Geometric Invariant Theory and Einstein-Weyl Geometry

Mustafa Kalafat

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Abstract

In this article, we review the Geometric Invariant Theory for Toric Varieties and present an application of it to Einstein-Weyl Geometry

1 Introduction

In §2 we review the basics of Einstein-Weyl geometry and minitwistorspaces. In §3 and §4 we give a review of GIT and Toric Varieties, finally in §5 we present our application.

Acknowledgments. I want to thank to Claude LeBrun for his directions, and Alastair Craw for teaching me the GIT.

2 Einstein-Weyl Geometry

Let (M,g) be a self-dual Riemannian 4-manifold. Suppose it admits a free isometric circle(S^1) action. Then the quotient manifold M/S^1 is naturally equipped with a so-called *Einstein-Weyl Geometry*. That is to say we have a triple $(M/S^1, [h], D)$ where $[h]$ is a conformal class, here for the induced metric of the quotient, and D is a torsion-free affine connection. The conditions

$$Ric_{(ij)} = \lambda h \qquad \text{(Einstein-like)}$$

more precisely $Ric(u,v) + Ric(v,u) = 2\lambda h(u,v)$, and

$$Dh = \alpha \otimes h \text{ for some 1-form } \alpha \qquad \text{(Weyl Connection)}$$

are to be satisfied. More interestingly, this action can naturally be extended to a holomorphic $\mathbb{C}^*$-action over the twistor space. And we call the corresponding quotient $Z/\mathbb{C}^*$ as the $Minitwistorspace$ of the self-dual manifold. If the twistor space is algebraic or Moishezon this quotient space becomes a complex surface with singularities in general.

So, whenever one has a self-dual metric with an isometric circle action, it is a very natural question to ask what is the minitwistor space.

In the march of 2004, Honda gave an explicit description for the twistor space of some self-dual metrics on $3\mathbb{CP}_2$ admitting a free isometric circle action, equivalently a nowhere zero Killing Field as follows :

Theorem 2.1 (Nobuhiro Honda,[Ho04])**.** *Let g be a self-dual metric on $3\mathbb{CP}_2$ which admits a non-trivial Killing Field. Suppose further that it is of positive scalar curvature type,*

and not conformally equivalent to the explicit self-dual metrics constructed by LeBrun's hyperbolic ansatz[Le91].
Then the twistor space is bimeromorphic to(or small resolution of) the double cover of $\mathbb{CP}_3$ *branched along a quartic whose equation according to some homogeneous coordinates is given by*

$$(Z_2Z_3 + Q(Z_0, Z_1))^2 - Z_0Z_1(Z_0 + Z_1)(Z_0 - aZ_1) = 0$$

where $Q(Z_0, Z_1)$ *is a quadratic form of* Z_0 *and* Z_1 *with real coefficients, and* $a \in \mathbb{R}^+$ *moreover, the naturally induced real structure on* $\mathbb{CP}_3$ *is given by*

$$\sigma(Z_0 : Z_1 : Z_2 : Z_3) = \left(\bar{Z}_0 : \bar{Z}_1 : \bar{Z}_3 : \bar{Z}_2\right),$$

and the naturally induced $U(1)$*-action on* $\mathbb{CP}_3$ *is given by*

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto \left(Z_0 : Z_1 : e^{i\theta}Z_2 : e^{-i\theta}Z_3\right) \quad for \quad e^{i\theta} \in U(1).$$

To construct the minitwistor space of the Honda metrics, we appeal to the Geometric Invariant Theory(GIT) for Toric Varieties. This celebrated theory is developed by D. Mumford around 1970's to understand the quotients of group actions on manifolds.

We will try to compute the image under the double branched cover, so that we could be able to recover the original minitwistor space by taking a double cover along the related branch locus. GIT computes the quotients according to some linearizations. It takes out some bad orbits(unstable) and give a toric variety as a result.

3 Action of a torus on an affine space

In this section we will analyze the actions of the group $T = \mathbb{C}^{*r}$ on the affine space $\mathbb{C}^n$ and understand the quotients arisen this way. We call $T = \mathbb{C}^{*r} = (\mathbb{C}^*)^r$ as an *algebraic torus* for each positive integer r. We begin by

Proposition 3.1 ([Do]p73,[Mu]p119)**.** *Any character*

$$\chi : T \longrightarrow \mathbb{C}^*$$

is given by

$$\chi(t) = \chi(t_1 \cdots t_r) = t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r} = \prod_{i=1}^{r} t_i^{a_i}$$

for $t_i \in \mathbb{C}$ *and* $a_i \in \mathbb{Z}$. *So we have the isomorphism* $\quad \chi(T) \approx \mathbb{Z}^r$

Recall that a character χ of a group with values in a field is a homomorphism from the group to the multiplicative group, i.e. satisfying $\chi(gh) = \chi(g)\chi(h)$. Moreover $\chi(T)$ stands for the group of characters of T.

Consequently, after diagonalizing, a T action on $\mathbb{C}^n$ is written as

$$t \cdot \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} \chi_1(t)Z_1 \\ \vdots \\ \chi_n(t)Z_n \end{pmatrix} = \begin{pmatrix} t^{a_1}Z_1 \\ \vdots \\ t^{a_n}Z_n \end{pmatrix} = \begin{pmatrix} t_1^{a_{11}} \cdots t_r^{a_{r1}} Z_1 \\ \vdots \\ t_1^{a_{1n}} \cdots t_r^{a_{rn}} Z_n \end{pmatrix}$$

So the matrix $A = [a_{ij}] \in M_{r\times n}(\mathbb{Z})$ encodes the action.

More generally, let T be a group acting on the complex manifold X by the map $\sigma : T \times X \to X$. For a holomorphic line bundle $\pi : L \to X$, we define

Definition 3.2. *A* linearization *of L with respect to the action of T is an action $\overline{\sigma} : T\times L \to L$ satisfying*
(1) The following diagram commutes

$$\begin{array}{ccc} T\times L & \xrightarrow{\overline{\sigma}} & L \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ T\times X & \xrightarrow{\sigma} & X \end{array}$$

(2) The zero section $X \approx L_0 \subset L$ is T- invariant

So this is the extension of the action σ to L, preserving the fibers, i.e. all point on a fiber maps onto the same fiber under the action of an element. It follows from the definition that this action on a fiber $\overline{\sigma}_t : L_p \to L_{tp}$ for any $t \in T$ and any $p \in X$ is a linear isomorphism.

In our case, the action of $\mathbb{C}^{r*}$ on $\mathbb{C}^n$ is given by the matrix $A = (a_1 \cdots a_r) \in M_{n\times r}(\mathbb{Z})$. Consider the trivial line bundle $\underline{\mathbb{C}} \to \mathbb{C}^{n*}$. Fix $\alpha = (\alpha_1 \cdots \alpha_r) \in \mathbb{Z}^r$. Extend the action over to the bundle $\underline{\mathbb{C}}$ as

$$t\cdot(Z,W) = (t\cdot Z, t^{\alpha}W) = (t\cdot Z, t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_r^{\alpha_r}W) \ \text{ where } \ Z\in\mathbb{C}^n, W\in\mathbb{C}$$

We denote this linearized line bundle by L_α. So any $a \in \mathbb{Z}^r$ gives an extension or a linearization.

Recall that the sections of the trivial line bundle are identified with the polynomials $F \in \mathbb{C}[Z_1\cdots Z_n]$, like the homogenous polynomials for bundles over $\mathbb{P}^n$. A section F is an invariant section of L_α if

$$t\cdot(Z,F(Z)) = (t\cdot Z, t^{\alpha}\cdot F(Z)) = (t\cdot Z, F(t\cdot Z))$$

$$\text{i.e. } \ t^{\alpha}\cdot F(Z) = F(t\cdot Z)$$

$$t_1^{\alpha_1}\cdots t_r^{\alpha_r}F(Z_1\cdots Z_r) = F(t^{a_1}Z_1\cdots t^{a_r}Z_r)$$

The action of $\overline{\sigma}$ on L induces an action on $L^{\otimes d}$ as for a decomposable $l \in L_p$, $\overline{\sigma}_t(l) = \overline{\sigma}_t(l_1\otimes\cdots\otimes l_d) = \overline{\sigma}_t(l_1)\otimes\cdots\otimes\overline{\sigma}_t(l_d) \in L_{tp}$

Likewise, G is an invariant section of $L_\alpha^{\otimes d}$ if for $G = F_1\cdots F_d$

$$\begin{aligned} G(t\cdot Z) &= F_1(t\cdot Z)\cdots F_d(t\cdot Z) = (t^{\alpha}\cdot F_1)\cdots(t^{\alpha}\cdot F_d) \\ &= t^{\alpha d}\cdot F_1\cdots F_d = t^{\alpha d}\cdot G(Z) \end{aligned}$$

Proposition 3.3. *$G \in H^0(\mathbb{C}^n, L_\alpha^{\otimes d})^T$ i.e. G is an invariant section of the linearized line bundle $L_\alpha^{\otimes d}$ iff it is a linear combination of monomials $Z^m (= Z_1^{m_1}\cdots Z_n^{m_n})$ such that*

$$[A \ , -\alpha\ ]\begin{bmatrix} m \\ d \end{bmatrix} = \Theta$$

where $A \in M_{r\times n}(\mathbb{Z})$ is the action matrix, $\alpha \in \mathbb{Z}^r$ is the tuple for the extension

Proof. Say $G = Z^m$, then

$$\begin{aligned}
G(t\cdot Z) &= t^{\alpha d}\cdot G(Z)\\
G(t^{a_1}Z_1\cdots t^{a_n}Z_n) &= (t_1^{\alpha_1}\cdots t_n^{\alpha_n})^d Z^m\\
(t^{a_1}Z_1)^{m_1}\cdots(t^{a_n}Z_n)^{m_n} &= \quad ''\\
t^{a_1m_1}\cdots t^{a_nm_n}Z^m &= \quad ''\\
(t_1^{a_{11}}\cdot\cdot t_r^{a_{r1}})^{m_1}\cdots(t_1^{a_{1n}}\cdot\cdot t_r^{a_{rn}})^{m_n}Z^m &= \quad ''
\end{aligned}$$

comparing the powers of t_i's from both sides we obtain the equality

$$\begin{aligned}
a_{i1}m_1+\cdots+a_{in}m_n &= \alpha_i d\\
[a_{i1}\cdots a_{in}]\begin{bmatrix} m_1\\ \vdots\\ m_n\end{bmatrix} &= [\alpha_i]d \quad \text{for any} \quad 1\le i\le r\\
Am &= \alpha d
\end{aligned}$$

□

Example 3.4. *Consider the following action of $\mathbb{C}^{*2}$ on $\mathbb{C}^n$:*

$$(t_1,t_2)\cdot\begin{pmatrix} X\\ Y\\ Z\\ W\end{pmatrix} = \begin{pmatrix} t_1X\\ t_1^{-n}t_2Y\\ t_1Z\\ t_2W\end{pmatrix} \quad , \quad \alpha = \begin{pmatrix} 1\\ 1\end{pmatrix}$$

The action matrix is $A = \begin{bmatrix} 1 & -n & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$ *and the monomials for the invariant sections are obtained from the equation* $\left[\begin{array}{cccc|c} 1 & -n & 1 & 0 & -1\\ 0 & 1 & 0 & 1 & -1\end{array}\right]\begin{bmatrix} m_1\\ m_2\\ m_3\\ m_4\\ d\end{bmatrix} = \Theta$

Next we are going to give some definitions in the Geometric Invariant Theory(GIT). GIT deals with the actions of groups on manifolds, and figuring out their corresponding quotients.

Definition 3.5 (Stability[Do]p115)**.** *Let L be a T-linearized line bundle on the algebraic variety X and let $x \in X$*
(i) x is called semi-stable *with respect to L if it belongs to the set $X_s = \{y \in X : s(y) \neq 0\} = X\backslash\{s = 0\} \subset \mathbb{C}^n$ (affine) for some $m > 0$ and $s \in H^0(X, L^m)^T$.*
(ii) x is called unstable *with respect to L if it is not semi-stable.*

$X^{ss}(L)$ and $X^{us}(L)$ denotes respectively the set of semi-stable and unstable points in X.

Definition 3.6 (Categorical Quotient[Do]p92)**.** *A* categorical quotient *of a T-variety X is a T-invariant morphism $p : X \to Y$ such that for any T-invariant morphism $g : X \to Z$, there exist a unique morphism $\bar{g} : Y \to Z$ satisfying $\bar{g} \circ p = g$. Y is written sometimes as $X /\!\!/ T$ and also called the* categorical quotient *by the abuse of terminology.*

The GIT guarantees a (good) categorical quotient $X^{ss}(L_\alpha)/T$, see [Do]p118, denoted alternatively by $X(L) /\!\!/_\alpha T$. This is the quotient obtained by taking out the unstable orbits. So according to the GIT, semi-stable points has this well behaving quotient computable as follows.

Proposition 3.7 ([Do]p120)**.** *If X is projective and L is ample, we can compute the categorical quotient by*

$$X(L) /\!\!/_\alpha T = Proj\left(\bigoplus_{d \geq 0} \Gamma(X, L_\alpha^{\otimes d})^T\right)$$

4 Toric Varieties

We begin by the following definition

Definition 4.1. *Let $V \subset \mathbb{C}^n$ be an affine variety. We define its* Coordinate Ring *to be*

$$\mathbb{C}[V] = \mathbb{C}[z_1 \cdots z_n]|_V$$

This is to say the coordinate ring is the ring of *regular* functions according to the terminology of [Sha]p24. If we look at the restriction map

$$restr : \mathbb{C}[z_1 \cdots z_n] \longrightarrow \mathbb{C}[z_1 \cdots z_n]|_V$$

we see that its kernel is equal to I_V, the vanishing ideal of V. So the coordinate ring becomes

$$\mathbb{C}[V] = \mathbb{C}[z_1 \cdots z_n]/I_V$$

For any ring R define its *maximal spectrum* by

$$Specm(R) = \{I < R : I \text{ is a maximal ideal }\}$$

For any affine variety $V \subset \mathbb{C}^n$, defining the Zariski Topology on each side we have the homeomorphism $V \approx Specm(\mathbb{C}[V])$ between an affine variety and its coordinate ring. As the trivial case, $\mathbb{C}^n \approx Specm\mathbb{C}[z_1 \cdots z_n]$, where a point $a \in \mathbb{C}^n$ corresponds to its vanishing ideal $I_{\{a\}} = \mathbb{C}[z](z_1 - a_1) + \cdots + \mathbb{C}[z](z_n - a_n) = (z_1 - a_1, \cdots, z_n - a_n)$. The maximal ideals of the latter type consumes the maximal ideals of the polynomial ring $\mathbb{C}[z_1 \cdots z_n]$[Mu]p82 which is referred as the Weak Nullstellensatz in the literature [JPB]p59. The full spectrum is the larger space of prime ideals with which we do not deal here.

We first go into the definition of an affine toric variety. For that purpose we take a cone $\sigma \in \mathbb{R}^n$ satisfying the conditions of the following definition for the canonical lattice $N \approx \mathbb{Z}^n \subset \mathbb{R}^n$

Definition 4.2 (polyhedral,lattice,strongly convex). *Let* $A = \{x_1 \cdots x_n\} \subset \mathbb{R}^n$ *be a finite set of vectors. Then A cone* σ *is called*

- polyhedral *if it is of the form* $\{x \in \mathbb{R}^n : x = \lambda_1 x_1 \cdots \lambda_r x_r, \lambda_i \geq 0 \quad \text{and real}\}$
- lattice cone *if all the vectors* $x_i \in A$ *belong to* N
- strongly convex *if it does not contain any straight line going through the origin, i.e.* $\sigma \cap -\sigma = \{0\}$

then we define the *affine toric variety* corresponding to σ as

$$U_\sigma := Specm\mathbb{C}[\check{\sigma} \cap N^*]$$

where the duals are defined to be $\check{\sigma} = \{u \in \mathbb{R}^n : \langle u, \sigma \rangle \geq 0\}$ and $N^* = Hom_{\mathbb{Z}}(N, \mathbb{Z})$. One can abuse the notation and show it as $Spec\mathbb{C}[\check{\sigma}]$.

Similar to the way that the cones correspond to an affine toric variety, some collection of cones called fans correspond to a toric variety. More precisely

Definition 4.3 (Fan,[JPB]). *A* fan Δ *is a finite union of cones such that*

- *the cones are polyhedral,lattice and strongly convex*
- *every face of a cone of* Δ *is again a cone of* Δ
- $\sigma \cap \sigma'$ *is a common face of the cones* σ *and* σ' *in* Δ

Now for a fan Δ in N, we can naturally glue $\{U_\sigma : \sigma \in \Delta\}$ together to obtain a Hausdorff complex analytic space

$$X_\Delta := \bigcup_{\sigma \in \Delta} U_\sigma$$

which is irreducible and normal with dimension equal to $rk(N)$ and called the *Toric Variety*[Oda] associated to the fan (N, Δ). Topologically endowed with an open covering by the affine toric varieties $U_\sigma = Specm\mathbb{C}[\check{\sigma}]$.

Summarizing what we did in high brow terms [Do]$p189$: we constructed the $U_\sigma = Specm\mathbb{C}[\check{\sigma} \cap N^*]$ as the affine variety with $\mathcal{O}(U_\sigma)$ isomorphic to $C[\check{\sigma} \cap N^*]$. Since for any $\sigma, \sigma' \in \Delta$, $\sigma \cap \sigma'$ is a face in both cones, we obtain that $\mathbb{C}[(\check{\sigma \cap \sigma'}) \cap N^*]$ is a localization of each algebra $C[\check{\sigma} \cap N^*]$ and $C[\check{\sigma'} \cap N^*]$. This shows that $Specm\mathbb{C}[(\check{\sigma \cap \sigma'}) \cap N^*]$ is isomorphic to an open subset of U_σ and $U_{\sigma'}$. Which allows us to glue together the varieties U_σ's to obtain the toric variety X_Δ

By definition, X_Δ has a cover by open affine subsets U_σ. Since each algebra $\mathbb{C}[\sigma \cap N^*]$ is a subalgebra of $\mathbb{C}[N^*] \approx \mathbb{C}[Z_1^{\pm 1} \cdots Z_n^{\pm 1}]$ we obtain a morphism $\mathbb{C}^n \to X_\Delta$, which is $\mathbb{C}^n$ equivariant for the action of $\mathbb{C}^n$ on itself by left translations and on X_Δ by means of the $\mathbb{Z}^n$-grading of each algebra $\mathbb{C}[\sigma \cap N^*]$. If no cone $\sigma \in \Delta$ contains a linear subspace, which is the case, the morphism $\mathbb{C}^n \to X_\Delta$ is an isomorphism onto an open orbit. In general, X_Δ always contains an open orbit isomorphic to a factor group of $\mathbb{C}^n$. All toric varieties X_Δ are normal and rational. So we obtain

Theorem 4.4 ([Do]p189). *Let Δ be the N-fan formed by the cones $\sigma_j, j = 1\cdots s$. Then*

$$\mathbb{C}^n(L)/\!\!/_\alpha T = (\mathbb{C}^n)^{ss}(L_\alpha)/T \approx X_\Delta$$

Example 4.5. *The weighted projective space $\mathbb{P}_{1,1,2}$ is by definition the quotient of $\mathbb{C}^3 - 0$ by the $\mathbb{C}^*$-action given by the matrix $A = [1, 1, 2]$*
If we linearize the trivial bundle over $\mathbb{C}^3$ by $\alpha = 2$, the linear system $Am = \alpha$ is just $a + b + 2c = 2$, and nonnegative solutions for the triple (a, b, c) are generated by

$$(2, 0, 0) \quad (1, 1, 0) \quad (0, 2, 0) \quad (0, 0, 1)$$

so that the coordinate rings are obtained as

$$\begin{aligned}
&\mathbb{C}[\mathbb{N}^4 \cap \pi^{-1}(2)] = \mathbb{C}[X^2, XY, Y^2, Z]\\
&\mathbb{C}[U_1/\mathbb{C}^*] = \mathbb{C}[1, \tfrac{Y}{X}, \tfrac{Y^2}{X^2}, \tfrac{Z}{X^2}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{Z}{X^2}] = \mathbb{C}[a, b]\\
&\mathbb{C}[U_2/\mathbb{C}^*] = \mathbb{C}[\tfrac{X}{Y}, 1, \tfrac{Y}{X}, \tfrac{Z}{XY}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Y}{X}, \tfrac{Z}{XY}] = \mathbb{C}[a^{-1}, a, a^{-1}b]\\
&\mathbb{C}[U_3/\mathbb{C}^*] = \mathbb{C}[\tfrac{X^2}{Y^2}, \tfrac{X}{Y}, 1, \tfrac{Z}{Y^2}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Z}{Y^2}] = \mathbb{C}[a^{-1}, ba^{-2}]\\
&\mathbb{C}[U_4/\mathbb{C}^*] = \mathbb{C}[\tfrac{X^2}{Z}, \tfrac{XY}{Z}, \tfrac{Y^2}{Z}, 1] = \mathbb{C}[\tfrac{X^2}{Z}, \tfrac{XY}{Z}, \tfrac{Y^2}{Z}] = \mathbb{C}[b^{-1}, ab^{-1}, a^2b^{-1}]
\end{aligned}$$

if we assign $a = \frac{Y}{X}$ and $b = \frac{Z}{X^2}$.
Then since

$$\begin{gathered}
\bigcup_{i=1}^4 U_i = \mathbb{C}^3\backslash\{\{X^2 = 0\} \cap \{XY = 0\} \cap \{Y^2 = 0\} \cap \{Z = 0\}\} =\\
\mathbb{C}^3\backslash\{\{X = 0\} \cap \{XY = 0\} \cap \{Y = 0\} \cap \{Z = 0\}\} = \mathbb{C}^3\backslash\{X = Y = Z = 0\}
\end{gathered}$$

these are the coordinate rings of the stated weighted projective space.

5 Minitwistor Space

The image of the Honda Minitwistor space (2.1) is the quotient of $\mathbb{CP}_3$ by the $\mathbb{C}^*$ action

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto \left(Z_0 : Z_1 : \lambda Z_2 : \lambda^{-1} Z_3\right) \quad for \quad \lambda \in \mathbb{C}^*$$

On the other hand, to obtain $\mathbb{CP}_3$, we already have the classical $\mathbb{C}^*$ action

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto (\lambda Z_0 : \lambda Z_1 : \lambda Z_2 : \lambda Z_3) \quad for \quad \lambda \in \mathbb{C}^*.$$

Combining the two, the image equals to the quotient of the $\mathbb{C}^{*2}$ action by the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

on $\mathbb{C}^4$. Now extend this action canonically to the trivial line bundle over $\mathbb{C}^4$. Among all the linearization, one of them proves to have minimal number of unstable orbits :

Theorem 5.1. *The categorical quotient of* $\mathbb{C}^4$ *under the* $\mathbb{C}^{*2}$ *action*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

alternatively, the categorical quotient of $\mathbb{CP}_3$ *under the* $\mathbb{C}^*$ *action* $[\ 0 \quad 0 \quad 1 \quad -1\ ]$ *linearized by* $\alpha = (3, 1)$ *is the weighted projective space* $\mathbb{P}_{1,1,2}$

Proof. The linear system $Am = \alpha$ is

$$\left.\begin{array}{rcl} a+b+c+d & = & 3 \\ c-d & = & 1 \end{array}\right\} \quad or \quad \left\{\begin{array}{rcl} a+b+2d & = & 2 \\ c & = & d+1 \end{array}\right.$$

looking for nonnegative solutions, $1, 0$ are the only possibilities for d

$$\begin{array}{ll} d=1 : a=b=0, c=2 \ \text{ yields the solution} & (0 \quad 0 \quad 2 \quad 1) \\ & (2 \quad 0 \quad 1 \quad 0) \\ d=0 : a+b=2, c=1 \ \text{ yields the solutions} & (1 \quad 1 \quad 1 \quad 0) \\ & (0 \quad 2 \quad 1 \quad 0) \end{array}$$

the coordinate rings are

$$\mathbb{C}[\mathbb{N}^4 \cap \pi^{-1}(3,1)] = \mathbb{C}[X^2Z, XYZ, Y^2Z, Z^2W]$$

$$\mathbb{C}[U_1/\mathbb{C}^{*2}] = \mathbb{C}[1, \tfrac{XYZ}{X^2Z}, \tfrac{Y^2Z}{X^2Z}, \tfrac{Z^2W}{X^2Z}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{Y^2}{X^2}, \tfrac{ZW}{X^2}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{ZW}{X^2}]$$

$$\mathbb{C}[U_2/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2Z}{XYZ}, 1, \tfrac{Y^2Z}{XYZ}, \tfrac{Z^2W}{XYZ}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Y}{X}, \tfrac{ZW}{XY}]$$

$$\mathbb{C}[U_3/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2Z}{Y^2Z}, \tfrac{XYZ}{Y^2Z}, 1, \tfrac{Z^2W}{Y^2Z}] = \mathbb{C}[\tfrac{X^2}{Y^2}, \tfrac{X}{Y}, \tfrac{ZW}{Y^2}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{ZW}{Y^2}]$$

$$\mathbb{C}[U_4/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2Z}{Z^2W}, \tfrac{XYZ}{Z^2W}, \tfrac{Y^2Z}{Z^2W}, 1] = \mathbb{C}[\tfrac{X^2}{ZW}, \tfrac{XY}{ZW}, \tfrac{Y^2}{ZW}]$$

and these coordinate rings are isomorphic to the ones for the $\mathbb{P}_{1,1,2}$ as in (4.5). Realize the isomorphism by assigning $a = \frac{Y}{X}, b = \frac{ZW}{X^2}$ so that the coordinate rings respectively becomes

$$\mathbb{C}[a,b]\ ,\ \mathbb{C}[a, a^{-1}, a^{-1}b]\ ,\ \mathbb{C}[a^{-1}, a^{-2}b]\ ,\ \mathbb{C}[b^{-1}, ab^{-1}, a^2b^{-1}]$$

□

Since

$$\bigcup_{i=1}^{4} U_i = \mathbb{C}^4 \backslash \{\{X^2Z=0\} \cap \{XYZ=0\} \cap \{Y^2Z=0\} \cap \{Z^2W=0\}\} = \mathbb{C}^4 \backslash \{Z=0\}$$

so the orbits lying entirely on the hyperplane $\{Z=0\} \subset \mathbb{CP}_3$ are not counted under the $\mathbb{C}^*$ action.

Remark 5.2. *If we take* $\alpha = (1, 1)$ *to be the linearization, the quotient reduces to be a* $\mathbb{P}^1$ *as follows.*

$$[A|\alpha] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right] \approx \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

produces the solution space $S = \langle(-1, 1, 1, 0), (-2, 0, 2, 1)\rangle$*, so the coordinate rings are*

$$\mathbb{C}[\mathbb{N}^4 \cap \pi^{-1}(1,1)] = \mathbb{C}[X^{-1}YZ, X^{-2}Z^2T]$$
$$\mathbb{C}[U_1/\mathbb{C}^{*2}] = \mathbb{C}[1, \tfrac{X^{-2}Z^2T}{X^{-1}YZ}] = \mathbb{C}[1, X^{-1}Y^{-1}ZT] = \mathbb{C}[\beta]$$
$$\mathbb{C}[U_2/\mathbb{C}^{*2}] = \mathbb{C}[XYZ^{-1}T^{-1}, 1] = \mathbb{C}[\beta^{-1}]$$

Department of Mathematics, State University of New York, Stonybrook
E-mail address : kalafat@math.sunysb.edu

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