}) \text {-ModRGph } \hookrightarrow(T, \mathcal{V}) \text {-ModGph }
$$

consisting of modular $(T, \mathcal{V})$-categories, modular unitary $(T, \mathcal{V})$-graphs and modular right unitary ones.

Generalising to lax monads $T$ on an arbitrary thin equipment $\mathcal{K}$, we shall call $(\operatorname{Mod} T)$-graphs and $(\operatorname{Mod} T)$-categories modular $T$-graphs and modular $T$-categories respectively, while we write $T$-ModGph $:=(\operatorname{Mod} T)-\mathrm{Gph}$ and $T$-ModCat $:=$ $(\operatorname{Mod} T)$-Cat. It is shown in [31] that $T$-Cat forms a full reflective subcategory of $T$-ModCat, via the embedding

$$
\begin{equation*}
N: T \text {-Cat } \rightarrow T \text {-ModCat: }(A, \alpha) \mapsto\left(A, \alpha\left(\iota_{A}, \text { id }\right), \alpha\right) \tag{11}
\end{equation*}
$$

whose left adjoint is the forgetful functor. We will follow [9] in calling a modular $T$-category $A=(A, \bar{A}, \alpha)$ normalised if it lies in the image of $N$, that is $\bar{A}=$ $\alpha\left(\iota_{A}, \mathrm{id}\right)$.

Before describing modular $\mathcal{V}$-valued (pre-)closure spaces we state a couple of lemmas that describe relations between monoid structures and $T$-graph structures on a single object.

Lemma 4.1. Let $T$ be a lax monad on a thin equipment $\mathcal{K}$ and let $A=(A, \bar{A})$ be a monoid in $\mathcal{K}$. A bimodule $\alpha: T A \rightarrow A$ (Definition 1.7) forms a modular $T$-graph structure on $A$ in $\operatorname{Mod}(\mathcal{K})$ precisely if its underlying horizontal morphism forms a $T$-graph structure on $A$ in $\mathcal{K}$. In that case
(a) both the modular $T$-graph $(A, \bar{A}, \alpha)$ and the $T$-graph $(A, \alpha)$ are left unitary;
(b) $(A, \bar{A}, \alpha)$ is right unitary precisely if $(A, \alpha)$ is;
(c) $(A, \bar{A}, \alpha)$ is a modular $T$-category precisely if $(A, \alpha)$ is a $T$-category.

Proof. The main assertion states that the existences of the unitor cells (see Definition 3.4) for $(A, \bar{A}, \alpha)$ and $(A, \alpha)$ are equivalent, which amounts to proving that

$$
\bar{A} \leq \alpha\left(\iota_{A}, \mathrm{id}\right) \quad \Leftrightarrow \quad 1_{A} \leq \alpha\left(\iota_{A}, \mathrm{id}\right) .
$$

The implication $\Rightarrow$ follows from the unit axiom $1_{A} \leq \bar{A}$ for monoids (see Definition 1.7), while $\Leftarrow$ is shown by

$$
\bar{A}=1_{A} \odot \bar{A} \leq \alpha\left(\iota_{A}, \mathrm{id}\right) \odot \bar{A}=\iota_{A *} \odot \alpha \odot \bar{A} \leq \iota_{A *} \odot \alpha=\alpha\left(\iota_{A}, \mathrm{id}\right)
$$

where the second inequality follows from the fact that $\alpha$ is a bimodule.
For part (a) notice that ( $A, \bar{A}, \alpha$ ) is left unitary because Mod $T$ is normal. That $(A, \alpha)$ is left unitary is shown by

$$
T 1_{A} \odot \alpha \leq T \bar{A} \odot \alpha \leq \alpha
$$

Part (b) is a direct consequence of the fact that the cells expressing right unitarity for $(A, \bar{A}, \alpha)$ and $(A, \alpha)$ respectively coincide. The same holds for the cells expressing associativity, so that part (c) follows too.

In the setting of a Set-monad $T$ laxly extended to $\mathcal{V}$-relations the following was proved as Lemma 1 of [31]. The proof given there, which uses a result analogous to Proposition 1.14, applies verbatim to our setting.

Lemma 4.2 (Tholen). Let $T$ be a lax monad on a thin equipment $\mathcal{K}$. Consider both a monoid structure $\bar{A}: A \rightarrow A$ and a $T$-category structure $\alpha: T A \rightarrow A$ on a single object $A$. The triple $(A, \bar{A}, \alpha)$ forms a modular $T$-category precisely if $\bar{A} \leq \alpha\left(\iota_{A}, \mathrm{id}\right)$.

Example 4.3. Recall from Example 3.5 that the notions of left unitary $\mathcal{V}$-valued pseudoclosure space and $\mathcal{V}$-valued preclosure space coincide. Hence, by Lemma 4.1, the notions of modular $\mathcal{V}$-valued pseudoclosure space and modular $\mathcal{V}$-valued preclosure space coincide as well: both consist of a $\mathcal{V}$-category $A=(A, \bar{A})$ equipped with a $\mathcal{V}$-relation $\delta: P A \times A \rightarrow \mathcal{V}$ that satisfies reflexivity and modularity axioms:
(R) $k \leq \delta(\{x\}, x)$;
(M) $\left(\inf _{t \in T} \sup _{s \in S} A(s, t)\right) \otimes \delta(T, x) \otimes A(x, y) \leq \delta(S, y)$,
for all $x, y \in A$ and $S, T \in P A$. We write $\mathcal{V}$-ModPreCls $:=(P, \mathcal{V})$-ModGph for the category of modular $\mathcal{V}$-valued preclosure spaces, whose morphisms are both $\mathcal{V}$-functors and continuous maps.
$\mathcal{V}$-ModPreCls contains as a full subcategory the category $\mathcal{V}$-ModCls of modular $\mathcal{V}$-valued closure spaces $A=(A, \bar{A}, \delta)$, with $\delta$ a $\mathcal{V}$-valued closure space structure on $A$. In this case, by Lemma 4.2, the modularity axiom above reduces to
(M') $A(x, y) \leq \delta(\{x\}, y)$
for all $x, y \in A$. Under the embedding (11) any $\mathcal{V}$-valued closure space $A=(A, \delta)$ gives rise to a normalised modular $\mathcal{V}$-valued closure space $N A$, whose $\mathcal{V}$-category structure is given by $A(x, y):=\delta(\{x\}, y)$.

Taking $\mathcal{V}=2$ in the above we obtain the notion of a modular preclosure space: a preclosure space $A$ equipped with an ordering $\leq$ satisfying $\uparrow \uparrow \bar{\uparrow} \subseteq \bar{S}$ for all $S \subseteq A$, where $\uparrow S=\{x \in A \mid \exists s \in S: s \leq x\}$ is the upset generated by $S$. For a modular closure space $A$ the latter condition reduces to $x \leq y$ implies $y \in \overline{\{x\}}$ for all $x, y \in A$. Any closure space $A$ can be regarded as a normalised modular closure space by equipping it with the specialisation order: $x \leq y: \Leftrightarrow y \in \overline{\{x\}}$, that is $\overline{\{x\}}=\uparrow x$ for all $x \in A$.
Example 4.4. A modular $(U, \mathcal{V})$-graph is a $\mathcal{V}$-category $A=(A, \bar{A})$ equipped with a $\mathcal{V}$-valued pseudotopological space structure $\alpha: U A \rightarrow A$ (Example 3.6) satisfying the modularity axiom
(M) $\left(\inf _{T \in \mathfrak{y}}^{S \in \mathfrak{y}} \sup _{\substack{ \\t \in T}} A(s, t)\right) \otimes \alpha(\mathfrak{y}, x) \otimes A(x, y) \leq \alpha(\mathfrak{x}, y)$
for all $\mathfrak{x}, \mathfrak{y} \in U A$ and $x, y \in A$. We will call $A=(A, \bar{A}, \alpha)$ a modular $\mathcal{V}$-valued pseudotopological space. If the $\mathcal{V}$-valued convergence relation $\alpha$ corresponds to a $\mathcal{V}$-valued topological space structure under Theorem 3.9, so that $(A, \alpha)$ is a modular $(U, \mathcal{V})$-category, then by Lemma 4.2 the modularity axiom above reduces to
(M') $A(x, y) \leq \alpha(\iota x, y)$
for all $x, y \in A$, where $\iota x$ is the principal ultrafilter on $x$. Theorem 4.5 below shows that the correspondence of Theorem 3.9 restricts to one between modular $(U, \mathcal{V})$-categories and modular $\mathcal{V}$-valued topological spaces, by which we mean modular $\mathcal{V}$-valued closure spaces $A=(A, \bar{A}, \delta)$ whose structure $\mathcal{V}$-relations $\delta$ preserve finite joins (10). Modular $\mathcal{V}$-valued pretopological spaces are defined analogously; that they correspond to modular unitary $(U, \mathcal{V})$-graphs is proved by Theorem 4.5 as well. As with $\mathcal{V}$-valued closure spaces, any $\mathcal{V}$-valued topological space $A=(A, \delta)$ induces a normalised modular $\mathcal{V}$-valued topological space $N A$ with $\mathcal{V}$-category structure $A(x, y)=\delta(\{x\}, y)$.

In particular, by taking $\mathcal{V}=2$ in the above we recover the notion of modular topological space (Section 4 of [31]): a topological space $A$, with ultrafilter convergence $\alpha$, equipped with an ordering $\leq$ that is contained in the specialisation order of $\alpha$, i.e. $x \leq y$ implies $(\iota x) \alpha y$ for all $x, y \in A$.

Taking the Lawvere quantale $\mathcal{V}=[0, \infty]$ instead, a modular approach space $A$ (Section 6 of $[31]$ ) is a generalised metric space $A=(A, \bar{A})$ equipped with a point-set distance $\delta: P A \times A \rightarrow[0, \infty]$ such that $\delta(\{x\}, y) \leq A(x, y)$ for all $x, y \in A$.

We denote by $\mathcal{V}$-ModTop the category of modular $\mathcal{V}$-valued topological spaces (as defined in the example above), which forms a full subcategory of $\mathcal{V}$-ModCls. Likewise $\mathcal{V}$-ModPreTop denotes the full subcategory of $\mathcal{V}$-ModPreCls that consists of modular $\mathcal{V}$-valued pretopological spaces.

Theorem 4.5. Let $\mathcal{V}$ be a completely distributive quantale. The pair of adjunctions $\tilde{\varepsilon} \dashv R$ described in Theorem 3.9 and Theorem 3.11 lift as shown in the diagrams below, where $N$ is as defined in (11) and where forgetful functors are denoted by $U$. Except for the composites $N \circ R$ and $\bar{R} \circ N$, any two parallel composites between opposite corners coincide.


Leaving $\mathcal{V}$-category structures unchanged, the lifts above establish isomorphisms of categories

$$
(U, \mathcal{V})-\text { ModCat } \cong \mathcal{V} \text {-ModTop } \quad \text { and } \quad(U, \mathcal{V}) \text {-ModUGph } \cong \mathcal{V} \text {-ModPreTop. }
$$

Proof. Remember that the functors $\tilde{\varepsilon}$ and $R$ leave underlying sets $A$ unchanged while they act on structure $\mathcal{V}$-relations $\alpha: U A \rightarrow A$ and $\delta: P A \rightarrow A$ by $\alpha \mapsto \varepsilon_{A} \odot \alpha$ and $\delta \mapsto \varepsilon_{A} \triangleleft \delta$ respectively. Since the lifts of $\tilde{\varepsilon}$ and $R$ leave $\mathcal{V}$-category structures unchanged, it suffices to check that the latter assignments preserve modularity with respect to any given $\mathcal{V}$-category structure $\bar{A}: A \rightarrow A$ on $A$. That the first assignment does follows easily from Proposition 3.7(b), when applied to $J=\bar{A}$. To see that the second does too let $\delta: P A \rightarrow A$ be any $\mathcal{V}$-valued preclosure space structure on $A$ that is modular with respect to $\bar{A}$, i.e. $P \bar{A} \odot \delta \odot \bar{A} \leq \delta$. Then, writing $\varepsilon=\varepsilon_{A}$,

$$
\begin{aligned}
U \bar{A} \odot(\varepsilon \triangleleft \delta) \odot \bar{A} \leq \varepsilon \triangleleft & (\varepsilon \odot U \bar{A} \odot(\varepsilon \triangleleft \delta) \odot \bar{A}) \\
& =\varepsilon \triangleleft(P \bar{A} \odot \varepsilon \odot(\varepsilon \triangleleft \delta) \odot \bar{A}) \leq \varepsilon \triangleleft(P \bar{A} \odot \delta \odot \bar{A}) \leq \varepsilon \triangleleft \delta
\end{aligned}
$$

where the first two inequalities are given by the unit and counit of the adjunction $\varepsilon \odot-\dashv \varepsilon \triangleleft-$ while the equality follows from Proposition 3.7(b). We conclude that $\varepsilon_{A} \triangleleft \delta$ is again modular.

It remains to show the commutativity of the diagrams. It is clear that any two parallel composites containing $U$ coincide, leaving us to prove that

$$
(N \circ \tilde{\varepsilon})(A, \alpha)=(\overline{\tilde{\varepsilon}} \circ N)(A, \alpha)
$$

in the left-hand diagram, for any $(U, \mathcal{V})$-category structure $\alpha$ on $A$. For this it suffices to check that the $\mathcal{V}$-category structures coincide, which is shown by

$$
(\varepsilon \odot \alpha)\left(\iota_{A}^{P}, \mathrm{id}\right)=\iota_{A *}^{P} \odot \varepsilon \odot \alpha=\iota_{A *}^{U} \odot \alpha=\alpha\left(\iota_{A}^{U}, \mathrm{id}\right),
$$

where the second equality follows from Proposition 3.7(a).

## 5. T-cocomplete $T$-graphs

Let $T$ be a lax monad on a thin equipment $\mathcal{K}$. To be able to generalise the maximum theorem to Kan extensions $l: B \rightarrow M$ between $T$-graphs, we need either the Kan extension $l$ itself or its target $M$ to be 'well-behaved'. In Section 2 well-behaved Kan extensions were described: they are the ones that satisfy the Beck-Chevalley condition. By well-behaved $T$-graphs we mean ' $T$-cocomplete' ones as follows.

Definition 5.1. Let $T$ be a lax monad on a thin equipment $\mathcal{K}$. A $T$-graph $A=$ $(A, \alpha)$ is called $T$-cocomplete if $\alpha=a_{*}$ in $\mathcal{K}$ for some morphism $a: T A \rightarrow A$.

Applying the above to the induced lax monad $T=\operatorname{Mod}(T)$ on $\operatorname{Mod}(\mathcal{K})$, by the lemma below a modular $T$-graph $A=(A, \bar{A}, \alpha)$ is $T$-cocomplete whenever $\alpha=$ $\bar{A}(a$, id $)$ in $\mathcal{K}$, for some morphism $a: T A \rightarrow A$. We will see in Example 5.8 below that the isomorphisms of Theorem 4.5 fail to preserve $T$-cocompleteness in general.

The closely related but different notion of ' $T$-cocompleteness' for ( $T, \mathcal{V}$ )-categories $A=(A, \alpha)$, that is considered in Section III.5.4 of [17], can be rephrased in terms of the above as follows: $A$ is ' $T$-cocomplete' whenever its corresponding normalised modular $(T, \mathcal{V})$-category $N A=(A, \alpha(\iota, \mathrm{id}), \alpha)$ is $T$-cocomplete in our sense, that is $\alpha=\alpha(\iota \circ a$, id $)$ for some function $a: T A \rightarrow A$.

Lemma 5.2. Let $T$ be a lax monad on a thin equipment $\mathcal{K}$ and $A=(A, \bar{A}, \alpha) a$ modular $T$-graph. If $\alpha=\bar{A}(a, \mathrm{id})$ in $\mathcal{K}$ for some morphism $a: T A \rightarrow A$ then $a$ is $a$ homomorphism of monoids $(T A, T \bar{A}) \rightarrow(A, \bar{A})$ so that $\alpha=a_{*}$ in $\operatorname{Mod} \mathcal{K}$.

Proof. By Lemma 1.6 the existence of the structure cell exhibiting $a$ as a homomorphism of monoids can be deduced from

$$
T \bar{A} \odot a_{*}=T \bar{A} \odot a_{*} \odot 1_{A} \leq T \bar{A} \odot a_{*} \odot \bar{A}=T \bar{A} \odot \alpha \leq \alpha=\bar{A}(a, \mathrm{id})
$$

where we use the unit axiom for $\bar{A}$ and the fact that $\alpha$ is a bimodule.
Example 5.3. Let $P$ be the powerset monad extended to $\mathcal{V}$-relations taking values in a quantale $\mathcal{V}$. A modular $\mathcal{V}$-valued preclosure space $A=(A, \bar{A}, \delta)$ (see Example 4.3) is $P$-cocomplete whenever for each subset $S \in P A$ there exists a tacitly chosen generic point $x_{0} \in A$ such that

$$
\delta(S, y)=A\left(x_{0}, y\right)
$$

for all $y \in A$.
For a modular preclosure space $A=(A, \leq, S \mapsto \bar{S})$ (Example 4.3) the above reduces to $\bar{S}=\uparrow x_{0}$, that is all closed subsets of $A$ are principal upsets. For the normalised modular closure space $N A$ induced by a closure space $A$ the latter means $\bar{S}=\overline{\left\{x_{0}\right\}}$, that is every closed subset of $A$ contains a generic point.
Example 5.4. Let $U$ be the ultrafilter monad extended to $\mathcal{V}$-relations taking values in a completely distributive quantale $\mathcal{V}$. A modular $\mathcal{V}$-valued pretopological space $A$, regarded as a $\mathcal{V}$-category $(A, \bar{A})$ equipped with a convergence $\mathcal{V}$-profunctor $\alpha: U A \times A \rightarrow \mathcal{V}$ (see Example 4.4), is $U$-cocomplete whenever for each $\mathfrak{x} \in U A$ there exists a tacitly chosen generic point $x_{0} \in A$ such that

$$
\alpha(\mathfrak{x}, y)=A\left(x_{0}, y\right)
$$

for all $y \in A$. In Section III.5.6 of [17] $U$-cocomplete normalised modular topological spaces are characterised in terms of 'irreducible' closed subsets and, generalising this, in Section III.5.9 $U$-cocomplete normalised modular approach spaces are characterised in terms of 'irreducible' continuous maps.

The remainder of this section describes how every completely distributive quantale $\mathcal{V}$ can itself be regarded as a modular $\mathcal{V}$-valued topological space, that is both normalised and $U$-cocomplete. With this aim in mind, let $P$ be the powerset monad extended to $\mathcal{V}$-relations taking values in a completely distributive quantale $\mathcal{V}$. We consider the vertical part $(\operatorname{Mod} P)_{\mathrm{v}}$ of the induced monad $\operatorname{Mod} P$ (see Proposition 1.15) on the thin equipment $\operatorname{Mod}(\mathcal{V}$-Rel $)=\mathcal{V}$-Prof of $\mathcal{V}$-profunctors. Writing again $P:=(\operatorname{Mod} P)_{\mathrm{v}}$, this is the powerset monad on the category $\mathcal{V}$-Cat $=(\mathcal{V} \text {-Prof })_{\mathrm{v}}$ of $\mathcal{V}$-categories, whose Eilenberg-Moore category $(\mathcal{V} \text {-Cat })^{P}$ of algebras consists of $\mathcal{V}$-categories $(A, \bar{A})$ equipped with a $P$-algebra structure map $a: P A \rightarrow A$ that is a $\mathcal{V}$-functor.

We will consider the images of such algebras under the composite functor

$$
\begin{equation*}
(\mathcal{V} \text {-Cat })^{P} \xrightarrow{C} \mathcal{V} \text {-ModCls } \xrightarrow{\bar{R}}(U, \mathcal{V}) \text {-ModCat } \tag{12}
\end{equation*}
$$

where $\bar{R}$ is given in Theorem 4.5 and $C$ is the "composition functor" described in Section 4 of [31], when applied to the powerset monad. This composite maps a $P$-algebra $A=(A, \bar{A}, a)$ to the modular $(U, \mathcal{V})$-category $(\bar{R} \circ C)(A)=(A, \bar{A}, \alpha)$ whose $\mathcal{V}$-valued convergence relation is given by

$$
\alpha: U A \mapsto A:(\mathfrak{x}, x) \mapsto \inf _{S \in \mathfrak{x}} \bar{A}(a(S), x)
$$

The examples below describe two types of images under the composite (12). Both depend on the fact that a complete lattice $A$ (for instance $A=\mathcal{V}$ ) admits two algebra structures over the powerset monad $P$ on Set, given by

$$
\begin{equation*}
a_{\mathrm{inf}}: P A \rightarrow A: S \mapsto \inf S \quad \text { and } \quad a_{\text {sup }}: P A \rightarrow A: S \mapsto \sup S \tag{13}
\end{equation*}
$$

respectively.

Example 5.5. Let $A$ be a complete lattice. It is easily checked that the $P$-algebra structure $a_{\mathrm{inf}}$ above is an order preserving map $a_{\mathrm{inf}}:(P A, P \leq) \rightarrow(A, \leq)$, so that we may regard $\left(A, \leq, a_{\mathrm{inf}}\right)$ as an algebra in $(2-\mathrm{Cat})^{P}$. Applying the composite functor (12), where $\mathcal{V}=2$, we obtain a modular topology (Example 4.4) on $A$ whose ultrafilter convergence relation we denote by $\alpha_{\mathrm{inf}}: U A \rightarrow A$; it is given by

$$
\mathfrak{x} \alpha_{\mathrm{inf}} x \quad: \Leftrightarrow \quad \sup _{S \in \mathfrak{x}} \inf S \leq x
$$

for all $\mathfrak{x} \in U A$ and $x \in A$. Dually the $P$-algebra structure $a_{\text {sup }}$ given in (13) induces a modular topology on the complete lattice $A^{\circ}=(A, \geq)$ that is obtained by reversing the order on $A$.

The proposition below shows that if $A$ is completely distributive then the topology corresponding to the convergence relation $\alpha_{\mathrm{inf}}$ is the Scott topology [28] as follows. The open subsets $O \subseteq A$ are the downsets satisfying

$$
\begin{equation*}
\underset{D \in \operatorname{DnDir} A}{\forall} \inf D \in O \Rightarrow D \cap O \neq \emptyset \tag{14}
\end{equation*}
$$

where $D$ ranges over all down-directed subsets of $A$ : a subset $D \subseteq A$ is down-directed whenever it is non-empty and every finite subset of $D$ has a lower bound in $D$, that is for all $x, y \in D$ there is $z \in D$ with $z \leq x$ and $z \leq y$.

For example the Scott topology on $[-\infty, \infty]$, with respect to the reversed order $\geq$, consists of the open subsets of the form $(x, \infty]$, where $x \in[0, \infty]$, together with $[-\infty, \infty]$ itself. A function $f: A \rightarrow[-\infty, \infty]$ that is continuous with respect to this topology is called lower semi-continuous; see e.g. Section IV. 8 of [3] or Section 2.10 of [1]. Dually, equipping $[-\infty, \infty]$ with the Scott topology with respect to the natural order $\leq$ instead, we obtain the notion of upper semi-continuous function $f: A \rightarrow[-\infty, \infty]$.

On $2=\{\perp \leq \top\}$ the Scott topology coincides with the Sierpinski topology, which has $\{\perp\}$ as its only non-trivial open subset.

Proposition 5.6. Let $A$ be a completely distributive complete lattice. The topology corresponding to the convergence relation $\alpha_{\mathrm{inf}}: U A \rightarrow A$, described in the example above, is the Scott topology.
Sketch of the proof. By Lemma 3.12 we have $\sup _{S \in \mathfrak{r}} \inf S=\inf _{S \in \mathfrak{x}} \sup S$ in the definition of $\alpha_{\mathrm{inf}}$ so that, by Example 3.10, the topology corresponding to $\alpha_{\mathrm{inf}}$ is given as follows: $O \subseteq A$ is open precisely if the equivalent conditions below hold.

$$
\underset{\substack{\mathfrak{x} \in U A \\ x \in O}}{\forall}\left(\sup _{S \in \mathfrak{x}} \inf S \leq x \Rightarrow O \in \mathfrak{x}\right) \quad \Leftrightarrow \quad \underset{\substack{\mathfrak{x} \in U A \\ x \in O}}{\forall}\left(\inf _{S \in \mathfrak{x}} \sup S \leq x \Rightarrow O \in \mathfrak{x}\right)
$$

To see that this implies that $O$ satisfies (14) consider, for any down-directed subset $D \subseteq A$ with $\inf D \in O$, any ultrafilter $\mathfrak{x} \in U A$ generated by the proper filter base $\{\downarrow x \cap D\}_{x \in D}$. Since $\downarrow x \in \mathfrak{x}$ for all $x \in D$ it follows that $\inf _{S \in \mathfrak{x}} \sup S \leq \inf D$, so that $O \in \mathfrak{x}$ by the above. Because $D \in \mathfrak{x}$ we conclude that $D \cap O \neq \emptyset$.

Conversely if $O \subseteq A$ is a downset satisfying (14) then it satisfies the equivalent conditions above. Indeed for any $\mathfrak{x} \in U A$ and $x \in O$ with $\inf _{S \in \mathfrak{r}} \sup S \leq x$, the set $D=\{\sup S\}_{S \in \mathfrak{r}}$ is down-directed with $\inf D \leq x \in O$. As $O$ is a downset $\inf D \in O$ follows so that $D \cap O \neq \emptyset$ by (14). Hence $\sup S \in O$ for some $S \in \mathfrak{x}$ which, again because $D$ is a downset, implies $S \subseteq O$; we conclude that $O \in \mathfrak{x}$.

Example 5.7. Here we consider the $P$-algebra structure $a_{\text {inf }}$ given by (13) on a completely distributive quantale $\mathcal{V}=A$. Since the inner hom $\multimap$ of $\mathcal{V}$ (see Example 1.8) is contravariant in the first variable and an inf-map in the second, we have

$$
\left(P \mathcal{V}_{\multimap}\right)(S, T)=\inf _{t \in T} \sup _{s \in S} s \multimap t \leq(\inf S) \multimap(\inf T)
$$

for all $S, T \in P \mathcal{V}$, showing that $a_{\text {inf }}$ is a $\mathcal{V}$-functor $a_{\text {inf }}: P \mathcal{V}_{-} \rightarrow \mathcal{V}_{-0}$. Hence $\left(\mathcal{V}_{-0}, a_{\text {inf }}\right)$ forms a $P$-algebra in $(\mathcal{V}-\mathrm{Cat})^{P}$ so that, under the composite functor (12), $\mathcal{V}_{-}$becomes a modular $(U, \mathcal{V})$-category (Example 4.4), whose $\mathcal{V}$-valued convergence relation we denote by $\nu_{\mathrm{inf}}: U \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$; it is given by

$$
\nu_{\inf }(\mathfrak{x}, x)=\left(\sup _{S \in \mathfrak{x}} \inf S\right) \multimap x .
$$

Notice that this defines a modular $(U, \mathcal{V})$-category structure on $\mathcal{V}_{-}$that is both normalised (see (11)) and $U$-cocomplete (Definition 5.1).

Analogous to the above, the dual $P$-algebra structure $a_{\text {sup }}$ on $\mathcal{V}$ given by (13) froms a $\mathcal{V}$-functor $a_{\text {sup }}: P \mathcal{V}_{\circ-} \rightarrow \mathcal{V}_{0-}$ and thus induces a modular $(U, \mathcal{V})$-category structure $\nu_{\text {sup }}$ on $\mathcal{V}_{0-}$, that is given by $\nu_{\text {sup }}(\mathfrak{x}, x)=\left(\inf _{S \in \mathfrak{x}} \sup S\right) \circ x$.
Example 5.8. By applying the previous example to the Lawvere quantale $[0, \infty]$, equipped with the generalised metric $[0, \infty]_{-\circ}(x, y)=y \ominus x$ (Example 1.10), we obtain the metric convergence relation $\nu_{\text {sup }}: U[0, \infty] \rightarrow[0, \infty]$ given by

$$
\nu_{\sup }(\mathfrak{x}, x)=x \ominus\left(\inf _{S \in \mathfrak{x}} \sup S\right)
$$

for all $\mathfrak{x} \in U[0, \infty]$ and $x \in[0, \infty]$. Dually, equipping $[0, \infty]$ with the reversed metric $[0, \infty]_{\circ}(x, y)=x \ominus y$ instead, we obtain the metric convergence relation $\nu_{\mathrm{inf}}$ that is given by

$$
\nu_{\inf }(\mathfrak{x}, x)=\left(\sup _{S \in \mathfrak{x}} \inf S\right) \ominus x
$$

Under the isomorphisms of Theorem 4.5 the above metric convergence relations correspond to the point-set distances given by

$$
\delta_{\text {sup }}(S, x)=\left\{\begin{array}{ll}
x \ominus(\sup S) & \text { if } S \neq \emptyset ; \\
\infty & \text { if } S=\emptyset
\end{array} \quad \text { and } \quad \delta_{\inf }(S, x)= \begin{cases}(\inf S) \ominus x & \text { if } S \neq \emptyset \\
\infty & \text { if } S=\emptyset\end{cases}\right.
$$

respectively, for all $S \in P A$ and $x \in A$. The first of these is used throughout [25], see Examples 1.8.33 therein. While the metric convergence relations $\nu_{\text {sup }}$ and $\nu_{\mathrm{inf}}$ are $U$-cocomplete, notice that both point-set distances $\delta_{\text {sup }}$ and $\delta_{\text {inf }}$ fail to be $P$-cocomplete.

Proving that $\nu_{\text {sup }}$ corresponds to $\delta_{\text {sup }}$ amounts to showing that $\delta_{\text {sup }}(S, x)=$ $\left(\varepsilon_{[0, \infty]} \odot \nu_{\text {sup }}\right)(S, x)$ for all $S$ and $x$, where $\varepsilon_{[0, \infty]}$ is given in Proposition 3.7. If $S=\emptyset$ this follows from $\emptyset^{\sharp}=\emptyset$. If $S \neq \emptyset$ then, because $S \mapsto x \ominus(\sup S)$ preserves binary joins, the argument given in the second paragraph of the proof of Theorem 3.6 of 22] can be applied without change.
Remark 5.9. For a commutative and completely distributive quantale $\mathcal{V}$, Clementino and Hofmann describe in [6] a general construction that extends a 'suitable' monad $T$ on Set to the thin equipment $\mathcal{V}$-Rel. In [7] they show that in this setting $\mathcal{V}$ admits a $T$-algebra structure whose structure map is a $\mathcal{V}$-functor, thus generalising Example 5.7 above in the case that $\mathcal{V}$ is commutative.

## 6. Horizontal T-morphisms

The following definition generalises the notions of hemicontinuity for relations between topological spaces (see Section VI. 1 of [3] or Section 17.2 of [1]) to notions of 'open' and 'closed' horizontal morphism between $T$-graphs. In Definition 6.7 below these notions are extended to vertical morphisms.

Each of the generalisations of the maximum theorem given in the next section involves a Kan extension along either an open or closed horizontal morphism. Some of these generalisations provide conditions ensuring that the Kan extension itself is an open or closed morphism.

Definition 6.1. Let $T$ be a lax monad on a thin equipment $\mathcal{K}$. A horizontal morphism $J: A \rightarrow B$ between $T$-graphs $A=(A, \alpha)$ and $B=(B, \beta)$ is called

- T-open if $\alpha \odot J \leq T J \odot \beta$;
- T-closed if $T J \odot \beta \leq \alpha \odot J$.

Notice that if $T$ is a lax monad on the thin equipment $\mathcal{V}$-Rel of relations taking values in a quantale $\mathcal{V}$, and $T:=\operatorname{Mod}(T)$ is the induced lax monad on $\mathcal{V}$-Prof, then $T$-open $/ T$-closed horizontal morphisms in $\mathcal{V}$-Prof are precisely those $\mathcal{V}$-profunctors whose underlying $\mathcal{V}$-relations are $T$-open $/ T$-closed.
Example 6.2. Let $P$ be the powerset monad extended to $\mathcal{V}$-relations (Example 3.2). A $\mathcal{V}$-relation $J: A \rightarrow B$ between $\mathcal{V}$-valued pseudoclosure spaces $A=(A, \delta)$ and $B=(B, \zeta)($ Example 3.5$)$ is $P$-open if
(O) $\delta(S, x) \otimes J(x, y) \leq \sup _{T \in P B}\left(\inf _{t \in T} \sup _{s \in S} J(s, t)\right) \otimes \zeta(T, y)$
for all $S \in P A, x \in A$ and $y \in B$. Dually $J$ is $P$-closed if
(C) $\left(\inf _{t \in T} \sup _{s \in S} J(s, t)\right) \otimes \zeta(T, y) \leq \sup _{x \in A} \delta(S, x) \otimes J(x, y)$
for all $S \in P A, T \in P B$ and $y \in B$.
It is straightforward to show that if $J: A \rightarrow B$ is discrete, that is $\operatorname{im} J \subseteq\{\perp, k\}$, while $B$ is a $\mathcal{V}$-valued preclosure space (Example 3.5), then the axioms above reduce to
( ${ }^{\prime}$ ) $\delta(S, x) \otimes J(x, y) \leq \zeta\left(J_{k} S, y\right) ;$
$\left(\mathrm{C}^{\prime}\right) \zeta\left(J_{k} S, y\right) \leq \sup _{z \in J_{k}^{\circ} y} \delta(S, z)$,
for all $S \in P A, x \in A$ and $y \in B$. Here $J_{k}: A \rightarrow B$ is the ordinary relation defined by $x J_{k} y: \Leftrightarrow J(x, y)=k$ for all $x \in A$ and $y \in B$.

Choosing $\mathcal{V}=2$ in the above, the proposition below shows that a relation $J: A \rightarrow B$ between closure spaces is open precisely if, for any open $O \subseteq B$, the preimage $J^{\circ} O$ is $P$-open in $A$. Dually it is easy to show that $J$ is $P$-closed precisely if, for every closed $V \subseteq A$, the image $J V$ is closed in $B$.

Proposition 6.3. Let $P$ be the powerset monad on Rel. For a relation $J: A \rightarrow B$ between closure spaces the following are equivalent:
(a) $J$ is $P$-open;
(b) $J \bar{S} \subseteq \overline{J S}$ for all $S \subseteq A$;
(c) $J^{\circ} O$ is open in $A$ for all $O \subseteq B$ open.

Proof. (a) $\Leftrightarrow$ (b) follows immediately from axiom ( $\mathrm{O}^{\prime}$ ) in the previous example.
(b) $\Rightarrow$ (c). Assume that $O \subseteq B$ is open but its preimage $J^{\circ} O \subseteq A$ is not, that is $\overline{A-J^{\circ} O} \nsubseteq A-J^{\circ} O$. Thus $\overline{A-J^{\circ} O} \cap J^{\circ} O \neq \emptyset$ or, equivalently, $\overline{J-J^{\circ} O} \cap O \neq \emptyset$. But part (b) implies

$$
J \overline{A-J^{\circ} O} \subseteq \overline{J\left(A-J^{\circ} O\right)} \subseteq \overline{B-O}=B-O
$$

contradicting the latter.
(c) $\Rightarrow(\mathrm{b})$. Assuming (c), suppose that (b) does not hold, i.e. for some $S \subseteq A$ we have $J \bar{S} \nsubseteq \overline{J S}$ or, equivalently, $\bar{S} \cap J^{\circ}(B-\overline{J S}) \neq \emptyset$. But this is contradicted by

$$
\bar{S}=\overline{A-(A-S)} \subseteq \overline{A-J^{\circ}(B-J S)} \subseteq \overline{A-J^{\circ}(B-\overline{J S})}=A-J^{\circ}(B-\overline{J S})
$$

where the last equality follows from part (c).

The following theorem describes open and closed $\mathcal{V}$-relations $J: A \rightarrow B$ between $(U, \mathcal{V})$-categories $A$ and $B$ in terms of the corresponding $\mathcal{V}$-valued topological space structures on $A$ and $B$. In order to state it we need the following definition.

Definition 6.4. Let $U$ be the ultrafilter monad on $\mathcal{V}$-Rel, where $\mathcal{V}$ is a completely distributive quantale. Given a $(U, \mathcal{V})$-graph $A=(A, \alpha)$ and a set $B$ we will call a $\mathcal{V}$-relation $J: A \rightarrow B U$-compact whenever $(U J)\left(\mathrm{id}, \iota_{B}\right) \leq \alpha \odot J$, that is

$$
\inf _{S \in \mathfrak{x}} \sup _{s \in S} J(s, y) \leq \sup _{x \in A} \alpha(\mathfrak{x}, x) \otimes J(x, y)
$$

for all $\mathfrak{x} \in U A$ and $y \in B$.
If $J: A \rightarrow B$ is discrete, i.e. $\operatorname{im} J \subseteq\{\perp, k\}$, then the condition above reduces to

$$
k \leq \sup _{x \in J_{k}^{\circ} y} \alpha(\mathfrak{y}, x)
$$

for all $y \in B$ and $\mathfrak{y} \in U A$ with $J_{k}^{\circ} y \in \mathfrak{y}$. In particular if $\mathcal{V}=2$, so that $A$ is a pseudotopological space and $J$ is an ordinary relation, this means that every ultrafilter on $J^{\circ} y$ converges to some $x \in J^{\circ} y$; that is, for each $y \in B$ the preimage $J^{\circ} y$ is compact in $A$.

Theorem 6.5. Let $\mathcal{V}$ be a completely distributive quantale and let $U$ and $P$ be the ultrafilter and powerset monads on $\mathcal{V}$-Rel. Consider $(U, \mathcal{V})$-graphs $A=(A, \alpha)$ and $B=(B, \beta)$ as well as their induced $\mathcal{V}$-valued preclosure space structures $\delta=\varepsilon_{A} \odot \alpha$ and $\zeta=\varepsilon_{B} \odot \beta$; see Proposition 3.8. For a $\mathcal{V}$-relation $J: A \rightarrow B$ the following hold:
(a) if $J$ is $U$-open as a $\mathcal{V}$-relation of $(U, \mathcal{V})$-graphs then it is $P$-open as a $\mathcal{V}$-relation $J:(A, \delta) \rightarrow(B, \zeta)$ of $\mathcal{V}$-valued preclosure spaces;
(b) if $J$ is $U$-closed as a $\mathcal{V}$-relation of $(U, \mathcal{V})$-graphs then it is both $U$-compact, in the sense above, as well as $P$-closed as a $\mathcal{V}$-relation $J:(A, \delta) \rightarrow(B, \zeta)$ of $\mathcal{V}$-valued preclosure spaces.

The converse of (a) holds as soon as B is unitary and $U(U J \odot \beta)=U^{2} J \odot U \beta$; the converse of (b) holds whenever $A$ is a $(U, \mathcal{V})$-category and $U(\alpha \odot J)=U \alpha \odot U J$.

Proof. Part (a). Suppose that $J$ is $U$-open as a $\mathcal{V}$-relation between $(U, \mathcal{V})$-graphs, that is $\alpha \odot J \leq U J \odot \beta$. Using Proposition 3.7(b) we then have

$$
\delta \odot J=\varepsilon_{A} \odot \alpha \odot J \leq \varepsilon_{A} \odot U J \odot \beta=P J \odot \varepsilon_{B} \odot \beta=P J \odot \zeta
$$

showing that $J$ is $P$-open as a $\mathcal{V}$-relation between $\mathcal{V}$-valued preclosure spaces. For the converse assume that $B$ is unitary and that $U(U J \odot \beta)=U^{2} J \odot U \beta$. It follows that

$$
\begin{aligned}
(U(U J \odot \beta))\left(\mathrm{id}, \iota_{B}^{U}\right)= & U^{2} J \odot U \beta \odot \iota_{B}^{U *} \leq U^{2} J \odot \beta\left(\mu_{B}^{U}, \mathrm{id}\right) \\
& =U^{2} J \odot \mu_{B *}^{U} \odot \beta \leq \mu_{A *}^{U} \odot U J \odot \beta=(U J \odot \beta)\left(\mu_{A}^{U}, \mathrm{id}\right)
\end{aligned}
$$

where the inequalities follow from $B$ being unitary and from applying Lemma 1.6 to the naturality cell of $\mu$ at $J$. Thus by Proposition 3.7(c) we have $\varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot U J \odot \beta\right)=$ $U J \odot \beta$. Using this, assuming that $J$ is $P$-open, it follows that

$$
\begin{aligned}
\alpha \odot J \stackrel{(\mathrm{i})}{\leq} \varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot \alpha \odot J\right) & =\varepsilon_{A} \triangleleft(\delta \odot J) \leq \varepsilon_{A} \triangleleft(P J \odot \zeta) \\
& =\varepsilon_{A} \triangleleft\left(P J \odot \varepsilon_{B} \odot \beta\right) \stackrel{(\mathrm{ii})}{=} \varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot U J \odot \beta\right)=U J \odot \beta,
\end{aligned}
$$

where (i) is given by the unit of $\varepsilon_{A} \odot-\dashv \varepsilon_{A} \triangleleft-$ and (ii) follows from Proposition 3.7(b). This shows that $J$ is $U$-open.

Part (b). Assume that $J$ is $U$-closed as a $\mathcal{V}$-relation between $(U, \mathcal{V})$-graphs, that is $U J \odot \beta \leq \alpha \odot J$. Then $J$ is $U$-compact:

$$
(U J)\left(\operatorname{id}, \iota_{B}^{U}\right)=U J \odot \iota_{B}^{U *} \leq U J \odot \beta \leq \alpha \odot J,
$$

where the first inequality follows from applying Lemma 1.6 to the unit cell of $\beta$. That $J$ is $P$-closed as a $\mathcal{V}$-relation between the $\mathcal{V}$-valued preclosure spaces $(A, \delta)$ and $(B, \zeta)$ is shown by

$$
P J \odot \zeta=P J \odot \varepsilon_{B} \odot \beta=\varepsilon_{A} \odot U J \odot \beta \leq \varepsilon_{A} \odot \alpha \odot J=\delta \odot J
$$

where the second identity follows from Proposition 3.7(b). For the converse assume that $J$ is $U$-compact and $P$-closed while $A$ is a $(U, \mathcal{V})$-category and $U(\alpha \odot J)=$ $U \alpha \odot U J$. Using $U$-compactness of $J$ and the associativity axiom for $A$ it follows that
$U(\alpha \odot J)\left(\mathrm{id}, \iota_{B}^{U}\right)=U \alpha \odot U J \odot \iota_{B}^{U *} \leq U \alpha \odot \alpha \odot J \leq \alpha\left(\mu_{A}^{U}, \mathrm{id}\right) \odot J=(\alpha \odot J)\left(\mu_{A}^{U}, \mathrm{id}\right)$,
so that $\varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot \alpha \odot J\right)=\alpha \odot J$ by Proposition 3.7(c). We conclude that

$$
\begin{aligned}
U J \odot \beta \leq \varepsilon_{A} \triangleleft( & \left.\varepsilon_{A} \odot U J \odot \beta\right)=\varepsilon_{A} \triangleleft\left(P J \odot \varepsilon_{B} \odot \beta\right) \\
& =\varepsilon_{A} \triangleleft(P J \odot \zeta) \leq \varepsilon_{A} \triangleleft(\delta \odot J)=\varepsilon_{A} \triangleleft\left(\varepsilon_{A} \odot \alpha \odot J\right)=\alpha \odot J,
\end{aligned}
$$

showing that $J$ is $U$-closed.
Example 6.6. Let us choose $\mathcal{V}=2$ in the previous theorem, so that it applies to a relation $J: A \rightarrow B$ between topological spaces. Combined with the descriptions of $P$-open and $P$-closed relations between topological spaces given in Example 6.2, as well as the description of $U$-compact relations following Definition 6.4, we find that $J$ is $U$-open precisely if it is lower hemi-continuous in the classical sense, see e.g. Section VI. 1 of [3], while $J$ is $U$-closed precisely if its reverse $J^{\circ}: B \rightarrow A$ is upper hemi-continuous. More precisely, the notions of $U$-open and $U$-closed relation describe lower/upper hemi-continuity in terms of ultrafilter convergence. For a closely related description of hemi-continuity in terms of nets see Section 17.3 of [1].

In the definition below the notions of openness and closedness are extended to vertical morphisms. In the case that $T$ is a lax monad on $\mathcal{V}$-Rel this recovers the notions of 'open' and 'proper' morphism between ( $T, \mathcal{V}$ )-categories, as studied in Section V. 3 of [17], although there $T$ is not required to be normal.

Definition 6.7. Let $T$ be a normal lax monad on a thin equipment $\mathcal{K}$. A morphism $f: A \rightarrow C$ of $T$-graphs $A=(A, \alpha)$ and $C=(C, \gamma)$ is called

- T-open if its conjoint $f^{*}: C \rightarrow A$ is $T$-open, that is $\gamma(\mathrm{id}, f) \leq(T f)^{*} \odot \alpha$;
- T-closed if its companion $f_{*}: A \rightarrow C$ is $T$-closed, that is $\gamma(T f, \mathrm{id}) \leq \alpha \odot f_{*}$.

We remark that, in rewriting the inequalities of Definition 6.1 into those above, we use the fact that $T$ is normal, so that it preserves companions and conjoints. We shall only describe open and closed morphisms in ( $T, \mathcal{V}$ )-ModCat (see Section 4) where either $T=P$ is the powerset monad or $T=U$ is the ultrafilter monad. For a description of open and closed morphisms in $(U, \mathcal{V})$-Cat we refer to Section V.3.4 of 17].

Example 6.8. Let $P$ be the powerset monad on $\mathcal{V}$-Prof. A morphism $f: A \rightarrow C$ of modular $\mathcal{V}$-valued closure spaces $A=(A, \bar{A}, \delta)$ and $C=(C, \bar{C}, \zeta)$ is $P$-open if

$$
\zeta(T, f x) \leq \sup _{S \in P A}\left(\inf _{s \in S} \sup _{t \in T} C(t, f s)\right) \otimes \delta(S, x)
$$

for all $T \in P C$ and $x \in A$; dually $f$ is $P$-closed if

$$
\zeta(f S, z) \leq \sup _{x \in A} \delta(S, x) \otimes C(f x, z)
$$

for all $S \in P A$ and $z \in C$. If $\mathcal{V}$ is completely distributive so that the ultrafilter $U$ extends to $\mathcal{V}$-Prof as well (see Example 3.3) then, by Theorem 6.5, a morphism $f: A \rightarrow C$ of modular $\mathcal{V}$-valued topological spaces is $U$-open precisely if it is $P$-open, while it is $U$-closed precisely when it is $P$-closed and its companion $f_{*}: A \rightarrow C$ is $U$-compact.
Example 6.9. Taking $\mathcal{V}=2$ in the previous example, a monotone continuous map $f: A \rightarrow C$ between modular topological spaces (Example 4.4) is $U$-open if $\downarrow f O \subseteq C$ is open for all $O \subseteq A$ open; it is $U$-closed whenever it is $P$-closed, that is $\uparrow f V \subseteq C$ is closed for all $V \subseteq A$ closed, while $f^{-1}(\downarrow z) \subseteq A$ is compact for all $z \in C$.

In case $C=[-\infty, \infty]$ is equipped with the Scott topology with respect to $\geq$ (Example 5.5) then the first two of the conditions above weaken the classical notions of open and closed maps $f: A \rightarrow[-\infty, \infty]: f$ is $U$-open means $f O \subseteq[-\infty, \infty]$ does not have a minimum, for all $O \subseteq A$ open, while $f$ is $P$-closed means $f V \subseteq[-\infty, \infty]$ has a maximum, for all closed $V \subseteq A$. The third condition above means that $f^{-1}([z, \infty]) \subseteq A$ is compact for all $z \in[-\infty, \infty]$; functions $f: A \rightarrow[-\infty, \infty]$ with this property are called upper semi-compact or sup-compact, see e.g. Section 1 of [10].

We close this section with a couple of remarks.
Remark 6.10. Let $T$ be a lax monad on a thin equipment $\mathcal{K}$. Notice that the horizontal composite $J \odot H$ of $T$-open horizontal morphisms $J: A \rightarrow B$ and $H: B \rightarrow E$ is again $T$-open. In fact $T$-graphs, $T$-morphisms, $T$-open horizontal morphisms and the cells between them in $\mathcal{K}$ form a 'thin double category' $T$-Opn. While $T$-Opn has all companions $f_{*}$, the conjoint $f^{*}$ of a $T$-morphism $f$ will in general not be $T$-open, but $T$-closed instead. If the monad $T$ preserves horizontal composition strictly then we are able to compose $T$-closed horizontal morphisms as well, so that they form the horizontal morphisms of a thin double category $T$-Cls.
Remark 6.11. Let $T$ be a lax monad on $\mathcal{V}$-Rel. Weakening the notion of modular $(T, \mathcal{V})$-category considered in our Section 4, in Section 5 of [31] an 'open $\mathcal{V}$-structured $(T, \mathcal{V})$-category' $A$ is defined to be a $(T, \mathcal{V})$-category $(A, \alpha)$ equipped with a $\mathcal{V}$-category structure $\bar{A}: A \rightarrow A$ that is $T$-open in our sense. Similarly 'closed $\mathcal{V}$-structured $(T, \mathcal{V})$-categories' $A$ are defined to be triples $(A, \bar{A}, \alpha)$ with $(A, \alpha)$ a $(T, \mathcal{V})$-category and $(A, \bar{A})$ a $\mathcal{V}$-category, such that $\bar{A}$ is $T$-closed with respect to $\alpha$ and $T(\alpha \odot \bar{A})=T \alpha \odot T \bar{A}$.

## 7. Generalisations of the maximum theorem

We are now ready to state and prove generalisations of the maximum theorem for Kan extensions of $T$-morphisms between $T$-graphs. Starting with right Kan extensions the first of these generalisations, Theorem 7.1 below, assumes that the target of the Kan extension is $T$-cocomplete (Definition 5.1), while Theorem 7.6 instead assumes a Kan extension that satisfies the Beck-Chevalley condition (Theorem 2.6). Similarly left Kan extensions are considered in Theorem 7.8 and Theorem 7.9.

Theorem 7.1. Let $T$ be a normal lax monad on a thin equipment $\mathcal{K}$. Let $J: A \rightarrow B$ be a T-open horizontal morphism between T-graphs and e: B $\rightarrow$ a $T$-morphism into a $T$-cocomplete $T$-graph $M$. The right Kan extension $r: A \rightarrow M$ of e along $J$ in $\mathcal{K}$, if it exists, is a $T$-morphism.

Proof. We write $\alpha: T A \rightarrow A, \beta: T B \rightarrow B$ and $m_{*}: T M \rightarrow M$ for the horizontal structure morphisms of $A, B$ and $M$ respectively; because $M$ is $T$-cocomplete the last of these is the companion of a vertical morphism $m: T M \rightarrow M$. Consider the composite of cells on the left-hand side below, where $T \varepsilon$ denotes the ' $T$-image' of the cell $\varepsilon$ definining $r$ and where the cells denoted $\leq$, from top to bottom, exist because $J$ is $T$-open, $e$ is a $T$-morphism, and $m_{*}$ is the companion of $m$.


By the universal property of $\varepsilon$ the composite on the left factors as shown. Composing this factorisation with the appropriate cell among the pair of cells that defines $m_{*}$, we obtain the cell that exhibits $r$ as a $T$-morphism.

Example 7.2. If $T=P$ is the powerset monad on the thin equipment $\mathcal{K}=2$-Prof of modular relations (Example 1.9), so that $A, B$ and $M$ in the theorem above are modular preclosure spaces (Example 4.3), then the categorical proof above reduces to the following elementary proof. As $M$ is assumed to be $P$-cocomplete its closed subsets are principal upsets $\uparrow z$ (Example 5.3), so that for the continuity of $r$ it suffices to show that $r^{-1}(\uparrow z)$ is closed for all $\uparrow z \subseteq M$ closed.

To see this first notice that from the definition $r x=\inf _{y \in J x}$ ey of $r$ (Example 2.3) it follows that

$$
S \subseteq r^{-1}(\uparrow z) \quad \Leftrightarrow \quad J S \subseteq e^{-1}(\uparrow z)
$$

for any $S \subseteq A$. Using this we find

$$
\begin{aligned}
S \subseteq r^{-1}(\uparrow z) & \Leftrightarrow J S \subseteq e^{-1}(\uparrow z) \\
& \Leftrightarrow \overline{J S} \subseteq e^{-1}(\uparrow z) \quad \text { (because } e \text { is continuous) } \\
& \Rightarrow J \bar{S} \subseteq e^{-1}(\uparrow z) \quad \text { (because } J \text { is } P \text {-open; see Proposition 6.3) } \\
& \Leftrightarrow \bar{S} \subseteq r^{-1}(\uparrow z),
\end{aligned}
$$

so that taking $S=r^{-1}(\uparrow z)$ here proves the continuity of $r$.
Example 7.3. Consider topological spaces $A$ and $B$ and let $J: A \rightarrow B$ be a lower hemi-continuous relation, that is $J$ is $P$-open in our sense (see Example 6.2 and Example 6.6). Let $e: B \rightarrow[-\infty, \infty]$ be a lower semi-continuous map, that is $e$ is continuous with respect to the Scott topology on $[-\infty, \infty]$ with the reverse order $\geq$ (see Example 5.5). Regarding $A$ and $B$ as topological spaces with discrete orders, the previous theorem asserts that the right Kan extension $r: A \rightarrow[-\infty, \infty]$ of $e$ along $J$, given by

$$
r x=\sup _{y \in J x} e y
$$

for all $x \in A$, is lower semi-continuous. This recovers partly Theorem 1 of Section VI. 3 of [3] (or Lemma 17.29 of [1]), where more general maps of the form $e: A \times B \rightarrow[-\infty, \infty]$ are treated; see the final remark of the Introduction.
Example 7.4. Recall from Example 5.8 that the canonical normalised modular approach space structure on the Lawvere quantale $[0, \infty]$ is given by the point-set distance

$$
\delta_{\text {sup }}(S, x)= \begin{cases}x \ominus(\sup S) & \text { if } S \neq \emptyset \\ \infty & \text { otherwise }\end{cases}
$$

where $\ominus$ denotes truncated difference. Consider a metric relation $J: A \rightarrow B$ between approach spaces that is $P$-open (Example 6.2) or, equivalently by Theorem 6.5, $U$-open, as well as a continuous map $e: B \rightarrow[0, \infty]$. Regarding $A$ and $B$ as modular approach spaces with discrete metrics, the previous theorem asserts that the right Kan extension $r: A \rightarrow[0, \infty]$ of $e$ along $J$, given by the suprema

$$
r x=\sup _{y \in B} e y \ominus J(x, y)
$$

for all $x \in A$ (see Proposition 2.2), is continuous. The previous theorem likewise applies to right Kan extensions into $[0, \infty]$ equipped with the 'reversed' point-set distance $\delta_{\text {inf }}$ of Example 5.8.
Example 7.5. Analogous to the previous example, the above theorem applies to right Kan extensions of continuous maps $e: B \rightarrow \Delta_{\&}$ of modular probabilistic approach spaces. Here $\Delta_{\&}$ is the space of distribution functions (Example 1.4), equipped with either of the probabilistic approach space structures that are decribed in Example 5.7 for $\mathcal{V}=\Delta_{\&}$.
Theorem 7.6. Let $T$ be a normal lax monad on a thin equipment $\mathcal{K}$. Let $A, B$ and $M$ be T-graphs, e: B $\rightarrow$ a T-morphism and $J: A \rightarrow B$ a T-open horizontal morphism. The right Kan extension $r: A \rightarrow M$ of e along $J$ in $\mathcal{K}$, if it exists, is a $T$-morphism whenever it satisfies the Beck-Chevalley condition.

Moreover, in that case $r$ is $T$-closed as soon as both $e$ and $J$ are $T$-closed, provided that $T J \odot T e_{*}=T\left(J \odot e_{*}\right)$.
Proof. We write $\alpha: T A \rightarrow A, \beta: T B \rightarrow B$ and $\nu: T M \rightarrow M$ for the horizontal structure morphisms of $A, B$ and $M$. Consider the composite on the left-hand side below, where $T \varepsilon$ denotes the ' $T$-image' of the universal cell $\varepsilon$ that defines $r$ and where the other two cells exhibit $J$ as a $T$-open horizontal morphism and $e$ as a $T$-morphism respectively.


By assumption $r$ satisfies the Beck-Chevalley condition so that, by the horizontal dual of Theorem 2.6, the composite factors through $\varepsilon$ as shown. This factorisation exhibits $r$ as a $T$-morphism.

Now assume that both $e$ and $J$ are $T$-closed and that $T$ preserves the horizontal composite $J \odot e_{*}$. That $r$ is $T$-closed is shown by

$$
\begin{aligned}
\nu(T r, \mathrm{id})=T r_{*} \odot \nu \stackrel{(\mathrm{i})}{=} T\left(J \odot e_{*}\right) \odot \nu & =T J \odot T e_{*} \odot \nu \\
& \stackrel{\text { (ii) }}{\leq} T J \odot \beta \odot e_{*} \stackrel{\text { (iii) }}{\leq} \alpha \odot J \odot e_{*} \stackrel{(\mathrm{i})}{=} \alpha \odot r_{*},
\end{aligned}
$$

where the equalities marked (i) follow from the Beck-Chevalley condition for $r$ while the inequalities marked (ii) and (iii) follow from $e$ and $J$ being $T$-closed respectively.

Example 7.7. In the setting of Example 7.3 assume that the relation $J: A \rightarrow B$, besides being lower hemi-continuous, is upper hemi-continuous (see Example 6.6). Also assume that the right Kan extension $r$ of $e$ along $J$ satisfies the Beck-Chevalley condition, that is the suprema defining $r$ are attained as maxima (Example 2.7). The second assertion of the previous theorem states that $r$ is $P$-closed and upper semi-compact (Example 6.9) whenever the map $e$ is.

Next we turn to generalisations of the maximum theorem for left Kan extensions between $T$-graphs.

Theorem 7.8. Let $T$ be a normal lax monad on a thin equipment $\mathcal{K}$. Let $J: A \rightarrow B$ be a T-closed horizontal morphism between $T$-graphs and let $d: A \rightarrow M$ be a T-morphism into a $T$-cocomplete $T$-graph $M=\left(M, m_{*}\right)$, where $m: T M \rightarrow M$ (see Definition 5.1). The left Kan extension $l: B \rightarrow M$ of $d$ along $J$ in $\mathcal{K}$, if it exists, is a $T$-morphism whenever $m \circ T l$ is the left Kan extension of $m \circ T d$ along $T J$.

Proof. Writing $\alpha$ and $\beta$ for the $T$-graph structures of $A$ and $B$, consider on the left-hand side below the composite of the cells exhibiting $J$ as a $T$-closed horizontal morphism, $d$ as a $T$-morphism, $l$ as a left Kan extension and $m_{*}$ as a companion.


By assumption the first column in the right-hand side above defines $m \circ T l$ as a left Kan extension so that the left-hand side factors as shown. Composing this factorisation with the appropriate cell among the pair of cells that defines $m_{*}$, we obtain the cell that exhibits $l$ as a $T$-morphism.

Theorem 7.9. Let $T$ be a normal lax monad on a thin equipment $\mathcal{K}$. Let $A, B$ and $M$ be $T$-graphs, $d: A \rightarrow M$ a T-morphism and $J: A \rightarrow B$ a $T$-closed horizontal morphism. The left Kan extension $l: B \rightarrow M$ of $d$ along $J$ in $\mathcal{K}$, if it exists, is a T-morphism whenever it satisfies the Beck-Chevalley condition and $T d^{*} \odot T J=$ $T\left(d^{*} \odot J\right)$.

Moreover, in that case $l$ is $T$-open as soon as both $d$ and $J$ are $T$-open.
Proof. Denoting by $\alpha, \beta$ and $\nu$ the $T$-graph structures of $A, B$ and $M$, consider on the left-hand side below the composition of the cells exhibiting $J$ as a $T$-closed horizontal morphism, $d$ as a $T$-morphism and $l$ as the left Kan extension of $d$ along $J$.

Under the assumptions on $l$ it follows from Proposition 2.10 that $T l$ is the left Kan extension of $T d$ along $T J$, such that it satisfies the Beck-Chevalley condition. Hence, by Theorem 2.6 the composite on the left factors through the ' $T$-image' of $\eta$ as shown, and this factorisation exhibits $l$ as a $T$-morphism.

That $l$ is $T$-open whenever $d$ and $J$ are is shown by
$\nu(\mathrm{id}, l)=\nu \odot l^{*} \stackrel{(\mathrm{i})}{=} \nu \odot d^{*} \odot J \stackrel{(\mathrm{ii})}{\leq} T d^{*} \odot \alpha \odot J \stackrel{(\mathrm{iii})}{\leq} T d^{*} \odot T J \odot \beta=T\left(d^{*} \odot J\right) \odot \beta \stackrel{(\mathrm{i})}{=} T l^{*} \odot \beta$,
where the equalities (i) follow from the Beck-Chevalley condition for $l$ and the inequalities (ii) and (iii) follow from $d$ and $J$ being $T$-open respectively.

Example 7.10. Let $J: A \rightarrow B$ be an upper hemi-continuous relation (Example 6.6) between topological spaces, such that $J^{\circ} y \subseteq A$ is non-empty for each $y \in B$, and let $d: A \rightarrow[-\infty, \infty]$ be a upper semi-continuous map (Example 5.5). Regarding $A$ and $B$ as discrete ordered sets, by the extreme value theorem (see e.g. Theorem 8.1 below) the conditions on $J$ imply that the left Kan extension $l: B \rightarrow[-\infty, \infty]$ of $d$ along $J$ satisfies the Beck-Chevalley condition, that is it is given by the maxima

$$
l y=\max _{x \in J \circ y} d x
$$

for all $y \in B$. Applying the previous theorem we find that $l$ is upper semi-continuous, thus partly recovering Theorem 2 of Section VI. 3 of [3] (or Lemma 17.30 of [1]) which is stated in terms of the reverse $J^{\circ}: B \rightarrow A$. Moreover its second assertion means that $l$ is $U$-open (Example 6.9) whenever $d$ is $U$-open and $J$ is lower hemi-continuous.
Example 7.11. Let $J: A \rightarrow B$ be a relation between topological spaces that is both lower and upper hemi-continuous (Example 6.6), while $J x \subseteq B$ is non-empty for each $x \in A$, and suppose that $e: B \rightarrow[-\infty, \infty]$ is continuous. Since a function into $[-\infty, \infty]$ is continuous precisely if it is both lower and upper semi-continuous (Example 5.5), by combining Example 7.3 and the previous example (the latter applied to $J^{\circ}: B \rightarrow A$ and $e: B \rightarrow[-\infty, \infty]$ ) we find that the extension $m: A \rightarrow[-\infty, \infty]$ of $e$ along $J$, given by

$$
m x=\max _{y \in J x} e y
$$

for all $x \in A$, is continuous. This recovers the main assertion of Berge's maximum theorem, as stated in Section 4.3 of [3].
Example 7.12. Consider the Lawvere quantale [0, $\infty$ ] with the canonical normalised modular approach space structure with point-set distance $\delta_{\text {sup }}$ given in Example 5.8. Let $J: A \times B \rightarrow[0, \infty]$ be a $U$-closed metric relation (see Theorem 6.5(b) and Example 6.2) between approach spaces that is discrete, that is im $J \subseteq\{0, \infty\}$, such that $J_{0}^{\circ} y \neq \emptyset$ for all $y \in B$, and suppose that $d: A \rightarrow[0, \infty]$ is continuous. Regarding $A$ and $B$ as modular approach spaces with discrete metrics, the left Kan extension $l: B \rightarrow[0, \infty]$ of $d$ along $J$ is defined on $y \in B$ by

$$
l y=\inf _{x \in J_{0}^{\circ} y} d x
$$

see Example 2.3. By Theorem 8.2 below the conditions on $J$ ensure that $l$ satisfies the Beck-Chevalley condition. It thus follows from the previous theorem that $l$ is continuous, while it is $P$-open (Example 6.8) whenever $d$ and $J$ are $P$-open.
Example 7.13. A result for left Kan extensions $l: B \rightarrow \Delta_{\&}$ between probabilistic approach spaces, analogous to the previous example, can be derived from the theorem above as well. In this case however the hypothesis $U\left(d^{*} \odot J\right)=U d^{*} \odot U J$ may not be satisfied: while the extension of the ultrafilter monad $U$ to $[0, \infty]$-Rel
is strict (Example 3.3), I do not know if its extension to $\Delta_{\&}$-Rel is too. Moreover, in general Theorem 8.2 applies only in the cases where the convolution product on $\Delta_{\&}$ is induced by multiplication on $[0,1]$ or the Łukasiewicz operation, see Proposition 8.4 it may fail to apply in the case of the frame operation $p \& q=\min \{p, q\}$, see Example 8.5.

## 8. Generalisations of the extreme value theorem

In this last section we investigate Kan extensions that satisfy the Beck-Chevalley condition (Theorem 2.6). We treat two cases: the first concerning left Kan extensions between modular closure spaces (Example 4.3) and the second concerning a restricted class of left Kan extensions between modular $\mathcal{V}$-valued pseudotopological spaces (Example 4.4). Starting with the former, the theorem below is a straightforward generalisation of Weierstraß' extreme value theorem (see e.g. Corollary 2.35 of [1]).

A subset $S \subseteq A$ of a closure space $A$ is called compact if for every family $\left(V_{i}\right)_{i \in I}$ of closed subsets of $A$ we have

$$
\left(\underset{\substack{J \subseteq I \\ J \text { is finite }}}{\forall} S \cap \bigcap_{j \in J} V_{j} \neq \emptyset\right) \Rightarrow S \cap \bigcap_{i \in I} V_{i} \neq \emptyset .
$$

It is straightforward to check that this is equivalent to the definition of compact subsets in terms of finite open subcovers, which is often used in the case of topological spaces. In particular, continuous maps of closure spaces preserve compact sets.

Recall that a subset $S$ of an ordered set $M$ is called up-directed whenever it is non-empty and every finite subset of $S$ has an upper bound in $S$, that is for all $u, v \in S$ there is $w \in S$ with $u \leq w$ and $v \leq w$.

Theorem 8.1. Let $A$ and $M$ be modular closure spaces, $d: A \rightarrow M$ a monotone continuous function and $J: A \rightarrow B$ a modular relation into an ordered set $B$. The left Kan extension $l: B \rightarrow M$ of $d$ along $J$, if it exists, satisfies the Beck-Chevalley condition whenever $M$ is normalised and $d\left(J^{\circ} y\right) \subseteq M$ is compact and up-directed for each $y \in B$.

Proof. First recall from Example 4.3 that all principal upsets $\uparrow z$ in $M$ are closed because $M$ is normalised. As described in Example 2.7 we have to show that for each $y \in B$ the set $d\left(J^{\circ} y\right)$ has a maximum in $M$. To this end consider in $M$ the family of closed principal subsets

$$
(\uparrow z)_{z \in d\left(J^{\circ} y\right)}
$$

We claim that $d\left(J^{\circ} y\right) \cap \bigcap_{i=1}^{n} \uparrow z_{i} \neq \emptyset$ for any finite sequence $z_{1}, \ldots, z_{n} \in d\left(J^{\circ} y\right)$. Indeed, since $d\left(J^{\circ} y\right)$ is up-directed the set $\left\{z_{1}, \ldots, z_{n}\right\}$ has an upper bound $z$. As $\uparrow z \subseteq \uparrow z_{i}$ for each $i$ we conclude $z \in d\left(J^{\circ} y\right) \cap \uparrow z \subseteq d\left(J^{\circ} y\right) \cap \bigcap_{i=1}^{n} \uparrow z_{i}$. By compactness of $d\left(J^{\circ} y\right)$ it follows that there exists some $w \in d\left(J^{\circ} y\right) \cap \bigcap_{z \in d\left(J^{\circ} y\right)} \uparrow z$. But this means $z \leq w$ for all $z \in d\left(J^{\circ} y\right)$, showing that $w$ is a maximum of $d\left(J^{\circ} y\right)$.

The second generalisation of the extreme value theorem applies to a restricted class of left Kan extensions between modular $\mathcal{V}$-valued pseudotopological spaces (Example 4.4). Remember that, for a $\mathcal{V}$-profunctor $J: A \rightarrow B$ and $v \in \mathcal{V}$, we write $J_{v}: A \rightarrow B$ for the ordinary relation given by

$$
x J_{v} y \quad \Leftrightarrow \quad v \leq J(x, y)
$$

for all $x \in A$ and $y \in B$.

Theorem 8.2. Let $\mathcal{V}$ be a completely distributive quantale so that $\mathcal{V}=\mathcal{V}_{-}$is itself a modular $\mathcal{V}$-valued topological space, see Example 5.7. Consider a modular $\mathcal{V}$-valued pseudotopological space $A$ and a $\mathcal{V}$-category $B$, as well as a continuous $\mathcal{V}$-functor $d: A \rightarrow \mathcal{V}_{-}$and a $\mathcal{V}$-profunctor $J: A \rightarrow B$. The left Kan extension $l: B \rightarrow \mathcal{V}_{\infty}$ of $d$ along $J$ satisfies the Beck-Chevalley condition whenever $J$ satisfies the following conditions:
(a) $J$ is discrete, that is $\operatorname{im} J \subseteq\{\perp, k\}$;
(b) $J$ is $U$-compact, see Definition 6.4;
(c) for each $y \in B$ the set $d\left(J_{k}^{\circ} y\right)$ is up-directed in $\mathcal{V}$;
(d) for each $y \in B$

$$
k \leq \sup _{z \in d\left(J_{k}^{\circ} y\right)}\left(\sup d\left(J_{k}^{\circ} y\right) \multimap z\right)
$$

Notice that, by Example 2.3, condition (a) above means that the Kan extension $l$ is given by

$$
\begin{equation*}
l y=\sup d\left(J_{k}^{\circ} y\right) \tag{15}
\end{equation*}
$$

for all $y \in B$, so that condition (d) can be rewritten as

$$
k \leq \sup _{z \in d\left(J_{k}^{\circ} y\right)}(l y \multimap z)
$$

for each $y \in B$. If $\mathcal{V}=[0, \infty]$ then this inequality follows from the fact that the map $l y \multimap-$ preserves infima.

After giving the proof of the theorem above, Example 8.3 below shows that condition (a) on $J$ cannot be dispensed with. Proposition 8.4 shows that condition (d) holds for $\mathcal{V}=\Delta_{\times}$and $\Delta_{\&}$, where \& denotes the Łukasiewicz operation (see Example 1.4). Example 8.5 then shows that condition (d) does not generally hold in the case of $\mathcal{V}=\Delta_{\text {min }}$.

Proof. Using Proposition 2.9, first notice that discreteness of $J$ means that the Beck-Chevalley condition for $l$ reduces to the inequality

$$
\begin{equation*}
k \leq \sup _{x \in J_{k}^{\circ} y} \mathcal{V}_{-}(l y, d x) \tag{16}
\end{equation*}
$$

for all $y \in B$. While in general there might not be any $x \in J_{k}^{\circ} y$ with $k \leq$ $\mathcal{V}_{\rightarrow}(l y, d x)=l y \multimap d x$ notice that, if such a $x$ does exist then $l y \leq d x$ follows, so that the supremum $l y$ in (15) is attained as a maximum.

Discreteness of $J$ also means that $U$-compactness of $J$ (Definition 6.4) reduces to

$$
\begin{equation*}
k \leq \sup _{x \in J_{k}^{\circ} y} \alpha(\mathfrak{y}, x) \tag{17}
\end{equation*}
$$

for all $y \in B$ and $\mathfrak{y} \in U A$ with $J_{k}^{\circ} y \in \mathfrak{y}$, where $\alpha: U A \rightarrow A$ is the $\mathcal{V}$-valued convergence relation of $A$.

Let us fix $y \in B$. Using the above any $\mathfrak{y} \in U A$ with $J_{k}^{\circ} y \in \mathfrak{y}$ gives a lower bound for the right-hand side of (16) as follows, where $\nu:=\nu_{\mathrm{inf}}: U \mathcal{V}_{-\circ} \rightarrow \mathcal{V}_{-\circ}$ is
the $\mathcal{V}$-valued convergence relation of $\mathcal{V}_{-}$(see Example 5.7).

$$
\begin{aligned}
\sup _{x \in J_{k}^{\circ} y} \mathcal{V}_{\multimap}(l y, d x) & \stackrel{(\mathrm{i})}{=} \sup _{x \in J_{k}^{\circ} y} \nu((\iota \circ l)(y), d x) \\
& \stackrel{(\mathrm{ii})}{\geq} \sup _{\substack{x \in J_{k}^{\circ} y \\
\mathfrak{x} \in U A}}\left(U \mathcal{V}_{\multimap}\right)((\iota \circ l)(y),(U d)(\mathfrak{x})) \otimes \alpha(\mathfrak{x}, x) \\
& \geq\left(U \mathcal{V}_{\multimap}\right)((\iota \circ l)(y),(U d)(\mathfrak{y})) \otimes \sup _{x \in J_{k}^{\circ} y} \alpha(\mathfrak{y}, x) \\
& \stackrel{(\text { iii) }}{\geq}\left(U \mathcal{V}_{-}\right)((\iota \circ l)(y),(U d)(\mathfrak{y}))
\end{aligned}
$$

The equality denoted (i) here is a consequence of $\mathcal{V}_{-}$being normalised, see (11); (ii) follows from $(U d)^{*} \odot \alpha \leq \nu(\mathrm{id}, d)$, which is obtained by applying Lemma 1.6 to the cell exhibiting $d$ as a $U$-morphism; (iii) follows from (17).

We conclude that to prove the Beck-Chevalley condition for $l$, that is (16) holds, it suffices to construct an ultrafilter $\mathfrak{y}$ on $J_{k}^{\circ} y \subseteq A$ that satisfies

$$
\begin{equation*}
k \leq\left(U \mathcal{V}_{-}\right)((\iota \circ l)(y),(U d)(\mathfrak{y})) \tag{18}
\end{equation*}
$$

To construct the ultrafilter $\mathfrak{y}$ notice that condition (c) on $J$ implies that the sets

$$
X_{z}:=J_{k}^{\circ} y \cap d^{-1}(\uparrow z)
$$

where $z$ ranges over $d\left(J_{k}^{\circ} y\right)$, form a proper filter base; we choose $\mathfrak{y}$ to be any ultrafilter containing all of them. From the definition of the $X_{z}$ it follows that $\mathfrak{y}$ contains $J_{k}^{\circ} y$ as well as the preimages $d^{-1}(\uparrow z)$, for each $z \in d\left(J_{k}^{\circ} y\right)$. The latter implies that $\mathfrak{y}$ contains the sets $d^{-1}(\uparrow(l y \otimes(l y \multimap z)))$ too: this is a consequence of the inequalities $l y \otimes(l y \multimap z) \leq z$, which form the counit of the adjunction $(l y \otimes-) \dashv(l y \multimap-)$.

That $\mathfrak{y}$ satisfies (18) is shown by

$$
\begin{aligned}
\left(U \mathcal{V}_{-}\right)((\iota \circ l)(y),(U d)(\mathfrak{y})) & =\sup \left\{v \in \mathcal{V} \mid[(\iota \circ l)(y)]\left(U\left(\mathcal{V}_{-}\right)_{v}\right)[(U d)(\mathfrak{y})]\right\} \\
& =\sup \left\{v \in \mathcal{V} \mid d^{-1}(\uparrow(l y \otimes v)) \in \mathfrak{y}\right\} \\
& \geq \sup _{z \in d\left(J_{k}^{\circ} y\right)} l y \multimap z \geq k
\end{aligned}
$$

where the four (in-) equalities are consequences of respectively the equivalent definition (8) of $U \mathcal{V}_{-\infty}$, the equivalences below, the discussion above and, finally, condition (d) on $J$.

$$
\begin{aligned}
{[(\iota \circ l)(y)]\left(U\left(\mathcal{V}_{\multimap}\right)_{v}\right)[(U d)(\mathfrak{y})] } & \Leftrightarrow\left(\mathcal{V}_{\multimap}\right)_{v}(l y) \in(U d)(\mathfrak{y}) \\
& \Leftrightarrow\{x \in A \mid v \leq l y \multimap d x\} \in \mathfrak{y} \\
& \Leftrightarrow\{x \in A \mid l y \otimes v \leq d x\} \in \mathfrak{y} \\
& \Leftrightarrow d^{-1}(\uparrow(l y \otimes v)) \in \mathfrak{y}
\end{aligned}
$$

This completes the proof.
Example 8.3. To see that condition (a) of the theorem above, that is discreteness of $J: A \rightarrow B$, cannot be left out we consider the Sierpiński space $2=\{\perp, \top\}$ (see Example 5.5) as a normalised modular approach space (Example 4.4), by taking its image under the composite of embeddings

$$
\text { Top } \xrightarrow{I} \text { App } \xrightarrow{N} \text { ModApp. }
$$

The functor $N$ here is described in Example 4.4, while the functor $I$ maps any topological space $A$, with closure operation $S \mapsto \bar{S}$, to the approach space $I A=$ $(A, \delta)$ with point-set distance given by

$$
\delta(S, x)= \begin{cases}0 & \text { if } x \in \bar{S} \\ \infty & \text { otherwise }\end{cases}
$$

Regarding the Lawvere quantale $[0, \infty]$ as equipped with its canonical modular point-set distance $\delta_{\text {sup }}$ (Example 5.8), let $d: 2 \rightarrow[0, \infty]$ and $J: 2 \rightarrow *$, where $*=\{*\}$ denotes the singleton approach space, be defined by

$$
d(\perp)=2, \quad d(\top)=0, \quad J(\perp, *)=0 \quad \text { and } \quad J(\top, *)=1
$$

It is straightforward to check that $d$ is a non-expansive continuous map of modular approach spaces, that $J$ is a $U$-compact modular metric relation, and that their left Kan extension $l: * \rightarrow[0, \infty]$ is given by (see Proposition 2.2)

$$
l(*)=\min \{d(\perp)+J(\perp, *), d(\top)+J(\top, *)\}=\min \{2,1\}=1
$$

On the other hand we have

$$
\begin{aligned}
\max \{(d(\perp) \ominus l(*))+J(\perp, *),(d(\top) \ominus l(*)) & +J(\top, *)\} \\
& =\max \{2 \ominus 1+0,0 \ominus 1+1\}=1>0
\end{aligned}
$$

so that by Example $2.8 l$ fails to satisfy the Beck-Chevalley condition.
The following proposition shows that condition (d) of the previous theorem always holds in the cases $\mathcal{V}=\Delta_{\times}$and $\mathcal{V}=\Delta_{\&}$, the quantale of distance distribution functions equipped with multiplication $\otimes$ given by convolution with respect to either multiplication $\times$ or the Łukasiewicz operation on $[0,1]$; see Example 1.4. Remember that the unit $k:[0, \infty] \rightarrow[0,1]$ of $\Delta_{\&}$ is given by $k(t)=1$ for all $t>0$.

Proposition 8.4. Consider the quantale $\Delta_{\&}$ where either $\&=\times$ or $\&$ is the Eukasiewicz operation, see Example 1.4. For any up-directed set $\Phi \subseteq \Delta_{\&}$ :

$$
\sup _{\phi \in \Phi}(\sup \Phi \multimap \phi)=k
$$

Proof. Let us write $\sigma:=\sup \Phi$. Clearly if $\sigma=0$, the bottom element of $\Delta_{\&}$, then $\Phi=\{0\}$ so that the identity above reduces to $0 \multimap 0=k$ which immediately follows from the definition of - .

Hence we assume $\sigma>0$. For each $u \in(0, \infty)$ and $p \in(0,1)$ let us denote by $\pi_{(u, p)}$ the distance distribution function given by

$$
\pi_{(u, p)}(s):= \begin{cases}0 & \text { if } s \leq u \\ p & \text { if } s>u\end{cases}
$$

Because

$$
\sup _{\substack{u \in(0, \infty) \\ p \in(0,1)}} \pi_{(u, p)}=k \quad \text { and } \quad \sigma \multimap \psi=\sup \left\{\chi \in \Delta_{\&} \mid \sigma \otimes \chi \leq \psi\right\}
$$

(see Example 1.8), it suffices to prove that for every $u \in(0, \infty)$ and every $p \in(0,1)$ there is a $\psi \in \Phi$ with

$$
\sigma \otimes \pi_{(u, p)} \leq \psi
$$

Unpacking the convolution product $\otimes$ and using the left-continuity of $\sigma$, we find that this means

$$
\begin{equation*}
\sigma(s \ominus u) \& p \leq \psi(s) \tag{19}
\end{equation*}
$$

for all $s \in[0, \infty]$, where $\ominus$ denotes truncated difference.
To show that we can find such $\psi \in \Phi$ for any $u \in(0, \infty)$ and $p \in(0,1)$ we write

$$
B:=\sigma(\infty)=\sup _{s \in(0, \infty)} \sigma(s)>0
$$

and, for each $n \geq 1$,

$$
t_{n}=\inf \left\{s \in[0, \infty] \mid \sigma(s \ominus u / 2)>B \&^{n} p\right\}
$$

where

$$
B \&{ }^{n} p:=\overbrace{(\cdots((B \& p) \& p) \cdots \& p)}^{n \text { times }} .
$$

As $u / 2>0$ we have $\sigma\left(t_{n}\right)>B \&^{n} p$ so that, for each $n \geq 1$, there must be some $\phi_{n}:=\phi \in \Phi$ with $\phi\left(t_{n}\right)>B \&^{n} p$.

Now for $n=1$ we have for all $s \in\left[t_{1}, \infty\right]$ :

$$
\phi_{1}(s) \geq \phi_{1}\left(t_{1}\right)>B \& p \geq \sigma(s \ominus u) \& p
$$

where the last inequality follows from the definition of $B$. By definition of $t_{n}$ we have $\sigma(s \ominus u / 2) \leq B \&^{n} p$ for all $s<t_{n}$ so that, for all $n \geq 1$ and all $s \in\left[t_{n+1}, t_{n}\right)$ we have:

$$
\phi_{n+1}(s) \geq \phi_{n+1}\left(t_{n+1}\right)>B \& \&^{n+1} p \geq \sigma(s \ominus u / 2) \& p \geq \sigma(s \ominus u) \& p
$$

Finally, below we will show that in both cases of \& there is an integer $N \geq 1$ such that

$$
\phi_{N}(s) \geq \sigma(s \ominus u) \& p
$$

for all $s \in\left[0, t_{N}\right]$. We have thus found a finite number of distance distribution functions $\phi_{1}, \ldots, \phi_{N}$ in $\Phi$ such that for each $s \in[0, \infty]$ there is some $\phi_{i}$ with

$$
\phi_{i}(s) \geq \sigma(s \ominus u) \& p
$$

As $\Phi$ is up-directed it contains an upper bound $\psi$ of $\left\{\phi_{1}, \ldots, \phi_{N}\right\}$. From the above it follows that $\psi$ satisfies (19) for all $s \in[0, \infty]$, thus concluding the proof.

It remains to show the existence of $N$. In the case that \& is the Łukasiewicz operation, we take $N$ to be the minimal $n$ such that $B \& \&^{n} p=0$. Then $\sigma\left(t_{N}\right)=0$ by the left-continuity of $\sigma$ so that for all $s \in\left[0, t_{N}\right]$ :

$$
\phi_{N}(s) \geq 0=\sigma\left(t_{N}\right) \& p \geq \sigma(s \ominus u) \& p
$$

In the case that $\&=\times$ we define $l \in[0, \infty)$ and $N \geq 1$ by
$l:=\max \{s \in[0, \infty] \mid \sigma(s)=0\} \quad$ and $\quad N:=\min \left\{n \geq 1 \mid B \times p^{n}<\sigma(l+u / 2)\right\} ;$
that these extrema exist follows from the left-continuity of $\sigma$ and the fact that $\sigma(l+u / 2)>0$. By definition of $t_{N}$ we have $t_{N} \leq l+u$. Hence for all $s \in\left[0, t_{N}\right]$ we have $s \ominus u \leq l$ so that $\sigma(s \ominus u)=0$. Thus

$$
\phi_{N}(s) \geq 0=\sigma(s \ominus u) \times p
$$

Example 8.5. In the case of $\mathcal{V}=\Delta_{\min }$ (see Example 1.4), the assertion of the previous proposition is false in general. To see this consider the up-directed set $\Phi \subset \Delta_{\text {min }}$ consisting of the distance distribution functions

$$
\phi_{i}:[0, \infty] \rightarrow[0,1]: \phi_{i}(t)= \begin{cases}t & \text { if } t \leq i \\ i & \text { if } t>i\end{cases}
$$

where $i$ ranges over the real numbers in $\left(0, \frac{1}{2}\right)$, so that $\sup \Phi=\phi_{\frac{1}{2}}$. We claim that the assertion of the previous proposition fails, that is $\sup _{i \in\left(0, \frac{1}{2}\right)}\left(\phi_{\frac{1}{2}} \multimap \phi_{i}\right)<k$, where $k$ is the unit of $\Delta_{\text {min }}$. To see this remember that (see Example 1.8)

$$
\begin{equation*}
\phi_{\frac{1}{2}} \multimap \phi_{i}=\sup \left\{\chi \in \Delta_{\min } \left\lvert\, \phi_{\frac{1}{2}} \otimes \chi \leq \phi_{i}\right.\right\} \tag{20}
\end{equation*}
$$

For each $\chi \in \Delta_{\text {min }}$ with $\phi_{\frac{1}{2}} \otimes \chi \leq \phi_{i}$ we have

$$
i=\phi_{i}(\infty) \geq\left(\phi_{\frac{1}{2}} \otimes \chi\right)(\infty)=\min \left\{\phi_{\frac{1}{2}}(\infty), \chi(\infty)\right\}=\min \left\{\frac{1}{2}, \chi(\infty)\right\}
$$

where the second equality follows easily from the definition of the convolution product $\otimes$, see Example 1.4 Since $i<\frac{1}{2}$ we conclude that $\chi(\infty) \leq i<\frac{1}{2}$ for all $\chi$ in $(20)$. Because $k(\infty)=1$ it follows that $\sup _{i \in\left(0, \frac{1}{2}\right)}\left(\phi_{\frac{1}{2}} \multimap \phi_{i}\right)<k$, as claimed.

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